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Use of Euler Transformations in the Determination of Rocket Orientation

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NAVAL RESEARCH LABORATORY
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CONTENTS

ABSTRACT

An analytical treatment of data from rocket aspect sensors (such as magnetometers and solar sensors) was developed in terms of Euler angle coordinate systems and transformations, since the Euler angles represent the zenith, azimuth, and spin angular coordinates of the rocket. It was found that the relationships between the sensor outputs and the orientation of the rocket are exceedingly straightforward when considered in terms of these coordinate transformations.

Three cases of aspect sensor combinations were considered; complete magnetometer data were assumed in each of the three cases (except for consideration of failure of one of the three magnetometers) with supplementary information in case 1 from continuous sun sensor coverage, in case 2 from partial sun sensor data, and in case 3 from horizon scanners. The equations are derived for ideal experimental conditions (instrumental errors are neglected).

The solutions can easily be programmed on a computer. A general computer program was not developed, however, because the problems encountered in the reduction of aspect data vary considerably for each rocket flight.

PROBLEM STATUS

This is a final report on one phase of a continuing problem.

AUTHORIZATION

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USE OF EULER TRANSFORMATIONS IN THE DETERMINATION OF ROCKET ORIENTATION

INTRODUCTION

Many rocket experiments require accurate determination of the orientation of the rocket with respect to the earth. Instruments such as solar sensors, magnetometers, horizon scanners, star scanners, and gyroscopes can be incorporated in the vehicle payload to provide the required directional information.

A number of discussions involving the interpretation of rocket aspect data have been presented. A paper by Ott, Horstman, and Lahn (1) considered the case of a uniformly or rapidly rotating vehicle. Their methods can be modified to include more general modes of rocket motion. Asquith and Baxter (2) discussed attitude determination using spherical trigonometry for the interpretation of magnetometer and sun and earth sensor data. Their analysis was limited to the example of a rocket having a large spin to precession ratio. Morrison (3) considered the various methods of orientation determination in a qualitative discussion. Other authors (4,5) considered partial aspect reduction using magnetometers alone.

The purpose of the present report is to relate the outputs of attitude sensors to the orientation of a rocket in space through the use of Euler transformations. Euler angles are quite convenient for application to rocket aspect problems, since they physically represent the three angular coordinates of the vehicle: zenith, azimuth, and spin. In addition the transformation matrices are very straightforward, so that relationships between various coordinate systems are readily obtainable.

The three cases of attitude determination discussed herein are assumed to have complete magnetometer coverage supplemented (case 1) by complete solar sensor data, (case 2) by partial information from, say, a single sun detector, and (case 3) by horizon sensors. Gyroscopes and star scanners are not considered because of their different operating principles. A discussion of star scanning techniques has been given by Kupperian and Kreplin (6), with subsequent elaboration by Morrison (3).

No attempt is made to provide a general computer program to incorporate any of the present solutions, because the problems encountered in the reduction of aspect data vary considerably for each rocket flight. However, the present analyses are extended to a point where the results are easily programmable.

The effects of instrumental error are not taken into account in these calculations. Indeed, the calibration and resolution of the sensors play the major role in determining the accuracy of the solution. However, these effects are not within the scope of the present discussion. The special case of instrumental failure (or coverage by only two magnetometers instead of three) is considered.

In the present discussion the rocket experiment is assumed to be mounted laterally (perpendicular to the rocket axis). Experiments which look out through the nose cone have the same axis as the rocket. In this case, however, a lateral axis can be defined (arbitrarily) to provide a sensor reference axis.

It is assumed that all information about rocket altitude and subtrajectory position on the surface of the earth is provided by radar data.

SYMBOLS

The various symbols used in this report are as follows:

$\mathbf{R}$	Rocket vector.
$\mathbf{E}$	Experiment vector or direction of observation.
1, 2, 3	Rocket coordinate system axes; 1 is the experiment axis and 3 is the rocket axis.
Ea	Subscript representing the earth coordinate system.
I_1, I_3	Moments of inertia of the rocket along the 1 and 3 axes.
x, y, z	Right-handed, earth-fixed coordinate system.
θ, ϕ, ψ	Euler angles representing rocket zenith, azimuth, and spin angles in the earth coordinate system.
$\mathbf{V}_R$	A vector in the rocket coordinate system.
$\mathbf{V}_{Ea}$	A vector in the earth system.
$\mathbf{V}_Y$	A vector in the yaw system.
$\tilde{A}$	Transformation matrix from body-fixed coordinate system to space-fixed system.
A	Transpose of $\tilde{A}$.
$\mathbf{B}$	Magnetic field vector.
$\mathbf{S}$	Solar vector.
$\mathbf{Y}$	Yaw cone axis vector.
$Y1, Y2, Y3$	Yaw coordinate system axes; $Y3$ is along $\mathbf{Y}$.
β, α, σ	Euler angles representing the position of $\mathbf{R}$ in the yaw coordinate system.
ω	Apparent spin rate.
t	Time.
N	Average number of spins in a yaw cycle.
T	Yaw period.
R_0	Radius of earth.
H	Rocket altitude.
Z	Impact parameter.

A dot above a particular quantity represents differentiation with respect to time. All subscripts are, in general, self-explanatory and are defined in the text. In addition, all symbols which are used as simplifying substitutions are also defined therein. Note that yaw and precession are used interchangeably in this report.

THE EULER COORDINATE SYSTEM

An excellent discussion of Euler angles is given by Goldstein (7). He also includes a section on matrices and their transformation properties.

The coordinate systems used in the present calculations are shown in Fig. 1. $\boldsymbol{R}$ and $\boldsymbol{E}$ represent the rocket and experiment axes, and x, y, and z define an earth-fixed coordinate system. Note that the earth system axes x and y can represent (for example) south and east, with z representing the zenith direction. θ is the rocket zenith angle, ϕ - 90° is the rocket azimuth measured from the x axis, and ψ is the rocket spin angle referenced from the "line of nodes." Note that the projection of the rocket vector on the x-y plane is always perpendicular to the "line of nodes."

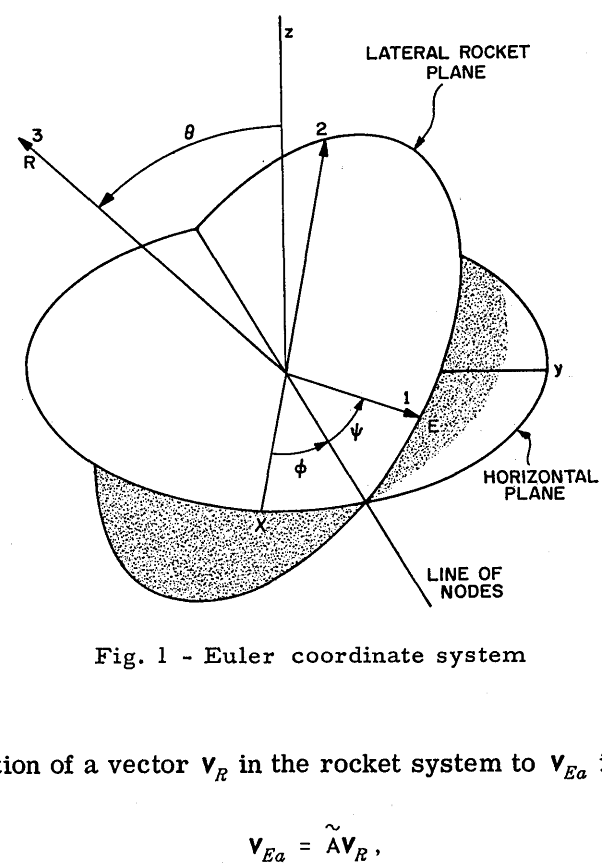


Fig. 1 - Euler coordinate system

The transformation of a vector $\mathbf{V}_R$ in the rocket system to $\mathbf{V}_{Ea}$ in the earth system is given by

$$\mathbf{V}_{Ea} = \tilde{A}\mathbf{V}_R, \tag{1}$$

where $\tilde{A}$ is the transformation matrix given (7) by

$$\tilde{A} = \begin{bmatrix} \cos\psi\cos\phi - \cos\theta\sin\phi\sin\psi & -\sin\psi\cos\phi - \cos\theta\sin\phi\cos\psi & \sin\theta\sin\phi \\ \cos\psi\sin\phi + \cos\theta\cos\phi\sin\psi & -\sin\psi\sin\phi + \cos\theta\cos\phi\cos\psi & -\sin\theta\cos\phi \\ \sin\theta\sin\psi & \sin\theta\cos\psi & \cos\theta \end{bmatrix}. \tag{2}$$

The relationship between the experiment orientation and the rocket coordinates is easily obtained using Eq. (1). In the rocket system the normalized experiment vector (in matrix form) is

$$\mathbf{E}_R = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \tag{3}$$

whereas in the earth system it is

$$\mathbf{E}_{Ea} = \begin{bmatrix} E_x \\ E_y \\ E_z \end{bmatrix} = \begin{bmatrix} \sin\theta_E \cos\phi_E \\ \sin\theta_E \sin\phi_E \\ \cos\theta_E \end{bmatrix}, \tag{4}$$

where ϕ_E is the experiment azimuth measured from the x axis and θ_E is the experiment zenith angle. Substituting Eqs. (2), (3), and (4) into Eq. (1) and writing out the results of the multiplication in component form, one has

$$\sin\theta_E \cos\phi_E = \cos\psi\cos\phi - \cos\theta\sin\phi\sin\psi, \tag{5}$$

$$\sin\theta_E \sin\phi_E = \cos\psi\sin\phi + \cos\theta\cos\phi\sin\psi, \tag{6}$$

and

$$\cos\theta_E = \sin\theta\sin\psi. \tag{7}$$

Thus once the rocket coordinates are determined, the experiment orientation can be obtained by use of the above equations.

ORIENTATION FROM MAGNETOMETERS WITH CONTINUOUS SOLAR COVERAGE

In the case of magnetometers with continuous solar coverage the sensors continuously measure the projections of the magnetic field and solar vectors in the rocket coordinate system throughout the flight. The advantage of this type of coverage is that attitude can be obtained even if the vehicle does not behave as a rigid body undergoing free motion, or if an attitude control system is employed.

The magnetic field and solar vectors are designated as $\mathbf{B}_R$ and $\mathbf{S}_R$ respectively in the rocket system. In matrix form they are

$$\mathbf{B}_R = \begin{bmatrix} B_1 \\ B_2 \\ B_3 \end{bmatrix} \quad \text{and} \quad \mathbf{S}_R = \begin{bmatrix} S_1 \\ S_2 \\ S_3 \end{bmatrix}, \tag{8}$$

where the 3 axis is along the rocket axis and 1 is along the experiment axis.

In the earth coordinate system the solar vector $\mathbf{S}_{Ea}$ can be found in standard tables (8), and the magnetic field vector $\mathbf{B}_{Ea}$ is obtainable by use of tabulated coefficients (9). For this discussion it is assumed that all of the vectors are normalized so that one must divide the sensor outputs by the absolute magnitude of $\mathbf{B}$ and $\mathbf{S}$.

Transformation of $\mathbf{B}$ and $\mathbf{S}$ from the rocket coordinate system to that of the earth involves the equations

$$\mathbf{B}_{Ea} = \tilde{A}\mathbf{B}_R \tag{9}$$

and

$$\mathbf{S}_{Ea} = \tilde{A}\mathbf{S}_R, \tag{10}$$

where $\tilde{A}$ is given by Eq. (2). If these equations are written out in component form, there will be six equations in six unknowns (the sines and cosines of θ, ϕ, and ψ). The solutions will provide a unique determination of the rocket coordinates and, subsequently, the experiment orientation.

Writing Eq. (9) in component form yields

$$B_x = B_1(\cos\psi\cos\phi - \cos\theta\sin\phi\sin\psi) + B_2(-\sin\psi\cos\phi - \cos\theta\sin\phi\cos\psi) + B_3(\sin\theta\sin\phi), \tag{11}$$

$$B_y = B_1(\cos\psi\sin\phi + \cos\theta\cos\phi\sin\psi) + B_2(-\sin\psi\sin\phi + \cos\theta\cos\phi\cos\psi) + B_3(-\sin\theta\cos\phi), \tag{12}$$

and

$$B_z = B_1(\sin\theta\sin\psi) + B_2(\sin\theta\cos\psi) + B_3\cos\theta. \tag{13}$$

Since the $\mathbf{S}$ equations are strictly analogous, they are not written.

To facilitate their solution, Eqs. (11), (12) and (13) can be made linear and put into matrix form. Note that Eq. (13) is immediately linear if it is divided by $\sin\theta$:

$$B_1\sin\psi + B_2\cos\psi + B_3\cot\theta - B_z\,csc\,\theta = 0. \tag{14}$$

Combining Eqs. (11) and (12) yields

$$B_2\sin\psi - B_1\cos\psi + B_y\sin\phi + B_x\cos\phi = 0. \tag{15}$$

Also, combining Eqs. (11) and (12) with Eq. (14) gives

$$B_x\sin\phi - B_y\cos\phi + B_z\cot\theta - B_3\,csc\,\theta = 0. \tag{16}$$

In matrix form, Eqs. (14), (15) and (16), along with the analogous solar equations, can be written

$$\begin{bmatrix} 0 & 0 & B_x & -B_y & B_z & -B_3 \\ 0 & 0 & S_x & -S_y & S_z & -S_3 \\ B_1 & B_2 & 0 & 0 & B_3 & -B_z \\ S_1 & S_2 & 0 & 0 & S_3 & -S_z \\ B_2 & -B_1 & B_y & B_x & 0 & 0 \\ S_2 & -S_1 & S_y & S_x & 0 & 0 \end{bmatrix} \begin{bmatrix} \sin\psi \\ \cos\psi \\ \sin\phi \\ \cos\phi \\ \cot\theta \\ csc\,\theta \end{bmatrix} = 0. \tag{17}$$

The solution of this equation will yield the rocket orientation. However, since this is a homogeneous matrix equation, five of the variables must be found as a function of the sixth. This poses no problem if the independent variable is chosen to be $\csc \theta$, since θ exists only in the first two quadrants.

The solution of the set of equations given in matrix form by Eq. (17) can easily be programmed on a computer having a library subroutine which solves simultaneous linear equations. However, a longhand solution is relatively simple because of the large number of zero elements in the matrix. The method is presented in Appendix A. The special cases of partial magnetometer coverage, magnetometer failure, and in-flight calibration are discussed in Appendix B.

ORIENTATION FROM MAGNETOMETERS WITH PARTIAL SOLAR COVERAGE

In the case of magnetometers with partial solar coverage the magnetometer information is supplemented by occasional aspect data. A typical example of this supplementation of data is when a single sun sensor is incorporated in the rocket payload. Then the solar vector is determined in the rocket system once per spin cycle (as long as the rocket vector direction is not the same as that of the sun). Between these points the motion is governed by rigid body dynamics. If the vehicle does not behave like a symmetric rigid body in free motion, and/or the moments of inertia are unknown, this method will not apply.

The assumption of free motion is generally applicable during the upper portion of the rocket trajectory. However, one must be careful in applying the methods of this section to a liquid fuel rocket, since some fuel may remain in the tanks after engine burnout, thus nullifying the rigid body requirement. This effect can be taken into account by introducing a correction factor during each spin cycle so that the attitude data is compatible for successive solar scans.

The method of this section will be to recognize that if the vehicle is a rigid body, its motion in the (earth-fixed) yaw coordinate system is constant. Then if the rocket position is known at some instant, the future orientation can be predicted. First, the spin rate, the precession rate, and the yaw cone angle of the rocket can be found from the magnetometer data. Second, the position of the yaw cone in the earth-fixed coordinate system is determined by the rocket position on the cone for at least three different times. Next, the phase or position of the rocket vector on the cone is found at one particular time and then combined with the constant spin and yaw rates to give the orientation at any later time. Last, by transforming from the known yaw coordinate system to that of the earth the required attitude can be found.

In general the motion of a symmetric rigid body under the action of no torques (ignoring the translation effects of gravity) can be described as a body-fixed vector rolling on a yaw cone. If the position of this fixed precession cone is known along with the position of the rocket on the cone at some instant, the orientation of $\mathbf{R}$ in the earth system can be found at all later times by use of rigid body dynamics.

Rigid Body Equations

The free motion of a symmetric rigid body can be depicted by considering a yaw coordinate system Y_1, Y_2, and Y_3 as shown in Fig. 2. The rocket is precessing about the Y_3 axis at a constant rate, $\dot{\alpha}$, and at a constant cone angle β. The vehicle is also spinning at a rate $\dot{\sigma}$, where $\dot{\sigma}$ is referenced to the Y_3 axis. The equations of rigid body dynamics (10) provide the relationship

$$\dot{\alpha} \cos \beta = \frac{I_3}{I_1} \omega , \quad (18)$$

where I_3 and I_1 are the moments of inertia along the **R** and **E** axes respectively and ω is the apparent spin rate produced by a combination of true spin and precession. It is given by

$$\omega = \dot{\sigma} + \dot{\alpha} \cos \beta . \quad (19)$$

Combining Eqs. (18) and (19) yields

$$\dot{\sigma} = \left(\frac{I_1 - I_3}{I_1} \right) \omega . \quad (20)$$

It will be shown later that $\dot{\sigma}$, $\dot{\alpha}$, and β are obtainable from magnetometer signals.

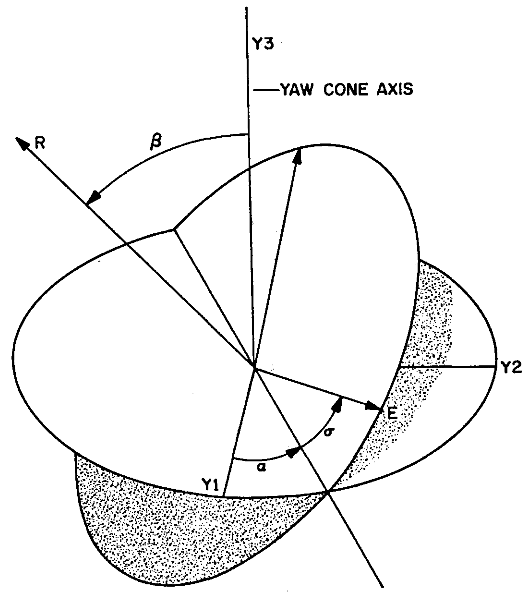


Fig. 2 - Yaw coordinate system

Rocket Position in the Yaw Coordinate System

If at some time, say $t = 0$, the position of the rocket in the yaw system is found to be α_0, σ_0, and β, then at later times the rocket coordinates will be

$$\alpha = \dot{\alpha} t + \alpha_0 , \quad (21a)$$

$$\sigma = \dot{\sigma} t + \sigma_0 , \quad (21b)$$

and

$$\beta = \beta . \quad (21c)$$

Then knowing the yaw axis coordinates, the **R** vector can easily be transformed to the earth system to give the rocket orientation at t.

To obtain $\dot{\alpha}$ and $\dot{\sigma}$ from the magnetometer data, consider what the magnetometers are measuring. Figure 3 shows the rocket vector in a magnetic field coordinate system. The magnetic-field direction **B** is arbitrarily depicted as being outside the yaw cone, although **B** within the yaw cone is also possible. The normalized longitudinal magnetometer signal (along the rocket axis) is

$$B_3 = \cos \theta_B , \quad (22)$$

and the lateral magnetometers measure

$$B_1 = \sin \theta_B \cos \psi_B \quad (23)$$

and

$$B_2 = \sin \theta_B \sin \psi_B , \quad (24)$$

where the phase reference is arbitrarily set equal to zero. Thus ψ is measured from the point where B_1 is a maximum. The problem is then to find $\dot{\alpha}$, $\dot{\sigma}$, and β from these equations.

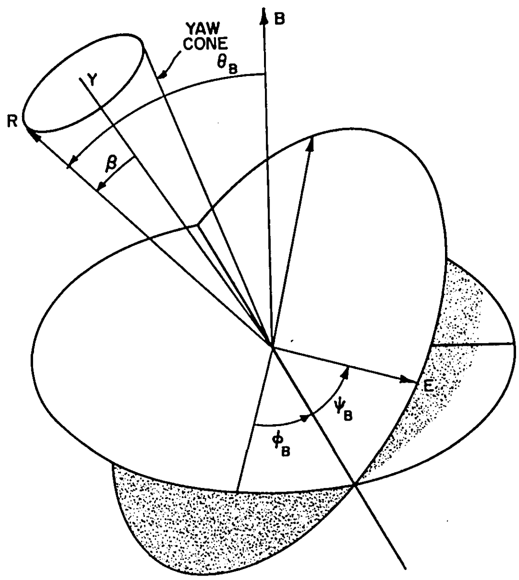


Fig. 3 - Magnetic field coordinate system with the magnetic field vector **B** outside the yaw cone

The yaw period can be determined by observing the cyclic behavior of B_3 (or the envelope of B_1 or B_2). Since the precession rate is the same with respect to either **B** or $Y3$, and since $\dot{\alpha}$ is a constant,

$$\dot{\alpha} = \frac{360^\circ}{\text{yaw period}} . \tag{25}$$

To determine the actual spin rate $\dot{\sigma}$ from the magnetometer data, one must know whether or not the yaw cone encloses the magnetic field vector. This is because the average spin period as measured from an axis outside the yaw cone is shorter than it would be if the axis were within the cone. In fact there will be one more spin per yaw period if the reference point is outside of the cone. Note, then, that if angular rates are considered, rather than periods, the yaw rate is the difference between the average spin rate, as measured from an axis inside of the cone, and the average rate measured with respect to an exterior axis.

To determine the position of the magnetic field vector with respect to the yaw cone, one can measure the varying magnetometer spin period throughout a complete yaw cycle. If the **B** vector is within the cone, the spin period is a maximum at the point of closest approach to **B** and a minimum at the farthest point. The reverse situation occurs when **B** is outside the cone: a minimum in the period is experienced at the point of closest approach. Thus by examining the spin period, one can determine whether or not the magnetic field vector is outside the yaw cone. Once this information has been obtained, the average (over a yaw cycle) spin period is the same as the true spin period if **B** is inside the cone. If **B** is outside, one spin rotation per yaw cycle must be subtracted from the average number of spins per cycle to obtain the true spin period. This can be summarized in the following way if T is the yaw period and N is the average number of spins in a time T:

$$\dot{\sigma} = 360^\circ \times \begin{cases} \dfrac{N}{T}, & B \text{ within the cone.} \\[2ex] \dfrac{N-1}{T}, & B \text{ outside the cone.} \end{cases} \tag{26}$$

Note that if B is outside the cone, $\dot{\sigma}$ can be written as

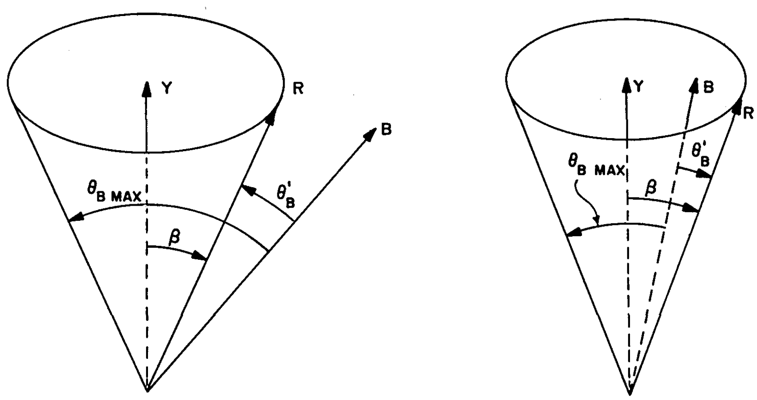
$$\dot{\sigma} = \text{average spin rate} - \dot{\alpha} . \tag{27}$$

Care must be exercised in the determination of N. The average spin period ought to be plotted over several yaw cycles. The result should be a straight line with no slope. If a slope is present, the rocket is not a rigid body, and if there are deviations from the straight line, an error has been made in the determination of N.

Now that $\dot{\alpha}$ and $\dot{\sigma}$ are known, β can be determined by the use of Eqs. (18) and (20). This of course assumes a knowledge of I_1 and I_3.

A second method can be employed to find β. Consider the output of the longitudinal magnetometer, i.e., $\cos\theta_B$. Define the minimum value of θ_B (or the angle of closest approach of **R** and **B**) as θ_B'. The maximum value of θ_B is $2\beta + \theta_B'$ if **B** is outside the yaw cone (Fig. 4a) and $\theta_B = 2\beta - \theta_B'$ if **B** is within the cone (Fig. 4b). Therefore,

$$\beta = 1/2 \begin{cases} \theta_{B\,\max} - \theta_B', & \textbf{B} \text{ outside the cone.} \\ \theta_{B\,\max} + \theta_B', & \textbf{B} \text{ inside the cone.} \end{cases} \tag{28}$$



(a) **B** outside the yaw cone

(b) **B** inside the yaw cone

Fig. 4 - Geometry for the determination of β from the minimum and maximum output θ_B of a longitudinal magnetometer

Yaw Cone Position in the Earth Coordinate System

For a solution of the rocket motion in the yaw coordinate system, it is necessary to know α_0 and σ_0. First, however, the position of the yaw cone in space (θ_Y, ϕ_Y) is determined. Note that ψ_Y is arbitrary and therefore can be set equal to zero. This merely alters the phases α_0 and σ_0.

The yaw coordinate system has the following form in the earth system:

$$\mathbf{Y}_{Ea} = \begin{bmatrix} \sin\theta_Y \sin\phi_Y \\ -\sin\theta_Y \cos\phi_Y \\ \cos\theta_Y \end{bmatrix}, \tag{29}$$

where ϕ_Y - 90° is the **Y** azimuth measured from the x axis, and θ_Y is the zenith angle of the yaw vector. Since **Y** and **B** are constants,

$$\mathbf{Y}\cdot\mathbf{B} = C_1 = \begin{cases} \cos(\beta + \theta_B'), & \textbf{B} \text{ outside the cone.} \\ \cos(\beta - \theta'), & \textbf{B} \text{ inside the cone.} \end{cases} \tag{30}$$

Then

$$B_x \sin\phi_Y - B_y \cos\phi_Y + B_z \cot\theta_Y = C_1 \csc\theta_Y, \tag{31}$$

which is an equation involving four unknowns. Now if the sun position (or some other earth-fixed vector) is known in the rocket system at a certain time, the rocket orientation can be determined by use of Eq. (17). Then with

$$\mathbf{Y} \cdot \mathbf{R} = \cos\beta = \text{constant} = C_2 \tag{32}$$

one can write

$$R_x \sin\phi_Y - R_y \cos\phi_Y + R_z \cot\theta_Y = C_2 \csc\theta_Y . \tag{33}$$

If the rocket position is known at two more points, equations similar to Eq. (33) can be written and uniquely solved for θ_Y and ϕ_Y. Otherwise one can plot θ_Y vs ϕ_Y from Eqs. (31) and (33) and test which values give a reasonable rocket motion. This may be possible if information is available from another source, such as the experiment itself.

If θ_Y and ϕ_Y have been determined, α_0 and σ_0 can be found at the time (say $t = 0$) when the rocket position is known. Since $\mathbf{R}_{Ea}$ and $\mathbf{E}_{Ea}$ are known, their projections in the yaw system are

$$\mathbf{R}_Y = \mathrm{A}_{YEa}\mathbf{R}_{Ea} = \begin{bmatrix} R_{Y1} \\ R_{Y2} \\ R_{Y3} \end{bmatrix}$$

and (34)

$$\mathbf{E}_Y = \mathrm{A}_{YEa}\mathbf{E}_{Ea} = \begin{bmatrix} E_{Y1} \\ E_{Y2} \\ E_{Y3} \end{bmatrix} ,$$

where A_{YEa} is the transposed matrix of $\tilde{\mathrm{A}}_{EaY}$. (Note that $\mathrm{A}\tilde{\mathrm{A}} = 1$ if the transformation matrices are orthonormal.) It is given by

$$\mathrm{A}_{YEa} = \begin{bmatrix} \cos\phi_Y & \sin\phi_Y & 0 \\ -\cos\theta_Y \sin\phi_Y & \cos\theta_Y \cos\phi_Y & \sin\theta_Y \\ \sin\theta_Y \sin\phi_Y & -\sin\theta_Y \cos\phi_Y & \cos\theta_Y \end{bmatrix} , \tag{35}$$

where ψ_Y has been set equal to zero. $\mathbf{R}_Y$ and $\mathbf{E}_Y$ are determined by rotating the rocket and experiment axes through angles α_0, σ_0, and β into the yaw system:

$$\mathbf{R}_Y = \tilde{\mathrm{A}}_{YR}\mathbf{R}_R$$

and (36)

$$\mathbf{E}_Y = \tilde{\mathrm{A}}_{YR}\mathbf{E}_R ,$$

where $\tilde{\mathrm{A}}_{YR}$ is given by Eq. (2) with α_0, σ_0, and β in place of ϕ, ψ, and θ. Since $\mathbf{E}_R$ is given by Eq. (3) and

$$\mathbf{R}_R = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} , \tag{37}$$

Eqs. (36) can be written in component form as

$$R_{Y1} = \sin\beta \sin\alpha, \tag{38a}$$

$$R_{Y2} = -\sin\beta \cos\alpha, \tag{38b}$$

and

$$R_{Y3} = \cos\beta \tag{38c}$$

and

$$E_{Y1} = \cos\sigma \cos\alpha - \cos\beta \sin\alpha \sin\sigma, \tag{39a}$$

$$E_{Y2} = \cos\sigma \sin\alpha + \cos\beta \cos\alpha \sin\sigma, \tag{39b}$$

and

$$E_{Y3} = \sin\beta \sin\sigma. \tag{39c}$$

With $\boldsymbol{R}_Y$ and $\boldsymbol{E}_Y$ known and at $t = 0$, $\sigma = \sigma_0$ and $\alpha = \alpha_0$, Eqs. (38) and (39) can be solved for σ_0 and α_0. Note that these angles are "overspecified," since β is measured independently. Then, at later times $\boldsymbol{R}_Y$ and $\boldsymbol{E}_Y$ are given by Eqs. (38) and (39) using Eq. (21). With $\tilde{A}_{EaY}$ as the transposed matrix of A_{YEa} the rocket and experiment attitudes in the earth system can be found at any time from the equations

$$\boldsymbol{R}_{Ea} = \tilde{A}_{EaY}\boldsymbol{R}_Y$$

and

$$\boldsymbol{E}_{Ea} = \tilde{A}_{EaY}\boldsymbol{E}_Y. \tag{40}$$

Another method of attitude interpretation is of interest for the combination of sensors discussed in this section, namely, the possibility of plotting the points of known orientation on a globe. By drawing curves through these points, approximate aspect solutions can be obtained. However, these solutions are often too crude and must be used with a smoothing procedure for practical applications.

ORIENTATION FROM MAGNETOMETERS PLUS HORIZON SENSORS

Two analyses of horizon sensor data are given in this section. The first consideration is the general case of rigid body motion, and the second is concerned with the simpler case of a large ratio of precession to spin rate. Of course, they both assume that the zenith angle of the rocket is large enough to permit detection of the horizons.

General Case

The first method is to find the position of the yaw cone for some particular time from the magnetometer and horizon data. Then one can follow the motion of the rocket on the yaw cone at later times via Eqs. (34) through (40).

Typically the signal from a horizon sensor has two intensity maximums per rotation with some low-level value between as the earth is scanned. When the time interval between the horizons is a maximum, the rocket has its largest zenith on the yaw cone. At this time, the rocket vector, the yaw cone axis, and the local vertical are all coplanar. Similarly, when the interval is a minimum, the rocket is at the point of closest approach to the local vertical. The rocket vector is again in the same plane as the zenith and the cone axis. Therefore, because the azimuth angle of the rocket is the same when θ is either maximum or minimum, ϕ can be eliminated from Eq. (16), provided the magnetic field of the earth has not varied appreciably over the region of translation of the rocket during half of a yaw period. The resulting equation is

$$B_z\,(\cot\,\theta_{max} - \cot\,\theta_{min}) = B_3\ csc\ \theta_{max} - B_3'\ csc\ \theta_{min}, \tag{41}$$

where B_3' is the magnetic field component along the rocket axis at a time corresponding to $\theta = \theta_{min}$.

Equation (41) can be solved for θ_{max} or θ_{min}, since

$$\theta_{max} = \begin{cases} 2\beta + \theta_{min} & \text{if the local vertical is outside the yaw cone,} \\ 2\beta - \theta_{min} & \text{if the local vertical is inside the yaw cone,} \end{cases} \tag{42}$$

where β, the half-angle of the yaw cone, can be found from the magnetometer data, as was shown in the preceding section. The position of the zenith with respect to the yaw cone can be found from the horizon sensor data. If the spin period measured by the sensors (as opposed to the time interval between horizon) has a minimum when the rocket zenith angle is maximum, the local vertical is inside the yaw cone (and vice versa). Thus, using Eq. (42), Eq. (41) has two solutions for θ_{min}. Usually one of them is obviously not the true case.

The position of the yaw cone axis can now be found. The zenith angle of $\mathbf{Y}$ is given by

$$\theta_Y = \begin{cases} \beta + \theta_{min}, & \text{if the vertical is outside the cone.} \\ \beta - \theta_{min}, & \text{if the vertical is inside the cone.} \end{cases} \tag{43}$$

The yaw axis azimuth is given by Eq. (16) with $\theta = \theta_{max}$ (or θ_{min}), since $\mathbf{Y}$ is coplanar with the rocket and the vertical. Again there are two solutions, but it is usually easy to discern the true azimuth. Finally, the rocket and experiment orientations in the earth coordinate system can be found at any later time with Eqs. (34) through (40).

Since the local vertical changes significantly for a rocket with a large horizontal range, the above procedure ought to be repeated periodically throughout the flight.

Rapidly Spinning Rocket

Next the case of a rapidly spinning (compared to precession) rocket is considered. The experiment zenith angle is easily found from the horizon sensor data. Then the azimuth angle can be found from the magnetic field equations.

The geometry of the situation is shown in Fig. 5. Circle C is the intersection of the lateral rocket plane and the earth. The spin angle phase is set at -90° so that ψ is zero when the zenith angle of the experiment direction is maximum. The rocket altitude is H, and R_0 is the earth's radius. The rocket and experiment zenith angles are θ and θ_E. Then Eq. (7) becomes

$$\cos\,\theta_E = -\sin\,\theta\,\cos\,\psi\,. \tag{44}$$

When the horizon is observed,

$$\sin\,\theta_E = \frac{R_0}{R_0 + H} \tag{45}$$

and

$$\cos\,\theta_E = -\sqrt{1 - \left(\frac{R_0}{R_0 + H}\right)^2} \approx -\frac{\sqrt{2R_0H}}{R_0 + H}\,. \tag{46}$$

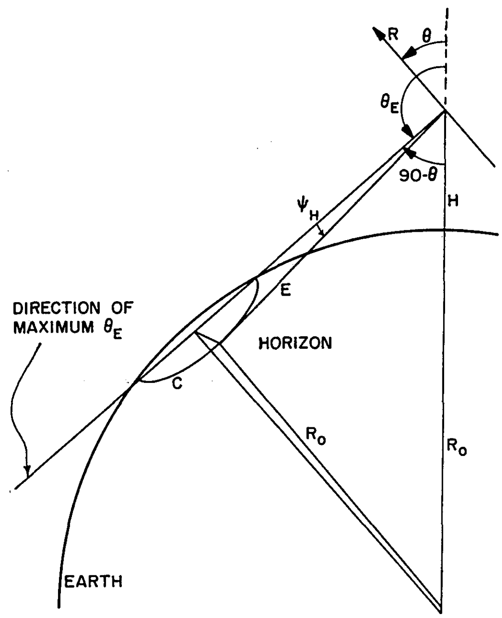


Fig. 5 - Geometry of a horizon scan

Then the rocket zenith angle is

$$\sin\,\theta = \frac{\sqrt{2R_0H}}{R_0 + H}\sec\,\psi_H\,, \tag{47}$$

where ψ_H is the spin angle at the horizon. Knowledge of the spin rate is necessary to find ψ. Since the horizon points are normally outside the yaw cone, the spin rate must be calculated by averaging the times between horizon scans. This involves some error, since only two horizon scans are available in each spin cycle. Then if this rate is $\dot{\psi}$,

$$\psi_H = \dot{\psi}\,t_H. \tag{48}$$

where t_H is the time it takes the sensor to scan from the minimum point on the lateral plane to the horizon. If the minimum point is not obvious from the sensor data, it must be calculated from a combination of the spin rate and two horizon points. The zenith angle of the experiment at any other time t in the spin cycle is then found via Eq. (44), with $\psi = \dot{\psi}t$.

The rocket azimuth can be determined by use of the magnetometer equations, specifically Eqs. (14), (15), and (16), with care exercised in referencing ψ to the line of nodes instead of the minimum point on the lateral plane. Then Eqs. (5) and (6) yield the experimental azimuth.

A useful quantity Z, defined as the distance above the surface of the earth (or impact parameter) that the experiment is looking, is shown in Fig. 6. Note that for any other point in the spin cycle, Eq. (47) can be replaced by

$$\sin\,\theta = \frac{\sqrt{2R_0(H-Z)}}{(R_0 + H)}\sec\,\psi\,. \tag{49}$$

Then combining Eqs. (44) and (46) gives

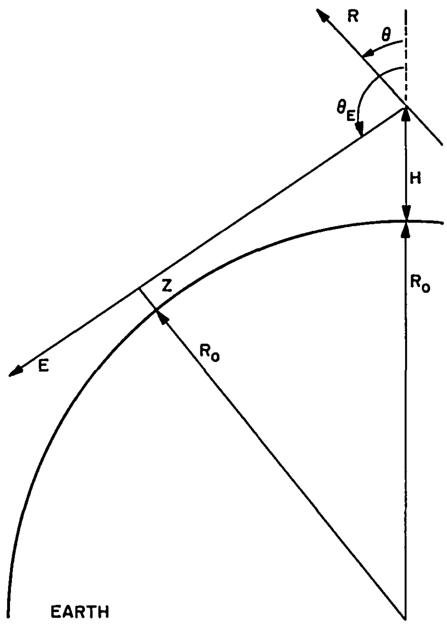


Fig. 6 - Geometry of the impact parameter Z

$$Z = H\left(1 - \frac{\cos^2\psi}{\cos^2\psi_H}\right). \tag{50}$$

Of course the impact parameter is not defined if θ_E is less than 90° (or Z is greater than H).

CONCLUDING REMARKS

Euler coordinate systems and transformations have been used in the interpretation of rocket aspect information. It has been found that they greatly simplify the use of aspect sensor data for three cases of interest. Formulas involving the attitude as a function of sensor outputs have been derived and are easily programmable on a computer.

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APPENDIX A

SOLUTION OF EQ. (17)

In this appendix the mechanics of solving Eq. (17) for the angular coordinates as functions of aspect sensor outputs are given.

Since Eq. (17) is a homogeneous matrix equation, one can divide the rows by $csc\ \theta$ and eliminate the sixth row, thus giving

$$\begin{bmatrix} 0 & 0 & B_x & -B_y & B_z \\ 0 & 0 & S_x & -S_y & S_z \\ B_1 & B_2 & 0 & 0 & B_3 \\ S_1 & S_2 & 0 & 0 & S_3 \\ B_2 & -B_1 & B_y & B_x & 0 \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \\ X_3 \\ X_4 \\ X_5 \end{bmatrix} = \begin{bmatrix} B_3 \\ S_3 \\ B_z \\ S_z \\ 0 \end{bmatrix}, \tag{A1}$$

where $X_1 = \sin \psi / csc\ \theta$, $X_2 = \cos \psi / csc\ \theta$, $X_3 = \sin \phi / csc\ \theta$, $X_4 = \cos \phi / csc\ \theta$, and $X_5 = \cos \theta$. Note that $csc\ \theta$ is always positive, since θ is defined as being between 0° and 180°.

To solve Eq. (A1) the following steps are taken. First, to produce a zero in place of B_1 in the first column, multiply the third row by B_2, multiply the fifth row by $-B_1$, combine the two, and substitute the result for the third row. Next, to produce a zero in place of S_1 in the first column, multiply the fourth row by B_2, multiply the fifth row by $-S_1$, combine the two, and substitute the result for the fourth row. This yields

$$\begin{bmatrix} 0 & 0 & B_x & -B_y & B_z \\ 0 & 0 & S_x & -S_y & S_z \\ 0 & Q_2 & -B_1B_y & -B_1B_x & B_2B_3 \\ 0 & Q_1 & -S_1B_y & -S_1B_x & B_2S_3 \\ B_2 & -B_1 & B_y & B_x & 0 \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \\ X_3 \\ X_4 \\ X_5 \end{bmatrix} = \begin{bmatrix} B_3 \\ S_3 \\ B_zB_2 \\ S_zB_2 \\ 0 \end{bmatrix}, \tag{A2}$$

where

$$Q_1 = B_1S_1 + B_2S_2$$

and

$$Q_2 = B_1^2 + B_2^2$$

The last row in Eq. (A2) is redundant and can be neglected, and the result is a four by four matrix equation:

$$\begin{bmatrix} 0 & B_x & -B_y & B_z \\ 0 & S_x & -S_y & S_z \\ Q_2 & -B_1 B_y & -B_1 B_x & B_2 B_3 \\ Q_1 & -S_1 B_y & -S_1 B_x & B_2 S_3 \end{bmatrix} \begin{bmatrix} X_2 \\ X_3 \\ X_4 \\ X_5 \end{bmatrix} = \begin{bmatrix} B_3 \\ S_3 \\ B_z B_2 \\ S_z B_2 \end{bmatrix} . \tag{A3}$$

Next, to produce zeros as the first three elements of the first column of Eq. (A3), multiply the third row by Q_1, multiply the fourth row by $-Q_2$, combine the two, and substitute the result for the third row. Again the last row can be neglected, and the result is the three by three matrix equation

$$\begin{bmatrix} B_x & -B_y & B_z \\ S_x & -S_y & S_z \\ Q_4 & Q_6 & Q_7 \end{bmatrix} \begin{bmatrix} X_3 \\ X_4 \\ X_5 \end{bmatrix} = \begin{bmatrix} B_3 \\ S_3 \\ Q_3 \end{bmatrix} ,$$

where

$$Q_3 = B_2 (B_z Q_1 - S_z Q_2) ,$$

$$Q_4 = B_y (-B_1 Q_1 + S_1 Q_2) ,$$

$$Q_6 = B_x (-B_1 Q_1 + S_1 Q_2) ,$$

and

$$Q_7 = B_2 (B_3 Q_1 - S_3 Q_2) .$$

Following the above procedure, one can again generate zeros on the first two elements in the first row of Eq. (A4), thus yielding the two by two matrix equation

$$\begin{bmatrix} Q_8 & Q_9 \\ Q_{10} & Q_{11} \end{bmatrix} \begin{bmatrix} X_4 \\ X_5 \end{bmatrix} = \begin{bmatrix} Q_5 \\ Q_{12} \end{bmatrix} \tag{A5}$$

where

$$Q_5 = B_3 S_x - B_x S_3 ,$$

$$Q_8 = -B_y S_x + S_y B_x ,$$

$$Q_9 = B_z S_x - S_z B_x ,$$

$$Q_{10} = -S_y Q_4 - S_x Q_6 ,$$

$$Q_{11} = S_z Q_4 - S_x Q_7 ,$$

and

$$Q_{12} = S_3 Q_4 - S_x Q_3 .$$

Equation (A5) then yields

$$X_5 = \frac{Q_5 Q_{10} - Q_8 Q_{12}}{Q_9 Q_{10} - Q_8 Q_{11}} \tag{A6}$$

and

$$X_4 = \frac{Q_{12} - Q_{11} X_5}{Q_{10}} . \tag{A7}$$

From the third row of Eq. (A4),

$$X_3 = \frac{Q_3 - Q_6 X_4 - Q_7 X_5}{Q_4} . \tag{A8}$$

From the fourth row of Eq. (A3),

$$X_2 = \frac{S_z B_2 + S_1 B_y X_3 + S_1 B_x X_4 - B_2 S_3 X_5}{Q_1} , \tag{A9}$$

and from the fifth row of Eq. (A2)

$$X_1 = \frac{B_1 X_2 - B_y X_3 - B_x X_4}{B_2} . \tag{A10}$$

Since $X_5 = \cos\theta$, then $\csc\theta$ can easily be found and the angular rocket coordinates determined. Then ϕ, ψ, and θ can be substituted into Eqs. (5), (6) and (7) of the main text to find the experiment orientation.

A special case of this method must be considered. If $\theta = 0$, then $\csc\theta$ becomes infinite and the above solutions are not correct. When this case occurs, the experimental zenith angle is 90° and the azimuth angle consists of a rotation about the x axis. The transformation matrix in this case becomes

$$\tilde{A} = \begin{bmatrix} \cos\phi_E & \sin\phi_E & 0 \\ -\sin\phi_E & \cos\phi_E & 0 \\ 0 & 0 & 1 \end{bmatrix} , \tag{A11}$$

where ϕ_E is the experimental azimuth measured from the x axis. Equation (9), in component form, yields

$$B_x = B_1 \cos\phi_E + B_2 \sin\phi_E , \tag{A12}$$

$$B_y = -B_1 \sin\phi_E + B_2 \cos\phi_E , \tag{A13}$$

and

$$B_z = B_3 . \tag{A14}$$

Solving these for $\cos\phi_E$ and $\sin\phi_E$ gives

$$\cos\phi_E = \frac{B_x B_1 + B_y B_2}{B_1^2 + B_2^2} \tag{A15}$$

and

$$\sin\phi_E = \frac{B_2 \cos\phi_E - B_y}{B_1} . \tag{A16}$$

Note, however, if $B_1 = 0$, then Eqs. (A12), (A13), and (A14) reduce to

$$\cos\,\phi_E = \frac{B_y}{B_2} \tag{A17}$$

and

$$\sin\,\phi_E = \frac{B_x}{B_2}. \tag{A18}$$

APPENDIX B

DETERMINATION OF MAGNETIC FIELD COMPONENTS USING LESS THAN THREE MAGNETOMETERS

Specific (although often-occurring) problems with respect to complete magnetometer and sun sensor coverage are considered in this appendix.

The magnetometer coverage is frequently limited to two perpendicular sensors with one aligned along the rocket axis and the other in the lateral plane. The magnetic field component, say B_2, along the third mutually perpendicular axis can be found from the fact that the scalar product of the magnetic field and solar vectors is a constant. Then with $\mathbf{S}$, B_1, and B_3 given,

$$B_2 = \frac{\mathbf{B} \cdot \mathbf{S} - B_1 S_1 - B_3 S_3}{S_2} . \tag{B1}$$

$\mathbf{B} \cdot \mathbf{S}$ can be determined by knowledge of the external field. Obviously Eq. (B1) does not hold if $S_2 = 0$. Then

$$B_2 = \pm \sqrt{B^2 - B_1^2 - B_3^2} , \tag{B2}$$

where the $\pm$ ambiguity is normally easy to resolve by examining the data. If two of the magnetometers are absent, Eqs. (B1) and (B2) can be combined to produce two possible solutions. One of the solutions often turns out to be obviously false.

It is noteworthy that three mutually perpendicular magnetometers are self-calibrating. However, if less than three are available, one must resort to the use of magnetic field tabulations.*

*J.C. Cain, S. Hendricks, W.E. Daniels, and D.C. Jensen, "Computation of the Main Geometric Field From Spherical Harmonic Expansions," NASA Report X-611-64-316, Oct. 1964.

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13. ABSTRACT

An analytical treatment of data from rocket aspect sensors (such as magnetometers and solar sensors) was developed in terms of Euler angle coordinate systems and transformations, since the Euler angles represent the zenith, azimuth, and spin angular coordinates of the rocket. It was found that the relationships between the sensor outputs and the orientation of the rocket are exceedingly straightforward when considered in terms of these coordinate transformations.

Three cases of aspect sensor combinations were considered; complete magnetometer data were assumed in each of the three cases (except for consideration of failure of one of the three magnetometers) with supplementary information in case 1 from continuous sun sensor coverage, in case 2 from partial sun sensor data, and in case 3 from horizon scanners. The equations are derived for ideal experimental conditions (instrumental errors are neglected).

The solutions can easily be programmed on a computer. A general computer program was not developed, however, because the problems encountered in the reduction of aspect data vary considerably for each rocket flight.

14. KEY WORDS	LINK A		LINK B		LINK C	
	ROLE	WT	ROLE	WT	ROLE	WT
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Orientation						
Determination						
Mathematical analysis						
Euler's angles						
Transformations (mathematics)						
Yaw						
Spin						
Magnetometers						
Sun sensors						
Horizon scanners						