

PREFACE

The ever-increasing demands for greater measurement accuracy require the careful investigation of effects previously treated heuristically. What would have been considered hairsplitting yesterday is expected today and may be taken for granted tomorrow. An example is the delay calibration problem, which is the subject of this little monograph.

For a full appreciation of the theory presented, the reader should be familiar with probability and statistics at least on the level of the book by Freund [1]. The practical application of the statistical methods described requires the extensive use of computer programs. The book by Afifi and Azen [2] provides a valuable exposition on the selection and interpretation of "packaged" statistical programs.

I have attempted to describe the measurement methods in a general way so that many variations can be included in the same general framework. The mathematical models are thought to be the simplest ones adequate to describe the physical reality. Despite this simplicity, the models yield a surprising richness of insight and mathematical content.

The measurement methods, the statistical techniques, and the theoretical results have been thoroughly tested. In this regard, I gratefully acknowledge the assistance of Hans Kuhr, Douglas Wahrenburger, and Dean Watkeys of Locus, Inc. Their contributions include the assembling of instrumentation, taking of data, writing of programs, and the performance of Monte Carlo studies which helped verify the theory.

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METHODS, STATISTICS, AND THEORY OF DELAY CALIBRATION

DIRECT ESTIMATION OF MEAN DELAY

Introduction

In radar and communication systems, it is often desirable to know when a pulse was actually received at the antenna, not just when it emerged from the receiver. Primarily because of temperature fluctuations in the receiving system, the delay through the system is constantly changing. Thus, there is a need for occasional delay calibration measurements.

Even at constant temperature, the delay is a random variable, due to the presence of random noise at the receiver input. Consequently, the calibration procedure must estimate the mean delay. Usually, the time available for calibration limits the number of measurements which can be made to estimate the mean delay. To determine the accuracy possible with limited measurements of the delay is one of the major purposes of this report.

A method of direct measurement of the delay is depicted in Fig. 1. The technique is to measure the time interval between an input mark pulse and the resulting output mark pulse. The input pulse is generated by the data simulator; the output pulse emerges from the receiver. The timing generator generates a time of day represented by a group of bits.

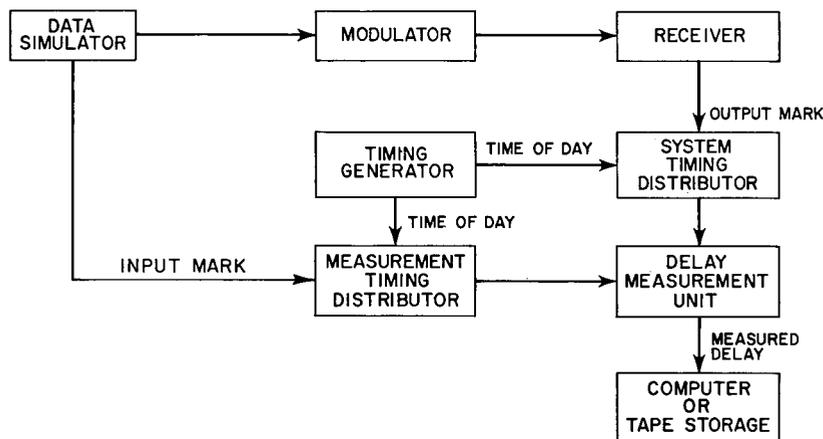


Fig. 1—Direct measurement method for delay calibration

In each timing distributor, a time of day is assigned to the received mark pulse. The difference between the assigned times is computed digitally in the delay measurement unit, the output of which is the measured delay. Computer software uses multiple measurements to estimate the mean delay.

This method is conceptually simple and direct. It can be highly automated, with most of the data recording, computation, and data-presentation functions handled by the computer. However, the very presence of the measuring apparatus disturbs the receiving

system and introduces inaccuracies into the mean-delay estimation. The primary disturbance is due to the quantizations introduced by the timing distributors. The effect of the quantizations shall be thoroughly analyzed in the next section.

Another source of error is the delay caused by the presence of the modulator in the calibration system. The modulator, which is only employed during calibration, generally can be temperature controlled sufficiently to insure that its time of passage is a constant. Thus, the modulator delay need only be measured accurately once. Subsequently, this constant can be subtracted from the measured system delay, either by the computer software or in the delay measurement unit. Thus, the modulator delay shall not be considered explicitly in the subsequent sections of this report.

We shall assume that the timing distributors are not part of the receiving system but are inserted during calibration only. In each distributor, there is an elapsed time between the arrival of the pulse at the input and the instant of time-of-day assignment. If the elapsed times for the two distributors are not equal, a constant bias is introduced into the mean-delay estimation. The preceding discussion concerning the modulator bias applies to this differential distributor bias. Hence, the latter shall not be considered further.

In most receiving systems, the mean delay is sensitive to variations in the signal-to-noise ratio. Thus, this ratio should be held reasonably constant during calibration. In a later section, a statistical test of the sensitivity shall be discussed.

General Theorems for a Single Measurement

We shall define the system delay d as the sum of the modulator and receiver delays, and the differential distributor delay. Let d_m indicate the measured delay, which is defined as the quantity actually measured and sent to the computer or tape storage. As shown in Fig. 2, we can write

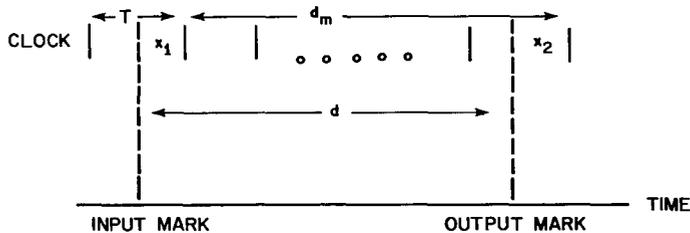


Fig. 2—Timing assignments in the direct measurement of delay

$$d_m = d - x_1 + x_2 \tag{1}$$

where x_1 and x_2 are the quantization errors due to the time tagging of the input and output marks, respectively. Immediately following the occurrence of a mark, the time tagging involves assigning a time of day corresponding to the next leading edge of a clock in the timing generator. It is assumed that x_1 is a random variable with a uniform distribution and is statistically independent of the random variable d . However, examination of

Fig. 2 indicates that x_2 is completely specified by the values of d and x_1 . Specifically, if T is the system clock period,

$$x_2 = T - T \operatorname{rem} \left(\frac{d - x_1}{T} \right), \quad (2)$$

where $\operatorname{rem}(z/T)$ is the remainder obtained when z is divided by T if $z \geq 0$. For example, if $z = 16.3T$, then $\operatorname{rem}(z/T) = 0.3$. When $z < 0$, we define $\operatorname{rem}(z/T) = \operatorname{rem}[(mT + z)/T]$, with m such that $mT + z \geq 0$.

If we define

$$y \equiv d - x_1, \quad (3)$$

it follows that

$$d_m = y + T - T \operatorname{rem} \frac{y}{T}. \quad (4)$$

Since x_1 is uniformly distributed between 0 and T and is statistically independent of d , it can be shown with elementary probability theory that the density function of the random variable y is given by

$$h(y) = \frac{1}{T} \int_0^T f(y + x) dx, \quad (5)$$

where $f(z)$ is the density function of the random variable d . We shall always assume that $f(z)$ is a well-behaved continuous function.

Observe that because of the time-tagging mechanism, the probability is zero that d_m is not equal to a positive integral multiple of the bit period T . From Eq. (4) it is seen that d_m is a function of the single random variable y . It follows from elementary probability theory that

$$P(d_m = nT) = P(nT - T \leq y < nT), \quad (6)$$

where n is a positive integer and $P(\)$ denotes the probability of the event in parentheses. Using Eq. (5) in Eq. (6), we conclude that the probability of the measured delay is given by

$$P(d_m = \tau) = 0 \quad \tau \neq nT$$

$$P(d_m = nT) = \int_{(n-1)T}^{nT} dy \int_0^T dx \frac{1}{T} f(y + x), \quad n = 1, 2, 3, \dots \quad (7)$$

If we regard d_m as an estimate of the expected value of d , the "true" delay, it is important to establish whether the estimate is biased. Let $E(\)$ denote the expected value of the quantity in parentheses. From the definition of expected value and Eq. (7), we

can write

$$E(d_m) = \sum_{n=1}^{\infty} n \int_{(n-1)T}^{nT} dy \int_0^T dx f(y+x). \quad (8)$$

A direct summation of this series for arbitrary $f(z)$ is clearly impossible. We shall instead use an indirect approach based on the expected-value theorem. The following theorem is fundamental in assessing the value of our measurement procedures.

Theorem 1: The expected value of the measured delay, given by equation (1), is equal to the expected value of the system delay; that is,

$$E(d_m) = E(d). \quad (9)$$

Thus, the measured delay is an unbiased estimate of the mean system delay.

Proof. From the expected-value theorem of probability, we can write

$$E(d_m) = \int_{-\infty}^{\infty} h(y) \left[y + T - T \operatorname{rem} \left(\frac{y}{T} \right) \right] dy. \quad (10)$$

Substituting Eq. (5) into Eq. (10), interchanging the order of integration, and performing a change of variable, we obtain

$$E(d_m) = \frac{1}{T} \int_0^T dx \int_{-\infty}^{\infty} dy f(y) \left[y - x + T - T \operatorname{rem} \left(\frac{y-x}{T} \right) \right]. \quad (11)$$

Since $f(y)$ is the density function of the random variable d , we have by definition

$$\int_{-\infty}^{\infty} y f(y) dy = E(d). \quad (12)$$

Combining Eqs. (11) and (12) and interchanging the order of integration, we get

$$E(d_m) = E(d) + \frac{T}{2} - \int_{-\infty}^{\infty} dy f(y) \left[\int_0^T dx \operatorname{rem} \left(\frac{y-x}{T} \right) \right]. \quad (13)$$

To evaluate the factor in brackets, we note that

$$\operatorname{rem} \left(\frac{y-x}{T} \right) = \begin{cases} \operatorname{rem} \left(\frac{y}{T} \right) - \frac{x}{T}, & 0 \leq x \leq T \operatorname{rem} \left(\frac{y}{T} \right); \\ \operatorname{rem} \left(\frac{y}{T} \right) - \frac{x}{T} + 1, & T \operatorname{rem} \left(\frac{y}{T} \right) < x \leq T. \end{cases} \quad (14)$$

Using Eq. (14), it follows that the factor in brackets in Eq. (13) is equal to the constant $T/2$. From the normalization property of density functions, the remaining integration is trivial. Collecting terms, we obtain Eq. (9), and the theorem is proved.

From the definition, Theorem 1, and Eq. (7), we can express the variance of the measured delay in the form

$$\sigma_T^2 = \sum_{n=1}^{\infty} n^2 T \int_{(n-1)T}^{nT} dy \int_0^T dx f(y+x) - [E^2(d)] \quad (15)$$

This equation is the most convenient form for numerical computation once the density function is specified. However, for analytical purposes and to determine approximate formulas, an integral expression is more useful. Furthermore, it is desirable to obtain bounds on the variance for an arbitrary continuous density function. The next theorem provides the needed information. We denote the variance of the system delay by σ^2 .

Theorem 2. The variance of the measured delay can be expressed as

$$\sigma_T^2 = \sigma^2 + T^2 \int_{-\infty}^{\infty} f(y) \left[\text{rem} \left(\frac{y}{T} \right) - \text{rem}^2 \left(\frac{y}{T} \right) \right] dy. \quad (16)$$

For any continuous density function, the following inequality must be satisfied:

$$\sigma^2 \leq \sigma_T^2 \leq \sigma^2 + \frac{T^2}{4}. \quad (17)$$

Proof: From the expected-value theorem, we write

$$E(d_m^2) = \int_{-\infty}^{\infty} h(y) \left[y + T - T \text{rem} \left(\frac{y}{T} \right) \right]^2 dy. \quad (18)$$

Using Eq. (5) and performing a change of variable, we obtain

$$E(d_m^2) = \int_{-\infty}^{\infty} f(y) I(y) dy, \quad (19)$$

where

$$I(y) = \frac{1}{T} \int_0^T \left[y - x + T - T \text{rem} \left(\frac{y-x}{T} \right) \right]^2 dx. \quad (20)$$

Substituting Eq. (14) into Eq. (20) and rearranging yields

$$I(y) = \frac{1}{T} \int_0^T \left[y + T - T \operatorname{rem} \left(\frac{y}{T} \right) - Tu \left(x - T \operatorname{rem} \left(\frac{y}{T} \right) \right) \right]^2 dx, \quad (21)$$

where $u(x)$ is the unit step function defined by

$$u(x) = \begin{cases} 0, & x \leq 0; \\ 1, & x > 0. \end{cases} \quad (22)$$

Squaring the term in brackets, simplifying, and noting that $u^2(x) = u(x)$, we have

$$\begin{aligned} I(y) = & \left[y + T - T \operatorname{rem} \left(\frac{y}{T} \right) \right]^2 \\ & + \left[-2y - T + 2T \operatorname{rem} \left(\frac{y}{T} \right) \right] \int_0^T u \left(x - T \operatorname{rem} \left(\frac{y}{T} \right) \right) dx. \end{aligned} \quad (23)$$

After performing the remaining integration and regrouping terms, we are left with

$$I(y) = y^2 + T^2 \operatorname{rem} \left(\frac{y}{T} \right) - T^2 \operatorname{rem}^2 \left(\frac{y}{T} \right). \quad (24)$$

Since $f(y)$ is the density function for the system delay, we know that

$$\int_{-\infty}^{\infty} y^2 f(y) dy = \sigma^2 + E^2(d). \quad (25)$$

Combining Eq. (19), (24), and (25) yields

$$E(d_m^2) = \sigma^2 + E^2(d) + T^2 \int_{-\infty}^{\infty} f(y) \left[\operatorname{rem} \left(\frac{y}{T} \right) - \operatorname{rem}^2 \left(\frac{y}{T} \right) \right] dy. \quad (26)$$

From the definition of σ_T^2 , Theorem 1, and Eq. (26), we obtain the final expression given by Eq. (16). The double inequality of Eq. (17) follows from Eq. (16) by noting that from elementary calculus

$$0 \leq \operatorname{rem} \left(\frac{y}{T} \right) - \operatorname{rem}^2 \left(\frac{y}{T} \right) \leq \frac{1}{4} \quad (27)$$

and employing the normalization property of density functions.

The variance σ_T^2 is a function of the mean system delay. Since it can be changed by inserting a variable delay line into the receiving system, the mean delay may be regarded as an adjustable parameter. We can increase the accuracy of any estimator of the mean

delay by setting the mean delay so that σ_T^2 is near its minimum. To this effect, we develop two theorems which help us to locate the appropriate mean delay, which we shall frequently denote by $\bar{y} = E(d)$ for simplicity. Since we are concerned only with physically realistic situations, we restrict consideration to density functions of the form

$$f(y) = g(|y - \bar{y}|), \quad (28)$$

where $g(|x|)$ is a continuously differentiable function of x .

Theorem 3. The variance of the measured delay is a periodic function of the mean delay. The period is equal to the clock period T.

Proof. Substitution of Eq. (28) into Eq. (16) and a change of variables yields

$$\sigma_T^2 = \sigma^2 + T^2 \int_{-\infty}^{\infty} g(|y|) \left[\text{rem} \left(\frac{y + \bar{y}}{T} \right) - \text{rem}^2 \left(\frac{y + \bar{y}}{T} \right) \right] dy. \quad (29)$$

Clearly, only the factor in brackets is a function of \bar{y} . From the definition of the rem function,

$$\text{rem} \left(\frac{y + \bar{y} + T}{T} \right) = \text{rem} \left(\frac{y + \bar{y}}{T} \right). \quad (30)$$

Thus, the factor in brackets in Eq. (29) is periodic in \bar{y} . It follows that σ_T^2 is periodic in \bar{y} , and the theorem is proved.

Theorem 4. The extrema of the variance of the measured delay occur at the two sets of points

$$\bar{y} = kT, \quad k = 1, 2, 3, \dots,$$

and

$$\bar{y} = kT + \frac{T}{2}, \quad k = 1, 2, 3, \dots. \quad (31)$$

The maxima occur at one of these sets; the minima occur at the other set.

Proof. Under the assumption that $g(|x|)$ is continuously differentiable, from Eqs. (16) and (28) we obtain

$$\frac{\partial \sigma_T^2}{\partial \bar{y}} = T^2 \int_{-\infty}^{\infty} \frac{\partial g(|y - \bar{y}|)}{\partial \bar{y}} \left[\text{rem} \left(\frac{y}{T} \right) - \text{rem}^2 \left(\frac{y}{T} \right) \right] dy. \quad (32)$$

Since $g(|y - \bar{y}|)$ is a symmetric function of $y - \bar{y}$, the partial derivative in Eq. (32) is an antisymmetric function of $y - \bar{y}$. At the sets of points indicated in Eq. (31), the function in brackets in Eq. (32) is a symmetric function of $y - \bar{y}$. Consequently, the integrand in Eq. (32) is antisymmetric, and

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$$\frac{\partial \sigma_T^2}{\partial \bar{y}} = 0 \quad (33)$$

at the points defined by Eq. (31). Thus, these points are the extrema. From Theorem 3 and the continuity of σ_T^2 , one set must be the maxima and the other, the minima.

To determine which set of points in Eq. (31) represents the minima, we can examine the second derivative of the variance. It is often easier to evaluate the variance explicitly at each set and determine the minima by direct comparison.

A special case of great interest occurs when the quantization is gross compared to the system standard deviation or when the signal-to-noise ratio is high. In this case, the integral expression of Theorem 2 can be simplified. We assume

$$\begin{aligned} f(y) &\geq 0, & E(d) - T < y < E(d) + T; \\ f(y) &\approx 0, & \text{otherwise.} \end{aligned} \quad (34)$$

We write $E(d)$ in the form

$$E(d) = K_0 T + R \quad (35)$$

where K_0 is the integral part obtained when $E(d)$ is divided by T , and R is defined by

$$R = T \operatorname{rem} \left[\frac{E(d)}{T} \right]. \quad (36)$$

If we use Eqs. (34) and (35), Theorem 2 yields

$$\sigma_T^2 = \sigma^2 + T^2 \int_{K_0 T + R - T}^{K_0 T + R + T} f(y) \left[\operatorname{rem} \left(\frac{y}{T} \right) - \operatorname{rem}^2 \left(\frac{y}{T} \right) \right] dy. \quad (37)$$

The integral can be divided into three regions. In these regions we have

$$\operatorname{rem} \left(\frac{y}{T} \right) = \begin{cases} \frac{y - K_0 T}{T} + 1, & K_0 T - T + R < y < K_0 T; \\ \frac{y - K_0 T}{T}, & K_0 T \leq y < K_0 T + T; \\ \frac{y - K_0 T}{T} - 1, & K_0 T + T \leq y < K_0 T + T + R. \end{cases} \quad (38)$$

On substitution of Eq. (38) into Eq. (37) and some manipulation, there results

$$\begin{aligned} \sigma_T^2 = & \sigma^2 + T^2 \int_{K_0 T+R-T}^{K_0 T+R+T} f(y) \left[\frac{y - K_0 T}{T} - \left(\frac{y - K_0 T}{T} \right)^2 \right] dy \\ & + T^2 \int_{K_0 T+R-T}^{K_0 T} f(y) \left(\frac{-2y + 2K_0 T}{T} \right) dy \\ & + T^2 \int_{K_0 T+T}^{K_0 T+R+T} f(y) \left(-2 + \frac{2y - 2K_0 T}{T} \right) dy. \end{aligned} \quad (39)$$

In view of Eq. (34), we can extend the limits of the first integral in Eq. (39) to infinity. Making use of Eqs. (12) and (25) and the normalization condition, the first two terms in Eq. (39) can be simplified. From Eq. (28) and a change of variables, the last two integrals in Eq. (39) can also be simplified. Our final result is

$$\sigma_T^2 = R(T - R) + 2T \int_R^T (y - R)g(|y|) dy + 2T \int_{T-R}^T (y - T + R)g(|y|) dy. \quad (40)$$

Note that σ_T^2 is symmetric about $R = T/2$. From the above, it is apparent that

$$\sigma_T^2(R = 0) = 2T \int_0^T yg(|y|) dy. \quad (41)$$

According to Theorems 3 and 4, Eq. (41) gives the value of σ_T^2 at one set of extrema. In view of Eqs. (28) and (34), we can write

$$\sigma_T^2(kT) \approx T E[|y - \bar{y}|], \quad k = 1, 2, 3, \dots \quad (42)$$

As we shall see in the next section, this expression usually gives the minimum variance. Evaluating Eq. (40) at $R = T/2$ and using the inequality in Theorem 2, we obtain the bounds:

$$\frac{T^2}{4} \leq \sigma_T^2 \left(kT + \frac{T}{2} \right) \leq \frac{T^2}{4} + \sigma^2, \quad k = 1, 2, 3, \dots \quad (43)$$

This variance is generally the maximum.

Normally Distributed System Delay

In this section, we obtain formulas for the probability distribution and the variance under the assumption that the system delay is approximately described by a normal density function. Thus, we take

$$f(y) = \frac{1}{\sqrt{2\pi}\sigma} \exp \left[-\frac{(y - \bar{y})^2}{2\sigma^2} \right], \quad (44)$$

where $\bar{y} = E(d)$ and σ^2 is the variance of the system delay. Since the system delay can never take a negative value, we must assume $\bar{y} \gg \sigma$ to use Eq. (44) for all y . Substituting into Eq. (7), we have

$$P(d_m = nT) = \frac{1}{T} \int_{(n-1)T}^{nT} \left[\operatorname{erfc} \left(\frac{y - \bar{y}}{\sigma} \right) - \operatorname{erfc} \left(\frac{T + y - \bar{y}}{\sigma} \right) \right] dy, \quad (45)$$

where

$$\operatorname{erfc}(x) = \frac{1}{\sqrt{2\pi}} \int_x^{\infty} \exp \left(-\frac{x^2}{2} \right) dx \quad (46)$$

is the complementary error function. By a suitable change of variable in each term, Eq. (45) can be written in the form,

$$P(d_m = nT) = \frac{\sigma}{T} \int_{a-T/\sigma}^a \operatorname{erfc} y \, dy - \frac{\sigma}{T} \int_a^{a+T/\sigma} \operatorname{erfc} y \, dy, \quad (47)$$

where for convenience we have defined

$$a = \frac{nT - \bar{y}}{\sigma}. \quad (48)$$

From tabulated integrals, it can be shown that

$$\int \operatorname{erfc} x \, dx = x \operatorname{erfc} x - N(x), \quad (49)$$

where for convenience we have defined the standard normal function,

$$N(x) = \frac{1}{\sqrt{2\pi}} \exp \left(-\frac{x^2}{2} \right) \quad (50)$$

Using Eq. (49) in Eq. (47), we obtain a closed-form expression for the probability distribution of the measured delay when the system delay has a normal density function:

$$\begin{aligned} P(d_m = \tau) &= 0, \quad \tau \neq nT; \\ P(d_m = nT) &= \frac{\sigma}{T} \left\{ 2a \operatorname{erfc} a - 2N(a) + N \left(a + \frac{T}{\sigma} \right) \right. \\ &\quad + N \left(a - \frac{T}{\sigma} \right) - \left(a - \frac{T}{\sigma} \right) \operatorname{erfc} \left(a - \frac{T}{\sigma} \right) \\ &\quad \left. - \left(a + \frac{T}{\sigma} \right) \operatorname{erfc} \left(a + \frac{T}{\sigma} \right) \right\}, \quad n = 1, 2, 3, \dots \quad (51) \end{aligned}$$

In two special cases, approximate closed-form expressions can be obtained for the variance of the measured delay. First consider the case where the quantization is fine. Thus, the system standard deviation is much greater than the clock period, that is, $\sigma \gg T$. We employ the Taylor series expansions valid for small values of b :

$$\begin{aligned} N(a + b) &\approx N(a)[1 - ab]; \\ \operatorname{erfc}(a + b) &\approx \operatorname{erfc} a - bN(a). \end{aligned} \quad (52)$$

Substituting Eq. (52) into Eq. (51) yields the approximate relation

$$P(d_m = nT) = \frac{T}{\sigma} N(a), \quad \sigma \gg T, \quad n = 1, 2, 3, \dots \quad (53)$$

Thus, in this case the limiting form of the envelope of the probability distribution is normal. It follows that

$$\sigma_T^2 \rightarrow \sigma^2, \quad \sigma \gg T, \quad (54)$$

which is intuitively satisfying.

The second special case to be considered occurs when Eq. (34) is applicable. For a normal distribution, the condition $T > 4\sigma$ will be accepted as sufficient for Eq. (34). We can then employ Eq. (40), with $g(y) = N(y)$. From a tabulated integral, it can be shown that

$$\int xN(x) dx = -N(x) \quad (55)$$

Using this equation and omitting terms which are negligible for all values of R , we obtain

$$\begin{aligned} \sigma_T^2 &\approx R(T - R) - 2T(T - R) \operatorname{erfc}\left(\frac{T - R}{\sigma}\right) - 2TR \operatorname{erfc}\left(\frac{R}{\sigma}\right) \\ &+ 2T\sigma \left[N\left(\frac{T - R}{\sigma}\right) + N\left(\frac{R}{\sigma}\right) \right], \quad T > 4\sigma. \end{aligned} \quad (56)$$

Taking the second derivative of this expression with respect to R , we get

$$\frac{\partial^2 \sigma_T^2}{\partial R^2} = \frac{2T}{\sigma} \left[N\left(\frac{T - R}{\sigma}\right) + N\left(\frac{R}{\sigma}\right) \right] - 2. \quad (57)$$

By direct substitution, it follows that σ_T^2 is a minimum when $R = 0$ and a maximum when $R = T/2$. Omitting negligible terms, Eq. (56) and Theorem 4 imply

$$\text{Minima: } \sigma_T^2(kT) \approx \sqrt{\frac{2}{\pi}} T\sigma, \quad k = 1, 2, 3, \dots; \quad (58)$$

$$\text{Maxima: } \sigma_T^2 \left(kT + \frac{T}{2} \right) \approx \frac{T^2}{4} + 4T\sigma N \left(\frac{T}{\sigma} \right) - 2T^2 \operatorname{erfc} \left(\frac{T}{2\sigma} \right), \quad T > 4\sigma,$$

$$k = 1, 2, 3, \dots \quad (59)$$

Of course, Eq. (58) could have been derived directly from Eq. (42).

Multiple Measurements

During the period of calibration, many delay measurements are made. If the mean delay is regarded as an unknown but nonrandom parameter, one might wish to make a maximum-likelihood estimate based on the sample data. However, even if the appropriate probability density function is known, the amount of computation required for the maximum-likelihood estimate is prohibitive. For example, if the system delay is normally distributed, the measured delay has a single sample density function given by Eq. (51). Clearly, the differentiation of the corresponding multiple sample density function and the solution of the likelihood equation is a formidable task.

Under the circumstances, a logical and simple procedure is to use the sample mean to estimate the mean delay. If we let the symbol \bar{d}_m denote the sample mean and d_{mi} denote a sample value of the measured delay, we have

$$\bar{d}_m = \frac{1}{N} \sum_{i=1}^N d_{mi} \quad (60)$$

Looking at Fig. 2, it is seen that the samples can be assumed to be independent if the interval between input marks is much larger than the system clock period T and if the system clock is independent of the data simulator clock. If the temperature and the signal-to-noise ratio are approximately constant during calibration, we can accurately assume that $E(d_{mi}) = E(d_m)$. Under these conditions, it follows from Eq. (60) and Theorem 1 that

$$E(\bar{d}_m) = E(d_m) = E(d) \quad (61)$$

Similarly, we can write the following expression for the variance of the sample mean:

$$\sigma_m^2 = \frac{\sigma_T^2}{N} \quad (62)$$

If the number of sample data points N is large, we can invoke the central limit theorem and assert that the sample mean is normally distributed.

Equation (62) indicates that we should make σ_T^2 as small as possible. With a variable delay line, we can adjust the mean delay so that σ_T^2 is a minimum. If the system delay is normally distributed, it follows from Eqs. (58) and (62) that we can adjust the mean delay so that

$$\sigma_m^2 = \sqrt{\frac{2}{\pi}} \frac{T\sigma}{N}, \quad T > 4\sigma \quad (63)$$

As an example, suppose $T = 100$ ns, $\sigma = 20$ ns, $N = 1000$, and $\bar{d}_m = 10.173$ μ s. With the insertion of a delay of 27 ns into the receiving system, we obtain $\bar{d}_m = 10.2$ μ s, which is an integral multiple of T . Our variance is now approximately a minimum, and from Eq. (63) we know that $\sigma_m \approx 1.3$ ns.

If the probability distribution of the system delay is unknown or possesses unknown parameters, the variance of the measured delay cannot be calculated using Eq. (42). We shall discuss the empirical estimation of the probability distribution in the next section. Another approach to determining σ_T^2 is to use the estimate provided by the sample variance, which is defined by

$$S^2 = \frac{1}{N-1} \sum_{i=1}^N (d_{mi} - \bar{d}_m)^2. \quad (64)$$

Under the same assumptions leading to Eq. (61), we have

$$E(S^2) = \sigma_T^2. \quad (65)$$

Thus, the sample variance provides an unbiased estimate of σ_T^2 . It follows from Eq. (62) that an unbiased estimate of the variance of the sample mean is given by

$$\hat{\sigma}_m^2 = \frac{S^2}{N}, \quad (66)$$

where the circumflex above the symbol on the left-hand side indicates an estimated quantity.

To precisely determine the accuracy provided by the estimate of Eq. (66), we need to know the probability distribution of the measured delay. However, an approximate computation of the variance of the estimate can be accomplished by assuming that d_m is approximately normally distributed. It then follows that $(N-1)S^2/\sigma_T^2$ has a chi-square distribution with $N-1$ degrees of freedom. From this fact and Eq. (66), we obtain the variance of the estimated variance:

$$\text{var}(\hat{\sigma}_m^2) \approx \frac{2\sigma_T^4}{N^2(N-1)}. \quad (67)$$

Another use for the sample variance is in estimating the standard deviation of the system delay, which can often be related theoretically to the signal-to-noise ratio at the receiver input. If the system delay has a normal distribution and the mean delay is appropriately adjusted, Eq. (58) is valid. If the left-hand side of Eq. (58) is estimated by the sample variance, we obtain the following estimate of the standard deviation of the system delay:

$$\hat{\sigma} = \sqrt{\frac{\pi}{2}} \frac{S^2}{T}. \quad (68)$$

The same crude approximation used to obtain Eq. (67) now yields

$$\text{var}(\hat{\sigma}) \approx \frac{\pi\sigma_T^4}{T^2(N-1)} \quad (69)$$

It was mentioned in the first section that the mean delay is sensitive to variations in the signal-to-noise ratio. Thus, we would like to know over what range of signal-to-noise ratio the mean delay can be considered a constant. Suppose that we measure the sample means at various fixed values of signal-to-noise ratio. Let \bar{x}_1 and \bar{x}_2 be the sample means from two random samples of size n_1 and n_2 at two different signal-to-noise levels. Assuming that the two random samples are independent, we test the null hypothesis that the two population means are equal against the alternative that they are not. If n_1 and n_2 are large enough that we can invoke the central limit theorem, the test statistic

$$Z = \frac{\bar{x}_1 - \bar{x}_2}{\sqrt{\frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}}} \quad (70)$$

has a standard normal distribution, where σ_1^2 and σ_2^2 are the population variances. To test the null hypothesis at the confidence level $\alpha = 0.05$, the critical region is $Z \geq 1.96$. When the variances are unknown, we substitute the sample variances S_1^2 and S_2^2 for σ_1^2 and σ_2^2 . We then accept the null hypothesis, if

$$|\bar{x}_1 - \bar{x}_2| \leq 1.96 \sqrt{\frac{S_1^2}{n_1} + \frac{S_2^2}{n_2}} \quad (71)$$

Repeating this test over all pairs in a range of signal-to-noise ratio, we can validate or reject the approximate constancy of the mean delay within the range.

EMPIRICAL DISTRIBUTIONS

Manual Distribution Method

To make use of the formulas and theorems derived in the prior sections, we must have at least some knowledge of the density function of the system delay. For example, Theorem 4 can be invoked only if it can be verified that $f(y)$ is symmetric about the mean, that is, if Eq. (28) can be shown to hold. We shall now examine the empirical estimation of the probability distribution function.

The manual distribution method (MDM) is implemented as shown in Fig. 3. It is assumed that the input marks constitute a stationary ergodic process during the time period of measurement. The two counters are set to count for a fixed time interval. The gate pulse of the digital time delay unit is triggered by the occurrence of the input mark. Throughout the duration of the gate pulse, the high-speed switch will not pass any output marks. Any output marks received after the fall time of the gate pulse will pass through the high-speed switch. Initially the duration of the gate pulse, τ , is set to the smallest value at which the number of events registered on the passed-pulse counter, N_p , is less than the total number of marks, N , registered on the mark

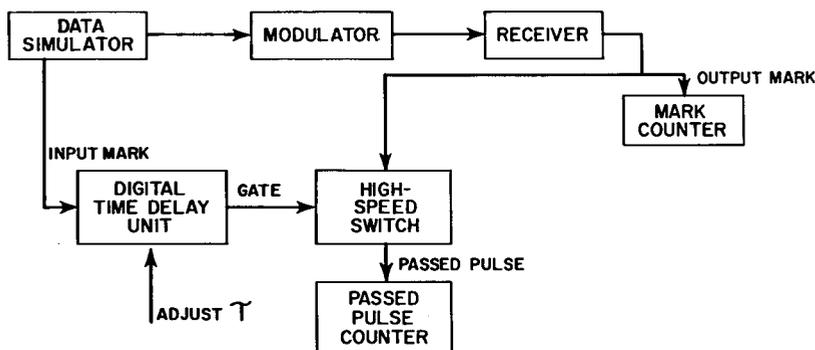


Fig. 3—Instrumentation of the manual distribution method

counter. The setting τ is then stepped in short intervals for successive measurements of N_p . Each measurement of N_p determines a point of the empirical distribution function, given by

$$\hat{F}(\tau) = 1 - \frac{N_p}{N}. \quad (72)$$

In other words, $\hat{F}(\tau)$ is an estimate of the probability that the measured delay is less than τ . Linear extrapolation ordinarily is used to determine $\hat{F}(\tau)$ for values of τ between settings.

The MDM is a time-consuming procedure. To insure ergodicity, temperature fluctuations and the signal-to-noise ratio at the receiving system input must be carefully controlled while the MDM is being employed. Furthermore, the time available for calibration may be insufficient for application of the MDM to an operating system. For these reasons, the technique is useful primarily during design, initial deployment, and malfunction correction.

We shall define the system delay as the delay due to the modulator and receiver. The quantity which actually is compared to the setting τ shall be called the measured delay. The MDM produces an empirical distribution function for the measured delay. An estimate of the mean value of the measured delay is provided by the value τ_0 at which $\hat{F}(\tau_0) = 1/2$. From the empirical distribution function, the corresponding density function can be calculated numerically. A numerical integration can then be performed to derive various moments, such as, the variance, skewness, and kurtosis. We shall examine next the accuracy of the MDM and establish a model for the relation of the measured delay to the system delay.

Because the high-speed switch cannot respond instantaneously to the gate pulse, there is a time lag between the fall time of the gate pulse and the actual time at which the output mark pulses begin to pass through the switch. Consequently, a constant bias is introduced into the measured delay. The rise time of the gate pulse coincides with a clock pulse internal to the digital delay unit. We shall denote this clock period by T . If the clock in the data simulator is nonsynchronous with the digital delay unit clock and if the marks are separated by many clock periods, the uncertainty in the start time of the

gate pulse can be considered a uniformly distributed quantization error, which ranges from 0 to T seconds. Thus, we have established the following model:

$$d_m = d - c + x, \tag{73}$$

where d_m is the measured delay, d is the system delay, c is the bias of the high-speed switch, and x is the quantization error.

Mathematical and Experimental Analysis of the MDM

Since the random variables d and x are independent and x is uniformly distributed, it follows from Eq. (73) that

$$E(d_m) = E(d) - c + \frac{T}{2} \tag{74}$$

and

$$\sigma_T^2 = \sigma^2 + \frac{T^2}{12}, \tag{75}$$

where σ_T^2 is the variance of d_m and σ^2 is the variance of d . Equation (74) tells us how to compute the mean system delay, which is what we are really trying to estimate, from the mean measured delay, which is estimated from the empirical distribution function supplied by the MDM. Equation (75) indicates that, generally speaking, the shape of $\hat{F}(\tau)$ will not be significantly affected by the quantization error if σ is at least several times the value of T .

A test of our model and an evaluation of the bias can be accomplished by the experimental procedure illustrated in Fig. 4. The two pulse generators are activated by the same external clock. Thus, the simulated input and simulated output pulses have a fixed difference in starting times, which we shall call the simulated delay. Because of the common external clock, the simulated delay must be an integral multiple of the clock period. If the estimated mean value of the delay is not an integral multiple of the clock period, the deviation provides a measure of the bias. If the external clock is stable during

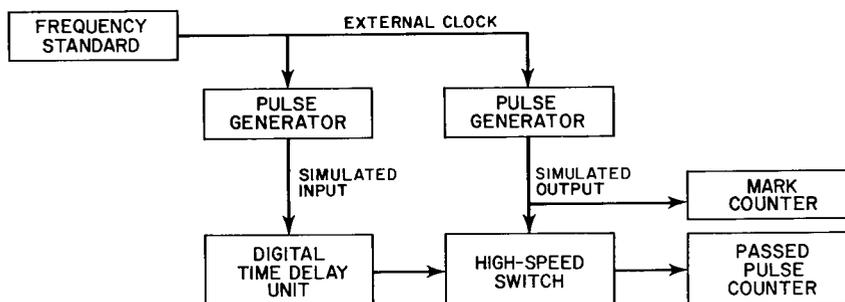


Fig. 4—Model test and bias measurement

the test, the simulated delay will have negligible variance. Thus, $\hat{F}(\tau)$ will show the distribution of the quantization error.

As an example, Fig. 5 shows some actual data taken from a model test using a Tektronics 7D11 Digital Time Delay Unit. This unit has an internal clock with a 2-ns period. The simulated input pulses were separated by 200 μ s, and we let $N = 34,247$. In Fig. 5 the straight line indicates a uniform distribution over a 2-ns interval. The actual data is seen to be quite close to the ideal.

The external clock feeding the pulse generators had a period of 1 μ s, and the simulated delay was 45 μ s. As shown in the figure, the mean measured delay is estimated as 44.9343 μ s. Using Eq. (74), we arrive at the estimated value of the bias as $c = 66.7$ ns.

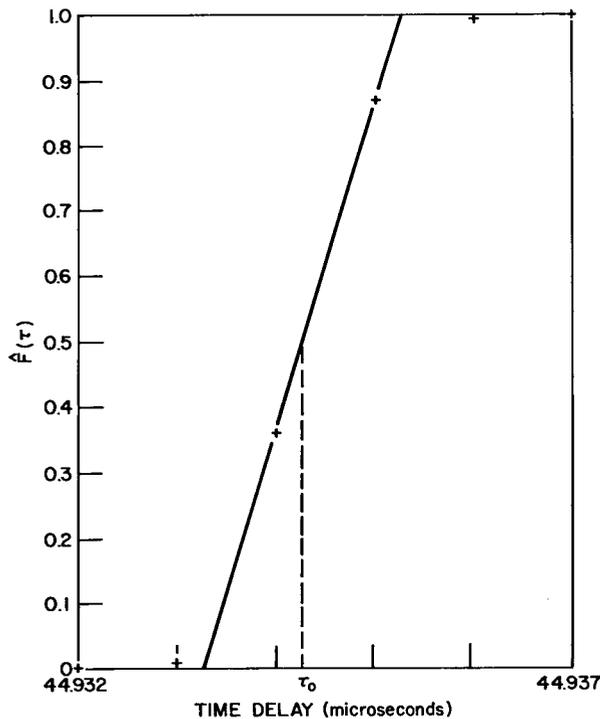


Fig. 5—Data for the MDM model test

For any fixed value of τ , $\hat{F}(\tau)$ is a random variable which can take only rational values. From Eq. (72), the probability of any particular rational number is

$$P\left[\hat{F}(\tau) = \frac{k}{N}\right] = P[N_p = N - k], \quad k = 0, 1, 2, \dots, N, \quad (76)$$

where $P []$ denotes the probability of the event enclosed in the brackets. The number N_p is determined by successive Bernoulli trials of the measured delay. Hence, N_p has a binominal distribution. Letting $F(\tau)$ denote the “true” distribution function of the measured delay, Eq. (76) then yields

$$P \left[\hat{F}(\tau) = \frac{k}{N} \right] = \binom{N}{k} [F(\tau)]^k [1 - F(\tau)]^{N-k} \quad (77)$$

From this relation, we obtain

$$E[\hat{F}(\tau)] = F(\tau) \quad (78)$$

and

$$\sigma_{\hat{F}}^2(\tau) = \frac{F(\tau)[1 - F(\tau)]}{N}, \quad (79)$$

where the left-hand side of Eq. (79) is the variance of $\hat{F}(\tau)$ at a fixed value of τ . Equation (78) states that the MDM provides an unbiased estimate of $F(\tau)$ at each value τ . Equation (79) gives a measure of the deviation of $\hat{F}(\tau)$ with respect to $F(\tau)$. Since $0 \leq F(\tau) \leq 1$, we obtain the upper bound given by

$$\sigma_{\hat{F}}^2(\tau) \leq \frac{1}{4N}, \quad (80)$$

where equality holds at the value τ_1 for which $F(\tau_1) = 1/2$.

We now examine the accuracy with which the mean measured delay can be estimated from the empirical distribution function. A rigorous derivation involves the adaptation of arrival-time estimation theory [3]. The derivation is so complicated that it deserves a report by itself. Hence, we shall content ourselves with an approximate calculation of the variance of the mean delay estimated from $F(\tau)$.

We assume that the resolution and the number of settings of τ are such that, with negligible error, $\hat{F}(\tau_0) = 1/2$ for some setting τ_0 of the digital time delay unit. In other words, we exclude the possibility that $\hat{F}(\tau_0) = 1/2$ for an extrapolated setting. If the number of points N is large, then it follows that, for the setting τ_0 , we have essentially measured the median of the sample. We can then apply the following theorem [1], which holds when the population density $f(y)$ is continuous and nonzero at the population median $\tilde{\mu}$: *For large n , the sampling distribution of the median for random samples of size $2n + 1$ is approximately normal, with the mean $\tilde{\mu}$ and the variance $1/8n[f(\tilde{\mu})]^2$.* If $f(y)$ is symmetric about the population mean $\mu = E(d_m)$, then $\tilde{\mu} = \mu$. Assuming that N is sufficiently large that $N \cong 2n$, we conclude that the estimated mean measured delay is approximately normal, with variance

$$\sigma_{\hat{m}}^2 \cong \frac{1}{4N[f(\mu)]^2}. \quad (81)$$

If $f(\mu)$ is unknown, we can estimate it by numerical differentiation of $\hat{F}(\mu)$. We can test whether two estimated means are equal by using Eq. (71), with the estimated mean in place of \bar{x} and $\sigma_{\hat{m}}^2$ in place of S^2/n .

Nonlinear Regression Analysis

When the "true" distribution function of the measured delay is known, except for some parameters, these parameters can be estimated from the empirical distribution function. If the parameters are expressed in terms of the moments of the density function, they can be estimated by the numerical procedure described in the MDM section. Alternatively, the parameters can be estimated by the least-squares method of nonlinear regression analysis. This method entails the estimation of the parameters by minimizing the sum of squares of deviations of $\hat{F}(\tau_i)$ with respect to $F(\tau_i)$, where the τ_i are the values of the discrete settings.

A more common situation is when one wishes to test a proposed regression model. The parameters of the model can be determined by the numerical procedure or the least-squares method. Then, the fit of the model is tested by an appropriate "goodness of fit" statistic.

The standard goodness of fit tests, such as the chi-square and the Kolmogorov-Smirnov [2], require a data format not supplied by the MDM. Thus special tests must be applied. We shall investigate two different techniques, both of which use the statistics

$$Z_i = \sqrt{N} \frac{\hat{F}(\tau_i) - F(\tau_i)}{\sqrt{F(\tau_i)[1 - F(\tau_i)]}}, \quad i = 1, 2, \dots, n, \quad (82)$$

where n is the number of discrete settings τ_i which are such that $F(\tau_i) \neq 0, 1$. As defined previously, N is the number of samples per fixed setting. It shall be assumed that $\hat{F}(\tau_i)$ is independent of $\hat{F}(\tau_j)$ when $j \neq i$. Thus, the Z_i are independent of each other. If N is sufficiently large, the binomial distribution of $\hat{F}(\tau_i)$ can be approximated by a normal one. From Eqs. (78) and (79), it then follows that the Z_i have standard normal distributions. The first test of the null hypothesis that $F(\tau)$ is the true distribution function is an immediate consequence of these considerations.

Modified Chi-Square Test:

Form the test statistic

$$\chi^2 = \sum_{i=1}^n Z_i^2 \quad (83)$$

Test this statistic as a chi-square distribution with $n - q$ degrees of freedom, where q is the number of parameters estimated from $\hat{F}(\tau)$.

The second test is based on the simple fact that either all of the $|Z_i|$ are less than a fixed number D or at least one of the $|Z_i|$ is greater than D . Thus, we may write

$$\begin{aligned} P(\text{some } |Z_i| > D) &= 1 - P(\text{all } |Z_i| \leq D) \\ &= 1 - [P(|Z| \leq D)]^n \\ &= 1 - (1 - 2 \operatorname{erfc} D)^n, \end{aligned} \quad (84)$$

where the second equality follows from the independence of the Z_i and the last equality results from using Eqs. (44) and (46). Equation (84) provides the justification for the following test.

Modified Kolmogorov-Smirnov Test:

Determine the maximum absolute value of the Z_i ; that is, determine the quantity

$$D = \max_i |Z_i|. \quad (85)$$

Compute the test value

$$p = 1 - (1 - 2 \operatorname{erfc} D)^n. \quad (86)$$

The null hypothesis is acceptable at the confidence level α if $p \geq \alpha$. Otherwise, the null hypothesis is rejected.

As mentioned previously, we can usually assume that the measured delay is distributed the same as the system delay if σ is at least several times the value of T . If the latter is not the case, we must calculate the known or proposed distribution of the measured delay by means of Eq. (73). If $f(y)$ is the density function of $d - c$ and x is uniformly distributed from 0 to T , it follows from elementary probability theory that

$$F(\tau) = \int_{-\infty}^{\tau-T} f(y) dy + \int_{\tau-T}^{\tau} f(y) \left(\frac{\tau - y}{T} \right) dy. \quad (87)$$

If the system delay is normally distributed, we can substitute Eq. (44) into Eq. (87) to obtain

$$F(\tau) = 1 - \frac{\sigma}{T} [a \operatorname{erfc} a - N(a) - (a - b) \operatorname{erfc} (a - b) + N(a - b)], \quad (88)$$

where we define

$$a = \frac{\tau - E(d) + c}{\sigma}$$

and

$$b = \frac{T}{\sigma}. \quad (89)$$

Since $F(\mu) = 1/2$ when $\mu = E(d_m) = E(d) - c + T/2$, the median of the measured delay is equal to its mean. Differentiating Eq. (88) with respect to τ , we obtain the density function,

$$f(\tau) = \frac{1}{T} [\operatorname{erfc} (a - b) - \operatorname{erfc} a]. \quad (90)$$

From this relation and Eq. (81), we have

$$\sigma_{\hat{m}}^2 \approx \frac{T^2}{4N \left[1 - 2 \operatorname{erfc} \frac{T}{2\sigma} \right]^2} \quad (91)$$

for the variance of the numerically estimated mean measured delay.

Recent Developments

With the production of Hewlett Packard's HP 5345A reciprocal counter [4], highly precise delay calibration is possible with a single measuring instrument. Whereas the conventional counter measures the number of input events during an interval, the reciprocal counter measures the time interval between events.

The reciprocal counter has two basic modes of operation. In the time-interval-averaging mode, it may be used in place of the timing distributors, the timing generator, and the delay measurement unit of Fig. 1. The sample mean and sample variance are determined according to Eqs. (60) and (64). A single delay measurement can be modeled by Eq. (1). However, band-limited noise is added to the counter's time base, causing x_1 and x_2 to be independent and uniformly distributed over the counter's clock period, which is $T = 2$ ns. The use of the noise-modulated clock removes any harmonic relationship between the counter's clock frequency and the repetition rate of the input pulses, helping to insure the independence of successive samples of the measured delay. Noting that we now have

$$\sigma_T^2 = \sigma^2 + \frac{T^2}{6}, \quad (92)$$

Eqs. (61), (62), (65), (66), and (67) are valid.

In another mode of operation, depicted in Fig. 6, the counter is used in conjunction with a calculator and a plotter. The sample mean and sample variance are determined

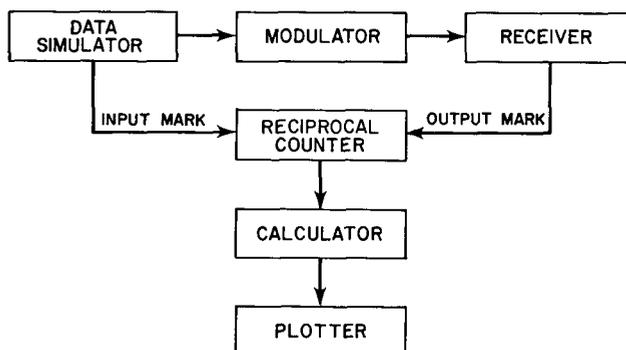


Fig. 6—Use of the reciprocal counter in delay measurement

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according to Eqs. (60) and (64). A single delay measurement can be modeled exactly as in the direct method of the first part of this report. Hence, Theorem 2 gives bounds on the variance of the measured delay. This mode is an alternative to the manual distribution method. Each sample value of the measured delay is sent to the calculator, where it is assigned to one of a set of disjoint class intervals. In view of Eq. (7), the partition between adjacent intervals should be located halfway between the possible values of the measured delay, which are integral multiples of $T = 2$ ns. After all the sample values are assigned to intervals, the calculator and plotter assemble a histogram (empirical probability distribution). The conventional chi-square or Kolmogorov-Smirnov tests can be applied to this histogram to determine the goodness of fit of a proposed probability distribution.

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