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Solitary Occupancy for Unequal Cell Probabilities with Application to Doppler Radars for Ocean Surveillance

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An occupancy problem in combinatorial theory is investigated whose solution has application in ocean surveillance. In particular, theory is developed for calculating the probability of solitary occupancy of N objects in M cells of unequal cell probabilities. For $t = 2, 3, \ldots, N$ the sum of the cell probabilities when all are raised to the tth power is defined as a new variable. These $N-1$ new variables are shown to be sufficient as arguments in representing the above probability as a polynomial equation. As an application, the theory is used to calculate

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20. Abstract (Continued)

the capability of a doppler radar to resolve N ship dopplers contained in the same range and azimuth cell. An integration method is also employed to calculate the same capability. The agreement is excellent between both methods.

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SOLITARY OCCUPANCY FOR UNEQUAL CELL PROBABILITIES WITH APPLICATION TO DOPPLER RADARS FOR OCEAN SURVEILLANCE

INTRODUCTION

There is a need in ocean surveillance to have information on surface ships of every type [1], and candidates for meeting this need are the doppler radars of continuous-wave and pulse types. A concise description of these radars is presented by Skolnik [2,3]. The doppler radar uses range, azimuth, and doppler gating to resolve ships. If more than one ship is contained in the same range and azimuth cell (termed here the ocean-surveillance resolution cell), then to resolve these ships they need to be separated in doppler by an amount equal to or greater than the doppler resolution capability of the radar. The doppler resolution capability is doubled to obtain a doppler segment that is used in decomposing the doppler dimension into disjoint doppler cells of equal width. The doubling accounts for the plus and minus differences needed to resolve in doppler.

The probability that a ship possesses a speed falling in one doppler cell may differ from that probability of falling in another. For a given density of ship speeds the probability associated with each doppler cell can be computed. In general these doppler-cell probabilities will not be equal. The problem of determining a doppler radar's capability to resolve N ships contained in the same range and azimuth cell is therefore a problem of calculating the probability of N ships' dopplers occupying solitary doppler cells when the cells have unequal probabilities. This is an occupancy problem in the category of combinatorial theory.

In combinatorial theory the solution to the solitary occupancy problem is well known for the case that all doppler-cell probabilities are equal [4]. Tractable results are lacking for the case that the doppler-cell probabilities are unequal [5]. The objective of this report is to investigate this latter case.

The investigation is conducted in the next section. Therein the probability of resolving N ships ($N = 2, 3, \ldots, 10$) contained in the same range and azimuth cell is given as a polynomial function of $N - 1$ quantities. The quantities are the sum of the doppler-cell probabilities when raised to the tth power, $t = 2, 3, \ldots, N$. With M representing the number of doppler cells and N the number of ships ($M \geqslant N$, since otherwise solitary occupancy is impossible), the theory of the next section transforms the problem involving M probabilities into a solution embracing $N - 1$ quantities. This transformation is very useful when M is large compared to N. The polynomial functions of the $N - 1$ quantities were obtained with the aid of the computer for $N = 2, 3, \ldots, 10$.

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The third section presents an application of the theory to doppler radars performing ocean surveillance. The obtained results are compared in the fourth section with those calculated by means of an integration method, and an excellent agreement is received.

SOLITARY OCCUPANCY FOR UNEQUAL CELL PROBABILITIES

Problem Statement

Let there be M distinct cells with probabilities $q_1, q_2, \ldots, q_M$ of an object occupying them. Suppose there are N distinct objects, with $N \leqslant M$. An object falling into the M cells must occupy some cell. Therefore

$$\sum_{N=1}^{M} q_i = 1. \tag{1}$$

Problem. Calculate the probability that the N objects occupy solitary cells when they are allowed to fall into the M cells according to the probabilities $q_1, q_2, \ldots, q_M$.

For $t = 1, 2, \ldots, N$ define

$$K_t = \sum_{i=1}^{M} q_i^t. \tag{2}$$

Note that $K_1 = 1$. Define P_N to be the probability that all N objects occupy solitary cells and refer to PF_N as the probability that at least a pair of the N objects will occupy a common cell. Thus,

$$PF_N = 1 - P_N. \tag{3}$$

The formulas for the probabilities P_N and PF_N, where $N = 2, 3, 4$ are derived in the next two subsections. Then the probabilities P_N and PF_N for arbitrary N is discussed. The approach adopted for deriving these formulas is to calculate PF_N first and then obtain P_N from Eq. (3). These formulas are shown to be functions only of the terms K_2, $K_3, \ldots, K_N$.

Solitary Occupancy When $N = 2, 3$

Let i be contained in the set $\{1, 2, \ldots, M\}$, and let C_i denote the ith cell. The probability $P(1 \in C_i, 2 \in C_i)$ that objects 1 and 2 are both contained in the ith cell C_i is given by

$$P(1 \in C_i, 2 \in C_i) = q_i^2. \tag{4}$$

Thus

$$PF_2 = \sum_{i=1}^{M} q_i^2 = K_2 \tag{5}$$

is the probability that the pair of objects will occupy the same cell. The probability P_2 that each object occupies a solitary cell is

$$P_2 = 1 - K_2. \tag{6}$$

For $N = 3$ we fail to have solitary occupancy in the following three mutually exclusive cases.

i. Objects 1 and 2 occupy the same cell. Object 3 occupies any cell.

ii. Objects 1 and 2 occupy separate cells. Objects 2 and 3 occupy the same cell.

iii. Objects 1 and 2 occupy separate cells. Objects 1 and 3 occupy the same cell.

The probability of case iii occurring is equal to that of case ii. The probability of case i is equal to K_2, as obtained in Eq. (5). Thus there is need to calculate only case ii.

Let i be contained in $\{1, 2, \ldots, M\}$. The probability that objects 2 and 3 occupy the ith cell and object 1 occupies a solitary cell is $q_i^2(1 - q_i)$.

Summing over all i, $i = 1, 2, \ldots, M$, we obtain the probability that case ii occurs:

$$\sum_{i=1}^{M} q_i^2(1 - q_i) = K_2 - K_3. \tag{7}$$

As a result

$$PF_3 = K_2 + 2(K_2 - K_3) \tag{8}$$

and

$$P_3 = 1 - 3K_2 + 2K_3. \tag{9}$$

Solitary Occupancy When $N = 4$

For $N = 4$ we fail to have solitary occupancy in the following $(4 - 1)!$ mutually exclusive cases:

a. Same as case i with the addition that object 4 occupies any cell.

b. Same as case ii with the addition that object 4 occupies any cell.

c. Same as case iii with the addition that object 4 occupies any cell.

d. Objects 1, 2, and 3 occupy separate cells. Objects 1 and 4 occupy the same cell.

e. Objects 1, 2, and 3 occupy separate cells. Objects 2 and 4 occupy the same cell.

f. Objects 1, 2, and 3 occupy separate cells. Objects 3 and 4 occupy the same cell.

The probability that either case a, b, or c occurs is equal to PF_3, given by Eq. (8). The probabilities of cases d, e, and f are equal. We therefore calculate only case d.

The probability that objects 1 and 4 occupy the same ith cell is given by q_i^2. The probability that objects 1 and 2 occupy separate cells conditioned on object 1 being located in cell i is

$$\sum_{\substack{j=1 \\ j \neq i}}^{M} q_j . \tag{10}$$

The probability that objects 1, 2, and 3 occupy separate cells conditioned on object 1 and 2 being located in cells i and j respectively is

$$\sum_{\substack{k=1 \\ k \neq j \\ k \neq i}}^{M} q_k . \tag{11}$$

Multiplying q_i^2 times the sums in (10) and (11) and then summing over i, $i = 1, 2, \ldots, M$, gives

$$\sum_{i=1}^{M} q_i^2 \left[\sum_{\substack{j=1 \\ j \neq i}}^{M} q_j \left(\sum_{\substack{k=1 \\ k \neq j \\ k \neq i}} q_k \right) \right] \tag{12}$$

as the probability that case d occurs. Note that

$$\sum_{\substack{k=1 \\ k \neq j \\ k \neq i}}^{M} q_k = 1 - q_j - q_i, \tag{13}$$

so that the two inner sums of (12) become

$$\sum_{\substack{j=1 \\ j \neq i}}^{M} q_j(1 - q_i - q_j) = (1 - q_i) \sum_{\substack{j=1 \\ j \neq i}}^{M} q_j - \sum_{\substack{j=1 \\ j \neq i}}^{M} q_j^2 = (1 - q_i)(1 - q_i) - (K_2 - q_i^2). \tag{14}$$

In view of Eq. (14) the sum in (12) becomes

$$\sum_{i=1}^{M} q_i^2(1 - 2q_i + 2q_i^2 - K_2) = K_2 - 2K_3 + 2K_4 - K_2^2. \tag{15}$$

Consequently, the probability that cases d, e, or f occur is equal to $3(K_2 - 2K_3 + 2K_4 - K_2^2)$, and the probability PF_4 is given by

$$PF_4 = PF_3 + 3(K_2 - 2K_3 + 2K_4 - K_2^2). \tag{16}$$

As a result

$$P_4 = P_3 - 3(K_2 - 2K_3 + 2K_4 - K_2^2) \tag{17}$$

or, in view of Eq. (9),

$$P_4 = 1 - 6K_2 + 8K_3 - 6K_4 + 3K_2^2. \tag{18}$$

Solitary Occupancy for Arbitrary $N \leqslant M$

Suppose we have N objects with $N \geqslant 4$. Assume that we have calculated PF_{N-1} and therefore P_{N-1}. Consider the following two mutually exclusive situations:

I. At least two objects out of the first $N-1$ objects share a cell together. The Nth object is contained in any cell.

II. None of the first $N-1$ objects share a common cell. Let $i \in \{1, 2, \ldots, N-1\}$. The Nth object shares a cell with the ith object.

The probability that case I occurs is PF_{N-1}. Observe that the probabilities of case II for $i \in \{1, 2, \ldots, N-1\}$ are equal. Thus we calculate the probability for case II to occur with $i = 1$. Consider the following sum defined as $PF(\mathrm{II}, N, 1)$:

$$PF(\mathrm{II}, N, 1) \equiv \sum_{j_1=1}^{M} q_{j_1}^2 \left(\sum_{\substack{j_2=1 \\ j_2 \neq j_1}}^{M} q_{j_2} \left\{ \sum_{\substack{j_3=1 \\ j_3 \neq j_2 \\ j_3 \neq j_1}}^{M} q_{j_3} \left[\cdots \left(\sum_{\substack{j_{N-1}=1 \\ j_{N-1} \neq j_i \\ \text{for } 1 \leqslant i < N-1}}^{M} q_{j_{N-1}} \right) \cdots \right] \right\} \right) \tag{19}$$

The first sum, $\Sigma q_{j_1}^2$, states that the Nth object and the 1st object occupy the same cell. The second sum is conditioned on the two objects 1 and N occupying the j_1th cell. It states that the 2nd object occupies a cell separate from object 1. The third sum is conditioned on objects 1 and N occupying the j_1th cell and the 2nd object occupying the j_2th cell. This sum corresponds to the 3rd object occupying a cell other than those occupied by the objects 1, 2, and N. The other sums are similar, with the last sum being conditioned on objects 1 and N occupying the j_1th cell, object 2 occupying the j_2th cell, ..., and the $(N-2)$th object occupying the j_{N-2}th cell.

We designate $PF(\mathrm{II}, N, i)$ to be the probability that case II occurs for $i = 1, 2, \ldots, N-1$. This notation corresponds with that given in Eq. (19). The probability $PF(\mathrm{II}, N)$ that case II occurs for any i is

$$PF(\mathrm{II}, N) = \sum_{i=1}^{N-1} PF(\mathrm{II}, N, i) = (N-1)PF(\mathrm{II}, N, 1), \tag{20}$$

since $PF(\mathrm{II}, N, i) = PF(\mathrm{II}, N, 1)$ for $i = 2, 3, \ldots, N-1$.

The probability PF_N of failing to have solitary occupancy for all objects is

$$PF_N = PF_{N-1} + (N-1)PF(\mathrm{II}, N, 1). \tag{21}$$

Thus the probability of having solitary occupancy is

$$P_N = P_{N-1} - (N-1)PF(\mathrm{II}, N, 1). \tag{22}$$

By induction it can be verified that P_N is a polynomial function of $K_2, K_3, \ldots, K_N$ Ref. 6. For example it can be shown that

$$P_5 = 1 - 10K_2 + 20K_3 - 30K_4 + 24K_5 - 20K_2K_3 + 15K_2^2. \tag{23}$$

Note that in each of the equations (6), (9), (18), and (23) the sum total of the coefficients of the negative terms is equal to the sum total of the coefficients of the positive terms.

A computer program has been used in Ref. 6 to derive the algebraic formulas for $P_6, P_7, \ldots, P_{10}$. The probabilities P_{11} and above can be derived by the procedure described therein. The computer program uses prime numbers as variables for the purpose of reducing the sum in Eq. (19) to a polynomial equation in terms of $K_2, K_3, \ldots, K_N$. Thus $PF(\mathrm{II}, N, 1)$ as a function of $K_2, K_3, \ldots, K_N$ is substituted into Eq. (22) to obtain P_N as a function of $K_2, K_3, \ldots, K_N$. In this manner we obtain

$$\begin{aligned} P_6 = {} & 1 - 15K_2 + 40K_3 - 90K_4 + 144K_5 - 120K_6 + 90K_4K_2 \\ & + 40K_3^2 - 120K_3K_2 - 15K_2^3 + 45K_2^2, \end{aligned} \tag{24}$$

$$\begin{aligned} P_7 = {} & 1 - 21K_2 + 70K_3 - 210K_4 + 504K_5 - 840K_6 + 720K_7 \\ & - 504K_5K_2 - 420K_4K_3 + 630K_4K_2 + 280K_3^2 \\ & + 210K_3K_2^2 - 420K_3K_2 - 105K_2^3 + 105K_2^2. \end{aligned} \tag{25}$$

$$\begin{aligned} P_8 = {} & 1 - 28K_2 + 112K_3 - 420K_4 + 1344K_5 - 3360K_6 \\ & + 5760K_7 - 5040K_8 + 3360K_6K_2 + 2688K_5K_3 \\ & - 4032K_5K_2 + 1260K_4^2 - 3360K_4K_3 - 1260K_4K_2^2 \\ & + 2520K_4K_2 - 1120K_3^2K_2 + 1120K_3^2 + 1680K_3K_2^2 \\ & - 1120K_3K_2 + 105K_2^4 - 420K_2^3 + 210K_2^2. \end{aligned} \tag{26}$$

The equations for P_9 and P_{10} are tabulated in Tables 1 and 2 respectively, since they are lengthy. To obtain P_9 from Table 1 multiply the sign, the coefficient and the factor together and then add all resulting products; that is,

$$P_9 = 1 - 36K_2 + 378K_2^2 - 1260K_2^3 + \cdots, \tag{27}$$

making use of $K_1 = 1$. Obtaining P_{10} from Table 2 is similar. The note immediately following Eq. (23) applies also to Eqs. (24) through (26) and to P_9 and P_{10} contained in Tables 1 and 2.

In obtaining $PF(\mathrm{II}, N, 1)$ as a function of $K_2, K_3, \ldots, K_N$ one must collect $(N-1)!$ terms to reduce the sum of Eq. (19) to an algebraic equation in these quantities. Thus in deriving P_{11} one would have the lifetime task of more than 3 million operations. The computer performs this task in less than 30 seconds with the computer program described in Ref. 6.

Table 1
Equation for P_9

Sign	Coefficient	Factor	Sign	Coefficient	Factor
+	1	K_1 (=1)	–	15120	K_4K_3
–	36	K_2	+	15120	$K_4K_3K_2$
+	378	K_2^2	+	11340	K_4^2
–	1260	K_2^3	+	3024	K_5
+	945	K_2^4	–	18144	K_5K_2
+	168	K_3	+	9072	$K_5K_2^2$
–	2520	K_3K_2	+	24192	K_5K_3
+	7560	$K_3K_2^2$	–	18144	K_5K_4
–	2520	$K_3K_2^3$	–	10080	K_6
+	3360	K_3^2	+	30240	K_6K_2
–	10080	$K_3^2K_2$	–	20160	K_6K_3
+	2240	K_3^3	+	25920	K_7
–	756	K_4	–	25920	K_7K_2
+	7560	K_4K_2	–	45360	K_8
–	11340	$K_4K_2^2$	+	40320	K_9

Table 2
Equation for P_{10}

Sign	Coefficient	Factor	Sign	Coefficient	Factor
+	1	K_1 (=1)	+	56700	K_4^2
-	45	K_2	-	56700	$K_4^2K_2$
+	630	K_2^2	+	6048	K_5
-	3150	K_2^3	-	60480	K_5K_2
+	4725	K_2^4	+	90720	$K_5K_2^2$
-	945	K_2^5	+	120960	K_5K_3
+	240	K_3	-	120960	$K_5K_3K_2$
-	5040	K_3K_2	-	181440	K_5K_4
+	25200	$K_3K_2^2$	+	72576	K_5^2
-	25200	$K_3K_2^3$	-	25200	K_6
+	8400	K_3^2	+	151200	K_6K_2
-	50400	$K_3^2K_2$	-	75600	$K_6K_2^2$
+	25200	$K_3^2K_2^2$	-	201600	K_6K_3
+	22400	K_3^3	+	151200	K_6K_4
-	1260	K_4	+	86400	K_7
+	18900	K_4K_2	-	259200	K_7K_2
-	56700	$K_4K_2^2$	+	172800	K_7K_3
+	18900	$K_4K_2^3$	-	226800	K_8
-	50400	K_4K_3	+	226800	K_8K_2
+	151200	$K_4K_3K_2$	+	403200	K_9
-	50400	$K_4K_3^2$	-	362880	K_{10}

APPLICATION TO DOPPLER RADARS

Consider a doppler radar system which has a doppler resolution capability of d_r knots for some fixed operating frequency and some coherent integration time of processing. We assume this radar system is providing surveillance of an ocean region containing merchant vessels as well as ships of naval interest. Let (α_1, β_1) represent the latitudinal and longitudinal coordinates of a position in this ocean region. Suppose there is a shipping lane several hundred nautical miles wide passing through the ocean region and containing (α_1, β_1). Let θ be the angle between the shipping lane direction and the radial direction to the radar site from the coordinate (α_1, β_1). The coordinate (α_1, β_1) is contained in some ocean-surveillance resolution cell whose size is determined by the beamwidth, pulsewidth, operating frequency, and range to (α_1, β_1), Refs. 7 and 8. If two ships are traveling with speeds s_1 and s_2 in the same direction along the shipping lane and have their positions in the ocean-surveillance resolution cell containing (α_1, β_1), then these two ships are resolvable in doppler provided

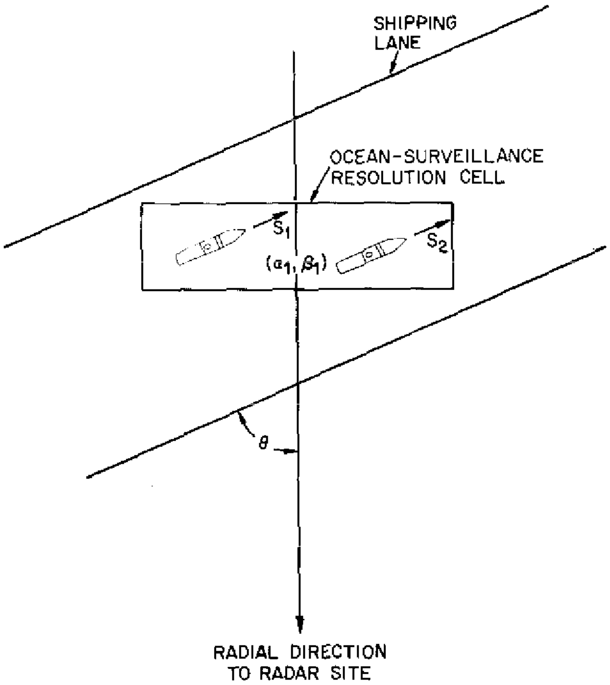

Fig. 1—Two ships traveling in the same ocean-surveillance resolution cell

$$|s_1 - s_2| \geqslant \frac{d_r}{\cos\theta} \text{ knots}, \quad 0 \leqslant \theta < \pi/2. \tag{28}$$

This follows since their speeds appear to be $s_1 \cos\theta$ and $s_2 \cos\theta$ with respect to the radial direction of the radar beam (Fig. 1). Two speeds viewed from the standpoint of the radar systems must be separated by a doppler distance of d_r knots to resolve speed ambiguities. Define $d_r(\theta)$ to be the doppler resolution capability of the radar as viewed in a direction that makes an angle θ, $0 \leqslant \theta < \pi/2$, with the radial line to the radar site. Thus

$$d_r(\theta) = \frac{d_r}{\cos\theta} \cdot \tag{29}$$

A statistical analysis of the speed of ships in the world's merchant fleets is tabulated in Ref. 9. Rounding off all speeds to whole numbers, this reference provides the number of ships that steam at any given speed between 6 and 32 knots. These values are presented in Table 3; they represent all types of merchant vessels of 1000 gross tons and over—combination passenger-and-cargo vessels, freighters, bulk carriers, and tankers. The number of ships in the category 1000 gross tons and over is 20,544. The cumulative distribution function of speed is obtained from the data of Table 3 and is plotted in Fig. 2 over the speed range from 5.5 to 32.5 knots. We denote this cumulative distribution function by f. Thus the probability that a randomly selected ship travels at a nominal speed between the speeds s_1 and s_2 is $|f(s_2) - f(s_1)|$. The density function of speed is plotted in Fig. 3

Table 3
Speed Distribution of the World's Merchant Fleets of 1000 Gross Tons and Over
(U.S. Department of Commerce, December 1971)

Nominal Speed (knots)	Number of Ships	Nominal Speed (knots)	Number of Ships
6	2	20	385
7	15	21	181
8	68	22	116
9	327	23	63
10	1074	24	6
11	1307	25	4
12	2486	26	7
13	2135	27	3
14	3250	28	2
15	3543	29	1
16	3041	30	0
17	1651	31	2
18	701	32	1
19	273		

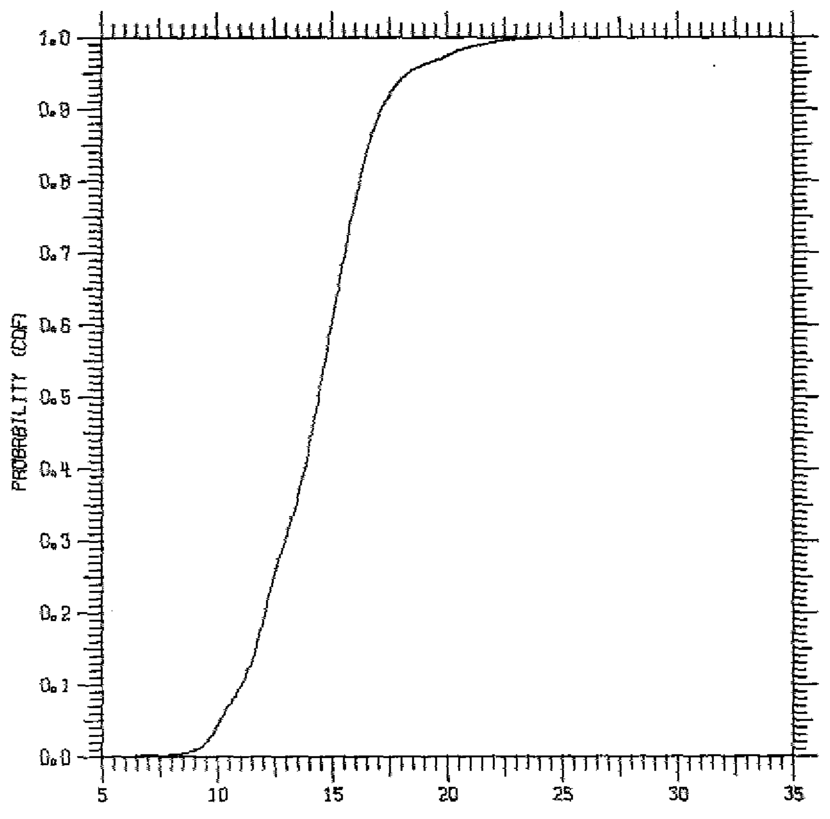

Fig. 2—Cumulative distribution function (CDF) of speed for the world's merchant fleets (obtained from Table 3)

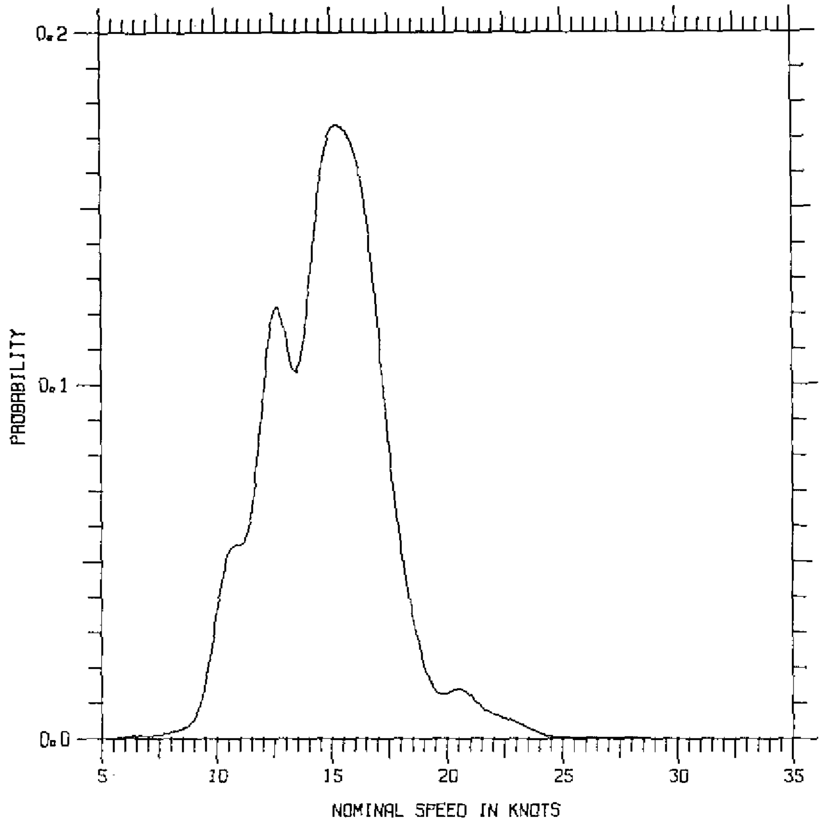

Fig. 3—Density function of speed for the world's merchant fleets

A doppler radar can distinguish between a speed that is advancing and one that is receding. Negative numbers will be used to denote receding ship speeds. Thus the doppler range of the radar as viewed along a shipping lane is

$$[-32.5, -5.5] \cup [5.5, 32.5].$$

This doppler range can be decomposed into M disjoint doppler cells of width $2d_r(\theta)$, where

$$M = \frac{2(27)}{2d_r(\theta)}. \tag{30}$$

If the right-hand side of Eq. (30) is not an integer, then we round it off to the next integer. The width of each cell is $2d_r(\theta)$, since a ship with speed s is resolvable provided no other ship steaming in the same ocean-surveillance resolution cell has its speed between $s - d_r(\theta)$ and $s + d_r(\theta)$. For $i = 1, 2, \ldots, M/2$ we denote the ith cell C_i by

$$C_i = [5.5 + (i - 1)d_r(\theta),\ 5.5 + id_r(\theta)]. \tag{31}$$

The probability q_i that a ship has an advancing speed in C_i is given by

$$q_i = \frac{f[5.5 + id_r(\theta)] - f[5.5 + (i-1)d_r(\theta)]}{2}. \tag{32}$$

For $j = (M/2) + 1, (M/2) + 2, \ldots, M$ we denote the jth cell C_j by

$$C_j = [-5.5 - id_r(\theta), -5.5 - (i-1)d_r(\theta)). \tag{33}$$

The probability q_j that a ship has a receding speed in C_j is given by

$$q_j = f\left[5.5 + \left(j - \frac{M}{2}\right)d_r(\theta)\right] - f\left[5.5 + \left(j - \frac{M}{2} - 1\right)d_r(\theta)\right]. \tag{34}$$

In writing Eqs. (31) and (33) we assumed that a ship has an equal likelihood of steaming in either direction along the shipping lane. Note that

$$q_i = q_{i+(M/2)} \tag{35}$$

for $i = 1, 2, \ldots, M/2$. Therefore Eq. (2) becomes

$$K_t = 2 \sum_{i=1}^{M/2} q_i^t. \tag{36}$$

Define for $i = 1, 2, \ldots, M/2$

$$\hat{q}_i = 2q_i, \tag{37}$$

$$\hat{K}_t = \sum_{i=1}^{M/2} \hat{q}_i^t, \tag{38}$$

$$\hat{C}_i = \left\{s\colon 5.5 + (i-1)d_r(\theta) \leqslant |s| \leqslant 5.5 + id_r(\theta)\right\}, \tag{39}$$

$$\hat{M} = \frac{M}{2}. \tag{40}$$

Note that

$$\hat{K}_t = 2^{t-1}K_t. \tag{41}$$

The quantities $\{\hat{q}_i, \hat{C}_i, \hat{K}_i, i = 1, 2, \ldots, M\}$ when substituted into the framework of the preceding main section lead to the probability of resolving all ships under the condition that all ships are headed in the same direction of the shipping lane. Since a doppler system distinguishes advancing ships separately from receding ships, the quantities $\{q_i, C_i, K_i, i = 1, 2, \ldots, M\}$ when substituted into the theory of the preceding section lead to the probability of resolving all ships located in the same ocean-surveillance resolution cell.

We qualify these ideas with an example. Let $d_r(\theta) = 1.5$ knots. From Eq. (30) we have $M = 18$ cells. The left and right endpoints of the nine positive cells $C_1, C_2, \ldots, C_9$

Table 4
Cell Probabilities With $d_r(\theta) = 1.5$ Knots

Indices i	Endpoint for Cell C_i		$2q_i$
	Left	Right	
1	5.5	8.5	0.0041
2	8.5	11.5	0.1318
3	11.5	14.5	0.3831
4	14.5	17.5	0.4009
5	17.5	20.5	0.0613
6	20.5	23.5	0.0173
7	23.5	26.5	0.0010
8	26.5	29.5	0.0003
9	29.5	32.5	0.0002

Table 5
Values of K_t With $d_r(\theta) = 1.5$ Knots

Indices t	K_t
2	0.16446
3	0.03079
4	0.00596
5	0.00117
6	0.00023
7	0.00004

are given in Table 4. The probabilities q_i are doubled (Table 4). The quantities K_2, $K_3, \ldots, K_7$ are listed in Table 5, and the probabilities $P_2, P_3, \ldots, P_7$ calculated using Eqs. (6), (9), (18), (23), (24), and (25) are listed in Table 6.

The cells C_i, $i = 1, 2, \ldots, M/2$, defined by means of Eq. (30) initiate at 5.5 knots and continue sequentially with constant width of $2d_r(\theta)$ up to 32.5 knots. This design procedure will usually bring forth biased results, since the best probabilities are obtained when two adjoining members of the cells meet exactly at the peak of Fig. 3 (nominal speed = 14.7 knots) and the worst probabilities occur when one member of the cells straddles the peak. This follows because the summit of Fig. 3 contains those speeds of highest likelihood. It is better designing to have the cells divide the summit equally rather than to have the entire summit contained in only one cell. Because of this, an averaging technique is employed to obtain unbiased results. Let n be an integer greater than or equal to 2. Define

$$\Delta \equiv \frac{2d_r(\theta)}{n} \tag{42}$$

Table 6
Probability of Resolving N Ships
With $d_r(\theta) = 1.5$ Knots

Indices N	P_N
2	0.836
3	0.568
4	0.305
5	0.125
6	0.038
7	0.008

and for each $k = 1, 2, \ldots, n$ translate all cells, $i = 1, 2, \ldots, M/2$, by the amount $(k-1)\Delta$:

$$C_i(k) = C_i - (k-1)\Delta. \tag{43}$$

Additional cells are needed when the right end of the speed spectrum [5.5, 32.5] is uncovered by the translation. Then for each $k = 1, 2, \ldots, n$ the probabilities $P_N(k)$, with $P_N(k)$ denoting the probabilities P_N calculated for the translation $(k-1)\Delta$, are obtained using the theory of the preceding section. Averaging these probabilities, we have

$$p_N \equiv \frac{1}{n} \sum_{k=1}^{n} P_N(k). \tag{44}$$

Since this probability is unbiased for large n ($n = 4$ is usually sufficient), we adopt it as the probability of resolving N ships located in the same ocean surveillance resolution cell. For $N = 2, 3, \ldots, 7$ this probability is calculated with $n = 4$ and $d_r(\theta) = 0.1, 0.2, \ldots, 4.9$. The cases $N = 2$, 3, and 4 are tabulated out to three significant figures in Tables 7a, 7b, and 7c respectively. From Table 7a the probability of resolving two ships is equal to 0.755 for a doppler capability of 2.5 knots. The cases $N = 5$, 6, and 7 are plotted in Fig. 4.

Table 7a
Probability of Resolving Two Ships
Calculated Using Eq. (44) With $d_r(\theta) = 0$ through 4.9 Knots

	0.0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9
0	1.000	0.988	0.976	0.965	0.953	0.942	0.930	0.919	0.908	0.897
1	0.887	0.876	0.866	0.856	0.846	0.837	0.828	0.819	0.810	0.801
2	0.793	0.785	0.777	0.769	0.762	0.755	0.748	0.741	0.735	0.729
3	0.722	0.716	0.710	0.705	0.699	0.694	0.690	0.686	0.682	0.678
4	0.675	0.671	0.668	0.665	0.662	0.658	0.655	0.652	0.649	0.645

Table 7b
Probability of Resolving Three Ships

	0.0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9
0	1.000	0.965	0.930	0.897	0.864	0.833	0.802	0.773	0.744	0.717
1	0.690	0.665	0.640	0.617	0.594	0.573	0.552	0.532	0.513	0.496
2	0.478	0.462	0.446	0.431	0.417	0.404	0.391	0.379	0.367	0.356
3	0.345	0.335	0.325	0.315	0.306	0.298	0.290	0.283	0.276	0.271
4	0.265	0.260	0.255	0.249	0.244	0.239	0.234	0.229	0.224	0.219

Table 7c
Probability of Resolving Four Ships

	0.0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9
0	1.000	0.931	0.865	0.803	0.745	0.690	0.639	0.591	0.547	0.505
1	0.467	0.431	0.397	0.366	0.338	0.312	0.288	0.265	0.245	0.226
2	0.209	0.193	0.178	0.165	0.152	0.141	0.131	0.122	0.113	0.106
3	0.099	0.092	0.086	0.081	0.076	0.072	0.068	0.065	0.062	0.059
4	0.057	0.055	0.053	0.051	0.049	0.047	0.046	0.044	0.043	0.042

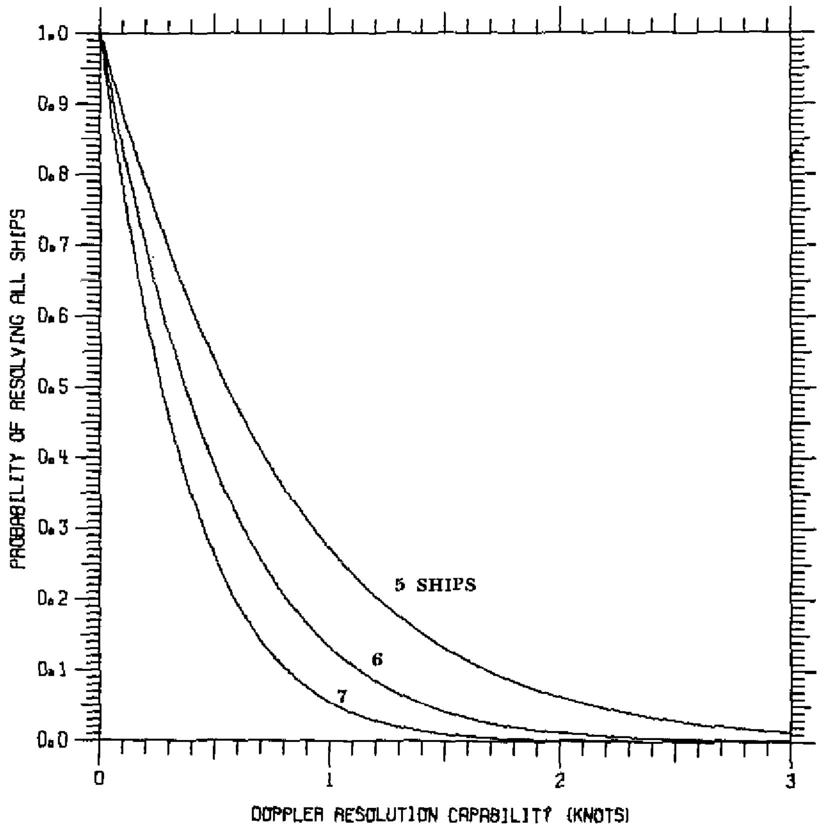

Fig. 4—Probability of resolving five, six, and seven ships calculated using Eq. (44)

COMPARISON WITH AN INTEGRATION METHOD

A check on the results given in the previous section can be made by tediously integrating over conditional probabilities. We will make such a check, maintaining the previous notation. We will first consider two ships per ocean-surveillance resolution cell and then consider three ships.

If two ships are traveling with speeds s_1 and s_2, both ships are resolved provided

$$|s_1 - s_2| \geqslant d_r(\theta), \tag{45}$$

where $d_r(\theta)$ is the doppler resolution capability of the radar. The likelihood $P_\ell(s_1)$ of a ship traveling at a speed s_1 is equal to the derivative of the cumulative distribution function f evaluated at s_1. This derivative is the density function of speed. Thus

$$P_\ell(s_1) = \frac{df(s_1)}{ds}. \tag{46}$$

For any given speeds s_1 and s_2 contained in $[5.5, 32.5]$ we define

$$F(s_1, s_2) \equiv 1, \text{ if } |s_1 - s_2| < d_r(\theta), \tag{47a}$$

$$\equiv 0, \text{ if } |s_1 - s_2| \geqslant d_r(\theta). \tag{47b}$$

Thus s_1 and s_2 are resolvable with respect to each other (Eq. 45 holds) if and only if $F(s_1, s_2) = 0$.

The probability $P_2[d_2(\theta), \text{s.d.}]$ of resolving two ships that are traveling in the same direction (s.d.), selected at random from the stockpile of merchant ships, is equal to 1 minus that of failing. Therefore

$$P_2[d_r(\theta), \text{s.d.}] = 1 - \int_{5.5}^{32.5} \int_{5.5}^{32.5} F(s_1, s_2) \frac{df(s_2)}{ds} \frac{df(s_1)}{ds} ds_2 ds_1. \tag{48}$$

It is convenient to let $f(s) = 0$ for all $s \leqslant 5.5$ and let $f(s) = 1$ for all $s \geqslant 32.5$. In view of this Eq. (48) becomes

$$P_2[d_r(\theta), \text{s.d.}] = 1 - \int_{5.5}^{32.5} \left\{ f[s_1 + d_r(\theta)] - f[s_1 - d_r(\theta)] \right\} \frac{df(s_1)}{ds} ds_1. \tag{49}$$

The probability $P_2[d_r(\theta), \text{s.d.}]$ is tabulated in Table 8 for $d_r(\theta) = 0.5, 1.0, \ldots, 5.0$ knots. These values were obtained by using an integration procedure in a computer program.

In calculating the probability of resolving two ships when each can travel in either direction, we need to consider the following mutually exclusive cases:

Table 8
Probability of Resolving Two Ships
Traveling in the Same Direction
Calculated Using Eq. (49)

Doppler Resolution $d_r(\theta)$ (knots)	$P_2[d_2(\theta), \text{s.d.}]$
0.5	0.883
1.0	0.774
1.5	0.674
2.0	0.586
2.5	0.510
3.0	0.445
3.5	0.391
4.0	0.348
4.5	0.312
5.0	0.282

1. First ship is advancing, second ship is receding
2. First ship is receding, second ship is advancing
3. Both ships are advancing
4. Both ships are receding.

The probability of case 1 or 2 occurring is 0.5, and the probability of case 3 or 4 occurring is also 0.5. So the probability $p_2[d_r(\theta)]$ of resolving two ships is

$$p_2[d_r(\theta)] = 0.5 + 0.5P_2[d_r(\theta), \text{s.d.}]. \tag{50}$$

The probability $p_2[d_r(\theta)]$ is given in Table 9 for $d_r(\theta) = 0.5, 1.0, \ldots, 5.0$. The agreement between these results and those contained in Table 7a is excellent.

Consider three ships traveling with speeds s_1, s_2, and s_3. We shall calculate first the probability of resolving them when they are traveling in the same direction. Three ships traveling in the same direction are failed to be resolved if the following three mutually exclusive events occur:

a. $|s_1 - s_2| < d_r(\theta), \quad \text{all } s_3,$ (51a)

b. $|s_1 - s_2| \geqslant d_r(\theta), \quad |s_3 - s_1| < d_r(\theta),$ (51b)

c. $|s_1 - s_2| \geqslant d_r(\theta), \quad |s_3 - s_1| \geqslant d_r(\theta), \quad |s_3 - s_2| < d_r(\theta).$ (51c)

The probability of failing due to case a is given by

Table 9
Probability of Resolving Two Ships
Using the Integration Method (Eq. (50))

Doppler Resolution $d_r(\theta)$ (knots)	p_2
0.5	0.942
1.0	0.887
1.5	0.837
2.0	0.793
2.5	0.755
3.0	0.723
3.5	0.696
4.0	0.674
4.5	0.656
5.0	0.641

$$\int_{5.5}^{32.5}\int_{5.5}^{32.5} F(s_1, s_2)\frac{df(s_2)}{ds}\frac{df(s_1)}{ds}\,ds_2 ds_1$$

$$= \int_{5.5}^{32.5}\left\{f[s_1 + d_r(\theta)] - f[s_1 - d_r(\theta)]\right\}\frac{df(s_1)}{ds}\,ds_1. \tag{52}$$

Define the function $S(.,.)$ by

$$S(s_1, s_2) = 1 - F(s_1, s_2). \tag{53}$$

for all s_1 and s_2 contained in [5.5, 32.5].

The probability of failing due to case b is

$$\int_{5.5}^{32.5}\int_{5.5}^{32.5}\int_{5.5}^{32.5} S(s_1, s_2)F(s_1, s_3)\frac{df(s_3)}{ds}\frac{df(s_2)}{ds}\frac{df(s_1)}{ds}\,ds_3 ds_2 ds_1$$

$$= \int_{5.5}^{32.5}\left\{f[s_1 + d_r(\theta)] - f[s_1 - d_r(\theta)]\right\}\left\{1 - f[s_1 + d_r(\theta)]\right.$$

$$\left. + f[s_1 - d_r(\theta)]\right\}\frac{df(s_1)}{ds}\,ds_1. \tag{54}$$

The probability of failing due to case c is

$$\int_{5.5}^{32.5}\int_{5.5}^{32.5}\int_{5.5}^{32.5} S(s_1, s_2)S(s_1, s_3)F(s_2, s_3)\,\frac{df(s_3)}{ds}\,\frac{df(s_2)}{ds}\,\frac{df(s_1)}{ds}\,ds_3 ds_2 ds_1$$

$$= \int_{5.5}^{32.5}\int_{5.5}^{s_1 - d_r(\theta)} \Big\{ f[B(s_1, s_2)] - f[s_2 - d_r(\theta)] \Big\} \frac{df(s_2)}{ds}\,\frac{df(s_1)}{ds}\,ds_2 ds_1$$

$$+ \int_{5.5}^{32.5}\int_{s_1 + d_r(\theta)}^{32.5} \Big\{ f[s_2 + d_r(\theta)] - f[A(s_1, s_2)] \Big\} \frac{df(s_2)}{ds}\,\frac{df(s_1)}{ds}\,ds_2 ds_1, \tag{55}$$

where $B(s_1, s_2)$ is the minimum of $s_1 - d_r(\theta)$ and $s_2 + d_r(\theta)$ and where $A(s_1, s_2)$ is the maximum of $s_1 + d_r(\theta)$ and $s_2 - d_r(\theta)$.

The probability $P_3[d_r(\theta), \text{s.d.}]$ of resolving three ships traveling in the same direction is given by

$$P_3[d_r(\theta), \text{s.d.}] = 1 - [\text{right-hand side of Eqs. (52), (54), and (55)}]. \tag{56}$$

Equations (52), (54), and (55) were integrated on a computer for $d_r(\theta) = 0.5, 1.0, \ldots, 5.0$. These values were then substituted into Eq. (56) to obtain the Table 10 values.

Table 10
Probability of Resolving Three Ships Traveling in the Same Direction Calculated Using Eq. (56)

Doppler Resolution $d_r(\theta)$ (knots)	$P_3[d_r(\theta), \text{s.d.}]$
0.5	0.689
1.0	0.455
1.5	0.296
2.0	0.195
2.5	0.135
3.0	0.101
3.5	0.082
4.0	0.073
4.5	0.068
5.0	0.066

Table 11
Probability of Resolving Three Ships
Using the Integration Method (Eq. (57))

Doppler Resolution $d_r(\theta)$ (knots)	Probability of Resolving Three Ships p_3
0.5	0.835
1.0	0.694
1.5	0.580
2.0	0.488
2.5	0.416
3.0	0.359
3.5	0.293
4.0	0.261
4.5	0.251
5.0	0.228

Since all three ships may travel in either direction along a shipping lane, there is a 0.75 probability that two ships will travel in the same direction with the third ship traveling in the opposite direction and a 0.25 probability that all three ships will travel in the same direction. As a result the probability $p_3[d_r(\theta)]$ of resolving three ships located in the same ocean surveillance cell is

$$p_3[d_r(\theta)] = 0.75P_2[d_r(\theta), \text{s.d.}] + 0.25P_3[d_r(\theta), \text{s.d.}]. \tag{57}$$

This probability is presented in Table 11 for the doppler resolution capabilities of $d_r(\theta) =$ 0.5, 1.0, ..., 5.0 knots. Note the excellent agreement between Tables 7b and 11.

CONCLUSION

Theory is developed in the second section for calculating the probability of solitary occupancy of N objects in M cells of unequal cell probabilities. It is shown that this probability is a function of $N-1$ quantities $K_2, K_3, \ldots, K_N$, where K_t, $t = 2, 3, \ldots, N$, is equal to the sum of the cell probabilities when raised to the tth power. Polynomial equations are given for $N = 2, 3, \ldots, 10$.

This theory is applied in the third section to determine the capability of a doppler radar to resolve N ships contained in a single ocean-surveillance resolution cell. The resulting probabilities of this combinatorial approach are presented for $N = 2, 3, \ldots, 7$ ships and the doppler resolution capability equal to 0.1, 0.2, ..., 5.0 knots.

A natural approach to calculating the probability of resolving N ships is the integration method employed in the preceding section. Using this alternate approach, the probabilities of resolving $N = 2, 3$ ships are given for various doppler resolution capabilities.

For $N = 3$ the integrals in the approach were cumbersome to implement, not to mention the tedious effort to derive them and the large computing time to perform the integration. Furthermore this difficulty increases exponentially as N increases. The reason for employing the integration method in view of this difficulty is to provide an acceptable check on the accuracy of the results given by the combinatorial approach. Importantly the combinatorial approach is simple to implement and is an inexpensive method of obtaining the results in a computer program. In view of this the excellent agreement between the integration and the combinatorial methods demonstrates the utility of the theory developed in the second section.

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