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The J-Code: A Minimum-Peak Distortion Code

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The transmission of a PCM symbol on a slowly fading Rayleigh channel was examined from the standpoint of minimizing peak distortion from transmission errors. An encoding technique was defined for M-ary DPSK (differential phase-shift keying) wherein multiple values were assigned to the guard band so as to cause peak distortion to be a linear function of the phase error. Comparisons were made of the use of a Gray code and the natural binary code, with a guard band evenly divided into two symbols. The newly devised J code resulted in a lower

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20. Abstract

probability of high peak distortion than either of these two encoding techniques. The mean square distortion of the J code for a slowly, fading Rayleigh channel is shown to be lower than that obtained for the natural binary code with a guard band of π, and at very low signal-to-noise conditions it is better than the natural binary code with an optimum guard band of 0.4π.

CONTENTS

THE J CODE: A MINIMUM-PEAK DISTORTION CODE

INTRODUCTION

In the development of modulation techniques for high-frequency radio transmission, consideration is being given to 8-phase and 16-phase DPSK. When the digital source is pulse-code-modulated (PCM) data, there is a need to consider the overall end-to-end performance of the communication system. This is in contrast to the general approach of minimizing the average bit error rate. To this end, a study has been made of how transmission errors are translated into distortion in the reconstructed PCM signal. The study included the use of different encoding techniques in the modulator. The natural binary code and the reflected binary code (Gray code) were examined for the condition that they were used to encode a PCM data sample of N bits into one of $M = 2^N$ symbols of the differential phase-shift keying (DPSK) code. The work included the use of guard bands to reduce the probability of circular foldover for the natural binary code. Three measurements of distortion caused by transmission errors were calculated. They were

1. Peak distortion
2. Probability of a given absolute distortion
3. Mean square distortion (MSD).

The absolute distortion and the MSD were calculated for a Rayleigh fading channel as a function of signal-to-noise ratio (S/N), the number of symbols (M), and the width of the guard band. The study resulted in the specification of an encoding technique (J code) that minimizes the probability of high distortion in the presence of transmission errors in a DPSK M-ary modulation system.

TRANSMISSION MODEL

A simple model of the communication system being considered is shown in Fig. 1. The source data is a sequence $\{u_i\}$ of PCM samples of N bits each. Each sample is encoded into a single DPSK symbol $\{v_i\}$ by either the Gray code, the natural binary code, or the J code. The transmission channel is an independently fading Rayleigh channel with additive Gaussian noise. The received symbol $\{r_i\}$ is decoded into the PCM output $\{z_i\}$, resulting in the distortion $D_i = |u_i - z_i|$.

The relative advantages of the J code are also applicable to other channels and types of modulation, such as the phase-shift keying (PSK) Gaussian channel.

Note: Manuscript submitted May 30, 1974.

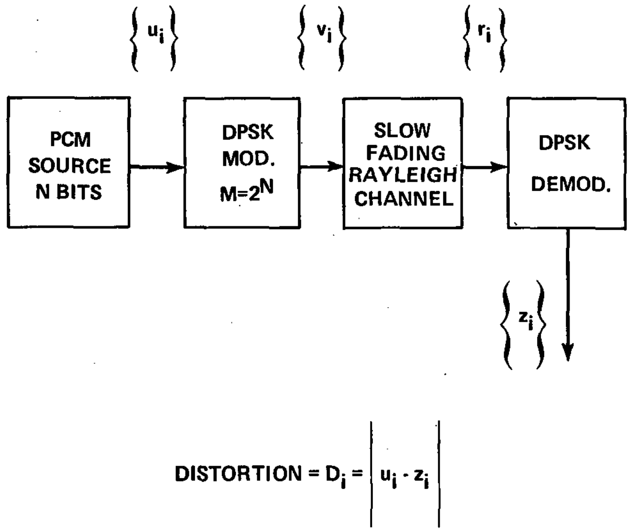


Fig. 1 — M-ary transmission model

PEAK DISTORTION

Peak distortion is defined here as

$$DP(e) = \text{MAX } |u_i - z_{i\pm e}| \quad \text{for } i = 1, \ldots, M$$
$$e = 1, \ldots, M/2.$$

That is, peak distortion is the maximum absolute value of the difference between the PCM data samples at the encoder and the reconstructed PCM samples at the decoder, for a given phase error in the demodulator. Phase error e is given in intervals of symbol widths $2\pi/M$. For this study, the encoder and decoder are restricted to an M-ary DPSK modulation system where each PCM data sample is mapped into a single symbol in the modem. To illustrate the peak distortion characteristics of different encoding techniques, 8-phase and 16-phase DPSK modulation will be used.

Figure 2 is a diagram of a 16-phase DPSK encoder using the Gray code. The Gray code is a minimum distance code. Its use results in minimum bit errors for small phase errors. For this reason the Gray code is generally used when the source bits either are, or must be considered to be, of equal weight (importance). The Gray code is not an optimum code for the transmission of binary data samples which are composed of different weight bits (PCM). The graph in Fig. 2 is a plot of the peak distortion characteristic of the Gray code as a function of phase error. That is, given that an absolute phase error of ϕ_e occurs, a peak distortion DP will occur if all symbols are equally likely to have been transmitted. The probability that DP will occur is a function of both the code structure and the probability that ϕ_e will occur. This aspect will be treated later for the case of the Rayleigh channel.

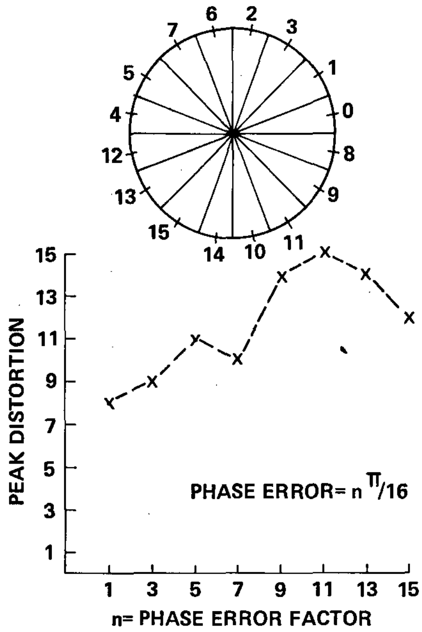


Fig. 2 — Peak distortion for Gray code, M = 16

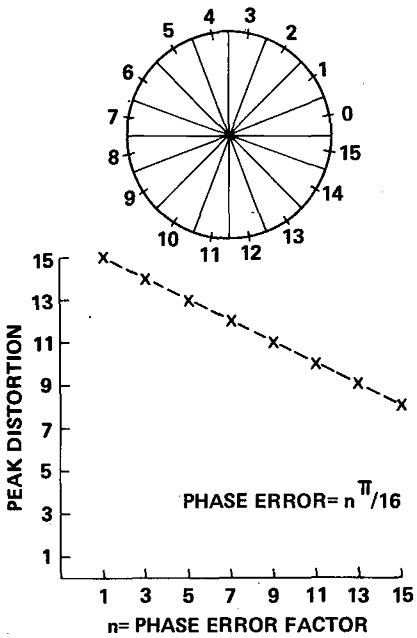


Fig. 3 — Peak distortion for natural binary code, M = 16

For the Gray code, the minimum value of DP is M/2, which is the value of the highest weight bit. This is because only the binary distance was considered in the development of the Gray code. For this reason, the natural binary code would be a more likely choice for encoding PCM data into a DPSK symbol.

The natural binary code, Fig. 3, is a minimum average distortion code. It is so because, except for the end points, the difference between adjacent symbols has the minimum absolute value of one. But, the peak distortion characteristic of this code is very poor because maximum peak distortion (DP = M − 1) is incurred for the minimum phase error condition.

The peak distortion characteristic of the natural binary code can be improved by the use of a guard band between the lowest magnitude symbol (000) and the highest magnitude symbol (111). Figure 4 shows DP as a function of ϕ_e for the natural binary code with a guard band (GB) of π. The use of a guard band improves the peak distortion characteristic for phase errors less than one-half the width of the guard band (i.e., DP increases linearly with ϕ_e for $\phi_e < GB/2$). But there is an undesirable result of using a guard band of π that is evenly divided into two sectors and assigned values of zero and M-1. Such an assignment results in a peak distortion of M-1 for phase errors between GB/2 and GB. It was for this reason that an attempt was made to define an encoding technique that would (a) produce minimum peak distortion for small phase error, and (b) require the maximum phase error to produce the maximum peak distortion. The encoding technique, which herein will be referred to as the J code, exhibits such characteristics.

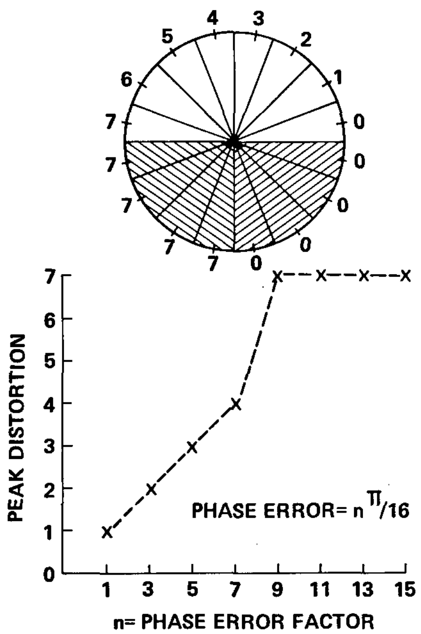


Fig. 4 — Peak distortion for natural binary code, M = 8, guard band = π

J CODE

The J code may be defined as an encoding technique for M-ary DPSK wherein multiple values are assigned to the guard band such that peak distortion is a linear function of phase error. Furthermore, when the guard band equals π, the guard band may be used for encoding data. Three examples of the J code have been examined. They will be referred to as the J1, J2, and J3 codes.

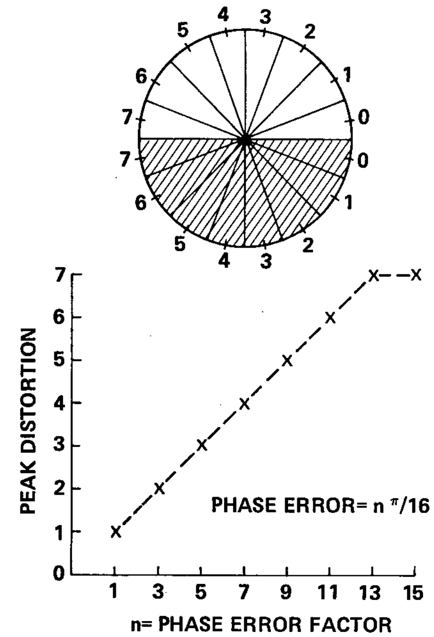


Fig. 5 — Peak distortion for J1 code, M = 8

The J1 code for M = 8 is shown in Fig. 5. In this case the guard band is π, and it is divided into symbols identical to the data sector. That is, the guard band is a mirror image of the data sector. The data sector is encoded with the natural binary code. This configuration minimizes the circular foldover problem. It makes peak distortion a linear function of the phase error. The graph in Fig. 5 shows DP vs ϕ_e for the J1 code, M = 8. With this particular encoding of the guard band, each hemisphere may be considered as the guard band for the other. Thus, an additional bit of information may be transmitted to indicate the intended hemisphere. Figure 5, therefore, is considered to be a J code configuration of an M = 16 DPSK system in which three bits indicate the absolute value of the phase angle and one bit indicates the sign of the phase angle. A simple model of such an encoder is shown in Fig. 6.

The J2 code is an example of applying the principle of the J code to a guard band of width less than π (Fig. 7). In this case, the guard band contains M/2 symbols. Each of these symbols is of the same width as the M data symbols. That is,

$$\begin{aligned} GB &= 2\pi\,[(M/2)/(M + M/2)] \\ &= 2\pi/3. \end{aligned}$$

The data sector is encoded with the natural binary code. The guard band is assigned values in steps of 2 from 1 to (M/2)-1 and from M/2 to M-2 (i.e., M = 8, GB values = 1, 3, 4, 6). (For an M = 16 configuration, the GB values would be 1, 3, 5, 7, 8, 10, 12, 14.) This assignment of values to the guard band complies with the J code principle of tapering the values to make the resultant distortion, caused by phase errors, independent of the sign of the phase error and proportional to the magnitude of the phase error. When the GB is less than π, peak distortion will always occur for phase error less than π. Figure 7 shows DP vs ϕ_e for the J2 code, M = 8.

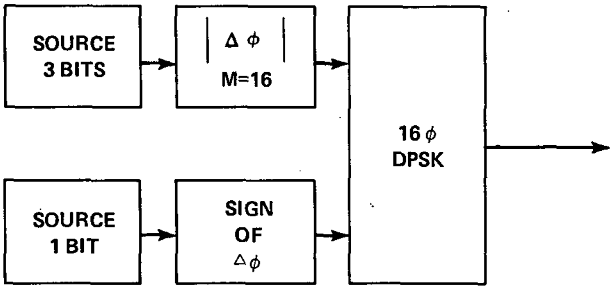


Fig. 6 — J1 encoder model, M = 16

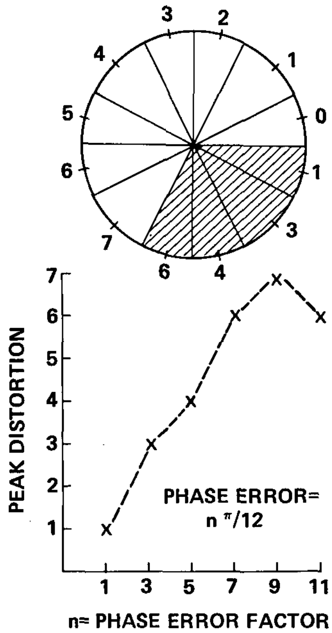


Fig. 7 — Peak distortion for J2 code, M = 8, guard band = $2\pi/3$

A third example of encoding to minimize peak distortion is the J3 code shown in Fig. 8. This may be viewed as a second example of how each hemisphere may be considered as the guard band for the other. In this case, different values are assigned to the symbols in the upper hemisphere compared to those in the lower hemisphere. Specifically, one hemisphere in encoded with even-numbered values (0 to M-2), and the other hemisphere is encoded with odd-numbered values (1 to M-1). Thus, with this configuration, all adjacent symbols differ by 2 except the two end points. They have a difference of 1. Figure 8 shows DP vs ϕ_e for the J3 code, M = 16. It is noted that the J1 code is identical to this J3 code for the condition that the least significant bit of a PCM sample is used to encode the sign of the phase shift in the J1 code.

Each of the DP vs ϕ_e plots presented considers only the peak distortion that could occur without considering the probability of occurrence. Therefore, to provide a means of comparison of the different codes, the slowly fading Rayleigh channel was selected as a model for the HF path.

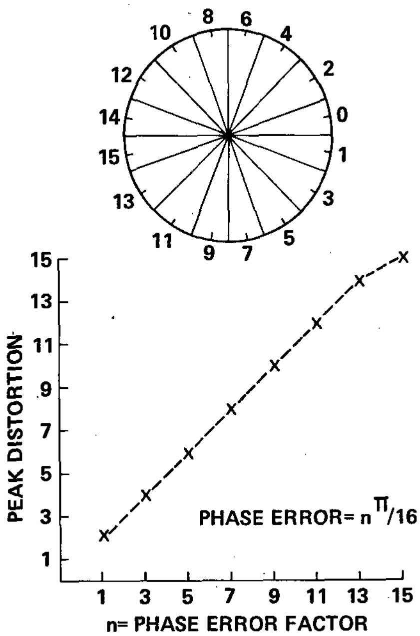


Fig. 8 — Peak distortion for J3 code, M = 16

RAYLEIGH CHANNEL

Proakis* derived an expression for the probability that a phase error will occur in the range θ_1 to θ_2.

$$P(\theta_1 \leqslant \theta \leqslant \theta_2) = \int_{\theta_1}^{\theta_2} P(\theta)\, d\,\theta .$$

The derivation was a function of S/N on a slowly fading Rayleigh channel with additive Gaussian noise. For a single-diversity DPSK system that expression is equivalent to

$$P(e) = (F_{\theta_2} - F_{\theta_1})/2\pi$$

where

$$F_{\theta_1} = \alpha\sqrt{z}\ \arctan\ \frac{1}{x}\ - \arctan\left(\frac{\alpha\sqrt{z}}{x}\right)$$

and

$$\alpha = E/(1+E) = 10^{S/10}/(1 + 10^{S/10})$$

S = Signal/noise in dB

*J. G. Proakis, "Probabilities of Error for Adaptive Reception of M-Phase Signals," IEEE Trans. COM-16, No. 1, 71-81 (Feb. 1968).

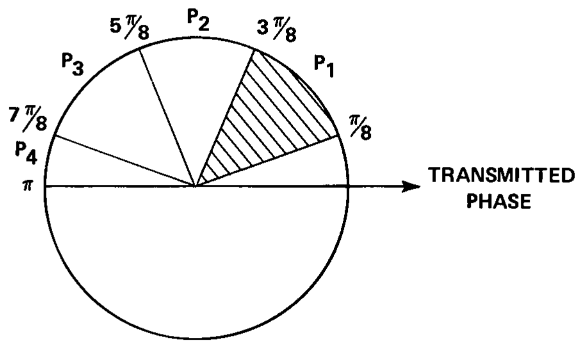
$$Z = |1 - [(1-\alpha^2)/\alpha^2]\ X^2|$$

$$X = \alpha \cos\theta / \sqrt{1 - \alpha^2 \cos^2\theta}.$$

P(e) is shown in Fig. 9 for M = 8, GB = 0. P(e) must be calculated for e = 1 to e = M/2. The values of P(e) have been calculated for the condition that each symbol of an M-ary system occupied $2\pi/(M+G)$ radians, where G was an even integer. The guard band was thus confined to the condition that

$$GB = 2\pi G/(M+G).$$

For systems with a guard band, the number of values of P(e) is (M+G)/2. The values computed for P(e) were the probabilities that the phase error were positive. Normally, these values must be multiplied by 2 if the probabilities of positive and negative phase errors are equal.



$$P_1 = \int_{\theta=\pi/8}^{\theta=3\pi/8} P(\theta)\, d\theta$$

Fig. 9 — Phase error intervals for M = 8, guard band = 0

PROBABILITY OF DISTORTION

The probability that distortion of absolute value D_i will occur for a given code and S/N condition was calculated as

$$P_i = (2/M) \sum_{e=1}^{e=(M+G)/2} N_i\, P(e),$$

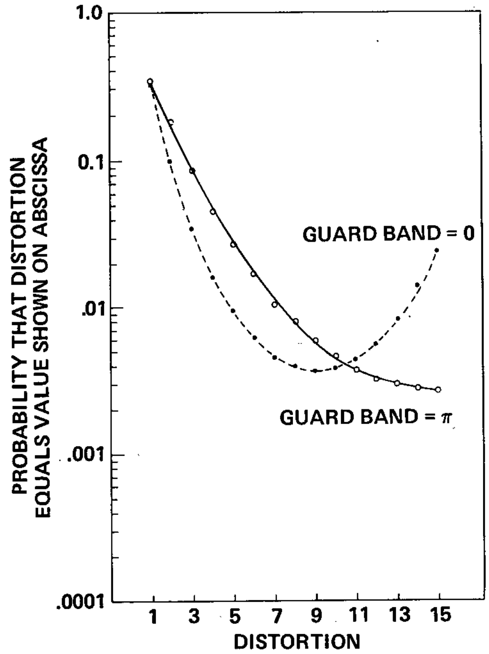


Fig. 10 — Probability of distortion for natural binary code, M = 16, guard band = 0 and π, independent Rayleigh fading channel, 10-dB S/N

where, N_i is the number of pairs of symbols separated by e symbols which produce a distortion of $|D_i|$. For each M-ary system, P_i was computed for values of i from one to M-1.

Figure 10 compares the probability of distortion of the natural binary code M = 16 for the conditions that GB = 0 and GB = π. The S/N was 10 dB. The bell-shaped nature of the GB = 0 curve is the result of circular foldover. The use of a guard band of π caused the value of P_i to decrease significantly for values of distortion greater than M/2. There was also a small degradation (increase) in P_i for distortion values less than M/2. When a guard band less than π was used, the resulting data points fell between the two conditions just described.

The probabilities of distortion for the three J codes for M = 16 and S/N = 10 dB are compared in Fig. 11 to the PD for natural binary code with GB = π. In each case, the value of P_i for distortion greater than M/2 is smaller for the J codes than for the reference system. For low values of distortion, the J1 code is essentially identical to the reference, whereas the J2 is slightly better. For the J3 code, P_i is larger for even values of i than for odd values because the differences between adjacent symbols are predominately 2.

Similar sets of comparisons were made for S/N = 0, 20, and 30 dB. Over this 30-dB range of signal levels the J codes exhibited the same relative advantage over the natural binary code with a guard band of π. That advantage is a lower probability of distortion of values greater than M/2. To determine whether this advantage was gained at the cost of some other desirable feature of an encoding technique, the same codes were compared on the basis of mean square distortion.

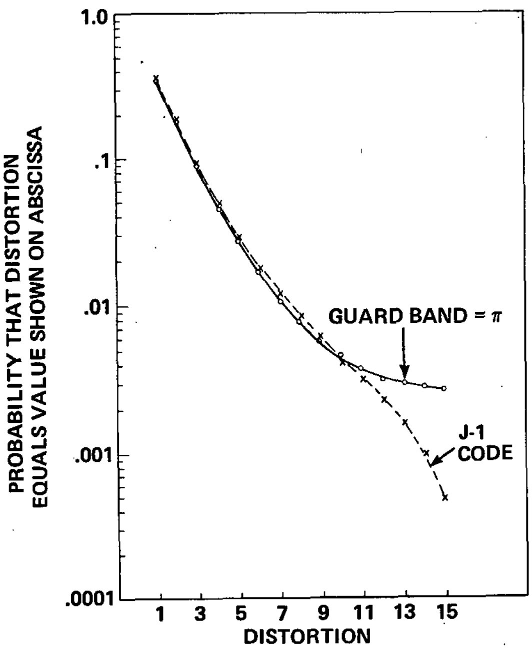


Fig. 11a — Probability of distortion for J1 code, M = 16, independent Rayleigh fading channel, 10-dB S/N

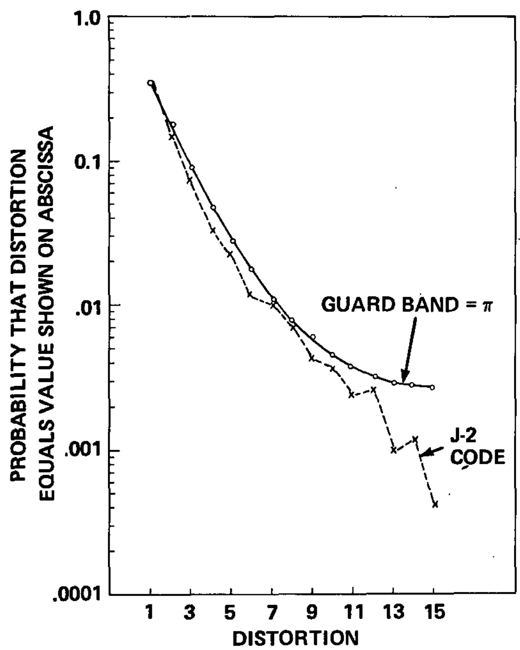


Fig. 11b — Probability of distortion for J2 code, M = 16, independent Rayleigh fading channel, 10-dB S/N

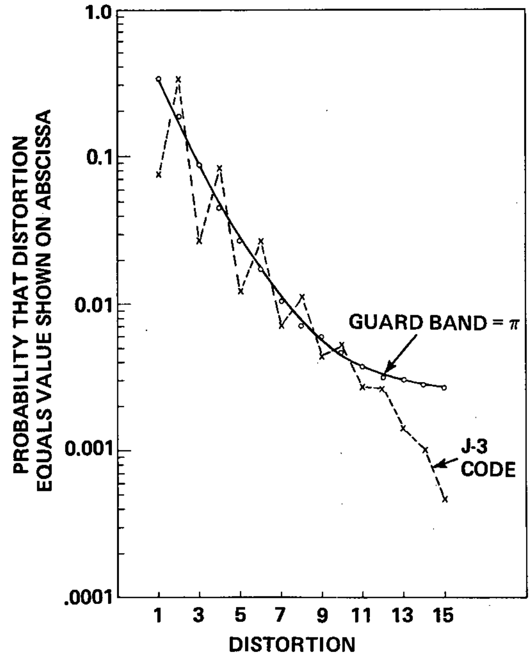


Fig. 11c — Probability of distortion for J3 code, M = 16, independent Rayleigh fading channel, 10-dB S/N

MEAN SQUARE DISTORTION

Mean square distortion is defined here as

$$\mathrm{MSD} = 2 \sum_{e=1}^{e=M/2} P(e) \sum_{i=1}^{i=M} \frac{(u_i - z_{i+e})^2}{M},$$

where

$P(e)$ = probability that a phase error will occur in the range corresponding to the eth symbol from the true symbol

u_i = transmitted PCM symbol

z_{i+e} = received PCM symbol.

This equation can be used to calculate MSD when a guard band is between the lowest and highest symbols, by defining the guard band in terms of symbol widths. If the same definition of guard band is used as given in the previous section, then M is replaced by M+G with G limited to even integers.

Figure 12 shows the resulting MSD as a function of the width of the guard band for the natural binary code, M = 8. Data are shown for S/N levels from 0 to 30 dB, in 5-dB steps. The area defined as the guard band was equally divided between the end points of the binary code. Thus, the width of each of these two symbols became

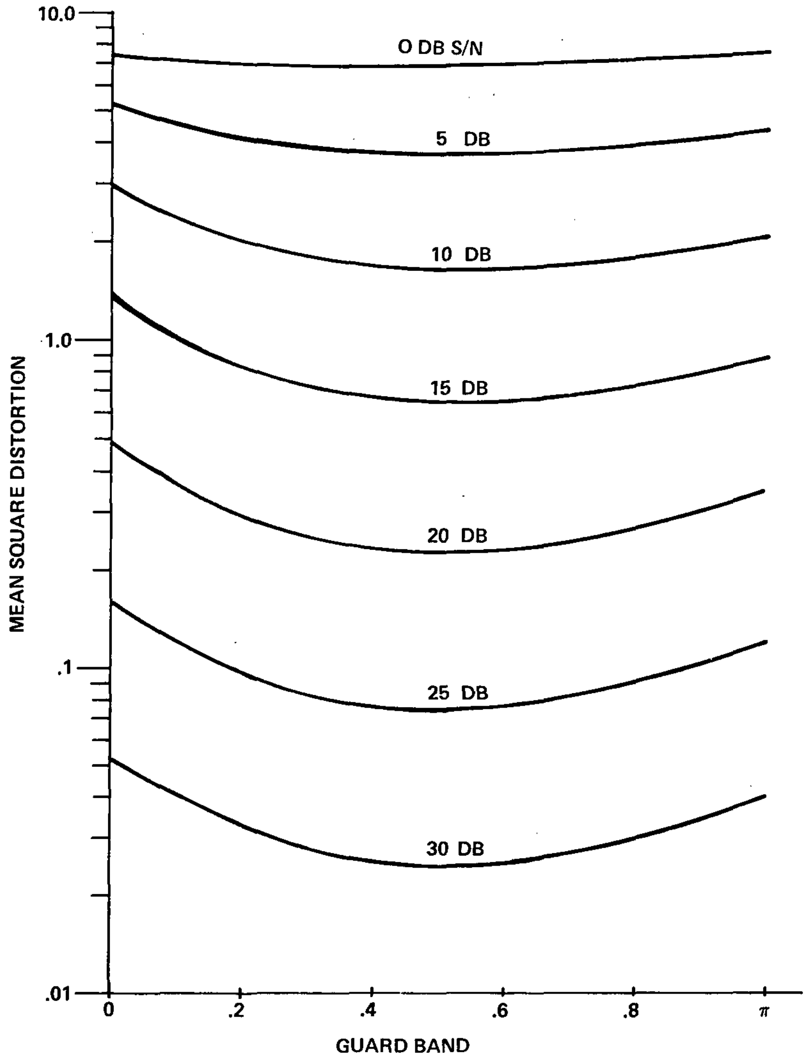


Fig. 12 — Mean square distortion for natural binary code, M = 8, as a function of the width of the guard band, for independent Rayleigh fading channel

$$W = \frac{G\pi}{M+G} \text{ rad.}$$

All other symbols of the M-ary code occupied $2\pi/(M+G)$ rad.

The data shown in Fig. 12 indicate that there is an optimum guard band width for the natural binary code when used on a slowly fading Rayleigh channel. That optimum width is between 0.4π and 0.5π. Similar data were obtained for other M-ary systems. Figures 13 and 14 show the results for 16-phase and 32-phase DPSK systems, respectively. These curves indicate that the advantages of the use of a minimum amount of guard band increase for large M-ary systems and for high S/N conditions. Furthermore, a guard band of 0.4π is near optimum for minimum MSD for any M-ary DPSK system likely to be used on a Rayleigh fading channel.

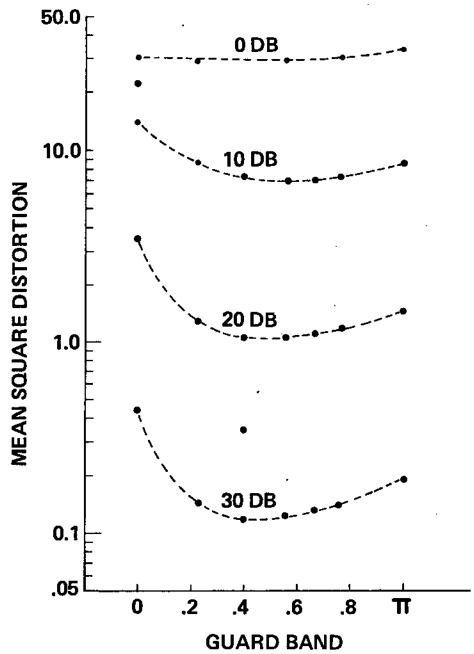


Fig. 13 — Mean square distortion for natural binary code, M = 16, as a function of the width of the guard band, for independent Rayleigh fading channel

Figure 15 compares the MSD as a function of S/N for the M = 16 natural binary code with a guard band of 0, π, and 0.4π. The 0.4π guard band represents the near-optimum condition when the guard band is evenly divided into two sectors and assigned values of zero and M-1. Also shown in Fig. 15 is MSD for the M = 16 Gray code. These data clearly reflect the superiority of the natural binary code with a guard band over the Gray code for transmitting PCM data samples by M-ary DPSK.

The MSD for the three J codes discussed herein were essentially identical. MSD for J1 is shown in Fig. 16. At high S/N the natural binary code with 0.4π guard band was slightly superior to these J code configurations. At very low S/N conditions the J codes were superior. The J codes were always superior to the natural binary code when a guard band of π was used.

CONCLUSIONS

An encoding technique that assigns multiple values to the guard band minimizes the probability of high distortion caused by transmission errors when a PCM data sample is encoded as a single M-ary DPSK symbol. This technique is superior to the use of a guard band of width π which is assigned values of zero and M-1. The mean square distortion of the J codes for a slowly fading Rayleigh channel is always lower than that obtained for the natural binary code with a guard band of π. But, for this channel, the optimum guard band width for minimum MSD is approximately 0.4π. This optimum guard band results in a MSD which at high S/N conditions is slightly lower than that obtained with the J codes.

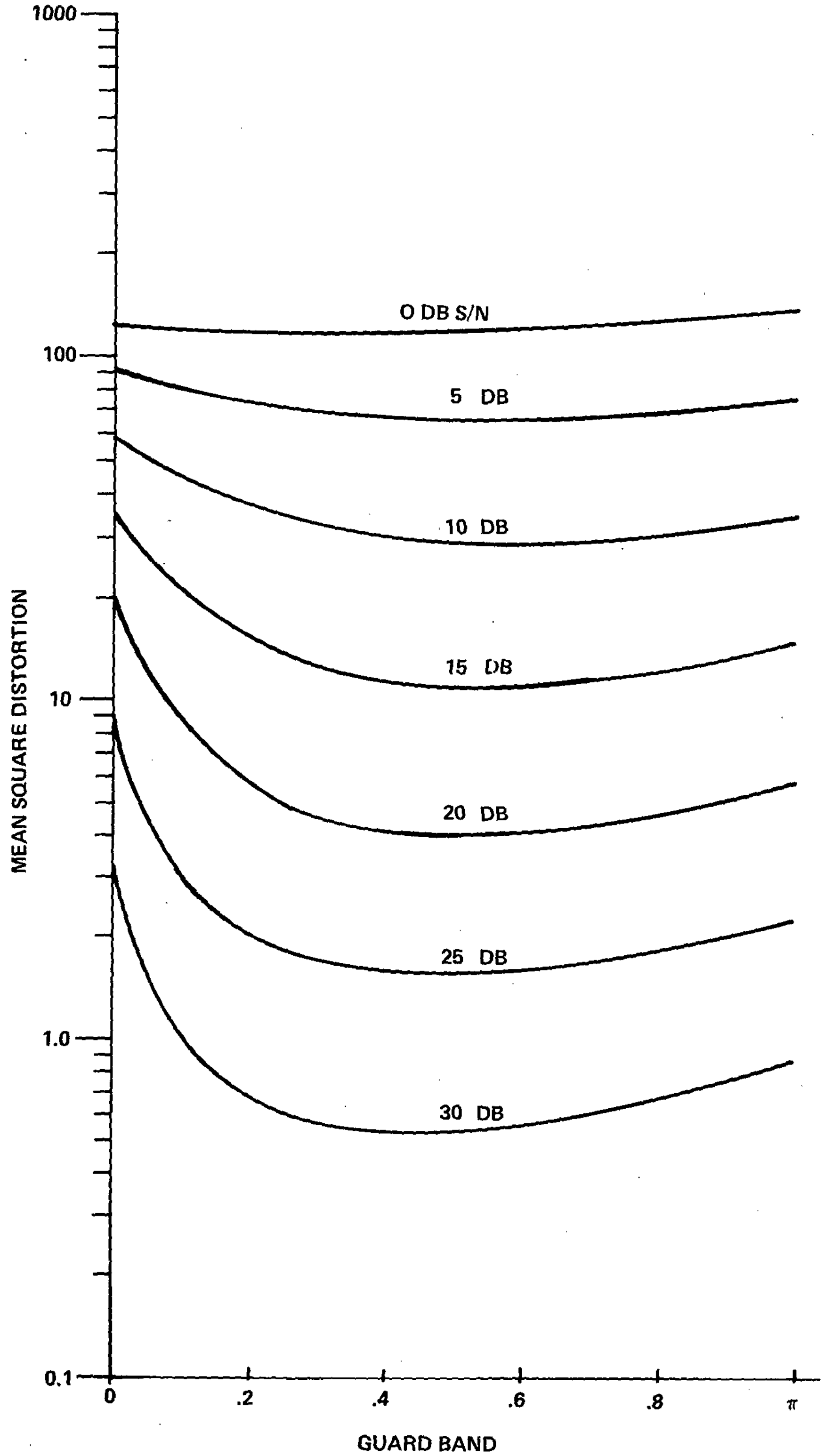


Fig. 14 — Mean square distortion for natural binary code, M = 32, as a function of the width of the guard band, for independent Rayleigh fading channel

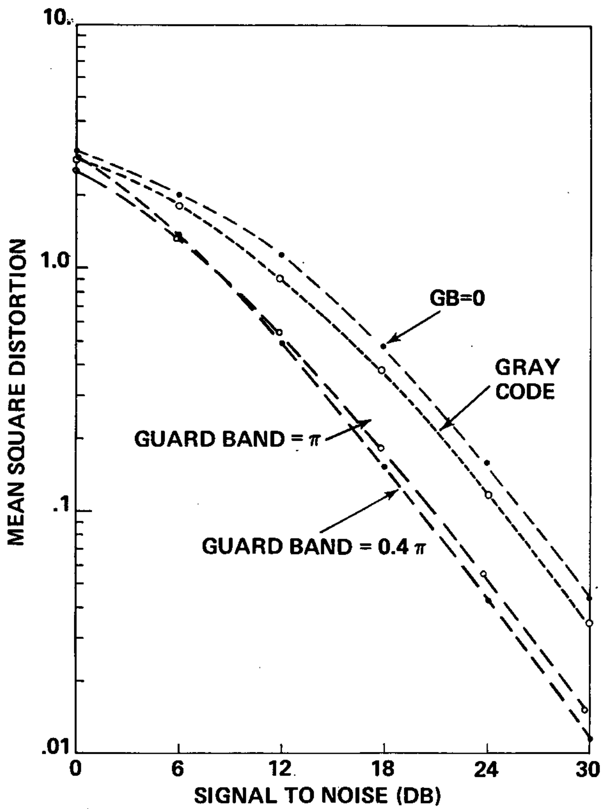


Fig. 15 — Mean square distortion for natural binary code, M = 16, and Gray code, M = 16, vs S/N for independent Rayleigh fading channel

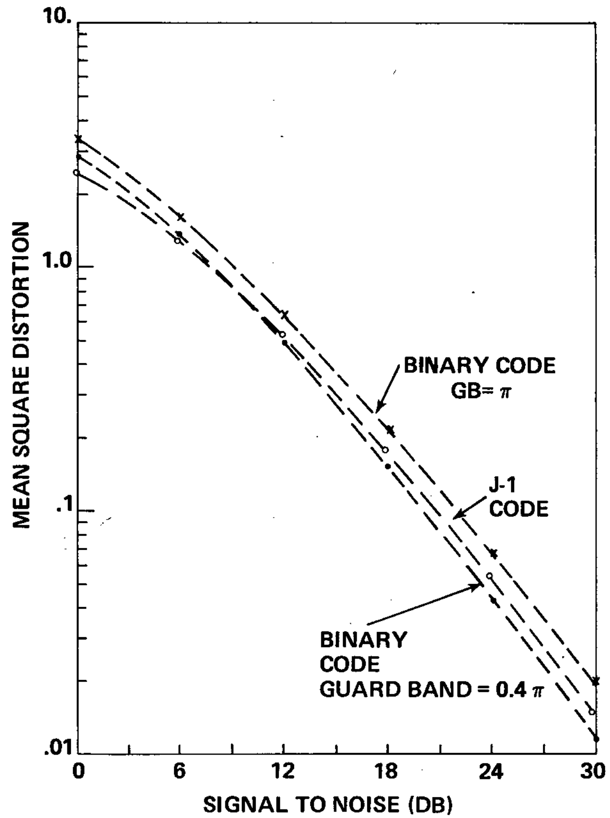


Fig. 16 — Mean square distortion for J1 code, M = 16, vs S/N for independent Rayleigh fading channel

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