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Electromagnetic Field in a Laser Resonator

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A New theory has been formulated whereby the electromagnetic field inside a laser resonator filled with an active medium is described through an expansion of the field into an angular spectrum of plane waves. The resonator is composed of two identical, circular reflectors of arbitrary size, focal length, and axial separation and contains an isotropic, homogeneous, saturable amplifying medium. This theory properly accounts for the vector properties of the field, is not limited by the usual paraxial approximations, and provides integral expressions for the spatial distribution of the field over the interior of the resonator in three dimensions. The integral

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expression for the transverse electric component of the field was evaluated numerically for several of the lower loss modes in confocal, spherical, and unstable resonators with linear dimensions the order of 10 wavelengths. These data indicate that some of the modes in confocal and spherical resonators are unstable, if the resonator has dimensions of a few wavelengths. The effect on the spatial distribution of the resonator field due to saturation of the amplifying medium is slight if spatial hole burning is neglected. The electromagnetic field given by this theory was also quantized, and the resulting formulation resembles the quantum theory for radiation in a closed resonator.

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CONTENTS

ELECTROMAGNETIC FIELD IN A LASER RESONATOR

INTRODUCTION

Since the invention of the laser the open resonator has been of considerable interest in applied physics. Our present theoretical understanding of the optical field distribution in an open resonator is founded on the work of Fox and Li [1] and of Boyd and Kogelnik [2]. Although their theories have been extended and refined [3], the approach of these authors to describing the field is still used and is the basis for nearly all of the subsequent theoretical work on the subject.

The Fox and Li analysis, the Boyd and Kogelnik analysis, and almost all subsequent treatments of the theory for open resonators employ the Fresnel-Kirchhoff integral of scalar diffraction theory [4]. However other approaches to describing electromagnetic wave propagation are superior in many respects. The Fresnel-Kirchhoff theory has numerous disadvantages. It is basically a scalar theory and can be extended to treat the electromagnetic properties of light only with some difficulty. Because it is an expansion of the field in terms of spherical (Huygens') wavelets, it usually gives rise to awkward expressions in rectangular coordinates. These expressions can frequently be handled only after making very restrictive approximations such as a paraxial approximation, the Fresnel approximation, or the Fraunhofer approximation. And lastly the Fresnel-Kirchhoff theory is not easily extended to consider wave propagation in amplifying or conducting media.

For certain classes of problems a much more straightforward, and in some respects more satisfactory, approach to wave propagation employs an expansion of the electromagnetic field into an angular spectrum of plane waves [5-7]. These expansions are particularly simple in rectangular coordinates, so that problems can often be solved more accurately. Also, in many cases all of the Cartesian components of an electromagnetic field can be represented very simply in terms of only two scalar plane-wave spectra, one for each state of plane polarization. Thus a full electromagnetic treatment often becomes no more difficult than a scalar treatment. For these and other reasons the angular-spectrum representation is frequently superior to the Huygens'-wavelet representation.

A scalar theory employing the angular-spectrum representation to describe the field in an empty resonator with plane parallel reflectors was given by Bergstein and Marom [8]. In the present report we will develop a closely related electromagnetic theory to describe the field in any resonator with two identical reflectors that have circular boundaries. These reflectors may have arbitrary radii, focal length, and axial separation. This theory is not limited to the paraxial approximation, is an electromagnetic theory which takes full account of the vector properties of light, and gives the field distribution in three dimensions over the interior of the resonator and not only over the surface of a reflector. A limitation to the theory is the use of an axial gain approximation to account for the active medium. This is similar, however, to the approximation that is used in the theories based on the Huygens-Fresnel principle.

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In the following (first) section we will derive the mode equation, which determines the angular spectrum for each mode of the resonator field, and we will give integral expressions for the Cartesian components of the electromagnetic field in terms of the angular spectrum. In the second section we will study the transverse electric component of the field in small resonators which are either confocal, spherical, or unstable, through numerical integration of the expression derived in the first section. These data indicate that these resonators all show signs of instability due to high diffraction losses and that the effects on the fields due to saturation of the amplifying medium are negligible if spatial hole burning is ignored. Finally in the third section we will quantize the field and show that the quantized field is in good agreement with that frequently assumed on the basis of closed resonator theory.

THE MODE EQUATION

This report is concerned with the distribution of the electromagnetic fields, in three dimensions, inside a laser resonator such as that shown schematically in Fig. 1. This resonator is formed by two identical reflectors with unit reflectivity, each of diameter $2a$ and focal length f and separated by an axial distance $2L$. Because the resonator is symmetric about the plane P in the figure, the fields in each half of the resonator are the same upon reflection about P (except for a possible change in sign). For this reason it is sufficient to determine the fields only over the region to the left of P.

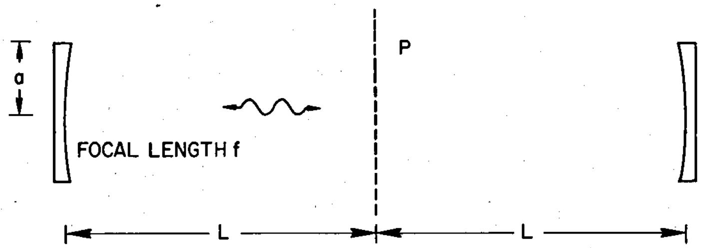


Fig. 1—Geometry of the resonator studied in this analysis

Incidentally it will become evident that the fields in one class of asymmetric resonators can also be determined directly from this analysis. These resonators are formed by replacing one of the reflectors by an infinite, plane reflector with unit reflectivity coinciding with the plane P.

The standing waves inside this symmetric resonator may be represented in terms of traveling waves by employing a technique used previously by Fox and Li [1]. This is done by replacing the two reflectors by an infinite sequence of equivalent lenses mounted in opaque stops and considering only traveling waves propagating through the lenses in the $+z$ direction as shown in Fig. 2. If we require that these traveling waves reproduce exactly the same fields between each pair of lenses, then the traveling waves in region A of Fig. 2 are the same as the component of the resonator standing waves which travels to the right within the half of the resonator to the left of P, and the traveling waves in region B are the mirror image of the complementary component of the resonator standing waves which travels to the left within the half of the resonator to the left of P. Thus the sequence of lenses is actually an unfolded replica of the resonator.

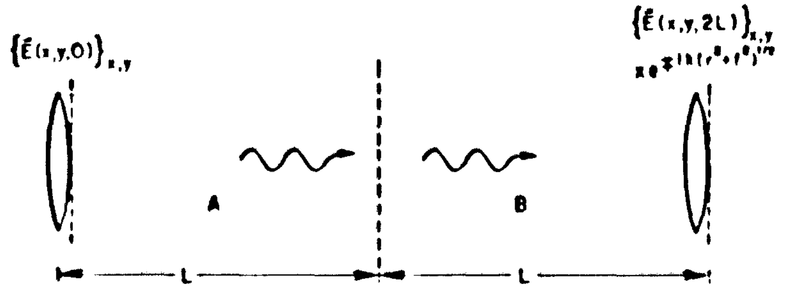


Fig. 2—The resonator from Fig. 1, which has been unfolded into an equivalent sequence of lenses. The mode equation follows from the requirement that the transverse electric field be equal, except for a possible change in sign, over the two planes as shown in this figure. This is given by Eqs. (1).

The traveling waves will reproduce the same fields between each pair of lenses if the following condition is met. The component of the electric field transverse to the resonator axis must have the same spatial distribution over the plane $z = 2L$, just after passing through the lens, as over the $z = 0$ plane (Fig. 2) except for a possible change in sign; that is,

$$\left\{\mathbf{E}(x, y, 0)\right\}_x = \pm\left\{\mathbf{E}(x, y, 2L)\right\}_x \exp\left[(\mp)ik(x^2 + y^2 + f^2)^{1/2}\right]\mathrm{circ}\left(\frac{x^2 + y^2}{a^2}\right), \quad (1a)$$

$$\left\{\mathbf{E}(x, y, 0)\right\}_y = \pm\left\{\mathbf{E}(x, y, 2L)\right\}_y \exp\left[(\mp)ik(x^2 + y^2 + f^2)^{1/2}\right]\mathrm{circ}\left(\frac{x^2 + y^2}{a^2}\right), \quad (1b)$$

where the symbols $\mathbf{A}_x$, $\mathbf{A}_y$, $\mathbf{A}_z$ represent the Cartesian x, y, z components of the vector $\mathbf{A}$. The fields here are assumed to be monochromatic with the time dependence $\exp(-i\omega t)$ suppressed and $k \approx \omega/c$. The upper sign inside the parenthesis is taken if the lenses are convex, the lower sign is taken if they are concave, and

$$\mathrm{circ}\,(r) = 1, \text{ if } 0 \leqslant r \leqslant 1, \quad (2a)$$

$$= 0, \text{ otherwise}. \quad (2b)$$

The resonator is assumed to contain an isotropic, homogeneous, nonconducting medium which contains sources which amplify the field by stimulated emission. Losses due to scattering and absorption by the medium are neglected. It is also assumed that the medium is Doppler broadened [9], as for a gas laser, so that the atomic populations coupled to the right- and left-traveling waves are different and can be separated into regions A and B of the unfolded resonator.

If in addition to these conditions an axial gain approximation is employed, as discussed in Appendix A, the traveling waves which propagate through the sequence of lenses shown in Fig. 2 can be expanded in the manner

$$\mathbf{E}(x, y, z) = g(z) \iint_{p^2+q^2<1} \underline{\mathcal{E}}(p, q) e^{ik(px+qy+mz)}\, dp\, dq. \quad (3a)$$

$$\mathbf{H}(x, y, z) = g(z) \iint\limits_{p^2+q^2 \leqslant 1} \underline{\mathcal{H}}(p, q) e^{ik(px+qy+mz)} dp\,dq, \tag{3b}$$

where

$$m = \sqrt{1 - p^2 - q^2}. \tag{4}$$

Each term in this expansion is a homogeneous plane wave modified by the complex gain coefficient $g(z)$ which arises due to the effect of the amplifying medium. As the amplification is reduced, $g(z)$ approaches unity but $\underline{\mathcal{E}}(x, y, z)$ and $\underline{\mathcal{H}}(x, y, z)$ do not change. In this limit Eqs. (3) become an angular spectrum expansion for a field in free space. The Cartesian components of $\underline{\mathcal{E}}(x, y, z)$ and $\underline{\mathcal{H}}(x, y, z)$ are angular spectra that are related by the well-known expressions [6]

$$\mathcal{E}_z(p, q) = -\left[\frac{p}{m}\, \mathcal{E}_x(p, q) + \frac{q}{m}\, \mathcal{E}_y(p, q)\right], \tag{5a}$$

$$\mathcal{H}_x(p, q) = \frac{-1}{Z_0}\left[\frac{pq}{m}\, \mathcal{E}_x(p, q) + \frac{1 - p^2}{m}\, \mathcal{E}_y(p, q)\right], \tag{5b}$$

$$\mathcal{H}_y(p, q) = \frac{1}{Z_0}\left[\frac{1 - q^2}{m}\, \mathcal{E}_x(p, q) + \frac{pq}{m}\, \mathcal{E}_y(p, q)\right], \tag{5c}$$

$$\mathcal{H}_z(p, q) = \frac{-1}{Z_0}\left[q\, \mathcal{E}_x(p, q) - p\, \mathcal{E}_y(p, q)\right], \tag{5d}$$

where

$$Z_0 = \sqrt{\frac{\mu}{\epsilon}}. \tag{6}$$

The complex gain coefficient in Eqs. (3) is given, as shown in Appendix A, by

$$g(z) = \exp\left\{\left[\left(\frac{\ell n|\gamma|}{2L}\right) + ik'\right] z\right\}, \tag{7}$$

if the medium is unsaturated, and by

$$g(z) = \sqrt{1 + \left[(|\gamma|^2 - 1)\, \frac{z}{(2L)}\right]}\, e^{ik'z}, \tag{8}$$

if it is completely saturated. The constants appearing in Eqs. (7) and (8) are defined by the single-pass gain

$$g(2L) = |\gamma| e^{ik'2L}. \tag{9}$$

The propagation constant is given by

$$k + k' = \frac{\eta\omega}{c} \tag{10}$$

where k' ($k' << k$) is a correction to an initially assumed value of k which is given from the following analysis and is used to determine the resonance frequency of the interferometer and where $\eta = \sqrt{\epsilon/\epsilon_0}$.

The axial gain approximation, used in Appendix A to derive Eqs. (3) through (10), limits this analysis to fields that satisfy two conditions. First, the field must be well collimated about the z axis (realistic if $L/a >> 1$) such that $\underline{\mathcal{E}}(p, q)$ and $\underline{\mathcal{H}}(p, q)$ are only appreciable if $p^2 + q^2 << 1$. Second, the field must propagate with sufficiently weak gain that $g(z)$ is slowly varying. Under this approximation the field distribution over any plane perpendicular to the z axis is the same as it would be in the absence of the amplifying medium, as evident from Eqs. (3), and the traveling waves are amplified only with increasing z.

The mode equation, a Fredholm equation with solutions giving the angular spectrum for the fields associated with each normal mode of the resonator, may now be obtained using this approximation. To proceed, we first transform Eqs. (3) into the cylindrical coordinates

$$p = \rho \cos\theta, \tag{11a}$$

$$q = \rho \sin\theta, \tag{11b}$$

$$x = r \cos\phi, \tag{11c}$$

$$y = r \sin\phi, \tag{11d}$$

as described in Appendix B, to obtain

$$\mathbf{E}(r, \phi, z) = 2\pi g(z) \sum_{n=-\infty}^{\infty} i^n e^{in\phi} \int_0^1 \mathbf{e}_n(\rho) e^{ikmz} J_n(k\rho r)\rho\, d\rho, \tag{12a}$$

$$\mathbf{H}(r, \phi, z) = 2\pi g(z) \sum_{n=-\infty}^{\infty} i^n e^{in\phi} \int_0^1 \mathbf{h}_n(\rho) e^{ikmz} J_n(k\rho r)\rho\, d\rho, \tag{12b}$$

where

$$\mathbf{e}_n(\rho) = \frac{1}{2\pi} \int_0^{2\pi} \underline{\mathcal{E}}(\rho \cos\theta, \rho \sin\theta) e^{-in\theta}\, d\theta, \tag{13a}$$

$$\mathbf{h}_n(\rho) = \frac{1}{2\pi} \int_0^{2\pi} \underline{\mathcal{H}}(\rho \cos\theta, \rho \sin\theta) e^{-in\theta} d\theta \tag{13b}$$

are coefficients in the Fourier series expansions with respect to θ and $n = 0, 1, 2, 3, ...$ Next we substitute Eq. (12a) into Eqs. (1) to obtain the mode equation in the form (Appendix C)

$$e_n^{(s)}(\rho) = \pm g(2L) \int_0^1 e_n^{(s)}(\rho) e^{ik\sqrt{1-\rho'^2}2L} \left[k^2 \int_0^a e^{(\mp)ik(r^2+f^2)^{1/2}} J_n(k\rho' r) J_n(k\rho r) r dr \right] \rho' d\rho', \tag{14}$$

where $e_n^{(s)}(\rho)$ represents the x component of $\mathbf{e}_n(\rho)$, if $s = +1$, or the y component, if $s = -1$.

The mode equation may be expressed in a more convenient form by making the transformation

$$e_n^{(s)'}(\rho) = e_n^{(s)}(\rho)\sqrt{\rho} \exp\left(ik\sqrt{1-\rho^2}L\right) \tag{15}$$

and rewriting Eq. (14) in the matrix notation (using the notation $\sum\limits_\rho A_\rho$ for a summation of A_ρ over the continuous index ρ)

$$\sum_{\rho'} M_{\rho\rho'} \left[e_{nn'}^{(s)'}\right]_{\rho'} = \gamma_{nn'} \left[e_{nn'}^{(s)'}\right]_\rho , \tag{16}$$

with the complex symmetric matrix

$$M_{\rho\rho'} = \sqrt{\rho\rho'} \exp\left[ik\left(\sqrt{1-\rho'^2} + \sqrt{1-\rho^2}\right)L\right] k^2 \int_0^a e^{(\mp)ik(r^2+f^2)^{1/2}} J_n(k\rho' r) J_n(k\rho r) r dr \tag{17}$$

and the eigenvalues

$$\gamma_{nn'} = \pm g(2L) \equiv |\gamma_{nn'}| e^{ik'_{nn'N}2L} \equiv e^{i\alpha_{nn'N}2L}. \tag{18}$$

The phase of $\gamma_{nn'}$ obtained from Eq. (16) is specified up to an arbitrary additive constant $N = 1, 2, 3, ...$ On solving Eq. (18) for $i\alpha_{nn'N}$, we find that

$$i\alpha_{nn'N} = \frac{\ln|\gamma_{nn'}|}{2L} + ik'_{nn'N} = \left\{\frac{\ln|\gamma_{nn'}| + i[\pi N - P(\gamma_{nn'})]}{2L}\right\}, \tag{19}$$

where $P(\gamma_{nn'})$ is the phase of $\gamma_{nn'}$ limited to the range $[0, 2\pi]$. In addition to the angular index n and the longitudinal index N, a radial index n', where $n' = 1, 2, 3, ...,$ has been added to descriminate among the many possible eigenvalues of Eq. (16). The state of polarization for a mode is given by s as described below. Thus a particular field distribution, or normal mode, is specified by a set of four integers: (n, n', N, s).

The axial gain approximation is not new to resonator theory. In fact it is used implicitly in all theories which use a free-space diffraction integral, such as the Fresnel-Kirchhoff integral, to account for the propagation of the waves between the reflectors. However an additional paraxial approximation, which is usually used to simplify the Fresnel-Kirchhoff integral, has been entirely avoided in the present theory by the use of the plane-wave representation. Therefore the present theory should be more accurate and should apply to resonator fields that do not satisfy the restrictions imposed by this paraxial approximation.

The Cartesian components of the electromagnetic fields associated with any mode given by (n, n', N, s) for an unsaturated medium are found by superimposing right- and left-traveling waves from Eqs. (3) to form the resonator standing waves, transforming to cylindrical coordinates using Eqs. (11), and making use of Eqs. (5), (7), and (13) as discussed in Appendix D. For the mode (n, n', N) which is linearly polarized in the x direction (for which we arbitrarily set the index $s = +1$), the field is given by

$$\left\{\mathbf{E}^{(+1)}_{nn'N}(r, \phi, z)\right\}_x = 4\pi i^{n+1} e^{in\phi} \int_0^1 e^{(+1)'}_{nn'}(\rho) e^{i\alpha_{nn'N}L} S[(k\sqrt{1-\rho^2} + \alpha_{nn'N})(z-L)] \times J_n(k\rho r)\sqrt{\rho}\, d\rho, \tag{20a}$$

$$\left\{\mathbf{E}^{(+1)}_{nn'N}(r, \phi, z)\right\}_y = 0, \tag{20b}$$

$$\left\{\mathbf{E}^{(+1)}_{nn'N}(r, \phi, z)\right\}_z = 2\pi i^{n-1} e^{in\phi} \int_0^1 e^{(+1)'}_{nn'}(\rho) e^{i\alpha_{nn'N}L} C[(k\sqrt{1-\rho^2} + \alpha_{nn'N})(z-L)] \times [J_{n+1}(k\rho r)e^{i\phi} - J_{n-1}(k\rho r)e^{-i\phi}] \frac{\rho\sqrt{\rho}\, d\rho}{\sqrt{1-\rho^2}}, \tag{20c}$$

$$\left\{\mathbf{H}^{(+1)}_{nn'N}(r, \phi, z)\right\}_x = \frac{\pi i^{n-1}}{Z_0} e^{in\phi} \int_0^1 e^{(+1)'}_{nn'}(\rho) e^{i\alpha_{nn'N}L} C[(k\sqrt{1-\rho^2} + \alpha_{nn'N})(z-L)] \times [J_{n+2}(k\rho r)e^{i2\phi} - J_{n-2}(k\rho r)e^{-i2\phi}] \frac{\rho^2\sqrt{\rho}\, d\rho}{\sqrt{1-\rho^2}}, \tag{20d}$$

$$\left\{\mathsf{H}_{nn'N}^{(+1)}(r,\phi,z)\right\}_y = \frac{\pi i^n}{Z_0}\, e^{in\phi} \int_0^1 e_{nn'}^{(+1)'}(\rho) e^{i\alpha_{nn'N}L} C[(km+\alpha_{nn'N})(z-L)]$$
$$\times \left\{ -[J_{n+2}(k\rho r)e^{i2\phi} + J_{n-2}(k\rho r)e^{-i2\phi}]\, \frac{\rho^2\sqrt{\rho}}{\sqrt{1-\rho^2}} + 4J_n(k\rho r)\sqrt{\rho}\, \frac{1-\frac{\rho^2}{2}}{\sqrt{1-\rho^2}} \right\} d\rho, \tag{20e}$$

$$\left\{\mathsf{H}_{nn'N}^{(+1)}(r,\phi,z)\right\}_z = \frac{2\pi i^{n-1}}{Z_0}\, e^{in\phi} \int_0^1 e_{nn'}^{(+1)'}(\rho) e^{i\alpha_{nn'N}L} S[(km+\alpha_{nn'N})(z-L)]$$
$$\times [J_{n+1}(k\rho r)e^{i\phi} + J_{n-1}(k\rho r)e^{-i\phi}]\,\rho\sqrt{\rho}\,d\rho, \tag{20f}$$

where

$$S(\xi) = -i\cos(\xi), \text{ if } N \text{ is odd}, \tag{21a}$$
$$= \sin(\xi), \text{ if } N \text{ is even}, \tag{21b}$$
$$C(\xi) = i\sin(\xi), \text{ if } N \text{ is odd}, \tag{21c}$$
$$= \cos(\xi), \text{ if } N \text{ is even}, \tag{21d}$$

and where $\alpha_{nn'N}$ is defined by Eq. (19). Modes of arbitrary polarization are found from Eqs. (20) by adding the field components for the complementary (n, n', N) mode which is plane polarized in the y direction $(s = -1)$. The field components for the y-polarized mode can be found simply by performing a coordinate rotation on Eqs. (20).

The fields in a saturated medium are somewhat different. For example the transverse electric component for the (n, n', N) mode which is plane polarized in the x direction is given by

$$\left\{\mathsf{E}_{nn'N}^{(+1)}(r,\phi,z)\right\}_x = 2\pi i^n e^{in\phi} \int_0^1 e_{nn'}^{(+1)'}(\rho) e^{ik'_{nn'N}L} \left\{ \sqrt{1 + (|\gamma_{nn'}|^2 - 1)\frac{z}{2L}} \right.$$
$$\times \exp[i(k\sqrt{1-\rho^2} + k'_{nn'N})(z-L)] \mp \sqrt{1 + (|\gamma_{nn'}|^2 - 1)\frac{(2L-z)}{2L}}$$
$$\left. \times \exp[i(k\sqrt{1-\rho^2} + k'_{nn'N})(L-z)] \right\} J_n(k\rho r)\sqrt{\rho}\,d\rho, \tag{22}$$

where the derivation follows in the same way as that for Eq. (20a), but by using Eq. (8) in place of Eq. (7), as discussed in Appendix D.

The mode field in Eqs. (20) or (22) is still arbitrary up to an overall constant amplitude, which is determined by normalization of the angular spectrum $e^{(+1)'}_{nn'N}(\rho)$. A particular normalization convention will be specified in a later section in a manner appropriate for the quantum theory of laser resonator fields.

The eigenvalues $\gamma_{nn'}$ given by Eq. (18) are equivalent to those appearing in the usual resonator theory and provide useful information about the modes. The fraction de of the electromagnetic energy lost by diffraction as the waves travel from one reflector to the other is given by

$$de = (1 - |\gamma_{nn'}|^{-2}), \tag{23}$$

just as in the conventional theory [3, Eq. (81)]. The phase of $\gamma_{nn'}$ specifies the resonant frequency for the mode. The frequency for the (n, n', N, s) mode, as given by substitution from Eq. (19) into Eq. (10), is

$$\omega_{nn'N} = c(k + k'_{nn'N}) = ck + \frac{c[\pi N - P(\gamma_{nn'})]}{2L} . \tag{24}$$

To employ Eq. (16), we must first assume a frequency ck close to resonance and then use Eqs. (16) and (24) to obtain the precise frequency $\omega_{nn'N}$. The assumed frequency is in effect pulled by the resonator an amount given by the frequency shift

$$df = \frac{\omega_{nn'N} - ck}{ck} = \frac{[\pi N - P(\gamma_{nn'})]}{2Lk}, \tag{25}$$

which follows directly from Eq. (24).

The longitudinal mode number N is not the same as the usual longitudinal mode number N', which is defined to be the number of half wavelengths in the length $2L$, but is related to it by

$$N' \equiv \frac{2L\omega_{nn'N}}{\pi c} = N + \left(\frac{2Lk - P(\gamma_{nn'})}{\pi}\right) \tag{26}$$

according to Eq. (24) [3, Eq. (84)]. For most resonator fields, N' as defined here is not an integer but approaches an integer with increasing $2L/a$.

The fields associated with the modes with even N produce transverse electric components which vanish over the $z = L$ plane according to Eqs. (20). Thus these are also the fields of an asymmetric resonator with an infinite, plane reflector of unit reflectivity in the $z = L$ plane.

NUMERICAL CALCULATION OF THE MODE FIELDS

The transverse electric component of the field for several of the lower loss modes in resonators that are either confocal, spherical, plane parallel, or unstable has been evaluated numerically through the use of the equations derived in the last section. It has been calculated only for modes which are plane polarized, circularly symmetric ($n = 0$), and of fixed longitudinal order ($N' \approx 40$). However, the same procedure can be employed to study other modes and other Cartesian components of the fields. Resonators were chosen for this study with dimensions $a = 5\lambda$ and $L = 10\lambda$ (where $\lambda = 2\pi/k$). These resonators could not be treated by the usual resonator theories, because they do not satisfy the restrictions imposed by the paraxial approximation.

The procedure used is relatively simple. First, the elements of the matrix given by Eq. (17) are determined for a particular set of resonator parameters using Simpson's rule to numerically evaluate the integrals. Second, the matrix is diagonalized using the IBM Share subroutine ALLMAT, which has been tested extensively in a similar application by Sanderson and Streifer [10]. The resulting eigenvalues $\gamma_{nn'}$ are used to determine $\alpha_{nn'N}$ from Eq. (19), and the eigenvectors $e_{nn'}^{(+1)'}(\rho)$ give the angular spectrum for the (n, n', N) mode. Finally, the field components associated with the (n, n', N) mode in an unsaturated medium are found by substituting $k'_{nn'N}$ and $e_{nn'}^{(+1)}('\rho)$ into Eq. (20) and integrating numerically using the trapezoidal rule. Similar modes for a completely saturated medium are obtained using Eq. (22) in place of (20a).

The matrix is represented to the computer by a 50 by 50, single-precision, complex array. Tests conducted by varying the array dimensions (up to 100 by 100) indicate that no serious sampling errors occurred in calculating the fields described here. However, for significantly larger resonators with larger Fresnel numbers,

$$\mathcal{F} \equiv \frac{\pi a^2}{\lambda L} > 5\pi, \tag{27}$$

serious sampling errors do occur.

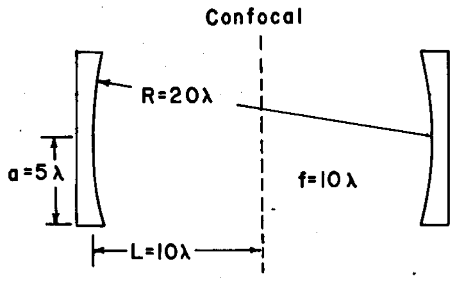


Fig. 3—Confocal resonator studied in this work

The three radial modes of lowest loss have been examined by this procedure for the small, confocal resonator shown in Fig. 3. Isometric drawings showing the transverse electric component, that is, the magnitude of $\{\mathbf{E}_{nn'N}(r, 0, z)\}_x$ over the region where $0 \leqslant z \leqslant L$, for these modes are shown in Fig. 4 for an unsaturated medium. Since the fields are symmetric about the $z = L$ plane, only the fields to the left of the plane are shown. The energy loss per pass de and the frequency shift df for these modes are also shown as determined from Eqs. (23) and (25). The energy loss is seen to increase anomalously fast with increasing n' from only 1.39% in the first (lowest loss) mode to 65.6% in the third. It is also evident that the modes become increasingly irregular as the loss increases. By irregular is meant the rapid irregular spatial variation of the envelope of the otherwise orderly standing wave pattern shown in Fig. 4c. This does not occur for instance in the standing-wave pattern shown in Fig. 4a, and it is very slight in the pattern shown in Fig. 4b.

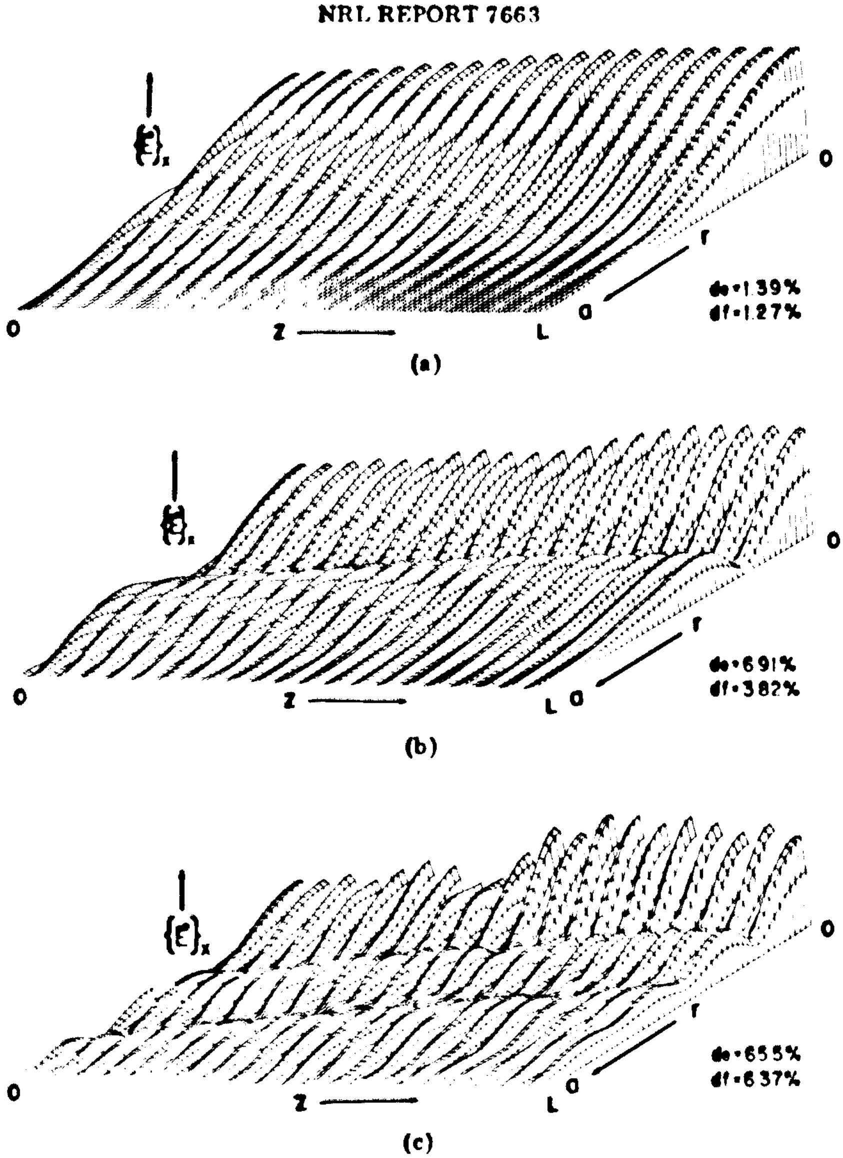


Fig 4—Isometric projections showing the spatial distributions for the transverse electric field over the confocal resonator shown in Fig 3. (a) The first (lowest loss, cylindrically symmetric) mode, with $n = 0$, $n' = 1$, $N \approx 40$, and $s = +1$. (b) The second (second lowest loss, cylindrically symmetric) mode, with $n = 0$, $n' = 2$, $N \approx 40$, and $s = +1$. (c) The third (third lowest loss, cylindrically symmetric) mode, with $n = 0$, $n' = 3$, $N \approx 40$, and $s = +1$.

Similar modes were examined for the spherical resonator shown in Fig. 5, and isometric drawings of the transverse electric components are shown in Fig. 6 for an unsaturated medium. The radial modes in this spherical resonator are seen to have losses that increase even more rapidly with n' than those for the confocal resonator. The associated irregularity of the field is evident in Figs. 6b and 6c. The calculations were repeated for a spherical resonator of the same proportions but of twice the size. For the larger resonator the modes are similar to those in the smaller resonator, but the losses increase much more slowly with n', and the fields are not as irregular.

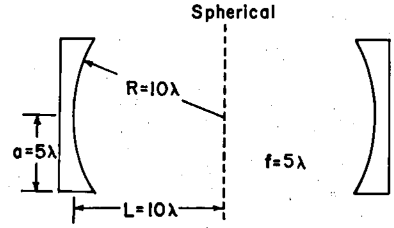


Fig. 5—Spherical resonator studied in this work

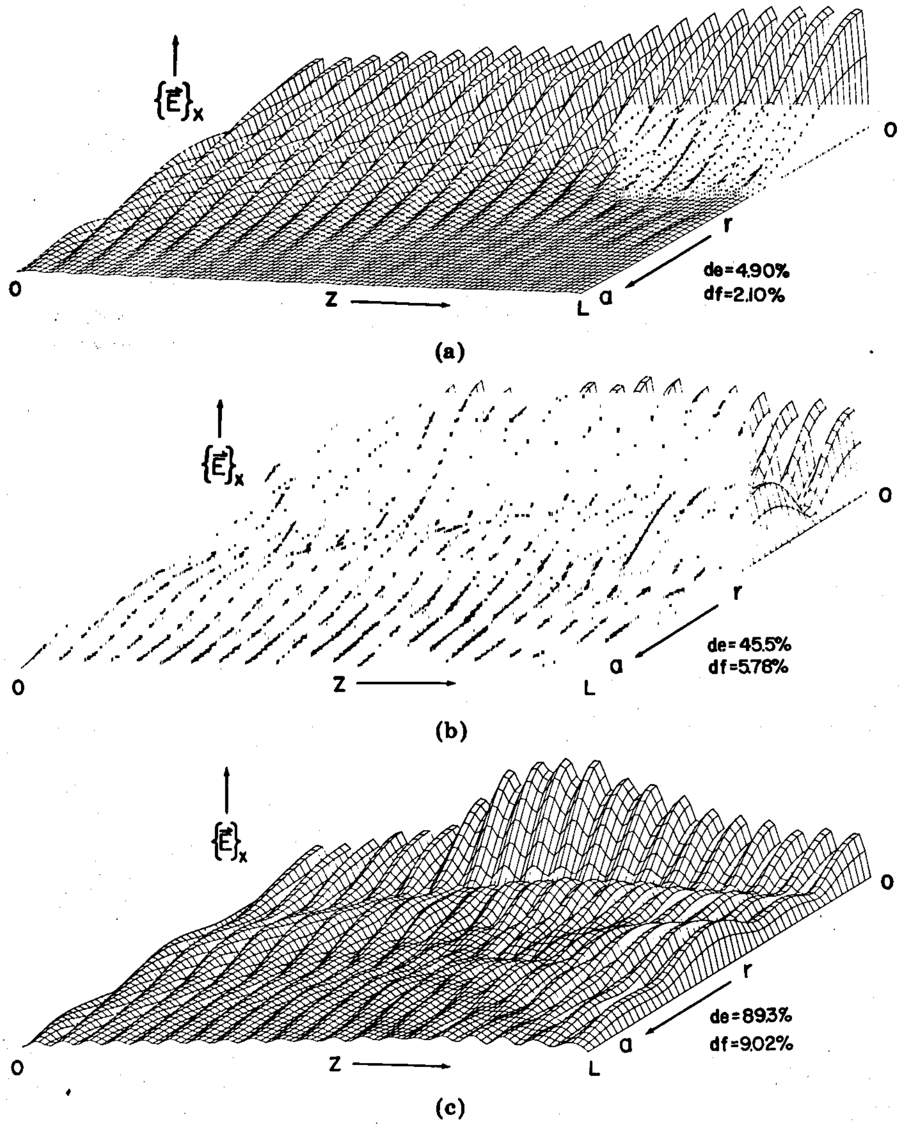


Fig. 6—Isometric projections showing the spatial distributions for the transverse electric field over the spherical resonator shown in Fig. 5. (a) The first (lowest loss, cylindrically symmetric) mode, with $n = 0$, $n' = 1$, $N' \approx 40$, and $s = +1$. (b) The second (second lowest loss, cylindrically symmetric) mode, with $n = 0$, $n' = 2$, $N' \approx 40$, and $s = +1$. (c) The third (third lowest loss, cylindrically symmetric) mode, with $n = 0$, $n' = 3$, $N' \approx 40$, and $s = +1$.

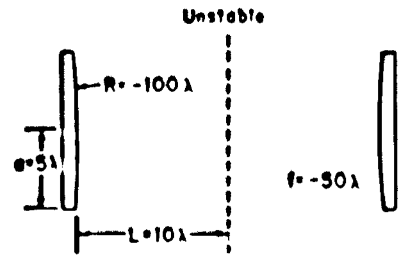


Fig. 7—Unstable resonator studied in this work

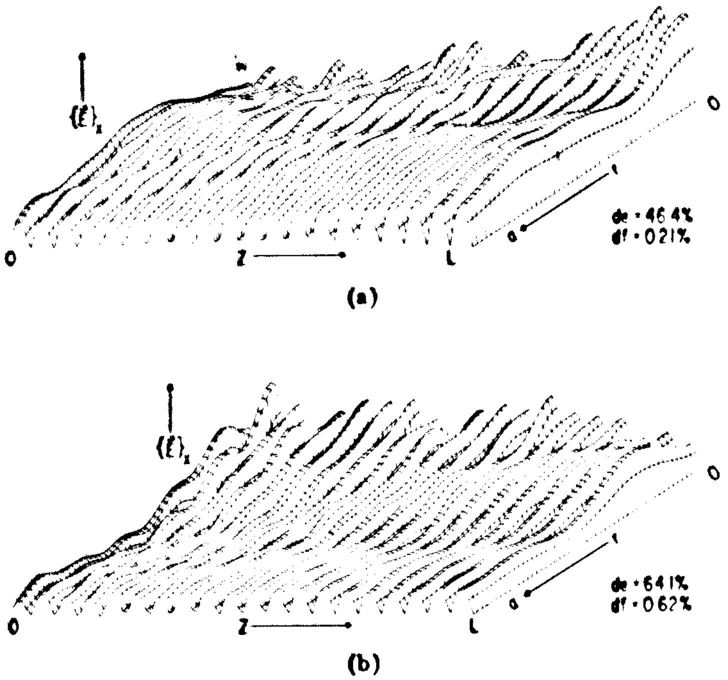


Fig. 8—Isometric projections showing the spatial distributions for the transverse electric field over the unstable resonator shown in Fig. 7. (a) The first (lowest loss, cylindrically symmetric) mode, with $n = 0$, $n' = 1$, $N \sim 40$, and $s = +1$. (b) The second (second lowest loss, cylindrically symmetric) mode, with $n = 0$, $n' = 2$, $N \sim 40$, and $s = +1$

The two lowest loss radial modes were also examined in the unstable resonator shown in Fig. 7. The isometric drawings of the transverse electric field for these modes are shown in Fig. 8 for an unsaturated medium. Both modes clearly show the rapid irregular amplitude variations which are well known to be characteristic of unstable resonators [11]. By comparison of Figs. 8a and 8b to Figs. 4c, 6b, and 6c, it is evident that the irregularities observed in the modes for the confocal and spherical resonators are similar to those known to occur in unstable resonators.

It appears likely that the anomalous behavior of all of these modes is associated with the onset of mode instability resulting from high diffraction loss in these small resonators. Although the confocal and spherical resonators are stable according to the usual geometrical theory used to describe mode stability, this theory does not account for diffraction loss. Therefore it is not surprising that when diffraction loss is taken into consideration, any open resonator may show some instability if it is sufficiently small relative to the wavelength of the field. This agrees with the observation that the modes were less irregular in the larger spherical resonator than in the similar resonator half the size.

Plane parallel resonators examined in this manner produced an interesting confirmation of the mode equation. For a plane parallel resonator with the dimensions $a = 5\lambda$ and $L = 10\lambda$ used in this work, the matrix in Eq. (17) is nearly diagonal. In fact, if we set

$$\exp[\mp ik(r^2 + f^2)] = 1 \tag{28}$$

in Eq. (17) and let $a \to \infty$, we obtain the diagonal matrix

$$M_{\rho\rho'} = \delta(\rho - \rho') \exp(i2k\sqrt{1 - \rho^2}L). \tag{29}$$

Thus plane parallel interferometers with infinite reflectors have modes that are plane waves in agreement with the early Fabry-Perot theory [12].

Many of the numerical calculations were repeated for the modes in a saturated medium using Eq. (22) in place of (20a). The distribution of the transverse electric component of the field is similar in each case to that obtained for the unsaturated medium. Therefore, within the axial gain approximation, where spatial hole burning is neglected, saturation appears to have very little effect on the fields. Typical results for the transverse component of the electric field, with and without saturation, are compared in Fig. 9.

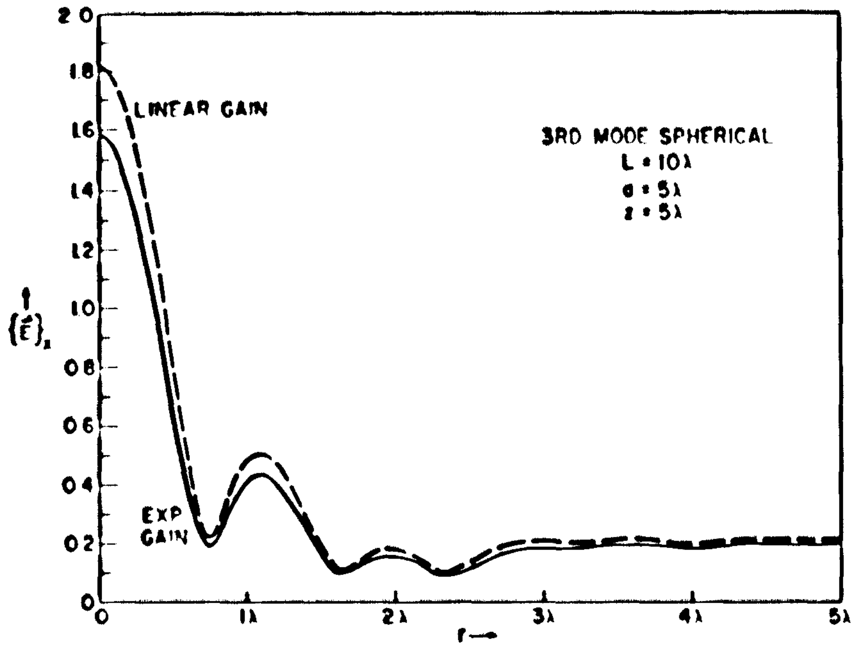
MODE QUANTIZATION AND NORMALIZATION

It is frequently assumed that the modes in an open resonator in the absence of the amplifying medium can be quantized in the same manner as those in a closed resonator [13, Eq. (2.1), or 14, Eq. (2.2)]. In this section this assumption is shown to be valid for paraxial fields. The mode amplitudes are determined by a normalization condition imposed on the angular spectrum by the quantum formulism.

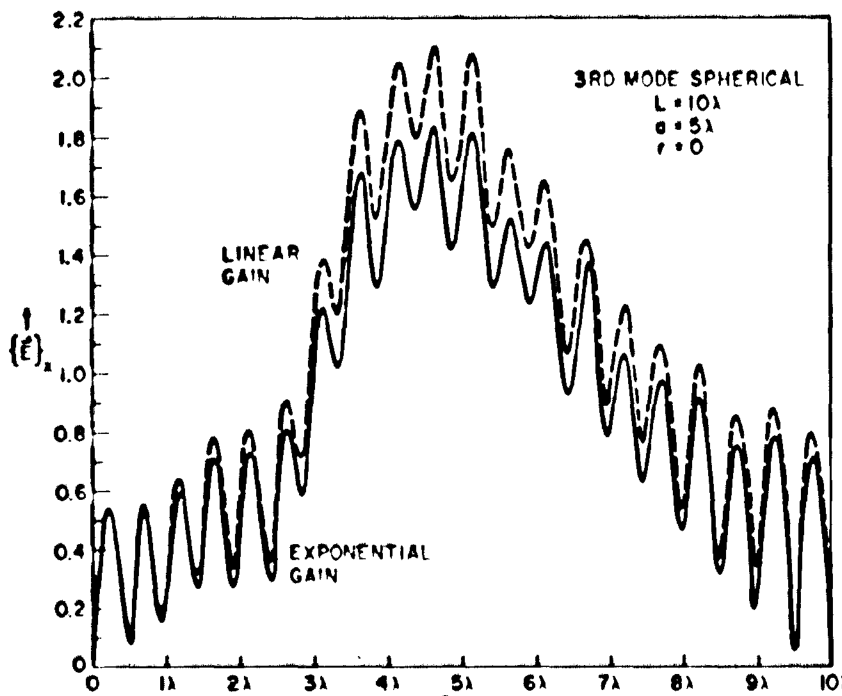
Physical fields are real functions of time; thus, to describe a field with precision, we must drop the analytic-function representation used in the earlier sections. Each Cartesian component associated with the (n, n', N) mode is described by an expression of the form

$$U(x, y, z, t) = \frac{1}{2} \sum_{nn'N} [a_{nn'N}(t)u_{nn'N}(x, y, z) + a^*_{nn'N}(t)u^*_{nn'N}(x, y, z)], \tag{30}$$

where $u_{nn'N}(x, y, z)$ represents the spatial distribution of a particular Cartesian component as given by Eqs. (20), and $a_{nn'N}(t)$ represents the time-dependent amplitude of the mode which is common to all of the Cartesian components. This amplitude is very nearly time harmonic with radial frequency $\omega_{nn'N}$.



(a)



(b)

Fig 9—The magnitude of the transverse electric field showing that saturation of the medium (exponential gain, given by Eq (22), rather than linear gain, given by Eq (20a)) has little effect on the field of a typical mode when hole burning effects are neglected (a) Plotted as a function of r. (b) Plotted as a function of z

To quantize the resonator field in the usual manner, we show that each mode behaves mathematically like a harmonic oscillator, so that the well-known quantum theory for this ocsillator can be used. We begin by calculating the total energy stored in the resonator,

$$\mathcal{H} = \frac{1}{8\pi}\int_0^{\infty}\int_0^{2\pi}\int_0^{2L} [\epsilon \mathsf{E}(x, y, z, t) \cdot \mathsf{E}(x, y, z, t) + \mu\, \mathsf{H}(x, y, z, t) \cdot \mathsf{H}(x, y, z, t)]\, dzrd\phi dr, \tag{31}$$

in which each Cartesian component must be expressed in the form given by Eq. (30). The vacuum resonator field for each mode, in the absence of the active medium, is obtained from Eqs. (20) by replacing $k\sqrt{1-\rho^2} + a_{nn'N}$ by $k_{nn'N}\sqrt{1-\rho^2}$ with $(k_{nn'N} = \omega_{nn'N}/c)$. The axial gain approximation is now discarded, so that these expressions for the vacuum resonator field follow rigorously from Maxwell's equations. For the most general resonator field the Hamiltonian $\mathcal{H}$ is obtained by substituting the vacuum field for each mode from Eqs. (20) into Eq. (30), then substituting from Eq. (30) into Eq. (31), and summing over all modes. After considerable algebra, and the use of paraxial approximations, as discussed in Appendix E, we obtain the expression

$$\mathcal{H} = \frac{1}{2}\sum_M \hbar\omega_M [a_M(t)a_M^*(t) + a_M^*(t)a_M(t)]. \tag{32}$$

where M is a simplified notation for (n, n', N, s) and the angular spectrum is normalized in the manner

$$\int_0^1 |e_M^{(s)\prime}(\rho)|^2 \left(2 + \frac{\rho^2}{1-\rho^2}\right) d\rho = \frac{\hbar\omega_M^3}{(2\pi^2\epsilon c^2 L)} . \tag{33}$$

By substitution from the definitions

$$a_M = \frac{1}{\sqrt{2\hbar\omega_M}}(\omega_M q_M + ip_M), \tag{34a}$$

$$a_M^* = \frac{1}{\sqrt{2\hbar\omega_M}}(\omega_M q_M - ip_M), \tag{34b}$$

Eq. (32) becomes

$$\mathcal{H} = \frac{1}{2}\sum_M p_M^2 + \omega_M^2 q_M^2 . \tag{35}$$

The Hamiltonian in Eq. (35) is identical with that for a collection of uncoupled harmonic oscillators, each with a displacement given by q_M and a momentum given by p_M [15a, 16]. It is clear from Eq. (35) that the fields in an open resonator obey the classical equation of motion for a harmonic oscillator:

$$\frac{\partial \mathcal{H}}{\partial p_M} = \frac{\partial q_M}{\partial t} = p_M, \tag{36a}$$

$$\frac{\partial \mathcal{H}}{\partial q_M} = -\frac{\partial p_M}{\partial t} = \omega_M^2 q_M. \tag{36b}$$

Now that we have established the similarity between each mode in the general resonator field and a classical harmonic oscillator, the field may be quantized in the usual manner. Consider any function f of q_M and p_M with the total derivative

$$\frac{df(q_M, p_M)}{dt} = \left[\frac{dq_M}{dt}\frac{\partial}{\partial q_M} + \frac{dp_M}{dt}\frac{\partial}{\partial p_M} + \frac{\partial}{\partial t}\right] f(q_M, p_M). \tag{37}$$

The equation of motion for f, upon substitution from Eqs. (36) into Eq. (37), becomes

$$\frac{df(q_M, p_M)}{dt} = \left\{ f(q_M, p_M), \mathcal{H} \right\} + \frac{\partial f(q_M, p_M)}{\partial t}, \tag{38}$$

where

$$\left\{ f, \mathcal{H} \right\} \equiv \frac{\partial f}{\partial q_M}\frac{\partial \mathcal{H}}{\partial p_M} - \frac{\partial f}{\partial p_M}\frac{\partial \mathcal{H}}{\partial q_M} \tag{39}$$

is the Poisson bracket of f and $\mathcal{H}$. The field is quantized by replacing the classical functions f and $\mathcal{H}$ in Eq. (38) by the Hilbert space operators $\hat{f}$ and $\hat{\mathcal{H}}$ and replacing the Poisson brackets by $1/i\hbar$ times the commutator brackets $[\hat{f}, \hat{\mathcal{H}}]$ [15b] to obtain the Heisenberg equation

$$\frac{d\hat{f}(\hat{q}_M, \hat{p}_M)}{dt} = \frac{1}{i\hbar}[\hat{f}(\hat{q}_M, \hat{p}_M), \hat{\mathcal{H}}] + \frac{\partial \hat{f}(\hat{q}_M, \hat{p}_M)}{\partial t}. \tag{40}$$

This is the quantum-mechanical equation of motion for any operator $\hat{f}$ which is a function of the operators $\hat{q}_M$ and $\hat{p}_M$.

In this formulation the mode amplitudes a_M and a_M^* in Eqs. (34) are replaced by the operators

$$\hat{a}_M = \frac{1}{\sqrt{2\hbar\omega_M}}(\omega_M \hat{q}_M + i\hat{p}_M), \tag{41a}$$

$$\hat{a}_M^\dagger = \frac{1}{\sqrt{2\hbar\omega_M}} (\omega_M \hat{q}_M - i\hat{p}_M), \tag{41b}$$

which obey the commutation relations

$$[\hat{a}_M, \hat{a}_{M'}^\dagger] = \delta_{MM'}, \tag{42a}$$

$$[\hat{a}_M , \hat{a}_{M'}] = [\hat{a}_M^\dagger, \hat{a}_{M'}^\dagger] = 0 \tag{42b}$$

and act as annihilation and creation operators, respectively, on the Mth mode of the Fock space vectors representing the energy eigenstates. That is [15c],

$$\hat{a}_M |\{n\}, n_M > = \sqrt{n_M}\, |\{n\}, n_M - 1 > , \tag{43a}$$

$$\hat{a}_M^\dagger |\{n\}, n_M > = \sqrt{n_M + 1}\, |\{n\}, n_M + 1 > . \tag{43b}$$

(The notation $|\{n\}, n_M >$ indicates a Fock space vector for a system of many energy levels. One particular level of interest, the Mth level, contains n_M quanta; the occupation of the other levels $\{n\}$ are not of immediate interest and are not indicated.) Each Cartesian component of the field in Eq. (30) is represented by the configuration space operator

$$\hat{U}(x, y, z, t) = \frac{1}{2} \sum_M [\hat{a}_M(t) u_M(x, y, z) + \hat{a}_M^\dagger(t) u_M^*(x, y, z)], \tag{44}$$

which may be used to calculate the various observable parameters associated with the field.

Scully and Lamb [17] have obtained a closely related expression for the electric field in a later resonator which they use in studying the detailed dynamics of the resonator field coupled to an amplifying medium. It is interesting to compare an expression derived from Eq. (44) with theirs. In their work they are concerned only with the field in a resonator with plane parallel reflectors, which is described using a simplified mathematical model. For example their electric field operator [17, Eq. (29)] is given by an expression which in our notation becomes

$$\left\{\hat{\mathsf{E}}(x, y, z, t)\right\}_x = \sum_{N'} \sqrt{\frac{\hbar\omega_{N'}}{2\epsilon L \pi a^2}}\, [\hat{a}_{N'}(t) + \hat{a}_{N'}^\dagger(t)] \sin\left[\frac{(N'\pi)z}{2L}\right]. \tag{45}$$

A more general expression for the x component of the electric field in a resonator is derived from our work by substituting Eq. (20a) into (44) giving

$$\left\{\mathsf{E}(x, y, z, t)\right\}_x = 4\pi \sum_M \left\{ i^{n+1} e^{in\phi} \hat{a}_M(t) \int_0^1 e'_M(\rho) S[(k_M\sqrt{1-\rho^2})(z-L)] \right.$$

$$\left. \times J_n(k\rho r)\sqrt{\rho}\, d\rho + h.c. \right\}, \tag{46}$$

where $h.c.$ signifies a term which is a Hermitian conjugate of the other term in the braces. By comparison of Eq. (45) with Eq. (46) we find that our theory leads to the field operator of Scully and Lamb, if we make the following approximations in Eq. (46). Let the reflectors be plane with large radius a and assume that only one axial plane wave contributes to each mode:

$$e'_{M}(\rho) = \sqrt{\frac{\hbar\omega_M^3}{4\pi^2\epsilon c^2 L}}\,\frac{\delta(\rho)}{\sqrt{\rho}}\,. \tag{47}$$

Also, we will neglect all modes which are not rotationally symmetric, so that $n = 0$, and we will use the paraxial approximation given by Eq. (E3):

$$(k_M\sqrt{1-\rho^2})L \approx \frac{N'\pi}{2}\,. \tag{48}$$

Equation (46) then becomes

$$\left\{\hat{E}_r(x, y, z, t)\right\}_x = 2\pi \sum_{N'} \sqrt{\frac{\hbar\omega_M^3}{4\pi^2\epsilon c^2 L}}\,[\hat{a}_M(t) + \hat{a}_M^\dagger(t)]\,\sin\left[\frac{(N'\pi)z}{2L}\right], \tag{49}$$

Equation (49) is identical with (45) except for a constant scale factor

$$s.f. = \frac{(\pi a^2)^{-1/2}}{(\sqrt{2}\ k_M)}\,, \tag{50}$$

which is due to differing field normalization. A problem arises with these idealized mode fields however which becomes evident upon substituting Eq. (47) into (33). The angular spectrum cannot be properly normalized due to the infinite energy of the single plane wave. This problem, which was overcome by Scully and Lamb by neglecting the field over the domain $r > a$, does not occur in our theory if the resonator fields decay away from the axis such that they have finite energy.

In our work only homogenous plane waves are included in the angular spectrum. For the most general fields evanescent plane waves should also be included [18].

CONCLUSIONS

The new open resonator theory developed in this report has several advantages over the theories which employ the Fresnel-Kirchhoff diffraction integral. Its primary advantage is that it applies to resonators in which the field cannot be treated in the paraxial approximation. This theory also leads more easily to a full electromagnetic treatment of the field. On the basis of this theory the components of all the field vectors may be determined numerically over the entire resonator in three dimensions. This has been demonstrated for the transverse electric field in several different resonators.

The primary limitation of this theory is the use of the axial gain approximation. Under this approximation the fields behave as free fields over every cross sectional area transverse to the axis. This same approximation appears, in one form or another, in all of the currently known open-resonator theories, and it is a serious limitation. This is especially true if saturation effects are to be considered.

We have considered the effect of complete uniform saturation on the mode and found it to be slight. Unfortunately the important effect of spatial hole burning could not be treated with this model. Thus we can only conclude that the primary cause of field changes in the presence of a saturable medium is the result of these holes.

The numerical data for the transverse components of the electric field obtained on the basis of this theory indicate that mode instability occurs in very small confocal and spherical resonators which have dimensions the order of a few wavelengths. This conclusion is supported by tests with larger resonators in which the apparent stability is improved. Since the geometrical theory by which these resonators are shown to be stable neglects all effects due to diffraction losses, it is not surprising that it appears to fail when the resonator has dimensions the order of a few wavelengths.

We have also shown how the resonator fields given by this model can be quantized. The quantized field is described by a formulation very similar to that usually used to describe the field in a finite closed resonator. The configuration-space field operators obtained here reduce to those obtained from the simplified mode theory of Lamb and Schully for a resonator with large, plane parallel reflectors. Although the axial gain approximation was used to obtain the fields in the presence of the active medium, the vacuum fields which were quantized do not depend on any gain approximations. The quantum theory is derived however using approximations which are valid only for paraxial fields.

The numerical data presented give a much clearer picture of the spatial distribution for the transverse electric field inside of such resonators than have been previously available.

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Appendix A
PROPAGATION OF TRAVELING WAVES IN AN AMPLIFYING MEDIUM

An approximate representation for a traveling wave propagating through an amplifying medium can be obtained as follows. The real electric field $\mathbf{E}_r(x, y, z, t)$, the polarization of the medium $\mathbf{P}(x, y, z, t)$, and the population inversion per unit volume $N(x, y, z, t)$ are related under frequently encountered conditions by the nonlinear differential equations*

$$\frac{\partial^2 \mathbf{P}(x, y, z, t)}{\partial t^2} + \frac{2}{T_2} \frac{\partial \mathbf{P}(x, y, z, t)}{\partial t} + \omega^2 \mathbf{P}(x, y, z, t) = \frac{-2\omega}{\hbar} L \frac{|\mu_{12}|^2}{3} N(x, y, z, t)\mathbf{E}_r(x, y, z, t), \tag{A1}$$

$$\frac{\partial N(x, y, z, t)}{\partial t} + \frac{N(x, y, z, t) - N^e}{T_1} = \frac{2}{\hbar\omega} \frac{\partial \mathbf{P}(x, y, z, t)}{\partial t} \cdot \mathbf{E}_r(x, y, z, t), \tag{A2}$$

$$\nabla \times (\nabla \times \mathbf{E}_r(x, y, z, t)) + \frac{\eta A}{c} \frac{\partial \mathbf{E}_r(x, y, z, t)}{\partial t} + \frac{\eta^2}{c^2} \frac{\partial^2 \mathbf{E}_r(x, y, z, t)}{\partial t^2} = -\mu_0 \frac{\partial^2 \mathbf{P}(x, y, z, t)}{\partial t^2}, \tag{A3}$$

where $\eta = \sqrt{\epsilon/\epsilon_0}$. The material constants T_1, T_2, L, $|\mu_{12}|$, N^e, and A, which Pantell and Puthoff* discuss in detail, are respectively the longitudinal relaxation time, the transverse relaxation time, the Lorentz correction factor, the matrix element of the dipole moment operator coupling the two states resonant with the field, the equilibrium population inversion per unit volume, and the power attenuation constant.

We begin by assuming that the field and the polarization are time harmonic and can be given by

$$\mathbf{E}_r(x, y, z, t) = \frac{1}{2} |g(x, y, z)| \mathbf{E}_0(x, y, z)e^{-\omega t} + c.c., \tag{A4}$$

$$\mathbf{P}(x, y, z, t) = \frac{1}{2} P_0(x, y, z)e^{-i\omega t} + c.c., \tag{A5}$$

where $\mathbf{E}_0(x, y, z)$ is the free field which remains if the coupling to the medium vanishes and where $c.c.$ indicates the complex conjugate of the preceding term. The population inversion per unit volume $N(x, y, z)$ is assumed to be time independent. Thus, upon substituting Eqs. (A4) and (A5) into (A2) and making use of the rotating wave approximation, we obtain

*R.H. Pantell and H.E. Puthoff, *Fundamentals of Quantum Electronics*, Wiley, New York, 1969, p. 86.

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$$\frac{N(x, y, z) - N^e}{T_1} = \frac{1}{2\hbar} [i|g(x, y, z)|\mathbf{E}_0(x, y, z) \cdot \mathbf{P}_0^*(x, y, z) + \text{c.c.}]. \quad \text{(A6)}$$

By similarly substituting Eqs. (A4) and (A5) into (A1), we have

$$\mathbf{P}_0(x, y, z) = \frac{-iT_2}{\hbar} L \frac{|\mu_{12}|^2}{3} N(x, y, z)|g(x, y, z)|\mathbf{E}_0(x, y, z). \quad \text{(A7)}$$

The population inversion can now be obtained by substituting Eq. (A7) into (A6) to obtain

$$N(x, y, z) = \frac{N^e}{1 + \left(\frac{T_1 T_2}{\hbar^2}\right) L \left(\frac{|\mu_{12}|^2}{3}\right)\left(\frac{2}{\eta\epsilon_0 c}\right) I(x, y, z)}, \quad \text{(A8)}$$

where $I(x, y, z)$ is the intensity of the light as defined by

$$I(x, y, z) = \frac{\eta\epsilon_0 c}{2} [|g(x, y, z)||\mathbf{E}_0(x, y, z)|]^2. \quad \text{(A9)}$$

By substituting Eqs. (A4), (A5), and (A7) into (A3), assuming that $g(x, y, z)$ is sufficiently slowly varying that terms containing its second derivatives can be neglected, and by separately equating the positive and negative frequency terms, we find that

$$\nabla|g(x, y, z)| \times [\nabla \times \mathbf{E}_0(x, y, z)] - [\nabla|g(x, y, z)| \cdot \nabla]\mathbf{E}_0(x, y, z)$$

$$= -i \frac{\omega\eta}{c} \left[\frac{\omega\mu_0 c T_2}{\hbar} L \frac{|\mu_{12}|^2}{3} N(x, y, z) - \frac{A}{2}\right] |g(x, y, z)|\mathbf{E}_0(x, y, z). \quad \text{(A10)}$$

In deriving Eq. (A10) use was made of the fact that the free field $\mathbf{E}_0(x, y, z)$ satisfies the usual wave equation

$$\nabla^2\mathbf{E}_0(x, y, z) + \frac{\omega^2}{c^2} \eta\mathbf{E}_0(x, y, z) = 0 \quad \text{(A11)}$$

and also the equation

$$\nabla \cdot \mathbf{E}_r(x, y, z, t) = 0. \quad \text{(A12)}$$

For the laser resonator under investigation we are concerned with a traveling wave propagating down the z axis:

$$\mathbf{E}_r(x, y, z) = \mathbf{e}(x, y, z)e^{ikz}. \quad \text{(A13)}$$

We assume for the purpose of representing the gain factor g that the vector $\mathbf{e}(x, y, z)$ can be treated as transverse to the z axis and slowly varying relative to the exponential in Eq. (A13). In addition we make the axial gain approximation by assuming g is a function of z only. Upon making these approximations and substituting Eq. (A8) into (A10), we obtain

$$\frac{\partial |g(x, y, z)|}{\partial z} = \left[\frac{\gamma_0}{2}\,\frac{I(0) + I_{sat}}{I_{sat} + I(x, y, z)} - \frac{A}{2}\right] |g(x, y, z)|, \tag{A14}$$

where I_{sat} is the intensity required to saturate the medium,

$$I_{sat} = \frac{\eta \epsilon_0 c}{\left(\frac{2T_1 T_2}{\hbar^2}\right) L \left(\frac{|\mu_{12}|^2}{3}\right)}, \tag{A15}$$

and

$$\gamma_0 = \frac{\hbar \omega N^e}{2T_1}\,\frac{1}{I(0) + I_{sat}} \tag{A16}$$

is the gain at $z = 0$.

If the medium is completely unsaturated such that $I(x, y, z) \ll I_{sat}$, then it follows by solution of Eq. (A14) that

$$|g(z)| = \exp\left\{\frac{\ell n |\gamma|}{2L} z\right\}, \tag{A17}$$

where we have taken the gain in the $z = 0$ plane to be $g(0) = 1$, and where the gain in the $z = 2L$ plane is given by the constant

$$|\gamma| = |g(2L)| = \exp\left\{\left[\frac{\gamma_0}{2}\,\frac{(I(0) + I_{sat})}{I_{sat}} - \frac{A}{2}\right] 2L\right\}. \tag{A18}$$

The modes of a resonator exist only for certain values of k. To determine these values of k from the eigenvalue analysis in this report, it is convenient to define the complex gain factor given by the expression

$$g(z) = \exp\left\{\left[\frac{\ell n |\gamma|}{2L} + ik'\right] z\right\}. \tag{A19}$$

Thus the magnitude of $g(z)$ is the true gain factor as given by Eq. (A14), and the phase represents a correction to k in equations of the form

$$\mathbf{E}_r(x, y, z, t) = g(z)\mathbf{e}(x, y, z)e^{i(kz-\omega t)} + c.c. \tag{A20}$$

The phase of $g(z)$ specified by the eigenvalue analysis in this report is such that $\mathbf{E}_r(x, y, z, t)$ as given by Eq. (A20) has the proper wavelength for a particular resonator mode.

If the medium is completely saturated such that $I(x, y, z) >> I_{sat}$, then a simple solution may still be found for the nonlinear differential equation in Eq. (A14) if further approximations are made. We assume that $I(x, y, z)$ in Eq. (A14) can be approximated by

$$I(x, y, z) = I_0|g(z)|^2. \tag{A21}$$

This is valid only if spatial hole burning (spatial variations in g due to spatial variations in $E_0(x, y, z)$) are ignored. If we also assume that the effects due to absorption can be ignored by setting $A = 0$, then Eq. (A14) has the approximate solution

$$|g(z)| = \sqrt{1 + \left[\frac{(|\gamma|^2 - 1)z}{(2L)}\right]} \tag{A22}$$

Here we have again taken $g(0) = 1$ and replaced Eq. (A18) by

$$|\gamma| = |g(2L)| = 1 + \frac{2L\gamma_0[I(0) + I_{sat}]}{I_0} . \tag{A23}$$

The linear intensity gain predicted by Eq. (A22) is in agreement with the experimental data of Allen and Peters*. To allow for the adjustment of the wavelength to satisfy the requirement of the resonator mode, we again define the complex gain factor by

$$g(z) = \sqrt{1 + \left[\frac{(|\gamma|^2 - 1)z}{(2L)}\right]}\, e^{ik'z}. \tag{A24}$$

We have found in this appendix that the fields are given within this approximation by the empty-resonator traveling waves multiplied by the complex gain factor $g(z)$. Therefore, by expanding the source-free fields of the empty resonator into an angular spectrum of vector homogeneous plane waves in the conventional manner†, we obtain the field equations employed in this report:

$$\mathbf{E}(x, y, z) = g(z) \iint_{p^2+q^2<1} \underline{\mathcal{E}}(p, q)e^{ik(px+qy+mz)}dp\,dq, \tag{A25a}$$

$$\mathbf{H}(x, y, z) = g(z) \iint_{p^2+q^2<1} \underline{\mathcal{H}}(p, q)e^{ik(px+qy+mz)}dp\,dq, \tag{A25b}$$

with the exp $(-i\omega t)$ time dependence suppressed.

*L. Allen and G.I. Peters, J. Phys. A4, 564 (1971).
†G. Borgiotti, Alta Frequenza 32, 808 (1963).

Appendix B
ANGULAR-SPECTRUM EXPANSION IN CYLINDRICAL COORDINATES

An angular-spectrum expansion, like the integral appearing in Eq. (3), can be transformed into cylindrical coordinates. In this appendix this procedure is described for the more general case where

$$\mathsf{U}(x, y, z) = \iint_{-\infty}^{\infty} \underline{\mathfrak{U}}(p, q) e^{ik(p\gamma + qy + mz)}\, dp dq, \tag{B1}$$

with

$$\begin{aligned} m &= \sqrt{1 - p^2 - q^2}, \text{ if } p^2 + q^2 \leqslant 1, \\ &= i\sqrt{p^2 + q^2 - 1}, \text{ if } p^2 + q^2 > 1. \end{aligned} \tag{B2}$$

Substituting Eqs. (11) into (B1), making the Fourier series expansion

$$\underline{\mathfrak{U}}(\rho \cos\theta, \rho \sin\theta) = \sum_{n=-\infty}^{\infty} \mathsf{u}(\rho, n) e^{in\theta} \tag{B3}$$

and interchanging the order of summation and integration, we have

$$\mathsf{U}(r, \phi, z) = \sum_{n=-\infty}^{\infty} \int_0^{\infty} \mathsf{u}(\rho, n) e^{ikmz} \int_0^{2\pi} e^{ik\rho r \cos(\theta - \phi) + in\theta}\, d\theta \rho d\rho. \tag{B4}$$

One integral appearing in Eq. (B4) is easily evaluated to give

$$\int_0^{2\pi} e^{ik\rho r \cos(\theta - \phi) + in\theta}\, d\theta = 2\pi i^{\pm n} J_{\pm n}(k\rho r) e^{\pm in\phi} \tag{B5}$$

by stubstituting the generating function*

$$e^{i\xi \cos(\theta - \phi)} = \sum_{n'=-\infty}^{\infty} i^{n'} e^{\mp n'(\theta - \phi)} J_{n'}(\xi), \tag{B6}$$

*See Eq. (a.1.41) in *Handbook of Mathematical Functions*, M. Abramowitz and I.A. Stegun, editors, Government Printing Office, Washington, D.C., 1964.

where ξ may be complex. Finally, by substituting Eq. (B5) into (B4), we obtain

$$\mathbf{U}(r, \phi, z) = 2\pi \sum_{n=-\infty}^{\infty} i^{\pm n} e^{\pm in\phi} \int_0^{\infty} \mathbf{u}(\rho, n) e^{\pm ikmz} J_{\pm n}(k\rho r)\rho d\rho, \tag{B7}$$

where

$$\begin{aligned} m &= \sqrt{1 - \rho^2}, \text{ if } \rho \leq 1, \\ &= i\sqrt{\rho^2 - 1}, \text{ if } \rho > 1, \end{aligned} \tag{B8}$$

and where either the upper or lower signs may be used. Equations (12) follow by substituting $\mathbf{U}(r, \phi, z)$ from (B7) for the integral in Eqs. (3), after selecting the upper sign and limiting the range of integration in (B7).

Appendix C
DERIVATION OF THE MODE EQUATION

In this appendix the mathematical procedures leading to Eq. (14) are described. Substituting Eq. (12a) into (1a) for the x component gives

$$2\pi \sum_{n'=-\infty}^{\infty} i^{n'} e^{in'\phi} \int_0^1 e_{n'}^{(+1)}(\rho') J_{n'}(k\rho' r)\rho' d\rho'$$

$$= \pm 2\pi g(2L) \sum_{n'=-\infty}^{\infty} i^{n'} e^{in'\phi} \int_0^1 e_{n'}^{(+1)}(\rho') e^{ik\sqrt{1-\rho'^2}2L} J_{n'}(k\rho' r)\rho' d\rho' e^{(\mp)ik(r^2+f^2)^{1/2}} \operatorname{circ}\left(\frac{r}{a}\right) . \tag{C1}$$

Next, we multiply Eq. (C1) by a function

$$e^{-in\phi} J_n(k\rho r) r, \tag{C2}$$

integrate over all $0 \leqslant r \leqslant \infty$ and $0 \leqslant \phi \leqslant 2\pi$, and interchange the order of the integrations to obtain

$$2\pi \sum_{n'=-\infty}^{\infty} i^{n'} \int_0^1 e_{n'}^{(+1)}(\rho') \left[\int_0^{2\pi} e^{i(n'-n)\phi} d\phi \int_0^{\infty} J_{n'}(k\rho' r) J_n(k\rho r) r dr \right] \rho' d\rho' = \pm 2\pi g(2L)$$

$$\times \sum_{n'=-\infty}^{\infty} i^{n'} \int_0^1 e_{n'}^{(+1)}(\rho') e^{ik\sqrt{1-\rho'^2}2L} \left[\int_0^{2\pi} e^{i(n'-n)\phi} d\phi \right.$$

$$\left. \times \int_0^a e^{(\mp)ik(r^2+f^2)^{1/2}} J_{n'}(k\rho' r) J_n(k\rho r) r dr \right] \rho' d\rho' . \tag{C3}$$

Finally we use the identities

$$\int_0^{2\pi} e^{i(n'-n)\phi} d\phi = 2\pi \delta_{nn'} \tag{C4}$$

and*

$$\int_0^{\infty} J_n(k\rho' r) J_n(k\rho r) r dr = \frac{\delta(\rho - \rho')}{(k^2 \rho)} \tag{C5}$$

*P.M. Morse and H. Feshbach, *Methods of Theoretical Physics*, McGraw-Hill, New York, 1953, p. 943.

in Eq. (C3) and obtain the mode equation for the x component of $\mathbf{e}_n(\rho)$:

$$e_n^{(+1)}(\rho) = \pm g(2L) \int_0^1 e_n^{(+1)}(\rho') e^{ik\sqrt{1-\rho'^2}\,2L} \left[k^2 \int_0^\infty e^{(\mp)ik(r^2+f^2)^{1/2}} J_n(k\rho' r) J_n(k\rho r) r\,dr \right] \rho' d\rho'. \tag{C6}$$

In the same manner an identical mode equation may be derived for the y component of $\mathbf{e}_n(\rho)$. Thus we generalize Eq. (C6) to the form of Eq. (14) by replacing $e_n^{(+1)}(\rho)$ by $e_n^{(s)}(\rho)$, where the x component of the angular spectrum is indicated if $s = +1$ and the y component if $s = -1$.

Appendix D
DERIVATION OF THE FIELD EQUATIONS

In this appendix we derive Eqs. (20) and (22) for the components of the resonator field.

The standing waves inside the resonator are obtained from the traveling waves as follows. The traveling waves propagating to the right are combined point by point with an identical set of traveling waves moving to the left. These traveling waves are combined such that the appropriate boundary conditions at the reflectors are satisfied. For example, if the plus sign is taken in front of Eqs. (1), the transverse electric component associated with the waves travling to the left must be subtracted from the similar component associated with the waves traveling to the right so that the transverse electric component vanishes at the reflectors. Similarly, if the negative sign is taken in front of Eqs. (1), these components must be added instead so that the transverse electric component of the standing wave vanishes at the reflectors. Thus, using Eq. (3a) we obtain the expression for the x component

$$\left\{ \mathsf{E}^{(s)}_{nn'N}(x, y, z) \right\}_x = \iint\limits_{p^2+q^2 \leqslant 1} \mathcal{E}_x(p, q) e^{ik(px+qy)} \left[g(z) e^{ikmz} \mp g(2L - z) e^{ikm(2L-z)} \right] dpdq, \tag{D1a}$$

where either the upper or lower sign is taken as the upper or lower sign in Eq. (1), or equivalently as N is even or odd respectively. This connection between the sign convention and the parity of N follows from Eqs. (18) and (19).

The other Cartesian components of the standing wave, which are also found using the appropriate boundary conditions at the reflectors, are given by

$$\left\{ \mathsf{E}^{(s)}_{nn'N}(x, y, z) \right\}_y = \iint\limits_{p^2+q^2 \leqslant 1} \mathcal{E}_y(p, q) e^{ik(px+qy)} \left[g(z) e^{ikmz} \mp g(2L - z) e^{ikm(2L-z)} \right] dpdq, \tag{D1b}$$

$$\left\{ \mathsf{E}^{(s)}_{nn'N}(x, y, z) \right\}_z = \iint\limits_{p^2+q^2 \leqslant 1} \mathcal{E}_z(p, q) e^{ik(px+qy)} \left[g(z) e^{ikmz} \pm g(2L - z) e^{ikm(2L-z)} \right] dpdq, \tag{D1c}$$

$$\left\{ \mathsf{H}^{(s)}_{nn'N}(x, y, z) \right\}_x = \iint\limits_{p^2+q^2 \leqslant 1} \mathcal{H}_x(p, q) e^{ik(px+qy)} \left[g(z) e^{ikmz} \pm g(2L - z) e^{ikm(2L-z)} \right] dpdq, \tag{D1d}$$

$$\left\{ H^{(s)}_{nn'N}(x, y, z) \right\}_y = \iint_{p^2+q^2<1} \mathcal{H}_y(p, q) e^{ik(px+qy)} [g(z) e^{ikmz} \pm g(2L - z) e^{ikm(2L-z)}] \, dp\, dq, \tag{D1e}$$

$$\left\{ H^{(s)}_{nn'N}(x, y, z) \right\}_z = \iint_{p^2+q^2<1} \mathcal{H}_z(p, q) e^{ik(px+qy)} [g(z) e^{ikmz} \mp g(2L - z) e^{ikm(2L-z)}] \, dp\, dq. \tag{D1f}$$

The validity of these six equations can be varified in the limit of zero gain ($g = 0$) by directly substituting them into Maxwell's equations for free space*.

Since the derivation of all of the Cartesian components in Eqs. (20) and (22) follow from Eqs. (D1) in essentially the same manner, we will give the derivation of only the z component of the electric component associated with the field in an unsaturated medium. To obtain this component, we assume that the fields are plane polarized in the x direction so that $\mathcal{E}_y = 0$, substitute Eq. (5a) into (D1c), substitute Eq. (7) into (D1c), and transform the result into cylindrical coordinates as given by Eqs. (11). We then find that

$$\left\{ E^{(+1)}_{nn'N}(r, \phi, z) \right\}_z = -i \int_0^1 e^{(+1)}_{nn'}(\rho) e^{i(km+\alpha_{nn'N})L} C[(km + \alpha_{nn'N})(z - L)]$$

$$\times \int_0^{2\pi} [e^{ik\rho r\cos(\theta-\phi)+i(n+1)\theta} + e^{ik\rho r\cos(\theta-\phi)+i(n-1)\theta}] \, d\theta \left(\frac{\rho}{\sqrt{1-\rho^2}} \right) d\rho, \tag{D2}$$

where $C(\xi)$ is given by Eqs. (21). The integral over θ is evaluated using Eq. (B5), so that upon substituting Eq. (15) we have

$$\left\{ E^{(+1)}_{nn'N}(r, \phi, z) \right\}_z = 2\pi i^{n-1} e^{in\phi} \int_0^1 e^{(+1)T}_{nn'}(\rho) e^{i\alpha_{nn'N}L} C[(km + \alpha_{nn'N})(z - L)]$$

$$\times [J_{n+1}(k\rho r) e^{i\phi} - J_{n-1}(k\rho r) e^{-i\phi}] \left(\frac{\rho\sqrt{\rho}}{\sqrt{1-\rho^2}} \right) d\rho. \tag{D3}$$

The other field components in Eqs. (20) are found in exactly the same way. The components in Eq. (22) are found in a similar manner, except that Eq. (8) is used in place of (7).

*See G. Borgiotti, Alta Frequenza 32, 808 (1963).

Appendix E
THE RESONATOR-FIELD HAMILTONIAN

The classical Hamiltonian for the most general resonator field is obtained by substituting Eqs. (20) into Eq. (31) and summing over all modes. In this appendix we will carry out this calculation.

To simplify the algebra it is useful to separate Eq. (31) into a sum of terms, each containing the energy contributed by one Cartesian component of the field. To obtain the energy contributed by the x component of the electric field associated with the field plane polarized in the x direction, we sum Eq. (20a) over all modes, set $k\sqrt{1-\rho^2} + k'_{nn'N} = k_{nn'N}\sqrt{1-\rho^2}$ (where $k_{nn'N}$ is real), and substitute it into Eq. (31). We then find that

$$\sum_{\substack{nn'N\\mm'M}} \frac{1}{8\pi}\int_0^{2L}\int_0^{2\pi}\int_0^{\infty} \epsilon\left\{\mathbf{E}^{(+1)}_{nn'N}(x, y, z, t)\right\}_x \left\{\mathbf{E}^{(+1)}_{mm'M}(x, y, z, t)\right\}_x rdrd\phi dz$$

$$= \frac{\pi}{2}\epsilon \sum_{\substack{nn'N\\mm'M}} \left\{ i^{n+m+2} a_{nn'N} a_{mm'M} \int_0^1 e^{(+1)'}_{nn'}(\rho)\sqrt{\rho}\, d\rho \int_0^1 e^{(+1)'}_{mm'}(\rho')\sqrt{\rho'}\, d\rho' \int_0^{2\pi} e^{i(n+m)\phi}\, d\phi \right.$$

$$\times \int_0^{2L} S[(k\sqrt{1-\rho^2} + k'_{nn'N})(z-L)]S[(k\sqrt{1-\rho'^2} + k'_{mm'M})(z-L)]\,dz$$

$$\left.\times \int_0^{\infty} J_n(k\rho r)J_m(k\rho' r)rdr + c.c.\right\}$$

$$+ \frac{\pi}{2}\epsilon \sum_{\substack{nn'N\\mm'M}} \left\{ i^{n-m} a_{nn'N} a^*_{mm'M} \int_0^1 e^{(+1)'}_{nn'}(\rho)\sqrt{\rho}\, d\rho \int_0^1 \left[e^{(+1)'}_{mm'}(\rho')\right]^* \sqrt{\rho'}\, d\rho' \int_0^{2\pi} e^{i(n-m)\phi} d\phi \right.$$

$$\times \int_0^{2L} S[(k_{nn'N}\sqrt{1-\rho^2})(z-L)]S[(k_{mm'M}\sqrt{1-\rho'^2})(z-L)]\,dz$$

$$\left.\times \int_0^{\infty} J_n(k\rho r)J_m(k\rho' r)rdr + c.c.\right\}. \tag{E1}$$

The θ integration is evaluated with the help of the identity in Eq. (C4). The integration over r is carried out using Eq. (C5). After the integration over ρ', only terms for which $\rho' = \rho$ are nonvanishing. The integration over z can be carried out using

$$\int_0^{2L} S[(k_{nn'N}\sqrt{1-\rho^2})(z-L)]S[(k_{mm'M}\sqrt{1-\rho^2})(z-L)]dz \approx L\delta_{nm}\delta_{n'm'}\delta_{NM}. \quad \text{(E2)}$$

This equation follows from Eq. (C4), provided that the integral extends over the domain of orthogonality for the sinusoids. This is not generally the case; however, if we assume that $e_{nn'}^{(s)'}(\rho)$ can be taken to be zero outside of some domain where $\rho << 1$, then we find from Eq. (26) that

$$k_{nn'N}\sqrt{1-\rho^2} \approx \frac{\pi N'}{(2L)} . \quad \text{(E3)}$$

By substituting Eq. (E3) into (E2) and assuming that N' can be approximated by an integer (valid for paraxial resonator fields), we find that the integral extends over the domain of orthogonality for the sinusoids.

After evaluation of the integrals in Eq. (E1) and after some reduction it takes the form

$$\frac{\pi^2 L\epsilon}{k^2} \sum_{\substack{nn'N \\ mm'M}} a_{nn'N}a^*_{mm'M} \int_0^1 e_{nn'}^{(+1)'}(\rho)\left[e_{mm'}^{(+1)'}(\rho)\right]^* d\rho\,\delta_{nm}\delta_{n'm'}\delta_{NM} + c.c.$$

$$= \frac{\epsilon L\lambda^2}{4} \sum_{nn'N} a_{nn'N}a^*_{nn'N} \int_0^1 |e_{nn'}^{(+1)'}(\rho)|^2 d\rho + c.c. \quad \text{(E4)}$$

The right-hand side of Eq. (E4) gives the energy contributed by the x component of the electric field for all modes which are plane polarized in the x direction. The energy contributed by the other Cartesian components associated with these modes can be found in the same way. The total energy contained in all of the x polarized modes, found by summing over all contributions, is

$$\mathcal{K} = \frac{\epsilon L\lambda^2}{4} \sum_{nn'N} (a_{nn'N}a^*_{nn'N} + a^*_{nn'N}a_{nn'N}) \int_0^1 |e_{nn'}^{(+1)'}(\rho)|^2 \left[2 + \frac{\rho^2}{1-\rho^2}\right] d\rho. \quad \text{(E5)}$$

It is evident from this expression that modes with different (n, n', N) do not couple; they contribute their energy independently to the total.

To find the energy contributed by the modes which are plane polarized in the y direction, it is first necessary to note that any two modes which are plane polarized in orthogonal directions do not couple. Consider any field with two plane-polarized components as given by Eq. (10). By substituting Eq. (5) into (10), and then substituting (10) into (31), we find that

$$\mathcal{H} = \frac{\lambda^2 L|g(z)|^2}{4\pi} \iint\limits_{p^2+q^2\leq 1} [\epsilon|\underline{\mathcal{E}}(p, q)|^2 + \mu|\underline{\mathcal{H}}(p, q)|^2]$$

$$= \frac{\epsilon\lambda^2 L|g(z)|^2}{2\pi} \iint\limits_{p^2+q^2\leq 1} \left[|\mathcal{E}_x(p, q)|^2 \left(1 - \frac{p^2}{m^2}\right) + |\mathcal{E}_y(p, q)|^2 \left(1 - \frac{q^2}{m^2}\right)\right] dpdq. \qquad \text{(E6)}$$

This result confirms our assumption that the two orthogonally polarized components of a general field do not couple but contribute their energy independently to the total.

Since all modes with different (n, n', N) are independent, and since the two modes with the same (n, n', N) but different s contain the same energy (the field distributions are the same but rotated 90° in space), we can obtain the total energy of the most general resonator field simply by summing Eq. (E5) over s. The Hamiltonian is put into the standard form given by Eq. (32) simply by normalizing the angular spectrum as given by (33) and simplifying the notation by replacing (n, n', N, s) by the single symbol M.

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