FR-3441

NRL REPORT P-3441

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ELECTRODYNAMICS OF A SUPERCONDUCTING TORUS

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April 5, 1949

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CONTENTS

ABSTRACT

Using Maxwell's electromagnetic equations and defining a superconductor as a conductor in which the magnetic induction is zero, expressions are obtained for the magnetic field and surface currents about a superconducting torus for several cases. First, two basic cases are solved, one involving a scalar potential only and the other a vector potential only. Then by superposing these two solutions appropriately, other interesting cases are studied. From these solutions are obtained expressions for the self-inductance, magnetic moments, and other characteristic quantities of the superconducting torus. These quantities are tabulated for a number of tori. To facilitate computation of quantities not adequately covered by the brief tables, approximate formulas are included in an appendix. Finally, to illustrate the practical value of these calculations, the behavior of a superconducting torus in a magnetic cycle is studied.

PROBLEM STATUS

This report presents the findings of one of the mathematical investigations being conducted on the behavior of superconducting systems.

AUTHORIZATION

NRL Problem P01-07R

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ELECTRODYNAMICS OF A SUPERCONDUCTING TORUS

INTRODUCTION

Experiments are frequently conducted using a superconducting toroidal ring of circular cross section, the ring being placed in a magnetic field applied perpendicularly to the plane of the torus. Consequently, a calculation is presented here of the field and current distributions, and other properties, of an ideal superconducting torus (with zero penetration depth).

In order that there be no resultant field inside the material of a superconducting torus, surface currents arise which are distributed so as to produce a magnetic field equal and opposite to the applied field within the material of the torus. Further, there may be present surface currents, in the absence of an applied magnetic field, for which the net current is not zero, but is produced by an external supply, or is induced to conserve the flux enclosed by the multiply-connected superconductor (torus) when the external field is removed. In general, the torus in a magnetic field applied perpendicularly to its plane will possess surface currents which are a superposition of these two types.

The basic electrodynamic problems solved here are the following two types:

Case I. A superconducting torus in a uniform field applied perpendicularly to the plane of the torus, with zero net current. This is the Meissner-Ochsenfeld effect giving zero resultant magnetic field within the material of the torus.

Case II. A superconducting torus in zero applied field with net current not zero. The current must be so distributed over the surface that the resultant magnetic field within the material of the torus is zero.

The results obtained for I and II can be utilized in the analysis of the following cases:

Case III. A net current is supplied by starting with an applied magnetic field and with zero net current (Case I), then removing the applied field leaving a net persistent current (Case II). This case differs from Case II in that the field and current distributions for Case III are expressed in terms of the initial applied field rather than the final net current, as in Case II.

Case IV. Beginning with zero applied field and no surface currents, a magnetic field is applied inducing a net current in such a manner that the enclosed magnetic flux remains zero. This is a superposition of Case I and Case II.

The basic cases, I and II are calculated rigorously with the assumption that the penetration depth ($\sim 10^{-5}$ cm) is negligible. From these cases, rigorous expressions in the form of infinite series are obtained for cases III and IV by superposition after a determination has been made of the enclosed flux for cases I and II. In addition, a rigorous expression is derived for the magnetic moment of the torus for each of the four cases, and for the self-inductance of a superconducting torus.

To make the calculations more immediately applicable, approximate formulae are given for the maximum and minimum fields at the surface of the torus for each of the four cases outlined above. Similarly, approximate formulae are given for the moments and the self-inductance. The error in each of these approximate formulae is easily evaluated by comparison with the numerical results obtained by tabulation of the rigorous formulae. This enables one to use the approximate formulae with greater confidence. These formulae are collected in the appendix.

Finally, the results of these calculations are used to determine the behavior of a superconducting torus subjected to a magnetic cycle involving the suppression of superconductivity.

SOLUTION OF THE FIELD EQUATIONS

A. METHOD OF CALCULATION OF CASES I AND II

The assumptions made are that the penetration depth is negligible relative to the thickness of the wire forming the torus, the normal component of the field vanishes at the surface of the superconductor, and the intermediate state is not involved.

Case I. In this case we compute the external field $\vec{H}$ and surface current distribution $\vec{J}$ from a scalar potential V, the equations being

$$\nabla^2 V = 0 \quad \text{, externally,} \tag{1}$$

$$\vec{H} = -\operatorname{grad} V \tag{2}$$

and the magnitude of the surface current J, in electrostatic units per cm.,

$$J = \frac{cH}{4\pi} \qquad \text{(H at toroidal surface)} \tag{3}$$

Case II. The field distribution $\vec{H}$ is calculated from the vector potential $\vec{A}$, the equations for this case being

$$\nabla \times \nabla \times \vec{A} = 0 \tag{4}$$

with $$\nabla \cdot \vec{A} = 0 \tag{5}$$

and $$\vec{H} = \nabla \times \vec{A} \tag{6}$$

The surface current is found from eq. (3) and the total current I is introduced in the usual way by taking the line integral of the magnetic field around a closed path threaded once by the current.

These calculations are made in the toroidal coordinate system.

B. THE TOROIDAL COORDINATE SYSTEM

In this section, we shall describe the toroidal coordinate system and give the differential operators in terms of these coordinates, without proof.[1, 2] The transformation equations connecting the familiar cylin-

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[1] H. Bateman, Partial Differential Equations of Mathematical Physics, Cambridge University Press (1932). Chapter X.

[2] E. W. Hobson, The Theory of Spherical and Ellipsoidal Harmonics, Cambridge University Press (1931), Sections 253-258.

drical coordinates (ρ, θ, z) with the toroidal coordinates (s, θ , φ) are the following, the polar angle θ being identical in the two systems

$$\rho = \frac{a\sqrt{s^2-1}}{s-\cos\varphi}\,, \tag{7}$$

$$z = \frac{a\sin\varphi}{s-\cos\varphi}\,, \tag{8}$$

and conversely,

$$\cot\varphi = \frac{\rho^2+z^2-a^2}{2az} \tag{9}$$

$$s = \frac{\rho^2+z^2+a^2}{\sqrt{(\rho^2+z^2+a^2)^2-4a^2\rho^2}} \tag{10}$$

It is customary to employ the coordinate $\mu = \cosh^{-1} s$ instead of s, as is done here, but the solutions of our problem occur as functions of $\cosh\mu$ and consequently s is more convenient to use. Moreover, the coordinate s_o = constant which defines the toroidal surface of the superconductor has a simple geometrical meaning, namely, the ratio of the mean radius of the toroidal loop to the radius of the toroidal wire.

Fig. 1 shows a section of the toroidal coordinate system in a θ = constant and $\theta \pm \pi$ = constant plane. The s = constant surfaces are tori, the tori for smaller values of s entirely enclosing those for larger values of s. The value s = 1 designates the torus which fills all of the coordinate space and which, physically, corresponds to a finite torus whose inner rim is reduced to a point. The value $s = \infty$ corresponds to an infinitely thin torus of radius a. Thus the symbol

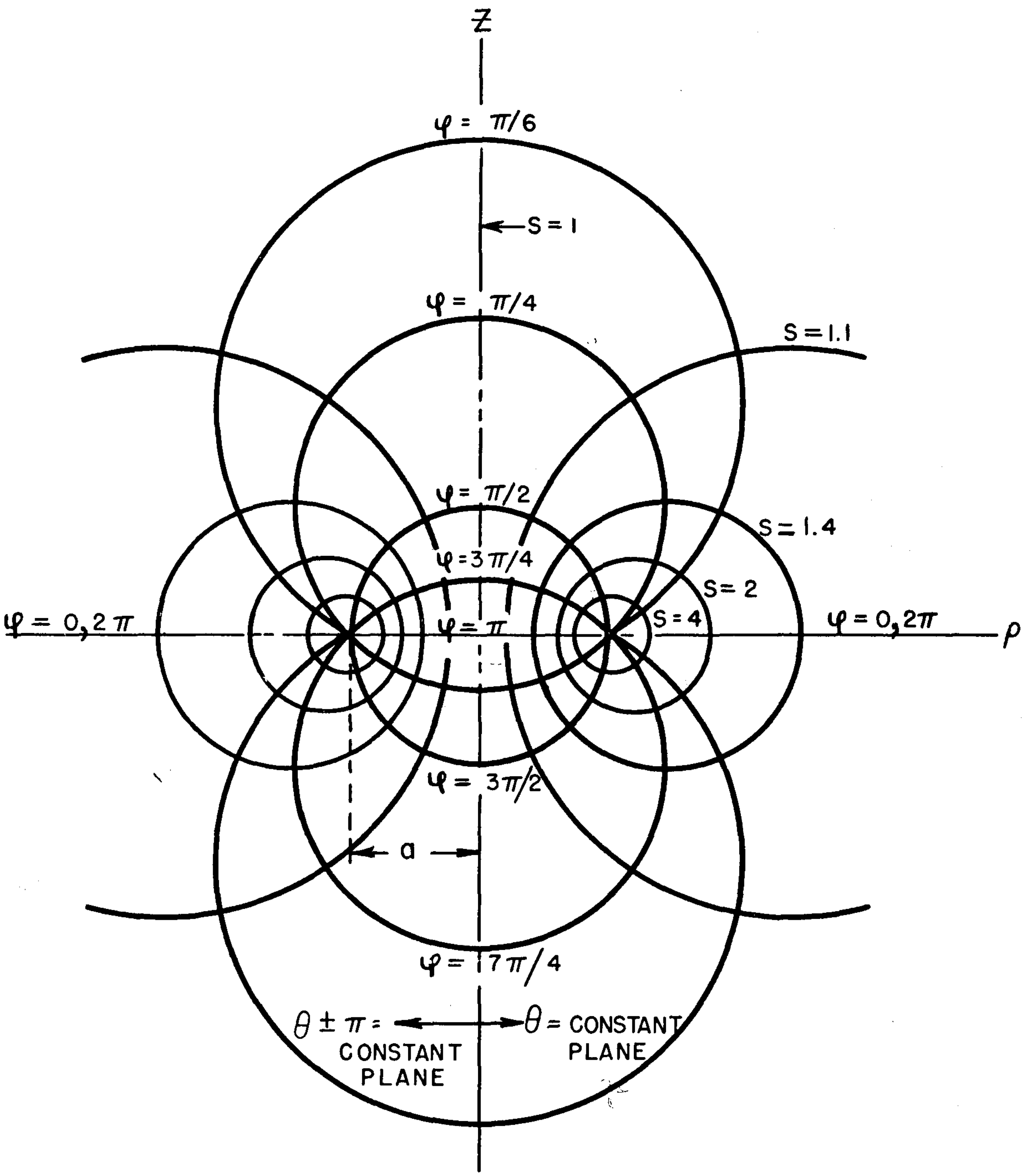

Fig. 1 - Toroidal coordinate system

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a which occurs in equations (7) through (10) is the scale factor of the system. If R is the distance from the center of the circular cross section of a torus to the center of the loop (mean radius of a toroidal surface) then a is connected to R by

$$a = R\sqrt{s_o^2 - 1}/s_o \tag{11}$$

where s_o is any particular toroidal surface (for example, the surface of the wire).

If we denote by r, the radius of the circular cross section of a torus (for example, the radius of the wire forming a torus), then

$$s_o = R/r \tag{12}$$

The two sets of surfaces orthogonal to the surfaces s = constant are the half planes θ = constant and the spherical bowls φ = constant. The θ = constant half planes are the same as the half planes of cylindrical coordinates. The φ = constant spherical bowls all have the fundamental circle $s = \infty$, of radius a, as their common rim. When $\varphi = 0$ or 2π, the surface becomes flattened to the infinite plane outside the fundamental circle and corresponds to the cylindrical coordinates $z = 0$ and $\rho > a$. When $\varphi = \pi$, the surface is again flattened and is the thin circular disc of radius a corresponding to the cylindrical coordinates $z = 0$ and $\rho < a$. In the upper half of Fig. 1, φ has the range, $0 < \varphi < \pi$, while in the lower half, $\pi < \varphi < 2\pi$.

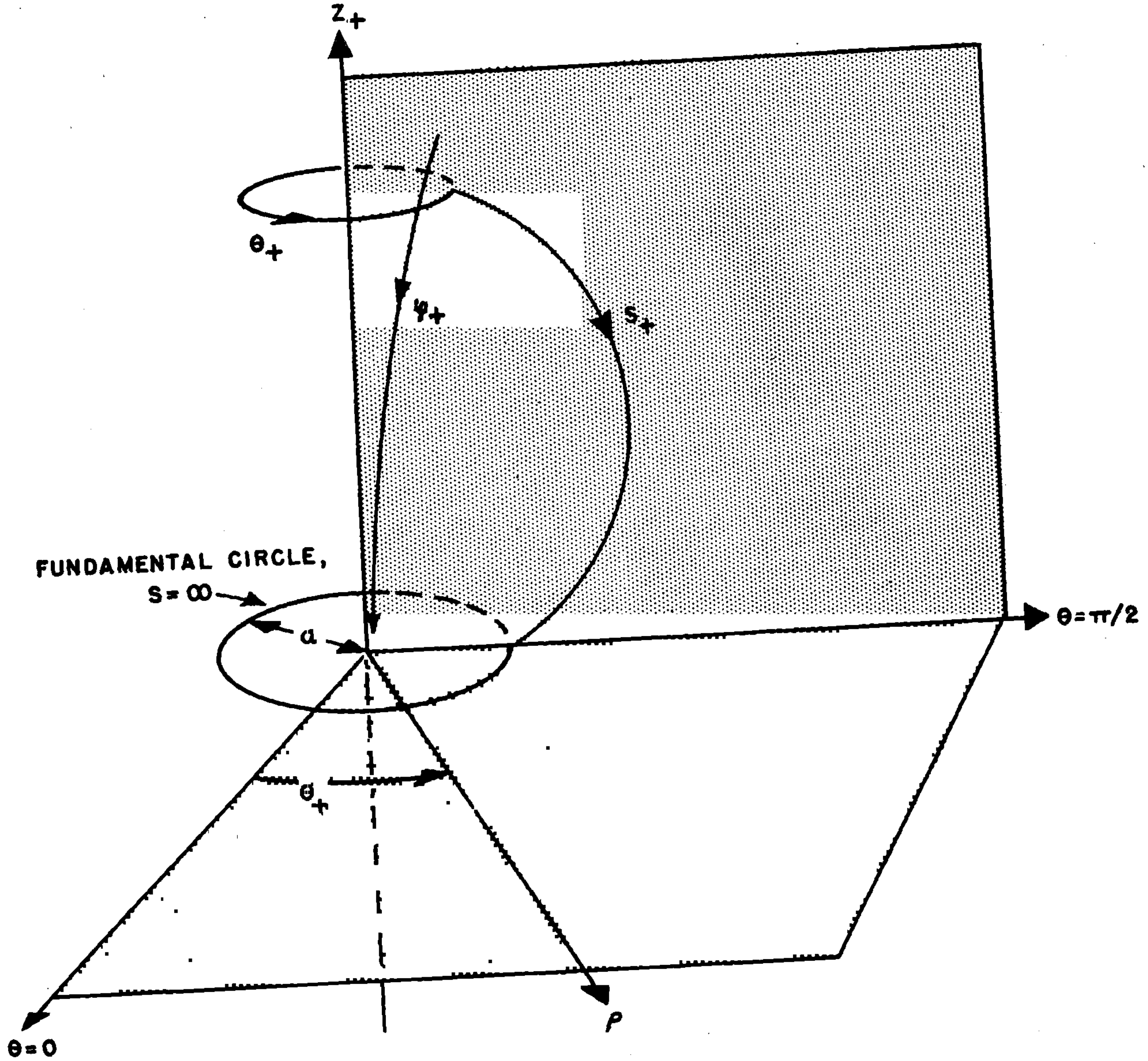

Fig. 2 - Directions of positively increasing s, θ, φ

The directions of positive s, θ, φ are shown in Fig. 2. The line element dl is given by

$$dl^2 = N^2\left[ds^2/(s^2-1) + d\varphi^2 + (s^2-1)\,d\theta^2\right] \tag{13}$$

where

$$N = a/(s - \cos\varphi) \tag{14}$$

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From eq's. (13) and (14) we can deduce the following operator relations:

gradV

$$(\mathrm{gradV})_s = \frac{\sqrt{s^2-1}}{N}\frac{\partial V}{\partial s} \tag{15}$$

$$(\mathrm{gradV})_\varphi = \frac{1}{N}\frac{\partial V}{\partial \varphi} \tag{16}$$

$$(\mathrm{gradV})_\theta = \frac{1}{N\sqrt{s^2-1}}\frac{\partial V}{\partial \theta} \tag{17}$$

$\nabla\cdot\vec{A}$

$$\nabla\cdot\vec{A} = \frac{1}{N^3}\left[\frac{\partial}{\partial s}\left(N^2\sqrt{s^2-1}\,A_s\right) + \frac{\partial}{\partial\varphi}\left(N^2 A_\varphi\right) + \frac{N^2}{\sqrt{s^2-1}}\frac{\partial A_\theta}{\partial\theta}\right] \tag{18}$$

$\nabla\times\vec{A}$

$$(\nabla\times\vec{A})_s = \frac{1}{N^2}\left[\frac{\partial}{\partial\varphi}(NA_\theta) - \frac{N}{\sqrt{s^2-1}}\frac{\partial A_\varphi}{\partial\theta}\right] \tag{19}$$

$$(\nabla\times\vec{A})_\varphi = \frac{1}{N^2}\left[\frac{N}{\sqrt{s^2-1}}\frac{\partial A_s}{\partial\theta} - \frac{\partial}{\partial s}\left(N\sqrt{s^2-1}\,A_\theta\right)\right] \tag{20}$$

$$(\nabla\times\vec{A})_\theta = \frac{1}{N^2}\left[\sqrt{s^2-1}\,\frac{\partial}{\partial s}(NA_\varphi) - \frac{\partial}{\partial\varphi}(NA_s)\right] \tag{21}$$

$\nabla^2 V$

$$\nabla^2 V = \frac{1}{N^3}\left[\frac{\partial}{\partial s}\left(N(s^2-1)\frac{\partial V}{\partial s}\right) + \frac{\partial}{\partial\varphi}\left(N\frac{\partial V}{\partial\varphi}\right) + \frac{N}{s^2-1}\frac{\partial^2 V}{\partial\theta^2}\right] \tag{22}$$

C. GENERAL SOLUTION OF THE DIFFERENTIAL EQUATIONS

<u>Case I:</u> A Superconducting Torus in a Uniform Field Applied Perpendicularly to the Plane of the Torus, with Zero Net Current -

The Laplace equation governing V is

$$\frac{\partial}{\partial s}\left(\frac{s^2-1}{s-\cos\varphi}\frac{\partial V}{\partial s}\right)+\frac{\partial}{\partial\varphi}\left(\frac{1}{s-\cos\varphi}\frac{\partial V}{\partial\varphi}\right)=0 \tag{23}$$

since the symmetry of the problem shows that V will be independent of the coordinate θ. The equation can be separated if we write

$$V=\sqrt{s-\cos\varphi}\;U \tag{24}$$

and then separate the equation for U by writing

$$U(s,\varphi)=S(s)\,\Phi(\varphi) \tag{25}$$

giving

$$\frac{d^2\Phi}{d\varphi^2}+n^2\Phi=0 \tag{26}$$

$$\frac{d}{ds}\left[(s^2-1)\frac{dS}{ds}\right]-\left(n^2-\tfrac{1}{4}\right)S=0 \tag{27}$$

where n^2 is the separation constant.

Since there is no net current in this case, the potential must be single valued so that

$$V=\sqrt{s-\cos\varphi}\left[b_0P_{-\frac{1}{2}}(s)+\sum_{n=1}^{\infty}a_nP_{n-\frac{1}{2}}(s)\sin n\varphi+\sum_{n=1}^{\infty}b_nP_{n-\frac{1}{2}}(s)\cos n\varphi\right] \tag{28}$$

The symbols $P_{n-\frac{1}{2}}(s)$ (n = 0, 1, 2, - - -) are half-odd integer Legendre functions, of the first kind satisfying the differential equation (27). Those of the second kind $Q_{n-1/2}(s)$ become infinite at s = 1 and are not suitable for representing the expansion of V. The properties of these functions will be discussed in a later section.

The components of $\vec{H}$ are, according to eq's. (2) and (15), (16), (17),

$$H_s=-\frac{(s-\cos\varphi)\sqrt{s^2-1}}{a}\frac{\partial V}{\partial s} \tag{29}$$

$$H_\varphi = -\frac{(s-\cos\varphi)}{a}\frac{\partial V}{\partial \varphi} \tag{30}$$

with $H_\theta = 0$.

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Case II: A Superconducting Torus in Zero Applied Field with Net Current Not Zero -

From symmetry, there will be only one component of the current vector, namely the θ-component. Therefore, the only component of $\vec{A}$ will be A_θ which we shall briefly denote by A. Eq. (4) becomes

$$\sqrt{s^2-1}\,\frac{\partial}{\partial s}\left[(s-\cos\varphi)\frac{\partial}{\partial s}\left(\frac{\sqrt{s^2-1}}{s-\cos\varphi}A\right)\right] + \frac{\partial}{\partial\varphi}\left[(s-\cos\varphi)\frac{\partial}{\partial\varphi}\left(\frac{A}{s-\cos\varphi}\right)\right] = 0 \tag{31}$$

We can separate the differential equation by writing

$$A = \sqrt{s-\cos\varphi}\; T(s)\,\Phi(\varphi) \tag{32}$$

and find that

$$\frac{d^2\Phi}{ds^2} + n^2\Phi = 0 \tag{33}$$

and

$$\frac{d}{ds}\left[(s^2-1)\frac{dT}{ds}\right] - \left(\frac{1}{s^2-1} + n^2 - \frac{1}{4}\right)T = 0 \tag{34}$$

If we write $T = \sqrt{s^2-1}\,\frac{dS}{ds}$ where S is the solution of eq. (27), we find that T satisfies (34) so that A is given by

$$A = \sqrt{s-\cos\varphi}\left[(b_0 + a_0\varphi)P^1_{-\frac{1}{2}}(s) + \sum_{n=1}^{\infty} a_n P^1_{n-\frac{1}{2}}(s)\sin n\varphi + \sum_{n=1}^{\infty} b_n P^1_{n-\frac{1}{2}}(s)\cos n\varphi\right] \tag{35}$$

where

$$P^{1}_{n-\frac{1}{2}}(s) = \sqrt{s^2-1}\, dP_{n-1/2}(s)/ds = \sqrt{s^2-1}\, P'_{n-\frac{1}{2}}(s) \tag{36}$$

and $P^{1}_{n-\frac{1}{2}}(s)$ is the associated half-odd integer Legendre function of the first kind and of first order. Those of the second kind become infinite at infinity and are therefore excluded.

The components of $\vec{H}$, according to eq's. (6) and (19), (20), (21) are

$$H_s = -\frac{\sin\varphi}{a} A + \frac{s-\cos\varphi}{a}\frac{\partial A}{\partial\varphi} \tag{37}$$

$$H_\varphi = \frac{s\cos\varphi - 1}{a\sqrt{s^2-1}} A - \frac{(s-\cos\varphi)\sqrt{s^2-1}}{a}\frac{\partial A}{\partial s} \tag{38}$$

with $H_\theta = 0$

D. THE HALF-ODD INTEGER LEGENDRE FUNCTIONS OF THE FIRST KIND

A convenient form for the representation of the half-odd integer Legendre functions $P_{n-\frac{1}{2}}(s)$, $s \geqslant 1$, is the Laplace second integral [3]

$$P_{n-\frac{1}{2}}(s) = \frac{1}{\pi}\int_0^\pi \frac{d\varphi}{(s+\sqrt{s^2-1}\cos\varphi)^{n+1/2}} \tag{39}$$

For the case n = 0, we get

$$P_{-\frac{1}{2}}(s) = \frac{2}{\pi}\sqrt{s-\sqrt{s^2-1}}\, K(k) \tag{40}$$

where $K(k) = \int_0^{\pi/2} d\xi/\sqrt{1-k^2\sin^2\xi}$ is the complete elliptic integral of the first kind and

$$k = \sqrt{\frac{2\sqrt{s^2-1}}{s+\sqrt{s^2-1}}} \tag{41}$$

[3] E. T. Whittaker and G. N. Watson, Modern Analysis, Cambridge University Press, (1927), section 15.23.

Similarly
$$P_{\frac{1}{2}}(s) = \frac{2\sqrt{s+\sqrt{s^2-1}}}{\pi}\,E(k) \tag{42}$$

where E is the complete elliptical integral of the second kind, and k is defined by eq. (41).

Values of $P_{n-\frac{1}{2}}(s)$ for n > 1 can be obtained from these by recurrence formulae.[4]

An alternative expression for $P_{-\frac{1}{2}}(s)$ and for $P_{\frac{1}{2}}(s)$ can be found in a few lines by an analysis of the differential equation (27). The Riemann P - equation for eq. (27) is the scheme (treating s as a complex variable for the moment),

$$P_{n-\frac{1}{2}}(s) = \mathbf{P}\left\{\begin{matrix} -1 & +1 & \infty & \\ 0 & 0 & n+\frac{1}{2} & s \\ 0 & 0 & -n+\frac{1}{2} & \end{matrix}\right\} \tag{43}$$

To find $P_{-\frac{1}{2}}(s)$ we set n = 0 and transform the Riemann P - equation (43) by making the homographic substitution

$$t = (s-1)/(s+1) \tag{44}$$

obtaining

$$P_{-\frac{1}{2}}(s) = \mathbf{P}\left\{\begin{matrix} 0 & 1 & \infty & \\ 0 & \frac{1}{2} & 0 & t \\ 0 & \frac{1}{2} & 0 & \end{matrix}\right\}$$

or
$$P_{-\frac{1}{2}}(s) = \sqrt{1-t}\;\mathbf{P}\left\{\begin{matrix} 0 & 1 & \infty & \\ 0 & 0 & \frac{1}{2} & t \\ 0 & 0 & \frac{1}{2} & \end{matrix}\right\} \tag{45}$$

Thus, the relation for $P_{-\frac{1}{2}}(s)$ becomes

$$P_{-\frac{1}{2}}(s) = \sqrt{\frac{2}{s+1}}\,F\left(\tfrac{1}{2}, \tfrac{1}{2}; 1; \tfrac{s-1}{s+1}\right) \tag{46}$$

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[4] Ref. 3, section 15.21.

where F is the hypergeometric function. But, the complete elliptic integral of the first kind K(k) is related to the hypergeometric function by [5]

$$K(k) = \frac{\pi}{2} F\left(\tfrac{1}{2}, \tfrac{1}{2}; 1; k^2\right) \tag{47}$$

so that

$$P_{-\frac{1}{2}}(s) = \frac{2}{\pi}\sqrt{\frac{2}{s+1}}\, K\left(\sqrt{\frac{s-1}{s+1}}\right) \tag{48}$$

From eq. (48) and the recurrence formula relating $P_{\frac{1}{2}}(s)$ with $P_{-\frac{1}{2}}(s)$ and $dP_{-\frac{1}{2}}(s)/ds$, [4]

$$P_{\frac{1}{2}}(s) = \frac{2}{\pi}\left[2\sqrt{\frac{s+1}{2}}\, E(k) - \sqrt{\frac{2}{s+1}}\, K(k)\right] \tag{49}$$

where

$$k = \sqrt{\frac{s-1}{s+1}} \tag{50}$$

and should not be confused with the k of eq's. (40), (41), (42).

Fig. 3 shows a plot of $P_{n-\frac{1}{2}}(s)$ for several values of n. For large values of s,

$$P_{n+\frac{1}{2}}(s) \sim \sqrt{\frac{2}{\pi}}\, \frac{2^n\,\Gamma(n+1)\, s^{n+\frac{1}{2}}}{\Gamma(n+\frac{3}{2})}, \qquad n \geqslant 0 \tag{51}$$

For s = 1, $P_{n+\frac{1}{2}}(1) = 1$, for all values of n. (52)

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[5] E. T. Copson, Theory of Functions of a Complex Variable, Oxford University Press (1935), pp. 245, 394.

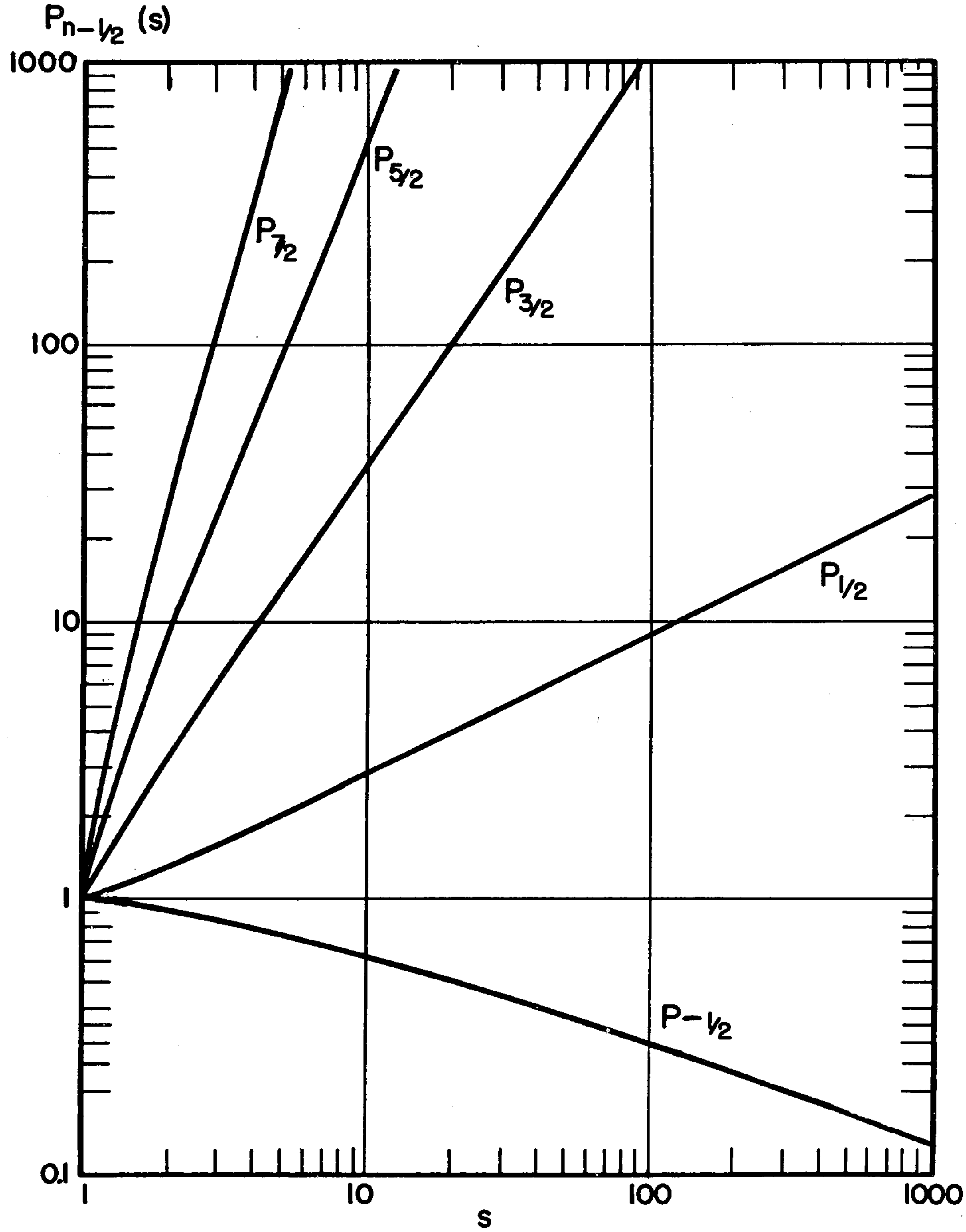

Fig. 3 - $P_{n-\frac{1}{2}}(s)$ functions

■. THE HALF-ODD INTEGER LEGENDRE FUNCTIONS OF THE SECOND KIND

The convenient expression for $Q_{n-\frac{1}{2}}(s)$, the half-odd integer Legendre function of the second kind, is [1]

$$Q_{n-\frac{1}{2}}(s) = \frac{1}{\sqrt{2}} \int_0^{\pi} \frac{\cos n\varphi \, d\varphi}{\sqrt{s - \cos\varphi}} \tag{53}$$

Putting n = 0, and transforming the integral appropriately, we can show that

$$Q_{-\frac{1}{2}}(s) = \sqrt{\frac{2}{s+1}}\, K\left(\sqrt{\frac{2}{s+1}}\right) \tag{54}$$

Using the recurrence formula [6]

$$Q_{\frac{1}{2}}(s) = s\, Q_{-\frac{1}{2}}(s) + 2(s^2-1)\, dQ_{-\frac{1}{2}}(s)/ds$$

we get from (54)

$$Q_{\frac{1}{2}}(s) = s\sqrt{\frac{2}{s+1}}\, K\left(\sqrt{\frac{2}{s+1}}\right) - 2\sqrt{\frac{s+1}{2}}\, E\left(\sqrt{\frac{2}{s+1}}\right) \tag{55}$$

Recurrence formulae can be established for these functions and are identical with those for $P_{n-\frac{1}{2}}(s)$, with $Q_{n-\frac{1}{2}}(s)$ replacing $P_{n-\frac{1}{2}}(s)$. [6]

Fig. 4 shows a plot of $Q_{n-\frac{1}{2}}(s)$ for several values of n. For large s,

$$Q_{n-\frac{1}{2}}(s) \sim \sqrt{\frac{\pi}{2}}\, \frac{\Gamma(n+\frac{1}{2})}{2^n\, \Gamma(n+1)}\, s^{-(n+\frac{1}{2})} \tag{56}$$

A relation between these Legendre functions of the first and second kinds which we shall often need is the following

$$(n+\tfrac{1}{2})\left[P_{n+\frac{1}{2}}(s)\,Q_{n-\frac{1}{2}}(s) - P_{n-\frac{1}{2}}(s)\,Q_{n+\frac{1}{2}}(s)\right] = 1 \tag{57}$$

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[6] Ref. 3, section 15.32.

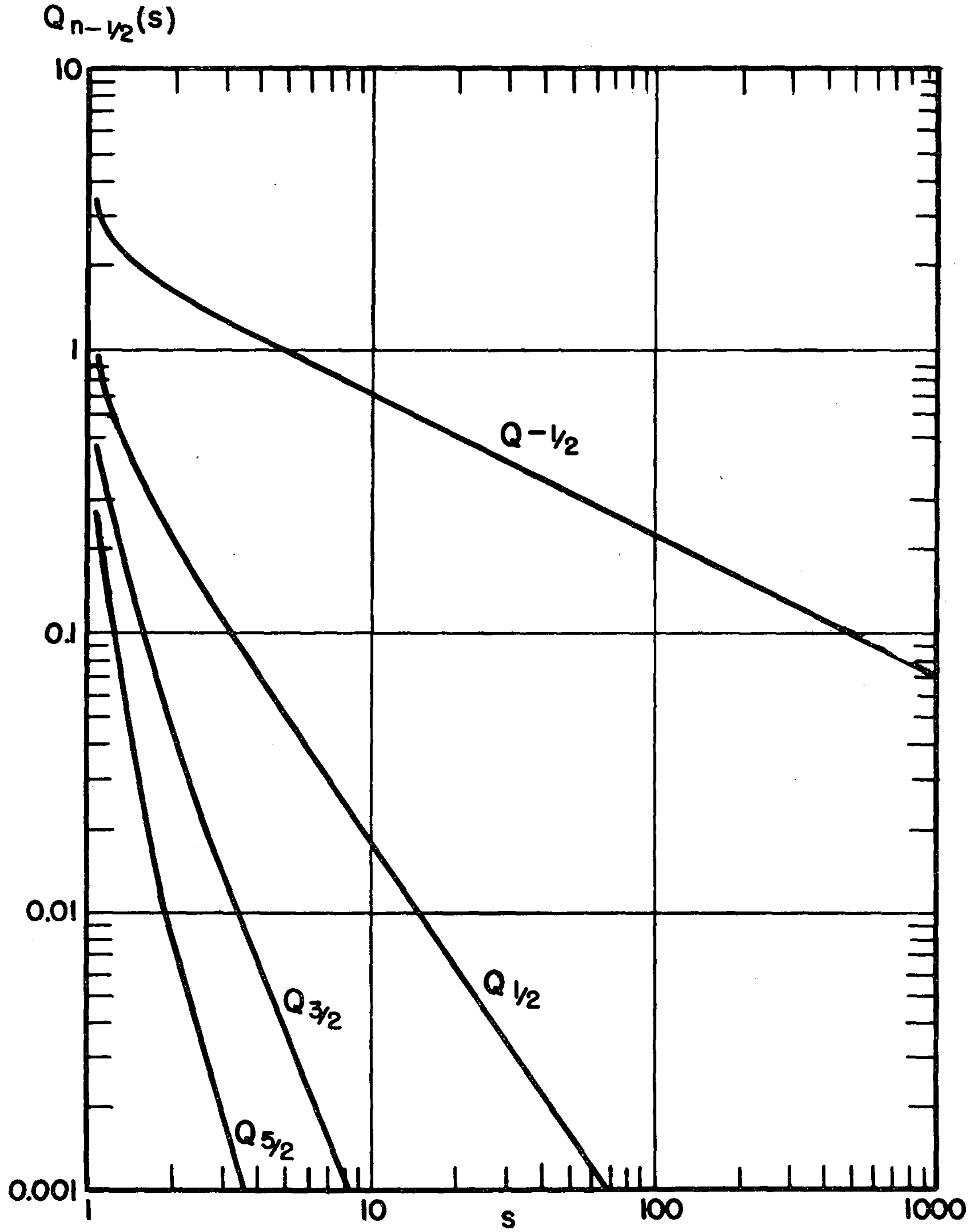

Fig. 4 - $Q_{n-\frac{1}{2}}(s)$ functions

for n = 0, 1, 2,- - -. This can be shown by proving that eq. (57) is true for n = 0 using eq's. (48), (49), (54) and (55), then using the recurrence formulae (loc. cit. 4, 6) to show that the expression in eq. (57) is equal to a similar expression with (n - 1) replacing n.

F. CASE I: FIELD AND CURRENT DISTRIBUTION

In the remainder of the report we shall simplify the writing by employing the abbreviated symbols $P_{n-\frac{1}{2}} \equiv P_{n-\frac{1}{2}}(s)$, $Q_{n-\frac{1}{2}} \equiv Q_{n-\frac{1}{2}}(s)$, $\mathring{P}_{n-\frac{1}{2}} \equiv P_{n-\frac{1}{2}}(s_o)$, $\mathring{Q}_{n-\frac{1}{2}} \equiv Q_{n-\frac{1}{2}}(s_o)$, and so forth, where s_o is the surface of the superconducting torus.

We begin by finding an expression for the uniform applied magnetic field. Let V_o be the potential of the uniform applied field H_o so that

$$\begin{aligned} V_o &= -H_o z = -H_o \frac{a \sin\varphi}{s - \cos\varphi} \\ &= -a H_o \sqrt{s-\cos\varphi}\, \frac{\sin\varphi}{(s-\cos\varphi)^{3/2}} \end{aligned} \tag{58}$$

Now $(s-\cos\varphi)^{-\frac{1}{2}}$ can be expanded in terms of the solutions of eq. (23). The expansion, given in terms of Legendre functions of the second kind, is [7]

$$(s-\cos\varphi)^{-\frac{1}{2}} = \frac{\sqrt{2}}{\pi} Q_{-\frac{1}{2}} + \frac{2\sqrt{2}}{\pi} \sum_{n=1}^{\infty} Q_{n-\frac{1}{2}} \cos n\varphi \tag{59}$$

By taking the derivative of both sides of eq. (59) with respect to φ, we find that

$$\frac{\sin\varphi}{(s-\cos\varphi)^{3/2}} = \frac{4\sqrt{2}}{\pi} \sum_{n=1}^{\infty} n\, Q_{n-\frac{1}{2}} \sin n\varphi$$

from which

$$V_o = -\frac{4\sqrt{2}\,a}{\pi} H_o \sqrt{s-\cos\varphi} \sum_{n=1}^{\infty} n\, Q_{n-\frac{1}{2}} \sin n\varphi \tag{60}$$

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[7] Ref. 2, p. 443

This expansion diverges at $s = 1$, but is to be used only as an auxiliary relation to enable us to match boundary conditions later, on the surface $s_o > 1$, where the $\mathring{Q}_{n-\frac{1}{2}}$ functions are finite. Then, the portion of the resultant field due to the applied field will be expressed again in closed form as in eq. (58).

The potential for Case I will be written as consisting of two parts, that due to the uniform applied field given by eq's. (58) or (60) and an added potential V_1 which is given by eq. (28) and whose coefficients are so chosen that the normal component of the field $H_s = -\operatorname{grad}_s (V_o + V_1)$ shall be zero at $s = s_o$, that is, at the surface of the torus.

Considerations of symmetry show that the field must be perpendicular to the $z = 0$ plane, that is, $H_s = 0$ when $\varphi = 0, \pi$. By taking the gradient of eq. (28), we see that this requires that $b_o = b_n = 0$, so that V_1 must be represented by a sine series just as V_o. Hence writing

$$V = V_o + V_1$$
$$B = \frac{4\sqrt{2}\, a H_o}{\pi}$$

we have

$$V = \sqrt{s - \cos\varphi} \sum_{n=1}^{\infty} \left(a_n P_{n-\frac{1}{2}} - n B Q_{n-\frac{1}{2}} \right) \sin n\varphi \tag{61}$$

and alternatively

$$V = \sqrt{s - \cos\varphi} \left\{ - \frac{a H_o \sin\varphi}{(s - \cos\varphi)^{\frac{3}{2}}} + \sum_{n=1}^{\infty} a_n P_{n-\frac{1}{2}} \sin n\varphi \right\} \tag{62}$$

We take $-(\text{grad})_s$ of eq. (61) using eq. (29) and reduce the result with the recurrence formulae

$$\frac{1}{2} P_{n-\frac{1}{2}} + s P'_{n-\frac{1}{2}} = \frac{1}{2}\left(P'_{n+\frac{1}{2}} + P'_{n-\frac{3}{2}}\right) \tag{63}$$

$$\frac{1}{2} Q_{n-\frac{1}{2}} + s Q'_{n-\frac{1}{2}} = \frac{1}{2}\left(Q'_{n+\frac{1}{2}} + Q'_{n-\frac{3}{2}}\right) \tag{64}$$

derivable from the usual recurrence formulae, where the prime denotes first derivative with respect to s. Also, we let

$$\begin{aligned} A_{n+1} &= a_{n+1} - a_n \\ A_n &= a_n - a_{n-1} \\ &\;\;\cdots\cdots \\ A_2 &= a_2 - a_1 \\ \text{and} \qquad A_1 &= a_1 \end{aligned} \tag{65}$$

The result for H_s is

$$H_s = \frac{\sqrt{s-\cos\varphi}}{2a} \sum_{n=1}^{\infty} \left\{ \left[A_{n+1} P^1_{n+\frac{1}{2}} - A_n P^1_{n-\frac{3}{2}} - B\left(Q^1_{n+\frac{1}{2}} - Q^1_{n-\frac{3}{2}}\right)\right] \sin n\varphi \right\} \tag{66}$$

At s_0, $H_s = 0$, for all values of φ, so that

$$A_{n+1} \overset{\circ}{P}{}^1_{n+\frac{1}{2}} - A_n \overset{\circ}{P}{}^1_{n-\frac{3}{2}} - B\left(\overset{\circ}{Q}{}^1_{n+\frac{1}{2}} - \overset{\circ}{Q}{}^1_{n-\frac{3}{2}}\right) = 0 \tag{67}$$

From the recurrence formula expressed by eq. (67), we can write A_{n+1} in terms of A_1. The result is

$$A_{n+1} = \frac{B \sum_{\ell=1}^{n} \left[\overset{\circ}{Q}{}^1_{\ell+\frac{1}{2}} - \overset{\circ}{Q}{}^1_{\ell-\frac{3}{2}}\right] \overset{\circ}{P}{}^1_{\ell-\frac{1}{2}}}{\overset{\circ}{P}{}^1_{n+\frac{1}{2}} \overset{\circ}{P}{}^1_{n-\frac{1}{2}}} + \frac{\overset{\circ}{P}{}^1_{\frac{1}{2}} \overset{\circ}{P}{}^1_{-\frac{1}{2}}}{\overset{\circ}{P}{}^1_{n+\frac{1}{2}} \overset{\circ}{P}{}^1_{n-\frac{1}{2}}} A_1 \tag{68}$$

A_1 can be determined from the condition that the line integral of the magnetic field around a curve threaded by the torus is zero, since the net current is zero. The easiest path along which to integrate is that for which s = constant < s_0 and in a θ = constant plane. The component of the magnetic field along such a path is just H_φ. It is evident that any line integral in the uniform applied field will be zero, so we consider only that part of the field $H_{1\varphi}$ due to V_1. In this case the line integral, using eq. (13) with $ds = d\theta = 0$, is

$$\oint_{s<s_0} H_{1\varphi}\, dl = \oint_{s<s_0} \frac{a H_{1\varphi}}{s - \cos\varphi}\, d\varphi = 0 \tag{69}$$

Now, from eq. (30), the V_1 part of eq. (62), and from eq. (65), we obtain for $H_{1\varphi}$,

$$H_{1\varphi} = \frac{\sqrt{s-\cos\varphi}}{2a}\left\{\frac{A_1 P_{\frac{1}{2}}}{2} + \sum_{n=1}^{\infty}\left[(n+\tfrac{1}{2})A_{n+1}P_{n+\frac{1}{2}} - (n-\tfrac{1}{2})A_n P_{n-\frac{3}{2}}\right]\cos n\varphi\right\} \tag{70}$$

Substituting eq. (70) into (69) and referring to the integral representation of $Q_{n-\frac{1}{2}}$ given by eq. (53) we get

$$\sum_{n=0}^{\infty} A_{n+1}(n+\tfrac{1}{2})\left(P_{n+\frac{1}{2}}Q_{n-\frac{1}{2}} - P_{n-\frac{1}{2}}Q_{n+\frac{1}{2}}\right) = 0$$

which, in view of (57) reduces to

$$\sum_{n=1}^{\infty} A_n = 0 \tag{71}$$

Equations (68) and (71) together give

$$A_1 = -\frac{4\sqrt{2}\,a\,H_0}{\pi} F^{-1} \sum_{n=1}^{\infty} \left[\frac{\sum_{\ell=1}^{n} [\mathring{Q}^1_{\ell+\frac{1}{2}} - \mathring{Q}^1_{\ell-\frac{3}{2}}] \mathring{P}^1_{\ell-\frac{1}{2}}}{\mathring{P}^1_{n+\frac{1}{2}} \mathring{P}^1_{n-\frac{1}{2}}} \right] \tag{72}$$

with

$$F = \mathring{P}^1_{\frac{1}{2}} \mathring{P}^1_{-\frac{1}{2}} \sum_{n=0}^{\infty} \frac{1}{\mathring{P}^1_{n+\frac{1}{2}} \mathring{P}^1_{n-\frac{1}{2}}} \tag{73}$$

Thus the problem of Case I is essentially solved. Before collecting the parts of the solution together, we introduce the further simplification

$$Q'_{\ell+\frac{1}{2}} - Q'_{\ell-\frac{3}{2}} = 2\ell\, Q_{\ell-\frac{1}{2}}$$

and

$$A_{n+1} = \frac{8\sqrt{2}\,a\,H_0}{\pi} \alpha_{n+1}$$

The magnetic field distribution and surface current distribution for Case I:

$$H_s/H_0 = -\frac{\sqrt{s^2-1}}{s-\cos\varphi}\sin\varphi + \frac{4\sqrt{2}}{\pi}\sqrt{s-\cos\varphi}\sum_{n=1}^{\infty}\left(\alpha_{n+1} P^1_{n+\frac{1}{2}} - \alpha_n P^1_{n-\frac{3}{2}}\right)\sin n\varphi \tag{74}$$

$$H_\varphi/H_0 = \frac{s\cos\varphi - 1}{s-\cos\varphi} + \frac{2\sqrt{2}}{\pi}\sqrt{s-\cos\varphi}\left\{\alpha_1 P_{\frac{1}{2}} + \sum_{n=1}^{\infty}\left[(2n+1)\alpha_{n+1} P_{n+\frac{1}{2}} - (2n-1)\alpha_n P_{n-\frac{3}{2}}\right]\cos n\varphi\right\} \tag{75}$$

$$J/\frac{cH_0}{4\pi} = \frac{s_0\cos\varphi - 1}{s_0 - \cos\varphi} + \frac{2\sqrt{2}}{\pi}\sqrt{s_0-\cos\varphi}\left\{\alpha_1 \mathring{P}_{\frac{1}{2}} + \sum_{n=1}\left[(2n+1)\alpha_{n+1}\mathring{P}_{n+\frac{1}{2}} - (2n-1)\alpha_n \mathring{P}_{n-\frac{3}{2}}\right]\cos n\varphi\right\} \tag{76}$$

where

$$\alpha_1 = -F^{-1}\sum_{n=1}^{\infty}\left[\frac{\sum_{l=1}^{n} l\,\mathring{Q}_{l-\frac{1}{2}}\,\mathring{P}'_{l-\frac{1}{2}}}{\mathring{P}'_{n+\frac{1}{2}}\,\mathring{P}'_{n-\frac{1}{2}}}\right] \tag{77}$$

(F is given by eq. (73))

and

$$\alpha_{n+1} = \frac{\sum_{l=1}^{n} l\,\mathring{Q}_{l-\frac{1}{2}}\,\mathring{P}'_{l-\frac{1}{2}}}{\mathring{P}'_{n+\frac{1}{2}}\,\mathring{P}'_{n-\frac{1}{2}}} + \frac{\mathring{P}'_{-\frac{1}{2}}\,\mathring{P}'_{\frac{1}{2}}}{\mathring{P}'_{n+\frac{1}{2}}\,\mathring{P}'_{n-\frac{1}{2}}}\,\alpha_1 \tag{78}$$

J is in electrostatic units per cm. To express the current density in amperes per cm., let J_A be the surface current density in amperes per cm. Then replace J/c in eq. (76) by $J_A/10$.

G. CASE II: FIELD AND CURRENT DISTRIBUTION

The detailed calculations for this case are somewhat more laborious than those of Section F. But as in Section F, we shall confine our remarks to the broad outlines of the calculation.

As before, we begin by calculating H_s, using eq's. (35) and (37). From symmetry H_s must be zero in the $\varphi = (0, \pi)$ plane, that is the field is perpendicular to this plane. This consideration taken together with the condition that H_s be single valued in φ, leads to $a_n = 0$, $n \geqslant 0$. This leaves

$$\begin{aligned} H_s = -\frac{\sqrt{s-\cos\varphi}}{2a}\Bigg\{ &\sum_{n=0}^{\infty} b_n P^1_{n-\frac{1}{2}} \sin\varphi \cos n\varphi \\ &+ 2s\sum_{n=1}^{\infty} n b_n P^1_{n-\frac{1}{2}} \sin n\varphi \\ &- 2\sum_{n=1}^{\infty} n b_n P^1_{n-\frac{1}{2}} \cos\varphi \sin n\varphi \Bigg\} \end{aligned}$$

The quantity enclosed in the braces must be expressed entirely in a sine series. Then, by appropriate use of the recurrence formulae and introducing the constants

$$B_{n+1} = (n+\tfrac{3}{2})\, b_{n+1} - (n-\tfrac{1}{2})\, b_n$$

$$B_n = (n+\tfrac{1}{2})\, b_n - (n-\tfrac{3}{2})\, b_{n-1}$$

$$\cdots\cdots$$

$$B_2 = \tfrac{5}{2}\, b_2 - \tfrac{1}{2}\, b_1$$

and $$B_1 = \tfrac{3}{2}\, b_1 + b_0 \quad ,$$

H_s becomes

$$H_s = \frac{\sqrt{s-\cos\varphi}}{2a} \sum_{n=1}^{\infty} \left[B_{n+1} P^1_{n+\frac{1}{2}} - B_n P^1_{n-\frac{1}{2}} \right] \sin n\varphi \tag{79}$$

When $s = s_o$, $\overset{o}{H}_s = 0$, for all values of φ. Thus

$$B_{n+1} = B_n \overset{o}{P}{}^1_{n-\frac{1}{2}} / \overset{o}{P}{}^1_{n+\frac{1}{2}}$$

which leads by successive application to

$$B_{n+1} = \frac{\overset{o}{P}{}^1_{\frac{1}{2}}\, \overset{o}{P}{}^1_{\frac{1}{2}}}{\overset{o}{P}{}^1_{n+\frac{1}{2}}\, \overset{o}{P}{}^1_{n-\frac{1}{2}}}\, B_1 \tag{80}$$

To find B_1, we proceed somewhat as in Section F. That is we express H_φ in terms of B_1 and take a line integral of H_φ around a circular path s = constant $< s_o$.

In view of the conventions adopted in Section B (see Fig. 2),

$$-\frac{4\pi I}{c} = \int_0^{2\pi} H_\varphi \, d\ell = a \int_0^{2\pi} \frac{H_\varphi \, d\varphi}{s - \cos\varphi} \tag{81}$$

with s = constant < s_0. I is in electrostatic units and is the net current in the torus.

To find H_φ , we use eq. (35) with $a_n = 0$ and eq. (38). Just as H_s was expressed as a sine series, so H_φ must be expressed as a cosine series. This trigonometric transformation and the successive reduction of the coefficients of the trigonometric terms by use of the recurrence formulae is much more tedious than in Section F. However, by using eq. (80) when the expression has been properly reduced, the end result is

$$H_\varphi = \frac{\sqrt{s - \cos\varphi}}{2a} B_1 \left\{ \tfrac{1}{2} P_{\frac{1}{2}} + \overset{\circ}{P}{}^1_{\frac{1}{2}} \overset{\circ}{P}{}^1_{-\frac{1}{2}} \sum_{n=1}^{\infty} \left[\frac{(n+\frac{1}{2}) P_{n+\frac{1}{2}}}{\overset{\circ}{P}{}^1_{n+\frac{1}{2}}} - \frac{(n-\frac{1}{2}) P_{n-\frac{3}{2}}}{\overset{\circ}{P}{}^1_{n-\frac{3}{2}}} \right] \frac{\cos n\varphi}{\overset{\circ}{P}{}^1_{n-\frac{1}{2}}} \right\} \tag{82}$$

To integrate (81) we write H_φ as

$$H_\varphi = \sqrt{s - \cos\varphi} \sum_{n=0}^{\infty} f_n(s) \cos n\varphi \tag{83}$$

and substitute into (81). With the help of eq. (53) the integration of eq. (81) yields

$$-\frac{2\pi I}{ac} = \sqrt{2} \sum_{n=0}^{\infty} f_n(s) \, Q_{n-\frac{1}{2}}(s)$$

Replacing $f_n(s)$ by the corresponding terms in eqs. (82) and (83) and using eq. (57) we get

$$B_1 = -\frac{2\sqrt{2}\,\pi I}{c} F^{-1} \tag{84}$$

where F is defined by eq. (73).

Therefore, the magnetic field and surface current distributions for Case II, when account is taken of eq. (11) and I/c is replaced by $I_A/10$, where I_A is the current in amperes, are

$$H_s R/I_A = -\frac{\sqrt{2}\,\pi}{10}\frac{s_0}{\sqrt{s_0^2-1}} F^{-1}\sqrt{s-\cos\varphi}\;\mathring{P}^1_{\frac{1}{2}}\,\mathring{P}^1_{-\frac{1}{2}}\sum_{n=1}^{\infty}\left[\frac{P^1_{n+\frac{1}{2}}}{\mathring{P}^1_{n+\frac{1}{2}}} - \frac{P^1_{n-\frac{3}{2}}}{\mathring{P}^1_{n-\frac{3}{2}}}\right]\frac{\sin n\varphi}{\mathring{P}^1_{n-\frac{1}{2}}} \tag{85}$$

$$H_\varphi R/I_A = -\frac{\sqrt{2}\,\pi}{20}\frac{s_0}{\sqrt{s_0^2-1}} F^{-1}\sqrt{s-\cos\varphi}\left\{P_{\frac{1}{2}} + 2\mathring{P}^1_{\frac{1}{2}}\mathring{P}^1_{-\frac{1}{2}}\sum_{n=1}^{\infty}\left[\frac{(n+\frac{1}{2})P_{n+\frac{1}{2}}}{\mathring{P}^1_{n+\frac{1}{2}}} - \frac{(n-\frac{1}{2})P_{n-\frac{3}{2}}}{\mathring{P}^1_{n-\frac{3}{2}}}\right]\frac{\cos n\varphi}{\mathring{P}^1_{n-\frac{1}{2}}}\right\} \tag{86}$$

$$J_A R/I_A = -\frac{\sqrt{2}\,s_0}{8\sqrt{s_0^2-1}} F^{-1}\sqrt{s_0-\cos\varphi}\left\{\mathring{P}_{\frac{1}{2}} + 2\mathring{P}^1_{\frac{1}{2}}\mathring{P}^1_{-\frac{1}{2}}\sum_{n=1}^{\infty}\left[\frac{(n+\frac{1}{2})\mathring{P}_{n+\frac{1}{2}}}{\mathring{P}^1_{n+\frac{1}{2}}} - \frac{(n-\frac{1}{2})\mathring{P}_{n-\frac{3}{2}}}{\mathring{P}^1_{n-\frac{3}{2}}}\right]\frac{\cos n\varphi}{\mathring{P}^1_{n-\frac{1}{2}}}\right\} \tag{87}$$

CHARACTERISTIC QUANTITIES

H. COMPUTATION OF FLUX AND SOLUTION OF CASES III AND IV BY SUPERPOSITION

In Case III, we wish to find expressions for $\vec{H}$ and $\vec{J}$ due to a persistent current remaining after removal of the applied field. We shall assume that the persistent current remaining is just the amount required to maintain the flux in the ring at the value which existed when the torus was in an applied field H_0 and had no net current. This means that the torus begins in a situation portrayed by Case I and ends in Case II. The solution of the problem will follow immediately after the flux for Cases I and II have been determined.

Starting with no applied field and no surface currents, a field is applied and a net current induced in such a manner that the enclosed-flux is zero. This is Case IV, and is solved as soon as we know the fluxes of Cases I and II since we need only oppose equal and opposite fluxes to complete the solution.

We shall use the symbol Φ to indicate flux (not to be confused with the symbol Φ in eq's. (26) and (33)). A superscript I, II, III, or IV on any of the physical quantities flux, field, current, moment, etc., will indicate the case which the quantity represents.

To compute the flux Φ for Cases I and II, we take the surface integral of the field over the $\varphi = \pi$ plane from s = 1 to s = s_0. The non-vanishing component of magnetic field is H_φ and since $\varphi = \pi$, we write this as H_π.

$$\Phi = \int H_\pi \, dA \tag{88}$$

where dA is the element of area. $dA = 2\pi \rho \, d\rho$ in the z = 0 plane. In toroidal coordinates, putting $\varphi = \pi$, we have

$$\Phi = 2\pi a^2 \int_{s=1}^{s_0} \frac{H_\pi \, ds}{(s+1)^2}$$

By inspection of eq's. (75) and (86), with $\varphi = \pi$, we see that the general term for H_π is of the form $\sqrt{s+1}\, P_{n-\frac{1}{2}}$ in both cases so that we have the following types of integrals to evaluate.

$$C_{n-\frac{1}{2}} = \int_1^s \frac{P_{n-\frac{1}{2}}(s)\, ds}{(s+1)^{\frac{3}{2}}} \tag{89}$$

When the upper limit is s_0, we write $\overset{\circ}{C}_{n-\frac{1}{2}}$ for the integral.

To integrate $C_{-1/2}$, we refer back to the elliptic integral expression for $P_{-1/2}$, eq. (48), with $k = \sqrt{\frac{s-1}{s+1}}$. Thus

$$C_{-\frac{1}{2}} = \frac{2\sqrt{2}}{\pi} \int_0^k k K(k)\, dk \tag{90}$$

on transforming from s to k. Integration by parts yields

$$2\, C_{-\frac{1}{2}} = \frac{2\sqrt{2}}{\pi} \left[k^2 K - \int_0^k k^2 \frac{\partial K}{\partial k}\, dk \right] \tag{91}$$

On the other hand, using the relation

$$K = \frac{1-k^2}{k} \frac{\partial K}{\partial k} - \frac{1}{k} \frac{\partial E}{\partial k}$$

which is one of the standard expressions relating K and E, we can deduce that

$$C_{-\frac{1}{2}} = \frac{2\sqrt{2}}{\pi} \left[(K-E) - \int_0^k k^2 \frac{\partial K}{\partial k}\, dk \right] \tag{92}$$

By subtracting eq. (92) from eq. (91) we can eliminate the integral. Then transforming from K and E to $P_{-\frac{1}{2}}$, $P_{\frac{1}{2}}$ by means of eq's. (48) and (49) we obtain as our final result

$$C_{-\frac{1}{2}} = \frac{P_{\frac{1}{2}} - P_{-\frac{1}{2}}}{\sqrt{s+1}} \tag{93}$$

Similarly, we can deduce that

$$C_{\frac{1}{2}} = \frac{(2s+1)P_{-\frac{1}{2}} - 3\, P_{\frac{1}{2}}}{\sqrt{s+1}} \tag{94}$$

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For $n \geqslant 1$ in $C_{n+\frac{1}{2}}$ we derive a recurrence formula from the standard recurrence formulae connecting the $P_{n+\frac{1}{2}}$ functions of various degrees. To evaluate

$$C_{n+\frac{1}{2}} = \int_{s=1}^{s} \frac{P_{n+\frac{1}{2}}\, ds}{(s+1)^{3/2}}$$

we begin by setting up the identity

$$\frac{d}{ds}\left(\frac{P_{n+\frac{1}{2}} - P_{n-\frac{1}{2}}}{\sqrt{s+1}}\right) = \frac{P'_{n+1/2} - P'_{n-\frac{1}{2}}}{\sqrt{s+1}} - \frac{1}{2}\,\frac{P_{n+1/2} - P_{n-1/2}}{(s+1)^{3/2}}$$

By means of the recurrence formulae

$$(s^2-1)\, P'_{n+\frac{1}{2}} = (n+\tfrac{1}{2})(s P_{n+\frac{1}{2}} - P_{n-\frac{1}{2}})$$

$$(s^2-1)\, P'_{n-\frac{1}{2}} = (n+\tfrac{1}{2})(P_{n+\frac{1}{2}} - s\, P_{n-\frac{1}{2}})$$

we can show that the identity we wrote transforms to

$$\frac{d}{ds}\left(\frac{P_{n+\frac{1}{2}} - P_{n-\frac{1}{2}}}{\sqrt{s+1}}\right) = \frac{n P_{n+\frac{1}{2}} + (n+1) P_{n-\frac{1}{2}}}{(s+1)^{3/2}}$$

Integrating this expression with respect to s, we obtain

$$\frac{P_{n+\frac{1}{2}} - P_{n-\frac{1}{2}}}{\sqrt{s+1}} = n\, C_{n+\frac{1}{2}} + (n+1)\, C_{n-\frac{1}{2}}$$

or finally

$$C_{n+\frac{1}{2}} = \frac{1}{n}\,\frac{P_{n+\frac{1}{2}} - P_{n-\frac{1}{2}}}{\sqrt{s+1}} - \left(1+\frac{1}{n}\right) C_{n-\frac{1}{2}} \qquad (95)$$

This holds for n = 1, 2, 3 - - - . It is indeterminate for n = 0.

Therefore

$$\frac{\Phi^{I}}{\pi R^2 H_o} = -\frac{(s_o-1)^2}{s_o^2} + \frac{4\sqrt{2}}{\pi}\frac{s_o^2-1}{s_o^2}\left\{\alpha_1 \mathring{C}_{\frac{1}{2}} + \sum_{n=1}^{\infty}(-1)^n\left[(2n+1)\alpha_{n+1}\mathring{C}_{n+\frac{1}{2}} - (2n-1)\alpha_n \mathring{C}_{n-\frac{3}{2}}\right]\right\} \tag{96}$$

Similarly,

$$\frac{\Phi^{II}}{I_A R} = -\frac{\sqrt{2}\,\pi^2}{10}\frac{\sqrt{s_o^2-1}}{s_o}F^{-1}\left\{\mathring{C}_{\frac{1}{2}} + 2\mathring{P}^1_{\frac{1}{2}}\mathring{P}^1_{-\frac{1}{2}}\sum_{n=1}^{\infty}(-1)^n\left[\frac{(n+\frac{1}{2})\mathring{C}_{n+\frac{1}{2}}}{\mathring{P}^1_{n+\frac{1}{2}}\mathring{P}^1_{n-\frac{1}{2}}} - \frac{(n-\frac{1}{2})\mathring{C}_{n-\frac{3}{2}}}{\mathring{P}^1_{n-\frac{1}{2}}\mathring{P}^1_{n-\frac{3}{2}}}\right]\right\} \tag{97}$$

To determine the field and current distributions for Case III, we set $\Phi^{I} = \Phi^{II}$. We denote by rhs(96) and rhs(97) the right hand sides of the respective equations. Then

$$I_A = RH_o\pi\,[\text{rhs}(96)]/[\text{rhs}(97)]$$

If we put

$$G = \frac{\pi\,[\text{rhs}(96)]}{[\text{rhs}(97)]} \tag{98}$$

we can write

$$I_A = RH_oG \tag{99}$$

The magnetic field distribution and surface current distribution for Case III are then obtained by substituting for I_A in eq's. (85), (86), (87) of Case II. It is not necessary to write the expressions explicitly here.

Case IV is a superposition of Cases I and III. The relation can be summarized as

$$\vec{H}^{IV} = \vec{H}^{I} - \vec{H}^{III} \tag{100}$$

Obviously, since $\vec{H}^{I}$ and $\vec{H}^{III}$ have been matched in regard to their fluxes, the flux due to $\vec{H}^{IV}$ within the ring is zero. Φ^{I}, Φ^{II} and G are tabulated in Table I.[8]

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TABLE I

FLUX FOR CASES I AND II

$s_0 = \frac{R}{r}$ (1)	$-\Phi^{I}/\pi R^2 H_0$ (2)	$-\Phi^{II}/I_A R$ (3)	G (4)
1.2	1.2562×10^{-1}	1.2721×10^{-1}	3.1024×10^{0}
1.4	2.6596 "	2.9864 "	2.7978 "
1.6	3.8156 "	4.6714 "	2.5660 "
1.8	4.7354 "	6.2421 "	2.3833 "
2.0	5.4680 "	7.6859 "	2.2350 "
3.0	7.5404 "	1.3353×10^{0}	1.7741 "
4.0	8.4411 "	1.7346 "	1.5288 "
5.0	8.9152 "	2.0395 "	1.3733 "
6.0	9.1972 "	2.2852 "	1.2644 "
7.0	9.3793 "	2.4907 "	1.1830 "
8.0	9.5043 "	2.6673 "	1.1194 "
9.0	9.5940 "	2.8220 "	1.0681 "
10.0	9.6608 "	2.9596 "	1.0255 "

I. SELF-INDUCTANCE

The self-inductance L of a superconducting torus in terms of the flux and current I_A is (in henries)

$$L = 10^{-8} \Phi^{II}/I_A \tag{101}$$

[8] $P^{m}_{n+\frac{1}{2}}$, $P^{m\prime}_{n+\frac{1}{2}}$ and the corresponding functions of the second kind are tabulated in Tables of Associated Legendre Functions, Columbia University Press, New York (1945). These functions are tabulated for values n = -1, 0, 1, 2, 3, 4, m = 0, 1, 2, 3, 4 and s from 1.0 to 10.0 (intervals of 0.1).

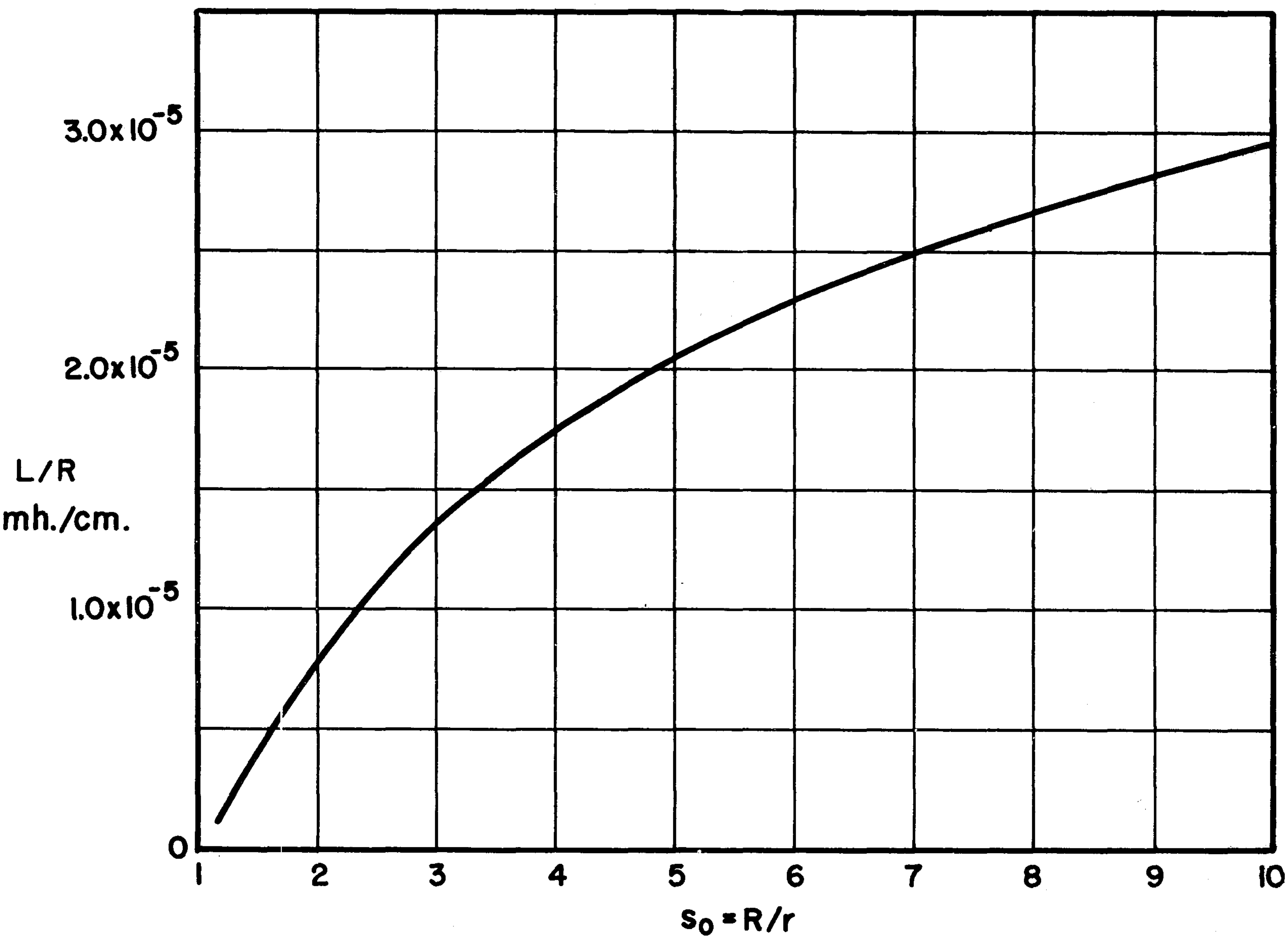

Fig. 5 - Self-inductance

This can be shown by beginning with energy relations connecting I_A, L and $\vec{H}$ and remembering that the resultant magnetic field inside the material of the torus is zero. Transformation of the volume integral of the magnetic energy by Green's theorem then leads to eq. (101).[9]

Thus, if we put L in millihenries,

$$L/R = \sqrt{2}\,\pi^2\,10^{-6}\frac{\sqrt{s_0^2-1}}{s_0}\,F^{-1}\left\{\mathring{C}_{\frac{1}{2}} + 2\mathring{P}^1_{\frac{1}{2}}\mathring{P}^1_{-\frac{1}{2}}\sum_{n=1}^{\infty}(-1)^n\left[\frac{(n+\frac{1}{2})\mathring{C}_{n+\frac{1}{2}}}{\mathring{P}^1_{n+\frac{1}{2}}\mathring{P}^1_{n-\frac{1}{2}}} - \frac{(n-\frac{1}{2})\mathring{C}_{n-\frac{3}{2}}}{\mathring{P}^1_{n-\frac{1}{2}}\mathring{P}^1_{n-\frac{3}{2}}}\right]\right\} \quad (102)$$

in millihenries/cm.

To obtain numerical values of L/R in millihenries/cm. for the values of $s_o = R/r$ shown in column (1) of Table I, multiply the (absolute) values recorded in column (3) of Table I by 10^{-5}. For example, when $s_o = 4$, $L/R = 1.7346 \times 10^{-5}$ mh./cm. Fig. 5 shows a graph of eq. (102).

J. MAGNETIC MOMENTS

Let M represent the moment of a torus in any one of the Cases I, II, III, or IV. Then

$$M = \int \pi \rho^2 J_{emu}\, dl_o$$

where dl_o is the line element, in the surface, perpendicular to J_{emu} there. M is in magnetic units (pole-cm. or dyne-cm./gauss).

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9 M. Abraham and R. Becker, The Classical Theory of Electricity and Magnetism, Blackie and Son Limited, London (1937), p. 169. The proof sketched there can be adapted easily to the superconducting torus.

Converting to toroidal coordinates

$$M = 2\pi R^3 \frac{(s_0^2-1)^{5/2}}{s_0^3} \int_0^{\pi} \frac{J_{emu}\, d\varphi}{(s_0 - \cos\varphi)^3} \tag{103}$$

Examination of eq's. (76) and (87) shows that the integrals involved on substitution for J_{emu} are

$$\int_0^{\pi} \frac{\cos n\varphi \, d\varphi}{(s_0 - \cos\varphi)^{5/2}} = \frac{4\sqrt{2}}{3(s_0^2-1)} \mathring{Q}^2_{n-\frac{1}{2}} \quad , \quad n = 0, 1, 2, \cdots \tag{104}$$

and

$$\int_0^{\pi} \frac{s_0 \cos\varphi - 1}{(s_0 - \cos\varphi)^4} d\varphi = \frac{\pi s_0}{(s_0^2-1)^{5/2}} \tag{105}$$

Eq. (104) can be derived by differentiating eq. (53) twice under the integral sign with respect to s and then multiplying by $(s^2 - 1)$ since

$$Q^2_{n-\frac{1}{2}} = (s^2-1)\, d^2 Q_{n-\frac{1}{2}}/ds^2$$

Similarly, eq. (105) can be derived from [10]

$$\int_0^{\pi} \frac{d\varphi}{s - \cos\varphi} = \frac{\pi}{\sqrt{s^2-1}}$$

Using (104) and (105) we have for the moment M^I of Case I,

$$M^I / \tfrac{4}{3}\pi R^3 H_0 = \frac{3}{8s_0^2} + \frac{2}{\pi^2}\left(1 - \frac{1}{s_0^2}\right)^{3/2} \Big\{ \alpha_1 \mathring{P}_{\frac{1}{2}} \mathring{Q}^2_{-\frac{1}{2}} + \sum_{n=1}^{\infty} \left[(2n+1)\alpha_{n+1} \mathring{P}_{n+\frac{1}{2}} - (2n-1)\alpha_n \mathring{P}_{n-\frac{3}{2}}\right] \mathring{Q}^2_{n-\frac{1}{2}} \Big\} \tag{106}$$

For Case II, the moment M^{II} is

$$M^{II}/\pi R^2 I_A = -\frac{1}{15}\left(1 - \frac{1}{s_0^2}\right) F^{-1} \left\{ \mathring{P}_{\frac{1}{2}} \mathring{Q}^2_{-\frac{1}{2}} + 2 \mathring{P}^1_{\frac{1}{2}} \mathring{P}^1_{-\frac{1}{2}} \sum_{n=1}^{\infty} \left[\frac{(n+\frac{1}{2}) \mathring{P}_{n+\frac{1}{2}}}{\mathring{P}^1_{n+\frac{1}{2}} \mathring{P}^1_{n-\frac{1}{2}}} - \frac{(n-\frac{1}{2}) \mathring{P}_{n-\frac{3}{2}}}{\mathring{P}^1_{n-\frac{1}{2}} \mathring{P}^1_{n-\frac{3}{2}}} \right] \mathring{Q}^2_{n-\frac{1}{2}} \right\} \tag{107}$$

---

[10] H. B. Dwight, Tables of Integrals, the Macmillan Company, New York (1947), Item No. 859.21.

The negative sign simply means that when I_A is in the direction of increasing θ (Fig. 2), M^{II} is in the direction of decreasing φ (or increasing z, Fig. 2).

For Case III,

$$\frac{M^{III}}{\frac{4}{3}\pi R^3 H_o} = \frac{3G}{4}\left(\frac{M^{II}}{\pi R^2 I_A}\right) \tag{108}$$

For Case IV

$$\left|\frac{M^{IV}}{\frac{4}{3}\pi R^3 H_o}\right| = \left|\frac{M^{I}}{\frac{4}{3}\pi R^3 H_o}\right| + \left|\frac{M^{III}}{\frac{4}{3}\pi R^3 H_o}\right| \tag{109}$$

Graphs of the moments M^{I}, M^{II}, M^{III}, and M^{IV} are shown in Figs. 6, 7, 8, and 9, respectively.

The moments are tabulated in Table II (magnitudes only)

TABLE II

MAGNETIC MOMENTS

$s_o = R/r$ (1)	$\frac{M^{I}}{\frac{4}{3}\pi R^3 H_o}$ (2)	$\frac{M^{II}}{\pi R^2 I_A}$ (3)	$\frac{M^{III}}{\frac{4}{3}\pi R^3 H_o}$ (4)	$\frac{M^{IV}}{\frac{4}{3}\pi R^3 H_o}$ (5)
1.2	5.400 $\times 10^{-1}$	1.256 $\times 10^{-2}$	2.922 $\times 10^{-2}$	5.692 $\times 10^{-1}$
1.4	3.9790 "	2.660 "	5.582 "	4.537 "
1.6	3.0411 "	3.8155 "	7.3430 "	3.7725 "
1.8	2.3960 "	4.7354 "	8.4643 "	3.2425 "
2	1.9349 "	5.4681 "	9.1659 "	2.8515 "
3	8.5001 $\times 10^{-2}$	7.5405 "	1.0033 $\times 10^{-1}$	1.8533 "
4	4.7508 "	8.4411 "	9.6787 $\times 10^{-2}$	1.4430 "
5	3.0293 "	8.9152 "	9.1825 "	1.2212 "
6	2.0988 "	9.1972 "	8.7216 "	1.0820 "
7	1.5395 "	9.3793 "	8.3220 "	9.8615 $\times 10^{-2}$
8	1.1774 "	9.5043 "	7.9797 "	9.1571 "
9	9.2955 $\times 10^{-3}$	9.5940 "	7.6853 "	8.6149 "
10	7.5247 "	9.6607 "	7.4302 "	8.1827 "

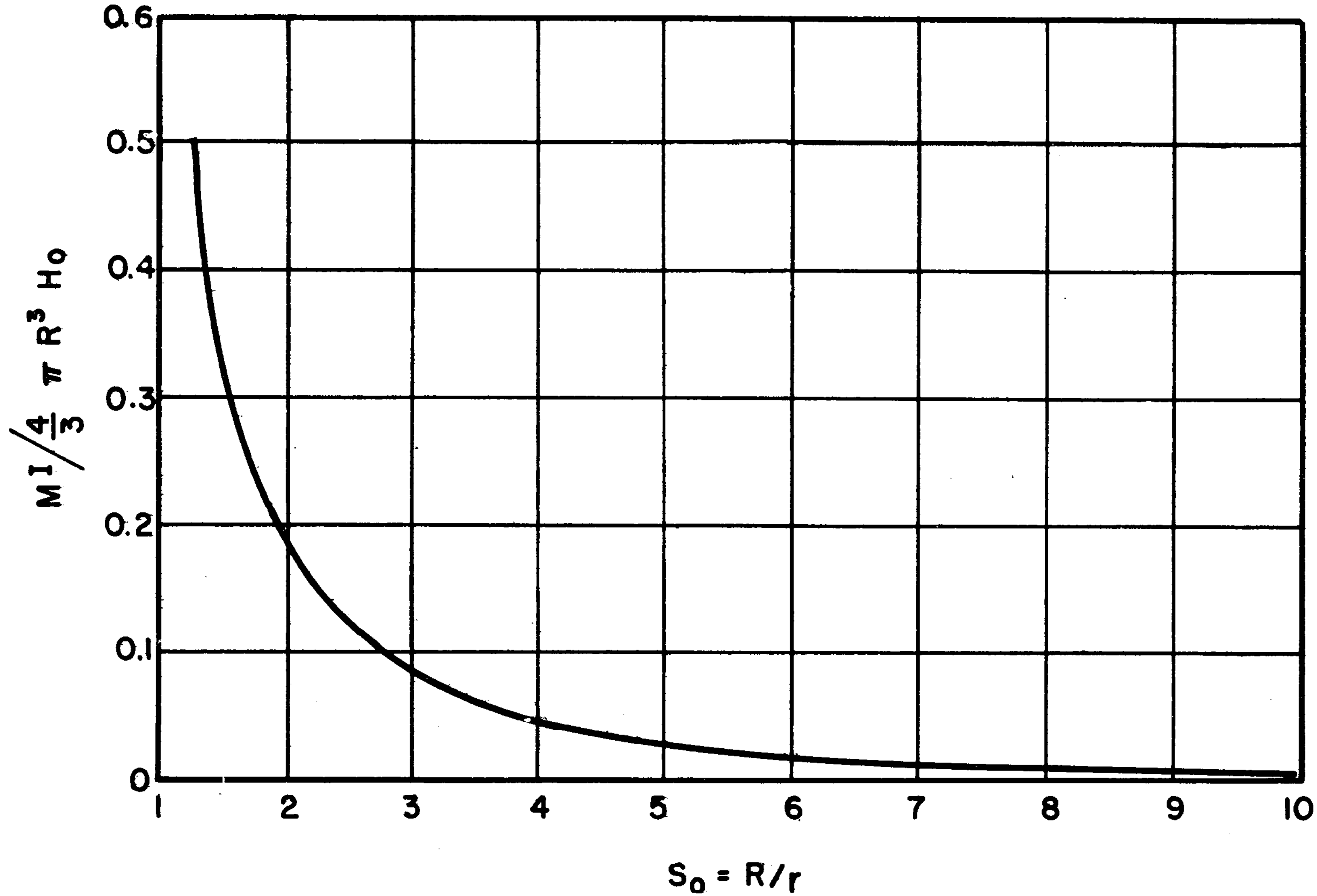

Fig. 6 - Magnetic moment: Case I (absolute velue)

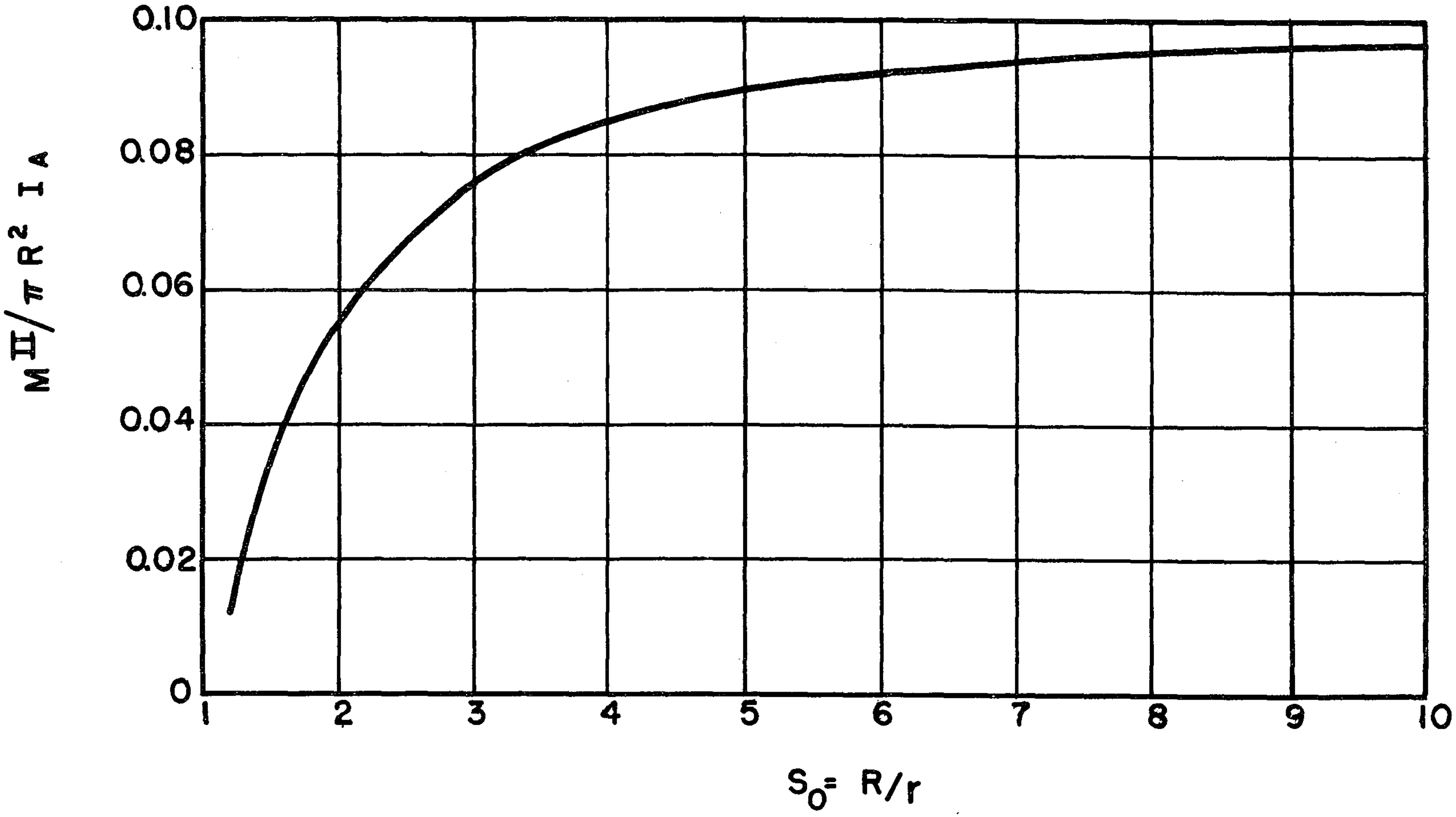

Fig. 7 - Magnetic moment: Case II (absolute value)

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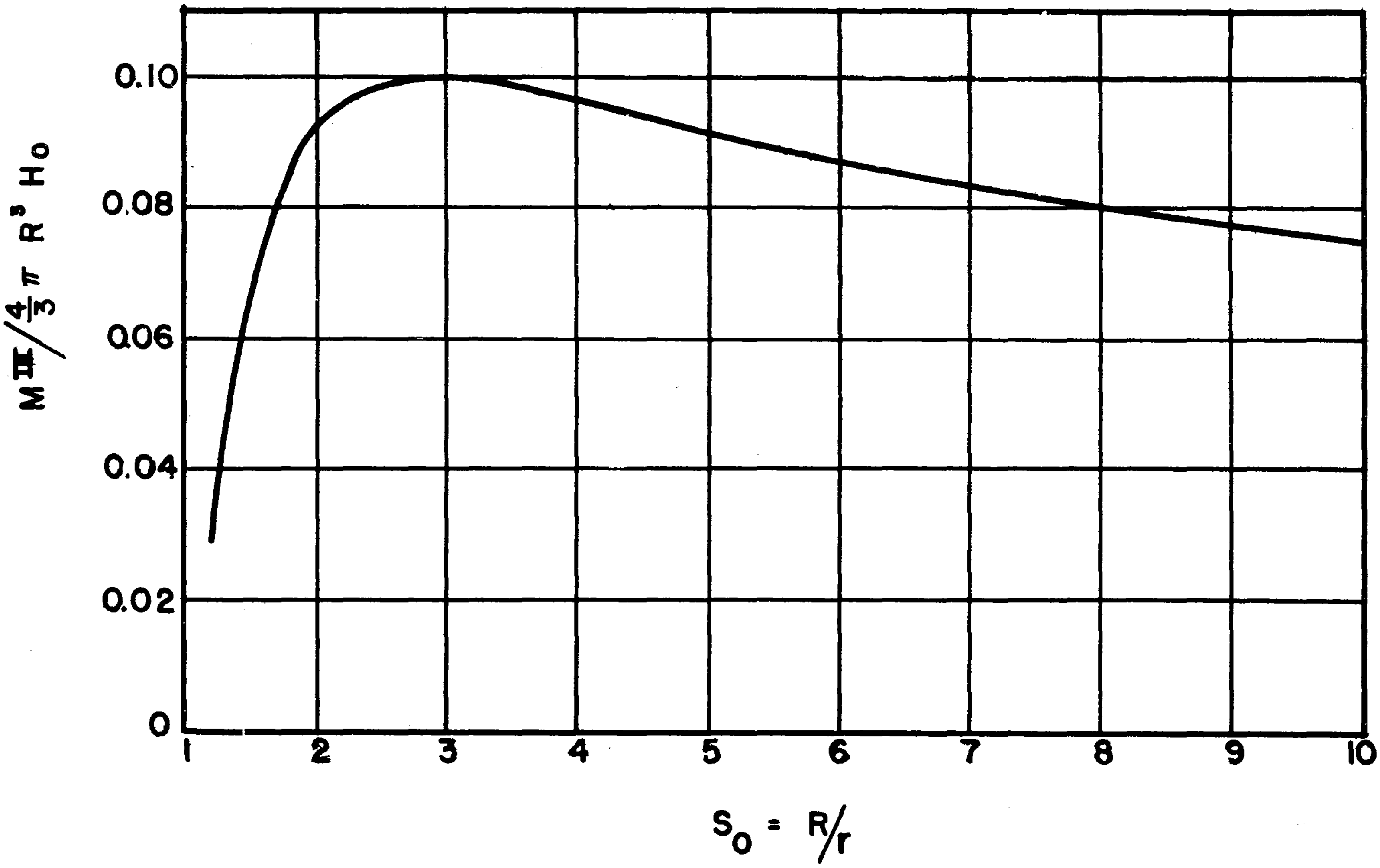

Fig. 8 - Magnetic moment: Case III (absolute value)

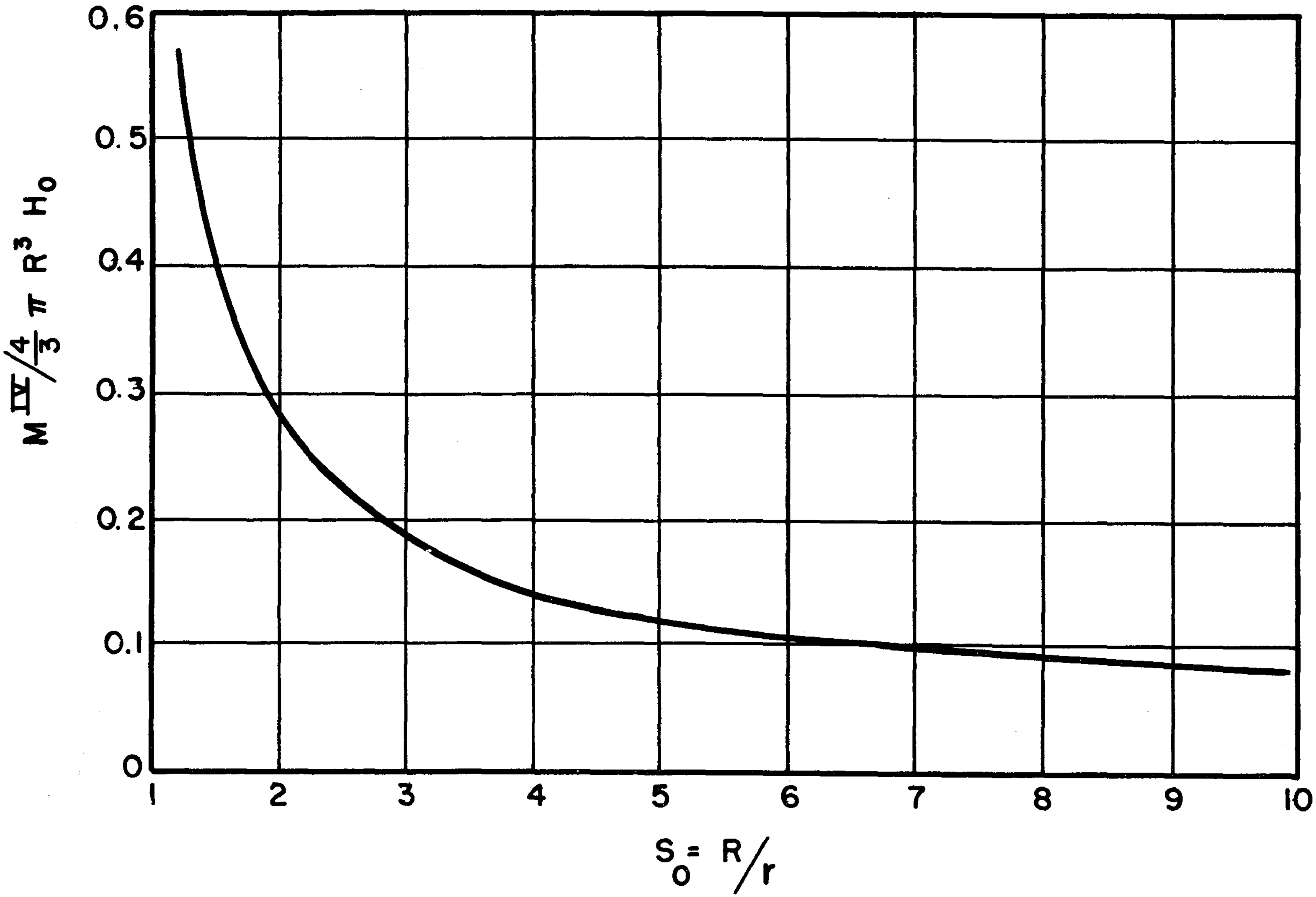

Fig. 9 – Magnetic moment: Case IV (absolute value)

K. MAXIMUM AND MINIMUM FIELDS

In experiments involving superconductors, it is desirable to know for what value of applied field, or of persistent current, the field at some place on the surface attains the critical value. Thus, if we know the maximum value of the resultant field at the surface of the torus in terms of the applied field or induced current, the condition for critical field at the surface of the torus can be computed. In addition, if we know the minimum field at the surface of the torus, we can obtain a rough idea of the field and current distributions, with the aid of qualitative drawings, without a detailed computation of the infinite series representations. For this purpose, Table III gives the minimum and maximum fields for Cases I and II and Table IV for Cases III and IV. The subscripts π and 2π refer to the coordinate φ and indicate the inner rim and outer rim of the torus, respectively. Only magnitudes are tabulated. Figs. 10, 11, 12 and 13 show plots for the outer and inner fields for Cases I, II, III and IV.

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TABLE III

OUTER AND INNER RIM FIELDS, CASES I AND II

$s_0 = R/r$ (1)	H^{I}_{π}/H_0 Inner (Max) (2)	$H^{I}_{2\pi}/H_0$ Outer (Neither) (3)	$H^{II}_{\pi}R/I_A$ Inner (Max.) (4)	$H^{II}_{2\pi}R/I_A$ Outer (Min.) (5)
1.2	4.7434	1.8082	1.5289	5.04 $\times 10^{-3}$
1.4	3.5688	1.8291	1.2757	1.461 $\times 10^{-2}$
1.6	3.0916	1.8449	1.2056	2.727 "
1.8	2.8304	1.8573	1.1902	4.2419 "
2	2.6659	1.8674	1.1987	5.9640 "
3	2.3236	1.9002	1.3544	1.6804 $\times 10^{-1}$
4	2.2089	1.9188	1.5594	3.0047 "
5	2.1528	1.9311	1.7740	4.4734 "
6	2.1198	1.9400	1.9906	6.0378 "
7	2.0983	1.9468	2.2072	7.6699 "
8	2.0832	1.9522	2.4231	9.3522 "
9	2.0720	1.9566	2.6381	1.1073 $\times 10^{0}$
10	2.0634	1.9602	2.8522	1.2825 "
15	2.0395	1.9718	3.9119	2.1881 "
20	2.0286	1.9781	4.9580	3.1225 "

TABLE IV

OUTER AND INNER RIM FIELDS, CASES III AND IV

$s_o = R/r$ (1)	H^{III}_{π}/H_o Inner (Max.) (2)	$H^{III}_{2\pi}/H_o$ Outer (Min.) (3)	H^{IV}_{π}/H_o Inner (Neither) (4)	$H^{IV}_{2\pi}/H_o$ Outer (Max.) (5)
1.2	4.743	1.562 $\times 10^{-2}$	~ 0	1.8238
1.4	3.569	4.087 "	3 $\times 10^{-4}$	1.8700
1.6	3.0936	6.9976 "	2.0 $\times 10^{-3}$	1.9148
1.8	2.8366	1.0110 $\times 10^{-1}$	6.2 "	1.9584
2	2.6791	1.3330 "	1.31 $\times 10^{-2}$	2.0007
3	2.4029	2.9813 "	7.93 "	2.1983
4	2.3841	4.5937 "	1.752 $\times 10^{-1}$	2.3781
5	2.4362	6.1434 "	2.834 "	2.5454
6	2.5168	7.6341 "	3.970 "	2.7034
7	2.6111	9.0736 "	5.128 "	2.8542
8	2.7125	1.0469 $\times 10^{0}$	6.293 "	2.9991
9	2.8177	1.1827 "	7.457 "	3.1393
10	2.9249	1.3152 "	8.616 "	3.2753
15	3.4681	1.9399 "	1.4286 $\times 10^{0}$	3.9116
20	4.0024	2.5206 "	1.9738 "	4.4987

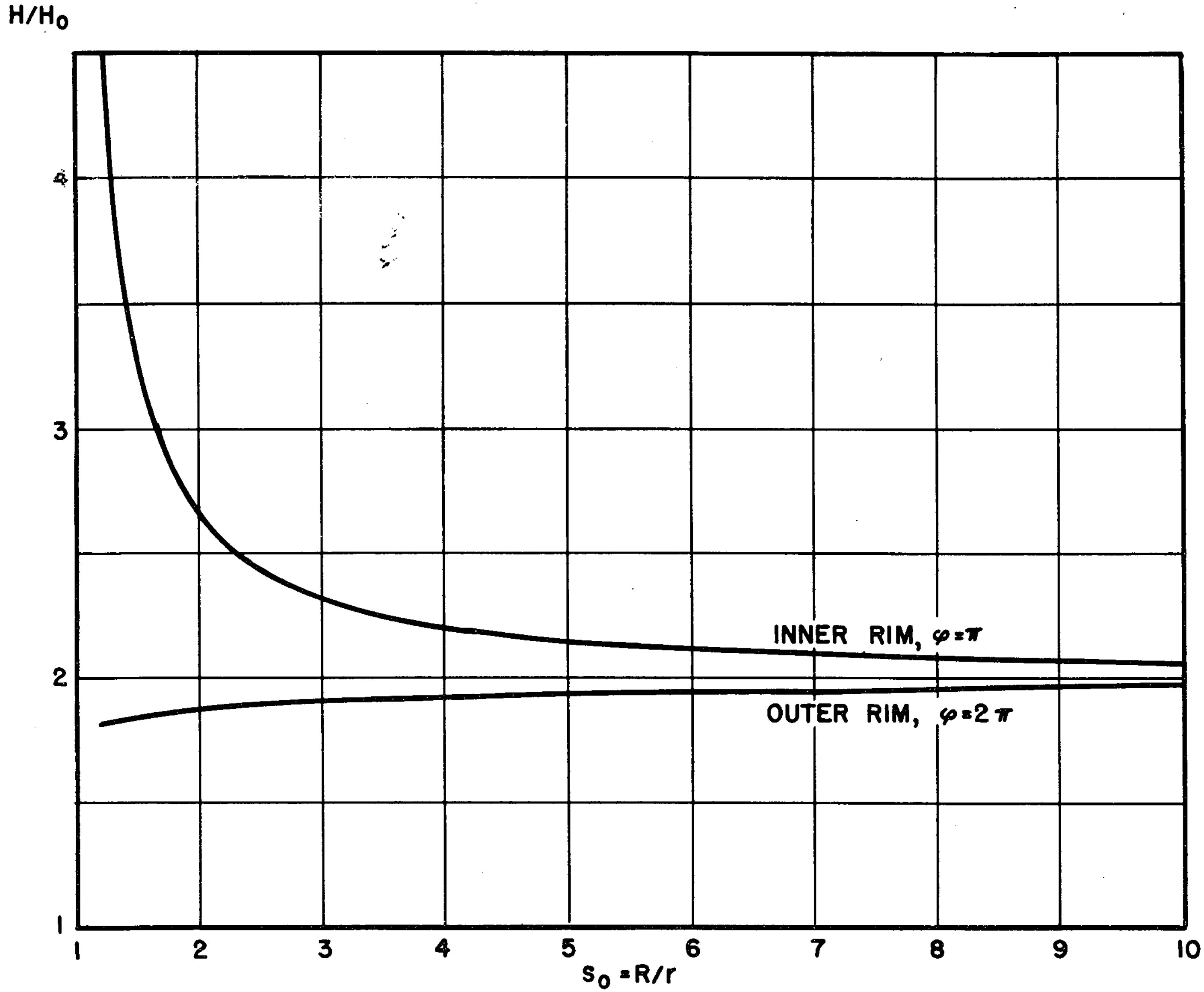

Fig. 10 - Plot of field strength at inner and outer rim of the toroidal surface: Case I

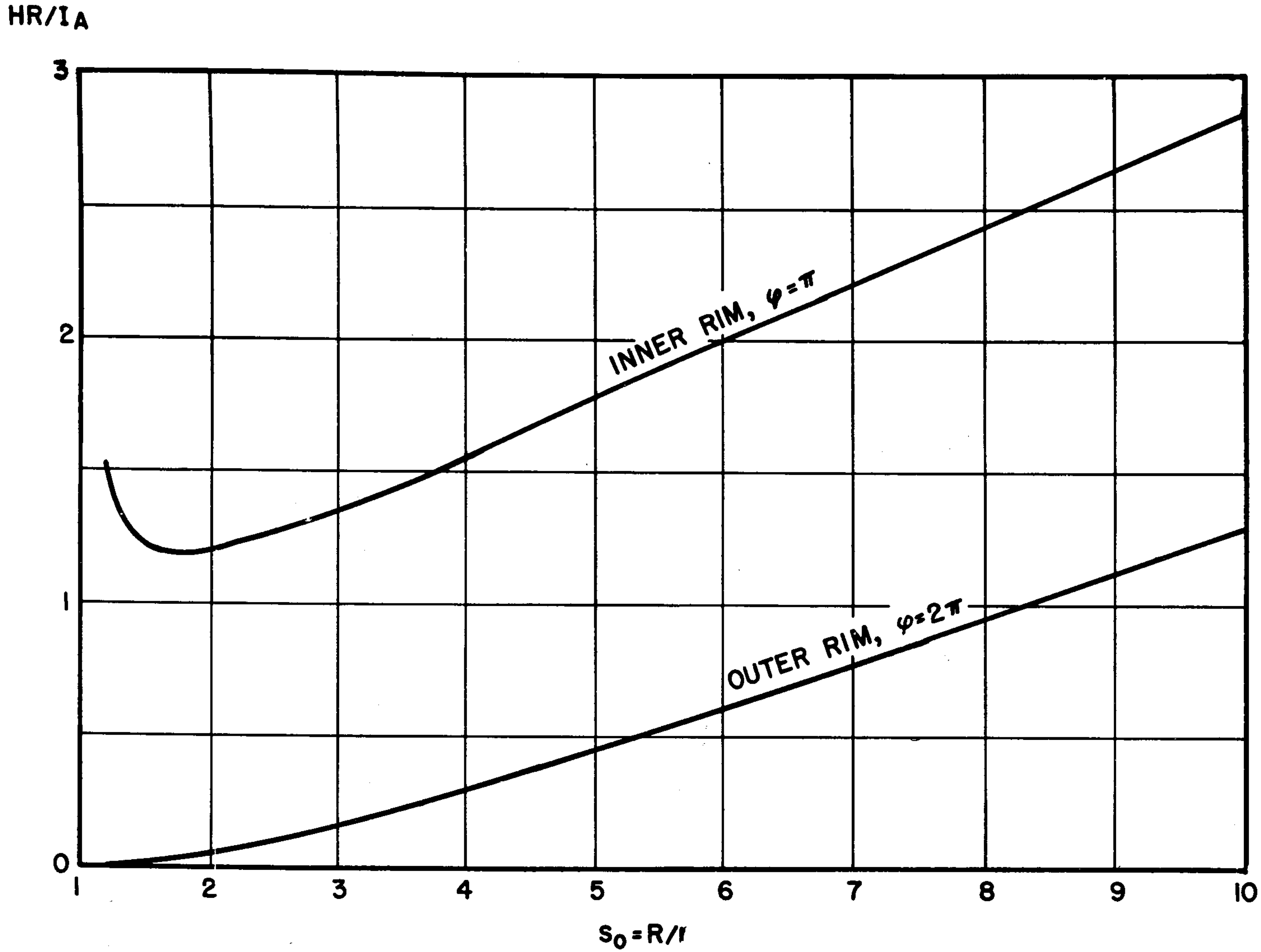

Fig. 11 - Plot of field strength at inner and outer rim of the toroidal surface: Case II

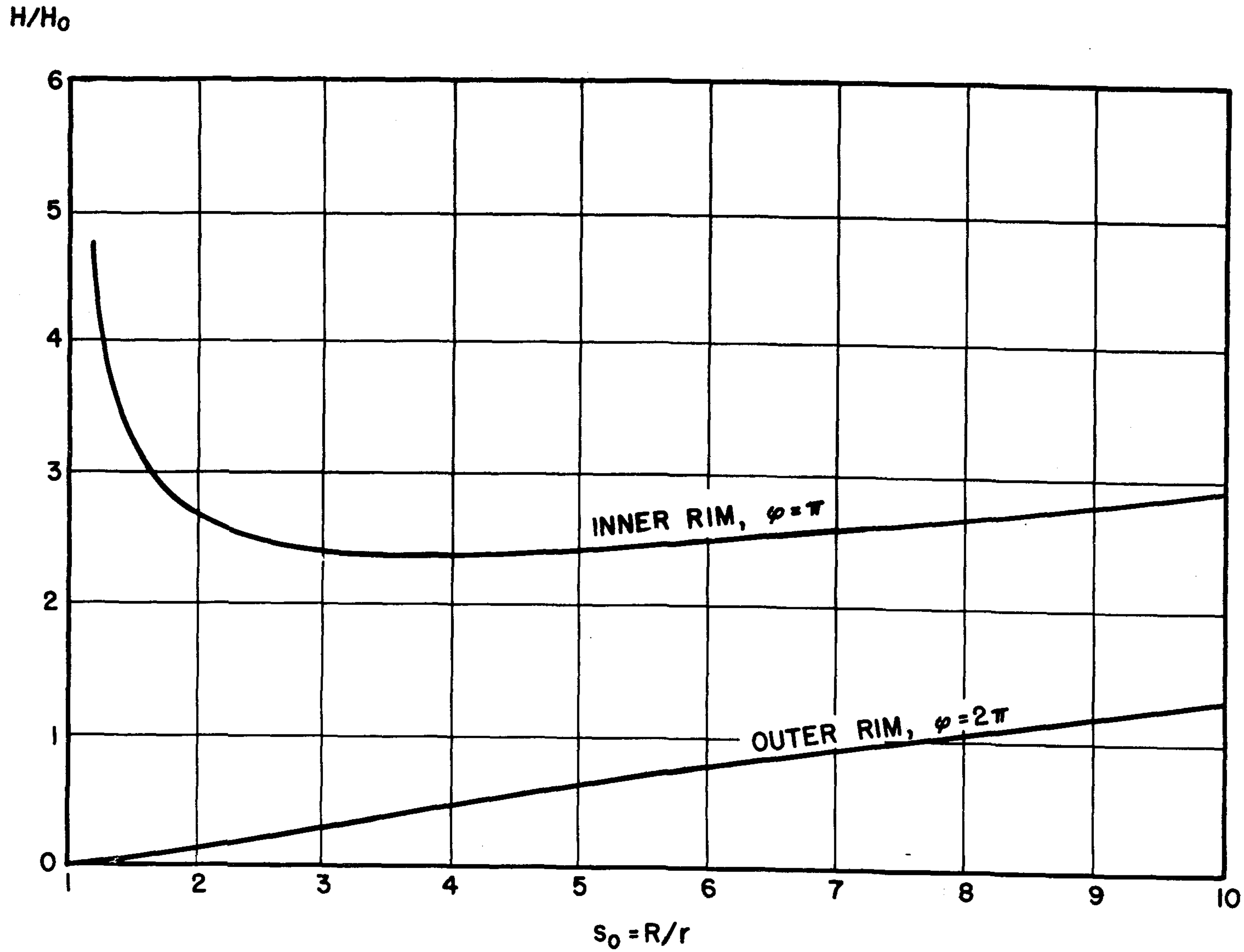

Fig. 12 - Plot of field strength at inner and outer rim of the toroidal surface: Case III

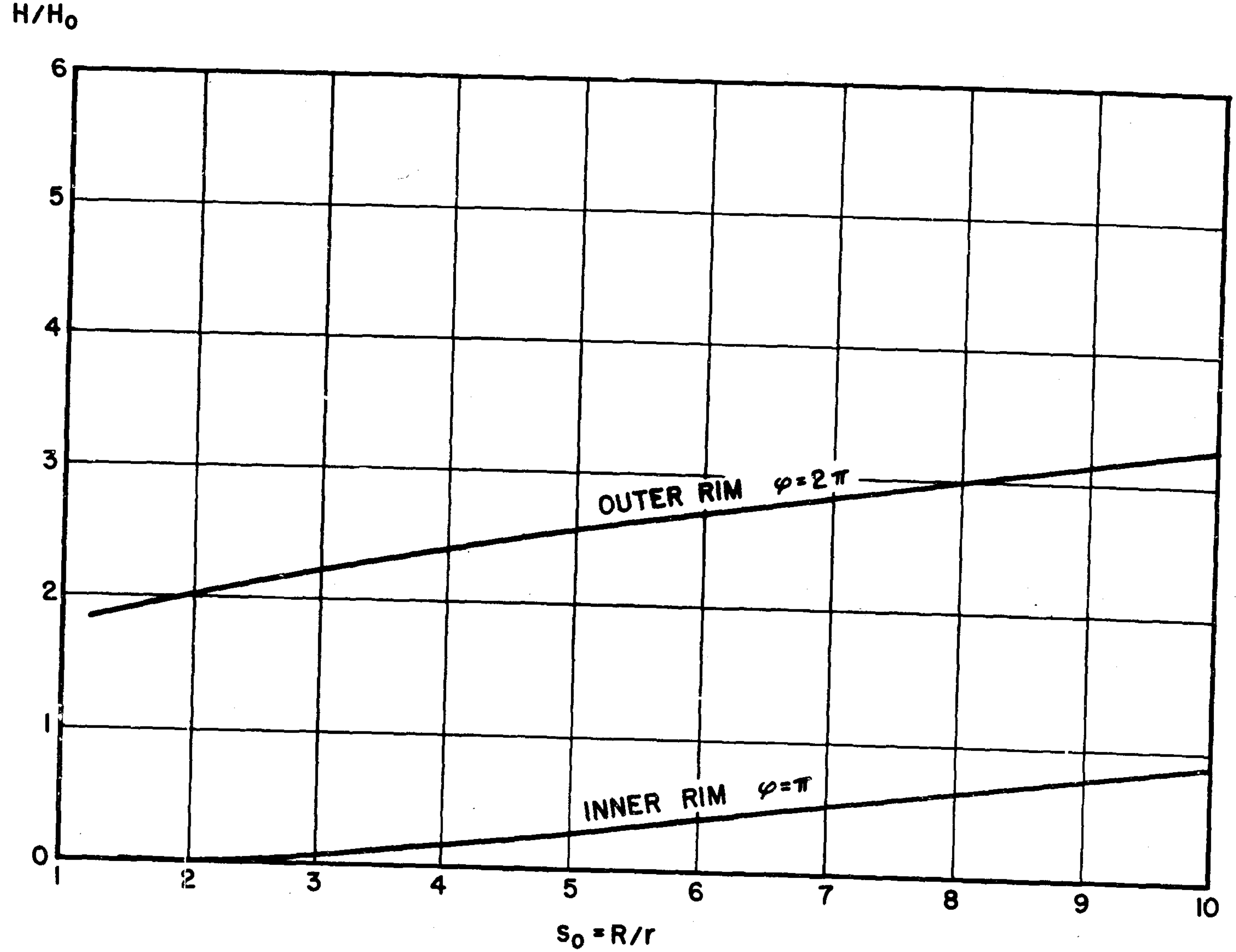

Fig. 13 - Plot of field strength at inner and outer rim of the toroidal surface: Case IV

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BEHAVIOR OF A SUPERCONDUCTING TORUS UNDER MAGNETIC CYCLING

A number of experiments and descriptions concerning the behavior of the moment of a superconducting torus in a variable applied magnetic field appear in the literature, but perhaps the clearest exposition is that which appears in Shoenberg's book on superconductivity.[11]

In Fig. 14, a hysteresis loop for a superconducting torus of ratio $s_o = R/r = 4$, is plotted, this being the size used in one of Shoenberg's papers.[12] The solid lines in Fig. 14 are for the superconductor in the ideal state and are subject to the calculations of this report when we introduce the additional provision that the surface currents are limited in such a manner that the maximum value of the magnetic field at the surface of the superconductor shall not exceed a certain critical value H_k. In view of eq. (3), this can also be considered as a restriction on the maximum allowable value for surface current density. This means that the persistent current will conserve flux, when the applied field changes, only to the extent that the critical field value is not exceeded at the surface of the superconductor. We shall use the same units as Shoenberg in our discussion.[11] $m = M/\frac{4}{3}\pi R^3 H_k$ is the moment M of the torus divided by the volume of a sphere whose radius is the same as the mean radius R of the torus and also divided by H_k. h is the ratio of the applied field H_o to the critical field H_k. The facts concerning the intermediate state incorporated in the broken lines of Fig. 14 are

[11] D. Shoenberg, Superconductivity, Cambridge University Press, (1938), Chapter IV.

[12] D. Shoenberg, Proc. Roy. Soc., 155, 712-726 (1936).

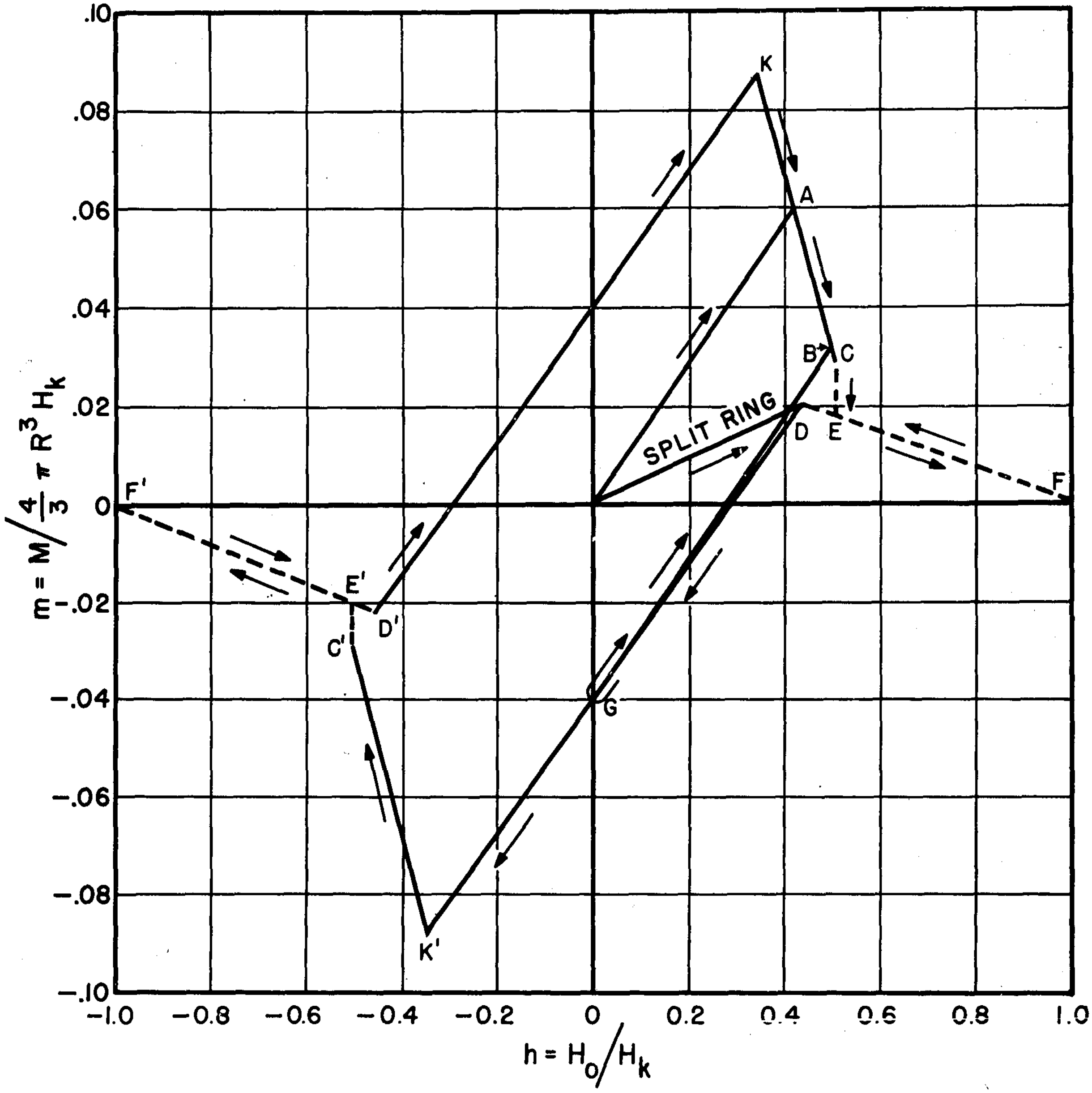

Fig. 14 - Behavior of torus in a magnetic cycle, $s_o = R/r = 4$

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those given by Shoenberg and others from experiment and are not amenable to the calculations of this report, but are accepted ad hoc.

Line OA: Beginning at 0, with no applied field or persistent current, h is gradually increased. From Table II, column (5), the eq. of OA is $m = 0.1443\ h$. This line ends at A when the critical field is reached. From Table IV, column (5), this is reached at the outer rim ($\varphi = 2\pi$) first, and actually at $h = (2.3781)^{-1} = 0.42050$, with m at A equal to 0.060676.

Line AC: If it were not for the critical field condition, an increase in h beyond 0.42050 would result in a prolongation of OA. But now, any current density on the surface of the superconductor tending to exceed the value associated with H_k is dissipated somehow. Thus the net persistent current will not increase, but will actually decrease in order that the field at $\varphi = 2\pi$ remain critical. In the meantime, the field at $\varphi = \pi$ will increase with increasing h until it likewise attains the critical value. Thus we have the line segment AC, terminating at C when the value of the field at $\varphi = \pi$ has likewise reached H_k so that the field at both inner and outer rim is critical.

(Such expressions as "the superconductivity is destroyed just long enough to permit some of the lines to cross the ring and increase the enclosed flux," have no meaning and are actually incorrect electrodynamics. The current merely fails to increase enough to compensate for the applied field increases and maintain zero flux. One <u>must</u> use the superposition principle. Zero field inside the material of a superconductor means an <u>equilibrium</u> of magnetic force fields, an equilibrium between applied field H_o and "spontaneously" induced field $-H_o$ due to induced surface currents. This means that the resultant $\vec{B}$ is zero. In electrodynamics, as in mechanics, "force equilibrium" has an entirely different meaning than "absence of forces.")

From Table III, column (3) and column (5), the field at the outer rim is given by

$$H_k = 1.9188\ H_o + 0.30047\ I_A/R\ (\varphi = 2\pi)$$

From Table II, column (2) and (3) we have

$$M = 0.047508\ H_o\ \frac{4}{3}\pi R^3 + 0.084411\ \pi R^2 I_A$$

From these two equations, it follows that

$$m = 0.21070 - 0.35677h$$

is the equation of line AC.

The terminus C is reached when $H_\pi = H_k$, or

$$H_k = 2.2089\ H_o - 1.5594\ I_A/R\ (\varphi = \pi)$$

On the way to C, the field at the inner rim reverses sign and becomes parallel to the applied field and the field at the outer rim.

At C, $m = 0.029194$

$h = 0.50874$

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Further increase in h must put the superconductor either in an "intermediate state" or into the "normal state." For <u>no</u> arrangement of surface currents, obeying our analysis for ideal superconducting tori, will result in a field less than or equal to the critical field at the outer and inner rims. Experiment shows that the superconductor enters a so-called "intermediate state" which endures until the applied field $h = 1$. The superconductor acts as though the permeability μ has some value between zero and unity, rather than zero as for the ideal superconductor.

<u>Lines OD, DF:</u> Consider the case of a torus split in a manner such that persistent (flux conserving) currents cannot occur. The relationship between the moment and applied field for this case is the line OD. The equation for OD is (Table II, column (2))

$m = 0.047508h$

D is reached when the field at the inner rim, where it is maximum, reaches H_k. From Table III, column (2), this occurs at

$h = (2.2089)^{-1} = 0.45271$

and for m at

$m = 0.021507$

Beyond D, the field at the surface of the torus would exceed H_k if the torus would remain in the ideal state. However, the magnetic moment, by experiment (Shoenberg [11, 12]) decreases linearly to zero as $h \to 1$. The path ODF is reversible.

Lines CE, EF: According to experiment, the torus, which was carried to the state C via OA, AC, drops to E and then follows the split torus path EF reversibly. This was taken by Shoenberg to mean that beyond C, the torus has no persistent current and is indistinguishable from a split torus. This, of course, neglects the effect of the split other than its prevention of persistent currents. At C, there still remains a net current, of amount $I_A/RH_k = 0.07936$ amperes per gauss cm. This accounts for the difference in m between the points C and E. Any attempt to increase h beyond its value at C immediately throws the torus into the intermediate state where persistent currents can no longer maintain themselves and it drops to E and follows the split ring course to F.

Line FD: Not until D is reached, on the return from F, can the superconductor exist in the ideal state. Any attempt to set up persistent currents to conserve flux somewhere along FD would result in the field at $\varphi = \pi$ exceeding H_k. The net current was such as to subtract field from the "case I" field at $\varphi = \pi$ along the path from A to C, but on the return from F, a net current would add field to the "Case I" field. When D is reached, the field at the inner rim is maximum and is equal to the critical field. Thus on the return path from F to D, the behavior of the torus does not differ from that of a split ring.

<u>Line DG:</u> One might hope that in decreasing the field h from the value at D, the flux value at D would be conserved. However, the current required to conserve the flux at its D value turns out to be such as to give rise to a field exceeding critical field at the inner rim. Only as much persistent current will flow as can occur without the magnetic field surpassing H_k at the surface of the torus. Referring to columns (2) and (4) of Table III, we see that the limiting condition on the net current which can flow is

$$H_k = 2.2089\ H_o + 1.5594\ I_A/R$$

For the moment, we have

$$\frac{M}{\frac{4}{3}\pi R^3} = 0.047508H_o - 0.063308\ I_A/R$$

which yields the following equation for the line DG,

$$m = 0.13718h - 0.040597$$

In the absence of an applied field h, the torus is left with a residual moment, at G,

$$m = -0.040597\ (h = 0)$$

due to the current

$$I_A = 0.64126\ H_k\ R \text{ amperes}$$

Corresponding to this residual current, the flux at G is only 92.7 per cent of the original flux at D. The point C lies on the extension of line DG.

Line GK: Beginning with the torus at G, we make the field negative. The current continues to increase in magnitude with $H_{\pi} = H_k$ all the way from G to K'. At G, $H_{2\pi}$ is 0.19260 H_k and increases to $H_{2\pi} = H_k$ at K'. This is a continuation of the line DG through G and stops at

$h = -0.34437$

$m = -0.087836$

Line GB: This is a superposition of a Case IV current maintaining zero flux, and the current already present at G. This yields the equations

$$H_{2\pi} = 0.19260H_k - 2.37812H_o$$

$$H_{\pi} = H_k - 0.17520H_o$$

As h increases from zero, H_{π} decreases to zero and $H_{2\pi}$ decreases to zero. But, $H_{2\pi}$ reaches zero first and increases to H_k, all the while, the field H_{π} remaining less than critical field. This occurs at the point B on the AC path at

$h = 0.50149$

$m = 0.031747$

Further increase now results in a retracing of BCEF. Having discussed the principal paths, it is now easy to follow the behavior of the torus along K'C', D'K and KA.

The ideas presented here are not new in principle, being the same in general point of view as those in Shoenberg's book.[11] However, the exact calculations given here, which were not available to Shoenberg, present an opportunity for a more careful experimental analysis of Shoenberg's description. For example, the extent to which the region around the

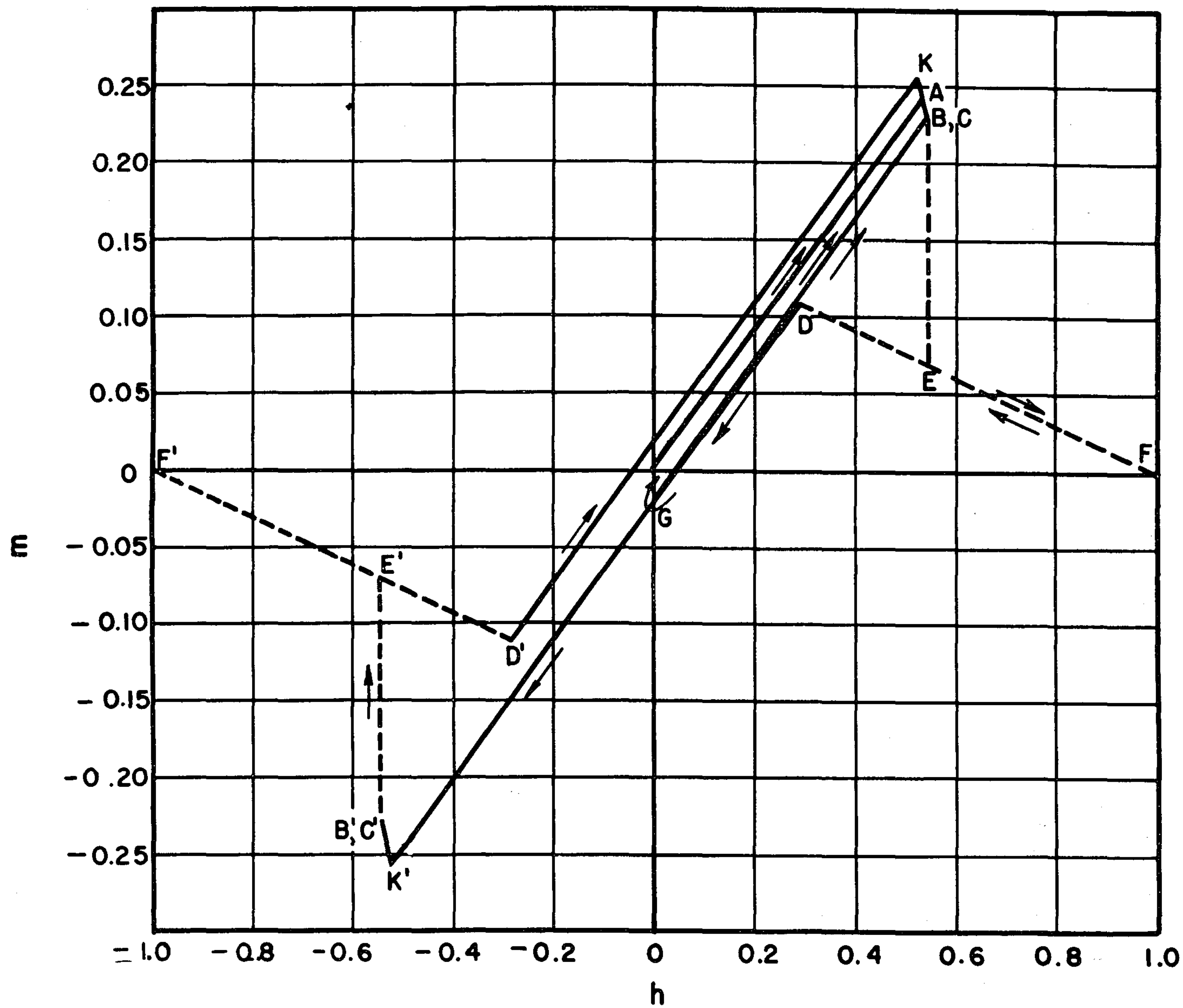

Fig. 15 - Behavior of torus in a magnetic cycle, $s_o = R/r = 1.4$

points B, C, D, E can be amplified by using smaller values of s_o is illustrated in Fig. 15 where, there, s_o has been chosen at the value 1.4. To be noted are: (a) the shift in the value of h at point D from a value near 0.5 to a value of 0.28; and (b) the increased change in the value of m between C and E.

APPENDIX

APPROXIMATE FORMULAE

To facilitate computation, a number of approximate formulae are listed here for some of the quantities calculated in the earlier sections. The error resident in the formulae are indicated along with each formula.

The symbol λ is used for an abbreviation of $\log_e 8s_0$, i.e. in these formulae

$$\lambda = \log_e 8s_0 = \log_e (8R/r)$$

Half-odd integer Legendre functions of the first kind:

$$P_{-\frac{1}{2}}(s) \sim \frac{\sqrt{2}}{\pi\sqrt{s}}\left[\log 8s + \frac{3\log 8s - 1}{16s^2}\right]$$

$$P_{\frac{1}{2}}(s) \sim \frac{2\sqrt{2s}}{\pi}\left[1 + \frac{2\log 8s - 3}{16s^2}\right]$$

$$n \geqslant 1 \qquad P_{n+\frac{1}{2}}(s) \sim \frac{\Gamma(n+1)\,(2s)^{n+\frac{1}{2}}}{\sqrt{\pi}\,\Gamma(n+\frac{1}{2})}\left[1 - \frac{n^2 - \frac{1}{4}}{4ns^2}\right]$$

For these three formulae, useful when s > 10, which is beyond the range of the tables [8], the error is less than 0.1% for s > 10 up to n = 8 and perhaps further.

First derivative of the half-odd integer Legendre functions of the first kind:

$$P'_{-\frac{1}{2}}(s) \sim -\frac{\sqrt{2s}}{2\pi(s^2-1)}\left[\log 8s - 2 + \frac{5 - \log 8s}{16s^2}\right]$$

$$P'_{\frac{1}{2}}(s) \sim \frac{\sqrt{2s}\ s}{\pi(s^2-1)}\left[1 - \frac{3(1+2\log 8s)}{16s^2}\right]$$

$n \geqslant 1$ $$P'_{n+\frac{1}{2}}(s) \sim \frac{\Gamma(n+1)(2s)^{n+\frac{3}{2}}}{2\sqrt{\pi}\,\Gamma(n+\frac{1}{2})(s^2-1)}\left[1 - \frac{(n+\frac{1}{2})(n+\frac{3}{2})}{4ns^2}\right]$$

The error in the above three formulae is less than 0.1% for $s > 10$ and for $n \leqslant 8$ and perhaps further.

Half-odd integer Legendre functions of the second kind

$$Q_{-\frac{1}{2}}(s) \sim \frac{\pi}{\sqrt{2s}}\left(1 + \frac{3}{16s^2}\right)$$

$$Q_{\frac{1}{2}}(s) \sim \frac{\pi}{2(2s)^{3/2}}\left(1 + \frac{15}{32s^2}\right)$$

$n \geqslant 1$ $$Q_{n-\frac{1}{2}}(s) \sim \frac{\sqrt{\pi}}{(2s)^{n+\frac{1}{2}}}\frac{\Gamma(n+\frac{1}{2})}{\Gamma(n+1)}\left(1 + \frac{(n+\frac{1}{2})(n+\frac{3}{2})}{4(n+1)s^2}\right)$$

the error at $s > 10$ being $< 0.1\%$ for $n \leqslant 5$.

The function F^{-1} defined by eq. (73)

$$F^{-1} \sim 1 + \frac{\lambda - 2}{8s_0^2}$$

The error in $F^{-1} < 0.1\%$ for $s_0 > 4$. F^{-1} is confined to the range

$$1 \leqslant F^{-1} \leqslant \frac{32}{3\pi^2} \sim 1.08076\,,$$

when

$$\infty \geqslant s_0 \geqslant 1\,.$$

The coefficients α_n defined by eq's. (77) and (78)

$$\alpha_1 \sim -\frac{\pi^2(s_0^2-1)}{32s_0^4}\left[1 + \frac{4\lambda+49}{32s_0^2}\right]$$

$$\alpha_2 \sim \frac{\pi^2(s_o^2-1)}{32\, s_o^4}\left[1 + \frac{4\lambda+37}{32\, s_o^2}\right]$$

$$n>1 \qquad \alpha_{n+1} \sim \frac{2\pi n\,\left[\Gamma(n+\frac{1}{2})\right]^2 (s_o^2-1)}{(2n-1)\,\left[\Gamma(n)\right]^2 (2s_o)^{2n+2}}\left[1 + \frac{1}{16 n\, s_o^2}\left(\frac{(2n+1)(4n^2-3)}{n-1} - \frac{4\lambda}{n} - \frac{8n^3+4n^2-10n+3}{n^2(n+1)}\right)\right]$$

For α_1 , and α_2 the error is of the order 0.03% at $s_o = 10$ but increases slowly with n.

The flux for cases I and II, and the ratio G defined by eq. (98):

$$\frac{\Phi^{I}}{\pi R^2 H_o} \sim -\left[\frac{(s_o-1)^2}{s_o^2} + 2\sqrt{\frac{s_o}{s_o+1}}\,\frac{(s_o^2-1)^2}{s_o^5}\left(1+\frac{1-\lambda}{2\, s_o}\right)\right]$$

Error ~ 0.1 % at $s_o=10$

$$\frac{\Phi^{II}}{I_A R} \sim -\frac{2\pi\sqrt{s_o-1}}{5\sqrt{s_o}}(\lambda-2)\left(1+\frac{1}{2s_o} - \frac{4\lambda^2-13\lambda+7}{16\, s_o^2\,(\lambda-2)}\right)$$

Error ~ 0.1 % at $s_o=10$

$$G \sim \frac{5}{2}\left(\frac{s_o-1}{s_o}\right)^{\frac{3}{2}}\frac{1}{\lambda-2}\left[1+\frac{3}{2s_o} - \frac{12\lambda^2-71\lambda+97}{16 s_o^2(\lambda-2)} - \frac{28\lambda^2-81\lambda+53}{8 s_o^3\,(\lambda-2)}\right]$$

Error ~ 0.1% at $s_o=10$

The self-inductance:

$$L/R \sim 4\pi\sqrt{\frac{s_o-1}{s_o}}\,(\lambda-2)\left(1+\frac{1}{2s_o} - \frac{4\lambda^2-13\lambda+11.7}{16\, s_o^2\,(\lambda-2)}\right)\times 10^{-6}$$

millihenries per cm.

Error ~ 0.1% at s_o = 2

The magnetic moments:

$$\frac{M^{I}}{\frac{4}{3}\pi R^3 H_o} \sim \frac{3}{(2s_o)^2}\left(1 + \frac{\lambda - 1.75}{8 s_o^2}\right)$$

Error $<$ 0.01% for $s_o \geqslant 2$

$$\frac{M^{II}}{\pi R^2 I_A} \sim \frac{1}{10}\left(1 - \frac{8\lambda - 7.7}{8 s_o^2}\right)$$

Error $\sim$ 0.1% at $s_o = 2$

$$\frac{M^{III}}{\frac{4}{3}\pi R^3 H_o} \sim \frac{3}{16(\lambda-2)}\left(1 - \frac{4\lambda - 6.11}{2 s_o^2}\right)$$

Error $\sim$ 0.1% in range $2 \leqslant s_o \leqslant 10$

$$\frac{M^{IV}}{\frac{4}{3}\pi R^3 H_o} \sim \frac{3}{16(\lambda-2)}\left(1 + \frac{31\lambda - 75}{16 s_o^2}\right)$$

Error $\sim$ 0.35% at $s_o = 2$

Error $<$ 0.2% for $4 < s_o < 10$

The maximum and minimum fields for the four cases I, II, III and IV:

$$H^{I}_{\pi}/H_o \sim 1 + \sqrt{\frac{s_o+1}{s_o}}\left(1 + \frac{8\lambda - 1}{32 s_o^2} + \frac{\lambda + 10}{8 s_o^3}\right)$$

Error $\sim$ 0.2% at $s_o = 2$

Error $<$ 0.1% at $s_o > 8$

$$H^{I}_{2\pi}/H_o \sim 1 + \sqrt{\frac{s_o-1}{s_o}}\left(1 + \frac{8\lambda - 1}{32 s_o^2} + \frac{\lambda + 1}{8 s_o^3}\right)$$

Error $<$ 0.1% for $s_o > 2$

$$H_{\pi}^{II} R/I_A \sim \frac{s_o}{5}\sqrt{\frac{s_o}{s_o-1}}\left(1+\frac{\lambda-1}{s_o}+\frac{\lambda+1}{4s_o^2}\right)$$

Error $< 0.65\%$ for $1.2 \leq s_o \leq 2$

Error $< 0.15\%$ for $s_o > 2$

$$H_{2\pi}^{II} R/I_A \sim \frac{s_o}{5}\sqrt{\frac{s_o}{s_o+1}}\left(1-\frac{\lambda-1}{s_o}+\frac{2\lambda-4.5}{4s_o^2}\right)$$

Error $\sim 2\%$ at $s_o \approx 2$

Error $\sim 0.6\%$ at $s_o = 3$

Error $< 0.1\%$ at $s_o > 6$

$$H_{\pi}^{III}/H_o \sim \frac{s_o-1}{2(\lambda-2)}\left(1+\frac{2\lambda+1}{2s_o}+\frac{23\lambda^2-69\lambda+88}{16s_o^2(\lambda-2)}\right)$$

Error $< 1\%$ at $s_o = 1.8$

Error $< 0.1\%$ at $s_o \geq 3$

$$H_{2\pi}^{III}/H_o \sim \frac{1}{2}\sqrt{\frac{s_o-1}{s_o+1}}\ \frac{s_o-1}{\lambda-2}\left(1-\frac{2\lambda-5}{2s_o}-\frac{2.5\lambda^2+3.8\lambda+2.5}{16s_o^2}\right)$$

Error $< 0.85\%$ at $s_o = 2$

Error $< 0.1\%$ for $s_o \geq 4$

$$H_{\pi}^{IV}/H_0 \sim \frac{s_0-1}{2(\lambda-2)}\left(1-\frac{6\lambda-17}{2s_0}-\frac{1.1\lambda^2+1.21\lambda+5.323}{4s_0^2}\right)$$

Error $\sim 0.1\%$ at $s_o = 2$

Error $\sim 0.4\%$ at $s_o = 4$

Error $< 0.03\%$ at $s_o \geqslant 6$

$$H_{2\pi}^{IV}/H_0 \sim 1+\frac{\sqrt{s_0(s_0-1)}}{2(\lambda-2)}\left(1+\frac{\lambda-3}{s_0}-\frac{3.125\lambda^2-18.5\lambda+14}{16s_0^2}\right)$$

Error $< 0.04\%$ for $s_o \geqslant 2$