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Third-Order Clairaut Equation for a Rotating Body of Arbitrary Density and its Application to Marine Geodesy

ZDENĚK KOPAL AND PAOLO LANZANO

Operations Research Branch
Mathematics and Information Sciences Division

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This report derives the Clairaut equations, which govern the deformation of the equipotential surfaces within a rotating mass in hydrostatic equilibrium, as ordinary differential equations containing up to third-order terms in a small parameter. This has been achieved by (a) eliminating the two integral terms which appeared in the original formulation, and (b) by expanding the equipotential surfaces into a power series of a small parameter which is essentially the ratio between the rotational and potential energy of the body.

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20. Abstract (Continued)

It is expected that the numerical integration of these newly obtained equations, which requires knowledge of the density profile within the rotating body, will contribute toward a solution of the following problems: (a) the determination of the geoid to a precision of one meter, (b) correction to the travel time of seismic waves, and (c) the determination of the exterior shape of the rapidly rotating planets Jupiter and Saturn.

The need for the third-order and higher order theories is dictated by our interest in ascertaining the planetary deformations to a higher degree of accuracy than heretofore available. In the case of the earth and the moon, the accuracy requirement is of the order of one meter. Such a stringent requirement is related to laser ranging and will provide more precise information on movement of the tectonic plates and on other questions of interest in marine geodesy.

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THIRD-ORDER CLAIRAUT EQUATION FOR A ROTATING BODY OF ARBITRARY DENSITY AND ITS APPLICATION TO MARINE GEODESY

INTRODUCTION

The fundamental problem of geodesy is the determination of an accurate shape of the geoid with respect to a chosen reference surface. For this purpose, geodetic activity in the last few decades has been primarily concerned with obtaining accurate gravity anomaly measurements in various locations on the earth. By introducing these measurements into Stokes' integral formula, one can evaluate the deviations of the geoid from the standard reference surface. This formula however requires contributions from all over the earth. To gravimetrically map extensive areas is an expensive and time-consuming proposition, which sometimes is unattainable because of the inaccessibility of certain areas of the globe. The theoretical refinements due to Molodenskii [1], although responsible for having brought a slight increase in accuracy, are predicated to the same limitations and pitfalls (for example, Rapp [2]).

To alleviate this situation, recourse has been made, from time to time, to theoretical considerations to provide alternate methods or more suitable methods or both. Recent research has been dealing with the representation of the geopotential by a series of functions different from spherical harmonics. Thus, for example, Hotine [3] and Walter [4] have considered ellipsoidal harmonics; Lundquist and Giacaglia [5] have introduced certain matrix transforms of harmonic functions which they have labeled sampling functions. Statistical models have also been introduced (for example, by Jordan [6]) to systematize the numerous data pertaining to gravity anomalies and deflections of the vertical. Elegant as they might be, these mathematical devices do not seem ultimately to help in the final numerical evaluation.

Satellite geodesy in the past has been quite successful in ascertaining the harmonics up to the tenth order and degree (Gaposchkin and Lambeck [7]), representing wavelengths longer than 4000 km. Claims have been made of having obtained undulations of only few meters. However, the determination of higher order harmonics (say, up to the sixteenth order and degree) by the same method has not met with great success, and one has to fall back to surface gravity measurements for reliable data.

Recently, sea-gravity measurements performed in the western North Atlantic ocean by Talwani [8] have revealed undulations of the geoid with a wavelength of a few hundred kilometers, having amplitudes of few tens of meters.

Note: Manuscript submitted July 18, 1974.

Thus one has the situation whereby undulations of long wavelengths (a few thousand kilometers) have amplitudes of a few meters, whereas undulations with a wavelength of a few hundred kilometers have amplitudes one order of magnitude higher.

This lack of convergence and the existing discrepancy between data obtained by sea or land measurements vs satellite data contribute to a situation which is far from being optimal and which calls for the introduction of new ideas/or working procedures or both if one wants to make use of all available gravity measurements for a more precise determination of the geoid.

It is our contention that there is a physical reason for the observed discrepancy and that any knowledge of the density distribution within our planet is fundamental for obtaining a more refined figure of equilibrium that should be used as the "spheroid" of reference. The inadequacy of the present reference surface (oblate ellipsoid of revolution) to represent a good global fit of the geoid is responsible, in our opinion, for the fact that one cannot accomodate at present all the available data. Both from a mathematical and a physical point of view, it is evident that if initially one does not start with a good fit to the physical surface, a loss of accuracy will ensue.

The theory for the equilibrium configuration of a rotating mass with arbitrary density distribution under hydrostatic equilibrium (the so-called theory of Clairaut) appears to be the most logical physical way to provide a good approximation to the geoid. This theory, relating the shape and gravitational potential of a heterogeneous earth deformed by diurnal rotation, was developed by A. C. Clairaut to first-order terms in the ratio of centrifugal force to surface gravity. It was capable of describing the shape of the earth within errors of about ±100 m. To increase the accuracy necessitated retention of second-order terms in the same parameter. The first steps in this direction were taken by Darwin [9], and the second-order theory was finally brought to a successful conclusion by DeSitter [10]. Perhaps the most general results are those obtained by Kopal [11] and published in a monograph which has since received extensive use. Applications of this second-order theory to geophysics [12, 13] have disclosed that a retention of second-order terms will approximate the surface of the earth by quantities of the order of a few tens of meters. However, should the current operational requirements call for an improvement of the precision to quantities of the order of one meter or less, a retention of third-order terms in the underlying theory becomes indispensable. The first steps toward developing such a theory were taken by Lanzano [14], and more recently both authors while at NRL have developed the Clairaut theory up to and including third-order terms in the small parameter. We are virtually certain that no similar work has been undertaken anywhere else. The numerical solution of the ensuing Clairaut equation should provide the sought spheroid of reference to a higher degree of approximation than heretofore available. The essential features of the method are described in what follows.

CLAIRAUT EQUATIONS

Consider any deformable, nonviscous body, and denote the density at any internal point by ρ, the pressure by p, and the sum of the self-gravitational and disturbing potential by ψ. Assuming hydrostatic equilibrium, we have

$$\operatorname{grad} p = \rho \operatorname{grad} \psi . \tag{1}$$

The compatibility conditions for this equation reveal that ψ must be a function of ρ alone; from Eq. (1) it follows that p is then also a function of ρ alone: the so-called equation of state. One can then deduce that the surfaces ψ = constant constitute equilibrium configurations.

Let us suppose that the mass in consideration rotates about an axis which remains fixed with respect to its surface. Because of this rotation, both the outermost surface and the other equipotential surfaces, which were originally spherical in shape, will be distorted into spheroidal configurations. These spheroids depend continuously on a parameter a, which represents the mean radius of the spheroid, and are such that through each point within the mass there passes one and only one spheroid.

Let us introduce a system of spherical polar coordinates (O; r, θ, ϕ) with the origin O coinciding with the centroid of the rotating mass, where the colatitude θ is measured from the rotational axis and the longitude ϕ is measured in the equatorial plane from an arbitrary axis. If we assume both axial symmetry and symmetry with respect to the equatorial plane, the equipotential surfaces can be represented as

$$\frac{r}{a} = 1 + \sum_{j=0}^{\infty} f_{2j}\left(\frac{a}{a_1}\right) P_{2j}(\cos\theta) , \tag{2}$$

where the P's are Legendre polynomials and a_1 denotes the mean radius of the outermost surface.

The distortion coefficients f can be determined from the total potential ψ expanded in terms of spherical harmonics of ascending order. For this purpose, replace in this potential expansion the radius-vector r by Eq. (2), and after having performed all the necessary operations, rearrange this expression into a new expansion of ψ in spherical harmonics. Since ψ must reduce to a constant, we equate the coefficients of $P_j (j = 1, 2, 3, \ldots)$ to zero and the coefficient of P_0 to a constant.

These steps give rise to Clairaut's integral equations for the f's which entail not only the density $\rho(a)$ within the mass but also its mean density

$$\overline{\rho}(a) = \frac{3}{a^3} \int_0^a \rho a^2 \, da . \tag{3}$$

Thus one would be compelled to ascertain the density distribution within the mass by making use of the Poisson equation associated with the given potential. For the earth this last step is not essential since an accurate model for its internal density can be deduced from seismological considerations.

To establish a recurrent approximation procedure, we introduce the rotational parameter

$$q = \frac{\omega^2 a_1^3}{3Gm_1} , \tag{4}$$

where G is the gravitational constant, m_1 is the total mass, ω is the rate of axial rotation, and we express the distortion coefficients f_j as power series in q:

$$f_{2j}(a, q) = \sum_{k=1}^{\infty} q^k f_{2j,k}(a) . \tag{5}$$

Notice that q is essentially the ratio between the rotational and the potential energy of the mass; for the earth we know that $q = 0.00115$. Preliminary computations, meant to establish order-of-magnitude relations in q, reveal that f_2 is of the first order, both f_0 and f_4 are of the second order, f_6 is of the third order, and f_8 is already of the fourth order in q. Thus, up to third-order terms, the equations of the equipotential surfaces can be written as

$$\frac{r}{a} = 1 + \sum_{j=0}^{3} \sum_{k=1}^{3} q^k f_{2j,k}\left(\frac{a}{a_1}\right) P_{2j}(\cos\theta) \tag{6}$$

with

$$f_{01} = f_{41} = f_{61} = f_{62} = 0 . \tag{7}$$

The first-order theory provides the function f_{21}; the second-order theory consists of determining f_{02}, f_{22}, and f_{42} in terms of the lower order results. All of these functions have to be used to ascertain f_{03}, f_{23}, f_{43}, and f_{63}, which constitute the third-order theory.

Let us consider the expression for the total potential. The only disturbing potential is the rotational potential, and this can be written as

$$W = \frac{1}{2}\omega^2 r^2 \sin^2\theta = q\frac{Gm_1}{a_1^3} r^2 [1 - P_2(\cos\theta)] . \tag{8}$$

The self-gravitational potential at an interior point P located on an equipotential surface of mean radius a consists of two terms:

$$V = \sum_{j=0}^{\infty} r^{-(1+j)} V_j(\theta, \phi),$$

due to the attraction on P by the mass located within said equipotential surface, and

$$U = \sum_{j=0}^{\infty} r^j U_j(\theta, \phi) ,$$

due to the attraction of the mass lying outside such surface and extending as far as the external boundary of the body. U_j and V_j are well-known triple integrals with respect to the mean radius and two angular variables [11]; their integrands consist of products of Legendre polynomials and powers of r; U_2 contains a logarithmic term in r/a.

To carry out the first two steps of the previously described procedure on $\psi = U + V + W$, we have to:

(1) Use the orthogonality condition for spherical harmonics in its integral form;

(2) Reduce the product of two or three Legendre polynomials to a linear expression of the same polynomials according to the Neumann-Adams formula [15]

$$P_j P_k P_r = \sum_h A^h_{jkr} P_h$$

for $h = j + k + r, j + k + r - 2, \ldots$.

Thus the nth power of an equipotential surface has been expressed as

$$\frac{\left(\frac{r}{a}\right)^n - 1}{n} = \sum_{j=0}^{3} \left\{ q f_{2j,1} + q^2 \left(f_{2j,2} + \frac{n-1}{2} A^{2j}_{22} f^2_{21} \right) + q^3 \left[f_{2j,3} + (n-1) B_{2j} + \frac{(n-1)(n-2)}{6} A^{2j}_{222} f^3_{21} \right] \right\} P_{2j}, \tag{9}$$

where

$$B_{2j} = \sum_{k=0}^{2} A^{2j}_{2k,2} f_{21} f_{2k,2} \tag{10}$$

and a comma is used to separate the sets of lower indices when they appear in literal form. Notice that $\ln(r/a)$, required within U_2, can be evaluated by taking the limit of Eq. (9) when n tends to zero.

Using these results, we have obtained the expansion of ψ into spherical harmonics of the form

$$\psi = \sum_{j=0}^{3} \sum_{k=1}^{3} q^k \psi_{2j,k}(a) P_{2j}(\cos\theta),$$

where the coefficients of the various harmonics are polynomials in q. The ψ's for $j = 0$ and $k = 1, 2, 3$ need not be written explicitly since they must be equated to arbitrary constants. The nonvanishing coefficients of the expansion are (primes denoting derivatives with respect to the mean radius a):

$$\psi_{21}(a) = -\frac{Gm_1}{a_1}\left(\frac{a}{a_1}\right)^2 + \frac{4\pi G}{a}\left[-f_{21} \int_0^a \rho a^2 \, da \right.$$

$$\left. + \frac{a^{-2}}{5} \int_0^a \rho(a^5 f_{21})' \, da + \frac{a^3}{5} \int_a^{a_1} \rho f_{21}' \, da \right] , \qquad (11)$$

$$\psi_{2j,2} = 2\frac{Gm_1}{a_1}\left(\frac{a}{a_1}\right)^2 \left(A_{20}^{2j} - A_{22}^{2j}\right) f_{21}$$

$$+ \frac{4\pi G}{a}\left\{\frac{a^{-2j}}{1 + 4j} \int_0^a \rho\left(a^{3+2j}F_{2j,2}\right)' da\right.$$

$$+ \frac{a^{1+2j}}{1 + 4j} \int_a^{a_1} \rho \left(a^{2-2j}E_{2j,2}\right)' da$$

$$+ \frac{1}{5}A_{22}^{2j}\left[2a^3 \int_a^{a_1} \rho f_{21}' \, da - 3a^{-2} \int_0^a \rho(a^5 f_{21})' \, da\right] f_{21}$$

$$\left. - \left(f_{2j,2} - A_{22}^{2j} f_{21}^2\right) \int_0^a \rho a^2 \, da \right\} , \qquad (12)$$

for $j = 1, 2$, with

$$E_{2j,2} = f_{2j,2} + \frac{1}{2}(1 - 2j)A_{22}^{2j} f_{21}^2$$

$$F_{2j,2} = f_{2j,2} + (1 + j)A_{22}^{2j} f_{21}^2 , \qquad (13)$$

and

$$\psi_{2j,3} = \frac{Gm_1}{a_1}\left(\frac{a}{a_1}\right)^2 \left[2f_{2j,2} + A_{22}^{2j} f_{21}^2\right.$$

$$\left. - \sum_{k=0}^{2} A_{2k,2}^{2j}\left(2f_{2k,2} + A_{22}^{2k} f_{21}^2\right)\right]$$

$$+ \frac{4\pi G}{a}\left\{\frac{a^{-2j}}{1 + 4j} \int_0^a \rho\left(a^{3+2j}F_{2j,3}\right)' da\right.$$

$$+ \frac{a^{1+2j}}{1 + 4j} \int_a^{a_1} \rho(a^{2-2j}E_{2j,3})' \, da$$

(Cont'd)

$$+\frac{1}{5}A^{2j}_{22}\left[2a^3\int_a^{a_1}\rho E'_{22}\,da - 3a^{-2}\int_0^a \rho(a^5F_{22})'\,da\right]f_{21}$$

$$+\frac{1}{9}A^{2j}_{42}\left[4a^5\int_a^{a_1}\rho(a^{-2}E_{42})'\,da - 5a^{-4}\int_0^a \rho(a^7F_{42})'\,da\right]f_{21}$$

$$-\frac{3}{5}a^{-2}\sum_{k=0}^{2}A^{2j}_{2k,2}\left(f_{2k,2} - 2A^{2k}_{22}f^2_{21}\right)\int_0^a \rho(a^5f_{21})'\,da$$

$$+\frac{a^3}{5}\sum_{k=0}^{2}A^{2j}_{2k,2}\left(2f_{2k,2} + A^{2k}_{22}f^2_{21}\right)\int_a^{a_1}\rho f'_{21}\,da$$

$$-\left(f_{2j,3} - 2B_{2j} + A^{2j}_{222}f^3_{21}\right)\int_0^a \rho a^2\,da\Bigg\}, \tag{14}$$

for $j = 1, 2, 3$, where

$$E_{2j,3} = f_{2j,3} + (1 - 2)B_{2j} - \frac{1}{3}j(1 - 2j)A^{2j}_{111}f^3_{21}$$

$$F_{2j,3} = f_{2j,3} + 2(1 + j)B_{2j} + \frac{1}{3}(1 + j)(1 + 2j)A^{2j}_{222}f^3_{21} \tag{15}$$

and the B's are defined by Eq. (10). It is also clear that

$$\psi_{41} = \psi_{61} = \psi_{62} = 0.$$

By equating ψ_{21} to zero, we obtain an integral equation in which the unknown function $f_{21}(a)$ appears twice under the integral sign: one integral vanishing at the center of mass ($a = 0$), the second vanishing at the external boundary ($a = a_1$). If in turn we isolate each integral and differentiate with respect to the variable a, we get an integral equation expressing the remaining integral of the unknown function in terms of the density and the deformation coefficients. These two expressions are needed for higher order approximations. Next, consider that expression which contains the integral vanishing at the boundary, and evaluate it at $a = a_1$; we find the boundary condition, valid at $a = a_1$:

$$2f_{21} + af'_{21} + 5 = 0. \tag{16}$$

A third differentiation of the previous results is required to reach an ordinary differential equation

$$a^2f''_{21} + 6D(f_{21} + af'_{21}) - 6f_{21} = 0, \tag{17}$$

where $D = \rho/\overline{\rho}$ and $\overline{\rho}$, the mean density, is given by Eq. (3). By introducing the logarithmic derivative

$$\eta = \frac{af'}{f}$$

the Clairaut equation will be taken into the Radau equation

$$a\eta_{21}' + 6D(1 + \eta_{21}) - (1 - \eta_{21})\eta_{21} - 6 = 0\,, \tag{19}$$

which is of the first order but nonlinear.

The conservation of total mass within the distorted configuration allows one to evaluate the function $f_0(a)$ in its various approximations. In fact we can write

$$m_1 = 4\pi \int_0^{a_1} \rho a^2\, da + 4\pi q^2 \int_0^{a_1} \rho(a)\left[a^3\left(f_{02} + \frac{1}{5}f_{21}^2\right)\right]' da$$

$$+ 4\pi q^3 \int_0^{a_1} \rho(a)\left[a^3\left(f_{03} + \frac{2}{5}f_{21}f_{22} + \frac{2}{105}f_{21}^3\right)\right]' da\,,$$

where primes, as usual, denote derivatives with respect to a. Since $\rho(a)$ is the density of the undistorted configuration and no change in total mass can occur because of the rotational motion, the first term alone appearing in the right-hand side shall represent the total mass. The other two integrands must vanish, thus yielding

$$f_{02} = -\frac{1}{5}f_{21}^2 \tag{20a}$$

and

$$f_{03} = -\frac{2}{5}f_{21}f_{22} - \frac{2}{105}f_{21}^3\,. \tag{20b}$$

By equating $\psi_{2j,2}$ (j = 1, 2), as given by Eq. (12), to zero and eliminating the two integrals of f_{21}, we get integral equations in which each unknown function $f_{2j,2}$ appears twice under an integral sign. Two differentiations are required to represent each one of these integrals in finite terms; these expressions are required for the next higher order approximation. One of these expressions provides the boundary conditions at $a = a_1$:

$$2f_{22} + af_{22}' = \frac{2}{7}(6 + 3\eta_{21} + \eta_{21}^2)f_{21}^2 + 2(5 + \eta_{21})f_{21}; \tag{21a}$$

$$4f_{42} + af_{42}' = \frac{18}{35}(6 + 5\eta_{21} + \eta_{21}^2)f_{21}^2\,. \tag{21b}$$

A third differentiation yields the second-order Clairaut equation

$$a^2 f''_{2j,2} + 6D(f_{2j,2} + a f'_{2j,2}) - 2j(2j + 1) f_{2j,2} = R_{2j,2} ,$$

valid for $j = 1, 2$ and where

$$R_{22} = \frac{2}{7}[(2 - 9D)\eta_{21} + 18(1 - D)]\eta_{21} f_{21}^2 + 12H(1 - D)(1 + \eta_{21}) f_{21} , \tag{22a}$$

$$R_{42} = \frac{18}{35}[(2 - 9D)\eta_{21}(2 + \eta_{21}) - 21D] f_{21}^2 , \tag{22b}$$

and

$$H(a) = \frac{\overline{\rho}(a_1)}{\overline{\rho}(a)} .$$

The corresponding Radau equation is easily obtainable.

To obtain third-order terms, we equate $\psi_{2j,3}$ ($j = 1, 2, 3$) to zero and follow basically the same steps as in the previous case, making use of the first and second-order results. One differentiation is required to get the boundary conditions, and a second one is required to obtain the third-order Clairaut equations. The operations of substitution and simplification entail the handling of approximately 200 literal terms. This work was done by hand and checked independently by both authors. A computer program is being envisaged in the near future to pursue expansions to higher powers of the parameter.

The third-order results can be summarized as follows:

- Boundary conditions at $a = a_1$:

$$\begin{aligned} 2f_{23} + a f'_{23} = {} & 2(5 + \eta_{22}) f_{22} - \frac{2}{7}(15 + 6\eta_{21} + 2\eta_{21}^2) f_{21}^2 \\ & + \frac{2}{7}(12 + 3\eta_{21} + 3\eta_{22} + 2\eta_{21}\eta_{22}) f_{21} f_{22} \\ & + \frac{2}{7}(26 + 3\eta_{21} + 3\eta_{42} + 2\eta_{21}\eta_{42}) f_{21} f_{42} \\ & - \frac{4}{35}(38 + 30\eta_{21} + 15\eta_{21}^2 + 2\eta_{21}^3) f_{21}^3 ; \end{aligned}$$

$$\begin{aligned} 4f_{43} + a f'_{43} = {} & 2(7 + \eta_{42}) f_{42} - \frac{18}{35}(21 + 10\eta_{21} + 2\eta_{21}^2) f_{21}^2 \\ & + \frac{18}{35}(12 + 5\eta_{21} + 5\eta_{22} + 2\eta_{21}\eta_{22}) f_{21} f_{22} \end{aligned} \tag{24}$$

(Cont'd)

$$+\frac{20}{77}(26+5\eta_{21}+5\eta_{42}+2\eta_{21}\eta_{42})f_{21}f_{42}$$

$$-\frac{36}{385}(54+44\eta_{21}+18\eta_{21}^2+3\eta_{21}^3)f_{21}^3; \tag{24}$$

$$6f_{63}+af'_{63}=\frac{5}{11}(26+7\eta_{21}+7\eta_{42}+2\eta_{21}\eta_{42})f_{21}f_{42}$$

$$-\frac{18}{77}(2+\eta_{21})(12+6\eta_{21}+\eta_{21}^2)f_{21}^3; \tag{25}$$

- Clairaut equations:

$$a^2f''_{2j,3}+6D(f_{2j,3}+af'_{2j,3})-2j(1+2j)f_{2j,3}=R_{2j,3} \qquad (j=1,2,3),$$

where

$$R_{23}=\frac{4}{7}[(2-9D)\eta_{21}\eta_{22}+9(1-D)(\eta_{21}+\eta_{22})]f_{21}f_{22}$$

$$+\frac{4}{7}[(2-9D)\eta_{21}\eta_{42}+3(10-3D)\eta_{21}+(16-9D)\eta_{42}+21D]f_{21}f_{42}$$

$$-\frac{4}{35}[(7+6D)\eta_{21}^3+3(11-15D)\eta_{21}^2+9(20-9D)\eta_{21}+3(22+5D)]f_{21}^3$$

$$+12H(1-D)(1+\eta_{22})f_{22}+\frac{36}{7}H(1-D)(2+\eta_{21})\eta_{21}f_{21}^2$$

$$+24H^2(1-D)(1+\eta_{21})f_{21}; \tag{26}$$

$$R_{43}=\frac{36}{35}[(2-9D)(\eta_{21}\eta_{22}+\eta_{21}+\eta_{22})-21D]f_{21}f_{22}$$

$$+\frac{40}{77}[(2-9D)\eta_{21}\eta_{42}+(23-9D)\eta_{21}+9(1-D)\eta_{42}]f_{21}f_{42}$$

$$-\frac{36}{385}[9D\eta_{21}^3-(10-27D)\eta_{21}^2+4(8+9D)\eta_{21}-6(1-2D)]f_{21}^3$$

$$+12H(1-D)(1+\eta_{42})f_{42}+\frac{108}{35}H(3\eta_{21}^2+6\eta_{21}+7)(1-D)f_{21}^2; \tag{27}$$

$$R_{63} = \frac{10}{11}[2 - 9D)\eta_{21}\eta_{42} + 3(4 - 3D)\eta_{21} - (2 + 9D)\eta_{42} - 33D]f_{21}f_{42}$$
$$- \frac{18}{77}[3D\eta_{21}^3 + (4 + 9D)\eta_{21}^2 - (4 - 45D)\eta_{21} - 24 + 15D]f_{21}^3. \qquad (28)$$

For a deeper insight into the mathematical details, see references 14, 16, and 17.

APPLICATIONS

Our principal result is the third-order approximation of the Clairaut equations. Two analytical developments of our results are the evaluation of the exterior potential and the various moments of inertia for the distorted configuration up to third-order terms in the parameter. We plan in the future to numerically integrate the Clairaut equations subject to the boundary conditions at the free surface expressed by Eqs. (23) - (25) and to the boundary conditions at the center of the configuration given by

$$f_{2j,3}(0) = f'_{2j,3}(0) \qquad (j = 1, 2, 3).$$

The numerical solution of this boundary-value problem provides values for the deformation of every equipotential surface within the rotating body. This naturally requires knowledge of the lower order approximations.

The numerical integration of the Clairaut equation necessitates a knowledge of the density distribution within the rotating body. In the case of the earth, a very plausible model which we are advocating is the HB_1 model of Haddon and Bullen [18] or a modification thereof. This model, which is characterized by its parametric simplicity, represents in our opinion one of the most advanced steps in our present knowledge of the earth's density distribution. In fact, not only is it compatible with seismic wave travel-time data and the revised value of the earth's moment of inertia, but it also incorporates the earth spheroidal and torsional free-oscillation data.

James and Kopal's numerical work [13] is based on the 1940 - 42 Bullen model and is limited to the second-order approximation.

For a rapidly rotating planet, like Jupiter or Saturn, third-order effects might not be sufficient to determine its shape. If this be the case, we are in a position to analytically extend our theory to the fourth or higher order by means of a computer program which allows one to perform all the algebraic and differential manipulations through the computer. For the fourth-order case this has already been done, but the results, though available, are too complicated to be included in this report.

In the terrestrial case, it is expected that by solving the third-order Clairaut equations, a precision of ±1 m for the geoid will be achieved and will represent the best "surface of interpolation" for isolating other factors not attributable to hydrostatic equilibrium. A more accurate determination of the geoid should also help to understand the rotational motion of the tectonic plates.

Another reason for wanting to know the equipotential surfaces within the earth to a higher precision is to be able to calculate corrections to seismic wave travel time, as mentioned recently by Bullen [19].

Future measurements via radar altimeters like the ones planned in the GEOS-C satellite will be very fruitful; however, to calibrate the altimeter, it is imperative that the geoid undulations be known by independent means.

In conclusion, we feel confident that our analytical procedure, which stems from a deep-rooted physical condition, if backed by valid density data, will provide a better approximation to the geoid and a better usage of available or future gravity measurements.

REFERENCES

1. M.S. Molodenskii, V.F. Eremeev, and M.I. Yurkina, *Methods for Study of the External Gravitational Field and Figure of the Earth*, OTS 61-31207, translated from the Russian, Israel Program for Scientific Translations, Jerusalem, 1962 (original Russian publication Moscow, 1960).
2. R.H. Rapp, *The Geoid: Definition and Determination*, Fourth GEOP Res. Conf. on Geoid and Ocean Surface, Boulder, Colo., 1973.
3. M. Hotine, *Mathematical Geodesy*, U.S. Gov. Printing Office, 1969.
4. H.G. Walter, "*The Use of Ellipsoidal Harmonics for the Representation of the Potential*," in *Dynamics of Satellites*, B. Morando, ed., Springer-Verlag, New York, 1970, p. 28.
5. C.A. Lundquist, G.E.O. Giacaglia, and R.H. Gay, *Application of Sampling Functions to Earth Gravity Models*, International Symposium on Earth Gravity, St. Louis, Mo., 1972.
6. S.K. Jordan, *J. Geophys. Res.* **77**, No. 20, 3660 (1972).
7. E.M. Gaposchkin and K. Lambeck, *J. Geophys. Res.* **76**, No. 20, 4855 (1971).
8. M. Talwani, H.R. Poppe, and P.D. Rabinowitz, *Gravimetrically Determined Geoid in the Western North Atlantic*, in "Sea Surface Topography from Space 2," NOAA Report ERL-228-AOML 7-2, 1972.
9. G.H. Darwin, *Mon. Not. R. Astron. Soc.* **60**, 82(1899).
10. W. DeSitter, *Bull. Astron. Inst. Neth.* **2**, 97 (1924).
11. Z. Kopal, *Figures of Equilibrium of Celestial Bodies*, University of Wisconsin Press, Madison, 1960.
12. E.C. Bullard, *Mon. Not. R. Astron. Soc. (Geophys. Suppl.)* **5**, 186 (1948).
13. R. James and Z. Kopal, *Icarus* **1**, 442 (1963).
14. P. Lanzano, *Icarus* **1**, 121 (1962).
15. W.N. Bailey, *Generalized Hypergeometric Series*, Stechert-Hafner, New York, 1964.
16. P. Lanzano, to appear in Astrophys. Space Sci. (1974).

17. Z. Kopal, *Astrophys. Space Sci.* **24**, 145 (1973).

18. R.A.W. Haddon and K.E. Bullen, *Phys. Earth Planet. Interiors* **2**, 35 (1969).

19. K.E. Bullen, and R.A.W. Haddon, *Phys. Earth Planet. Interiors* **7**, *199* (1973).