NRL Report 7661

A Success-Run Equation for Detection Applications

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December 21, 1973

NAVAL RESEARCH LABORATORY
Washington, D.C.

SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered)

REPORT DOCUMENTATION PAGE

READ INSTRUCTIONS BEFORE COMPLETING FORM

1. REPORT NUMBER: NRL Report 7661	2. GOVT ACCESSION NO.
3. RECIPIENT'S CATALOG NUMBER	
4. TITLE (and Subtitle): A SUCCESS-RUN EQUATION FOR DETECTION APPLICATIONS	5. TYPE OF REPORT & PERIOD COVERED: An interim report on a continuing NRL problem.
	6. PERFORMING ORG. REPORT NUMBER
7. AUTHOR(s): M. Potosky	8. CONTRACT OR GRANT NUMBER(s)
9. PERFORMING ORGANIZATION NAME AND ADDRESS: Naval Research Laboratory, Washington, D. C. 20375	10. PROGRAM ELEMENT, PROJECT, TASK AREA & WORK UNIT NUMBERS: NRL Problem D01-05, XF 11-111-402
11. CONTROLLING OFFICE NAME AND ADDRESS: Department of the Navy, Naval Electronic Systems Command, Washington, D. C. 20360	12. REPORT DATE: December 21, 1973
	13. NUMBER OF PAGES: 24
14. MONITORING AGENCY NAME & ADDRESS(if different from Controlling Office)	15. SECURITY CLASS. (of this report): Unclassified
	15a. DECLASSIFICATION/DOWNGRADING SCHEDULE

16. DISTRIBUTION STATEMENT (of this Report)

Approved for public release; distribution unlimited.

17. DISTRIBUTION STATEMENT (of the abstract entered in Block 20, if different from Report)

18. SUPPLEMENTARY NOTES

19. KEY WORDS (Continue on reverse side if necessary and identify by block number)

Detection
Success runs
Markov trials
Reliability

20. ABSTRACT (Continue on reverse side if necessary and identify by block number)

An exact expression for the probability of success runs with Bernoulli trials has been developed which is equivalent to Feller's approximation but does not have the constraints applicable to the approximate solution. Examples of applying the success-run expression are estimation of the design requirements for detection systems whose performance is specified by the probability of obtaining a sequence of consecutive detections over an observation span, computation of the performance of multiline barriers for obtaining the direction of

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target transit, computation of the reliability for barriers or parallel systems of any size, and classification of sources as targets and nontargets in regions of high false-alarm backgrounds. The results also extend to include Markov-dependent Bernoulli trials, applicable to detection trials in which alerting occurs.

CONTENTS

A SUCCESS-RUN EQUATION FOR DETECTION APPLICATIONS

INTRODUCTION

The design of underwater detection and communication systems often involves the structuring of sensor or data-transmission stations in a line or a barrier configuration with the sensors electrically in parallel. In this configuration stations may become inoperative in some random manner, and the probability of survival is usually described mathematically as an exponential function of time. In general the operation of groups of stations is without replacement, so that failures can be tolerated to some predetermined level. In multistation detection barriers, station failures result in a loss of coverage by portions of the barrier. When detection barriers are designed to provide overlapping coverage, the loss of single sensor stations does not significantly affect total system coverage. However the loss of adjacent stations generates gaps in coverage.

The probability that a system is in operation at a given time with no gaps caused by the failure of two or more adjacent stations may be predicted using the success-run theory developed by Feller [1]. Feller's derivation is an approximation subject to certain constraints on the probability of station failure and the number of consecutive failures in a line or barrier. In this report an exact expression will be developed for the probability of the occurrence of a success run, removing the constraints applicable to the approximate solution. Several applications of the results to the reliability and performance of undersea surveillance systems will be described. The success-run equation will be extended for Markov-dependent Bernoulli trials. The Markov model is suitable for detection trials in which alerting occurs.

DEFINITIONS OF SUCCESS RUNS

Feller's definition [1] of a success run of length r includes all disjoint runs of that length. For example a single run of length 3 is given by the sequence $FSSSF$ where F denotes a failure and S a success. The sequence $FSSSSSSF$ can be categorized as a single run of length 6, a run of length 5, a run of length 4 followed by one of length 2 or vice versa, as two runs of length 3, or as three runs of length 2. This definition is required by Feller to compute *recurrence* probabilities and parameters from which we can determine the probability that a given sequence will occur some given number of trials after a previous occurrence.

If we are concerned only with the probability that the *first* success run of length r will occur at a given time, or only with the probability that there is *no* success run of length r or greater in a given number of trials, then a success run can be defined as a sequence of successes of length r or greater. It will be demonstrated that this loose definition of a success run leads to the same results as Feller's definition in the computation of success-run probabilities, the mean time to the first success run, and mean recurrence times.

Manuscript submitted September 21, 1973.

FELLER'S SOLUTION

The probability that a success run of length r occurs at the nth trial is u_n, and f_n is the probability that the first run of length r occurs at the nth trial. Here a success run is defined as a run of length exactly r as described in the first paragraph of the preceding section. Following Feller the probability that all the r trials occurring in the interval $n - r + 1$ to n are successful is p^r (p is the probability of success on a single trial and $q = 1 - p$), and it is obvious that a success run can occur at at least one of these trials. Further it can be shown that

$$p^r = u_n + u_{n-1}p + \dots + u_{n-r+1}p^{r-1} \tag{1}$$

for $n - r$ and $u_1 = u_2 = \dots = u_{r-1} = 0$, with $u_0 = 1$. The generating function for u_r is shown [1] to be

$$U(s) = \frac{1 - s + qp^r s^{r+1}}{(1-s)(1-p^r s^r)} . \tag{2}$$

From the above Feller derives the generating function of the recurrence times:

$$F(s) = \frac{1}{1 - U(s)} = \frac{p^r s^r (1-ps)}{1 - s + qp^r s^{r+1}} . \tag{3}$$

Using the method of partial fractions, Feller derives the probability for the first success run at the nth trial as

$$f_n \approx \frac{(x-1)(1-px)}{(r+1-rx)q} \frac{1}{x^{n+1}} , \tag{4}$$

where x is the smallest root of the denominator of Eq. (3). Further the probability of no success runs of length r in n trials is given by

$$q_n = f_{n+1} + f_{n+2} + \dots \tag{5}$$

and is approximated using Eq. (4) by

$$q_n \approx \frac{1 - px}{(r+1-rx)q} \frac{1}{x^{n+1}} . \tag{6}$$

The smallest root is obtained by successive approximations of the solutions to the denominator of Eq. (3),

$$s = 1 + qp^r s^{r+1} , \tag{7}$$

and it can be shown to be

$$x = 1 + qp^2 + (r+1)(qp^r)^2 + \dots . \tag{8}$$

The condition that x is the smallest root is shown by Feller to be subject to the constraint

$$(r+1)q > 1 . \tag{9}$$

GENERAL REMARKS

The development of success-run probabilities by Feller assumes Bernoulli trials and includes an implicit assumption that the counting of trials begins from an initial condition of a failure. That is, the zeroth trial cannot be a success, since the first success run of length r can occur no sooner than the rth.

The development of the exact solution for Bernoulli-trial success runs continues with this assumption, since it is consistent with many models for detection processes and reliability, where the system has no output until the start of the sequence of trials. For some detection processes, where we are concerned with holding a target for a succession of detections, losing the target, and then regaining the target, initial conditions for a sequence of trials include the possibility of success or failure in the zeroth trial.

A generalization of the success-run theory is provided by Thiess [2] in developing his theory of intermittent processes. Thiess assumes that the sequence states are described by a Markov process. This leads to success-run probabilities where successive trials are correlated. As with Feller, Thiess assumes the initial or zeroth state to have no output. This is equivalent to the initial state reporting a failure. Loane, Richardson, and Boylan [3] extend Thiess' work to include the possibility of a success in the initial state for determining the probability of k successes in n trials of a Markov process but do not include the case of a success run of length r or greater in n trials.

Following the development of the recursive equation for independent Bernoulli trials, we will obtain a set of recursive equations for Markov-dependent Bernoulli trials. This solution leads to a generating function for the probability of no success runs of length r or greater in n trials. This generating function reduces, for zero state correlation, to the independent-Bernoulli-trial generating function with a general initial-state constraint.

RECURSIVE DEVELOPMENT

In developing a recursive solution for the probability of no success runs of length r or more in n trials, we consider each trial in a series of trials of length n which has no success run of length r or more to be a Bernoulli trial with probability p of success and probability $q = 1 - p$ of failure. The probability Q_n is defined as the probability that there is no success run of length r or greater during the n trials. A first success run may occur at the $(n+1)$th trial. The result of no success runs at the nth trial may be obtained if there were no success runs in the first $n-1$ trials and the nth trial resulted in a failure; or, if the nth trial resulted in a success, the $(n-1)$th trial resulted in a success, the first $n-3$ trials had no success runs, and the $(n-2)$th trial was a failure, and so on. This procedure is illustrated in Fig. 1. Thus the probability of no success runs of length r (or greater) in n trials is given by

$$Q_n = qQ_{n-1} + qpQ_{n-2} + qp^2Q_{n-3} + \dots + qp^{r-1}Q_{n-r} \tag{10}$$

or

$$Q_n = \sum_{i=1}^{r} qp^{i-1}Q_{n-i} . \tag{11}$$

PATTERN OF TRIALS n − r THROUGH n								PROBABILITY OF NO RUN AT TRIAL K	PRODUCT OF PATTERN
n − r	n − r + 1	n − r + 2	. . .	n − 3	n − 2	n − 1	n		
F or S	F or S	F or S	. . .	F or S	F or S	K	F	Q_{n-1}	qQ_{n-1}
F or S	F or S	F or S	. . .	F or S	K	F	S	Q_{n-2}	qpQ_{n-2}
F or S	F or S	F or S	. . .	K	F	S	S	Q_{n-3}	qp^2Q_{n-3}
F or S	F or S	F or S	. . .	F	S	S	S	Q_{n-4}	qp^3Q_{n-4}
. . .	. . .	. . .	. . .	. . .	. . .	. . .	. . .	. . .	. . .
F or S	K	F	. . .	S	S	S	S	Q_{n-r+1}	$qp^{r-2}Q_{n-r+1}$
K	F	S	. . .	S	S	S	S	Q_{n-r}	$qp^{r-1}Q_{n-r}$

Fig. 1—Recursive development of probability of no success runs *d* length *r* or greater in *n* trials

This equation is the exact equivalent of Feller's equation 7.11. The initial conditions are

$$Q_0 = Q_1 = \ldots = Q_{r-1} = 1; \quad Q_r = 1 - p^r. \tag{12}$$

There are no restrictions, the equation holding for all n, r, p, and $q\,(=1-p)$. For small r Eq. (11) can readily be used with hand calculators. However for large r the number of terms in each calculation can be large. Equation (11) can be used to obtain Feller's equation 7.11 as shown in Appendix A. Using z transforms, Eq. (11) may be solved to provide a closed solution form for Q_n when $r = 2$. This is developed in Appendices A and B.

A simpler recursive form is easily obtained. From Eq. (11) we subtract the equation for Q_{n-1}, giving

$$Q_n - Q_{n-1} = \sum_{i=1}^{r} qp^{i-1}Q_{n-i} - \sum_{i=1}^{r} qp^{i-1}Q_{n-(i+1)}\,. \tag{13}$$

Expanding, we have

$$Q_n - Q_{n-1} = qQ_{n-1} + pqQ_{n-2} + \ldots + qp^{r-1}Q_{n-r} - qQ_{n-2} - \ldots - qp^{i-2}Q_{n-r}\,.$$

Collecting terms,

$$\begin{aligned} Q_n - Q_{n-1} &= qQ_{n-1} - q\,[Q_{n-1} - qp^{r-1}Q_{n-(r+1)}] - qp^{r-1}Q_{n-(r+1)} \\ &= -qp^rQ_{n-(r+1)} \end{aligned}$$

or

$$Q_n = Q_{n-1} - qp^rQ_{n-(r+1)}\,. \tag{14}$$

The probability that there is at least one run of length r or greater is given by

$$P_n = 1 - Q_n = P_{n-1} + qp^r[1 - P_{n-(r+1)}]\,, \tag{15}$$

where

$$P_0 = P_1 = P_2 = \ldots = P_{r-1} = 0; \quad P_r = p^r\,. \tag{16}$$

Equation (15) has been programmed for a Hewlett-Packard 9100B calculator, and some results for various r, n, and p are given in Fig. 2.

The mean time to the first success run is the expected number of trials to achieve a success fun of length exactly r. At that point the count of trials is ended and a new count is begun. This corresponds to Feller's definition of success runs. The mean time to the first success run is shown in Appendix C to be

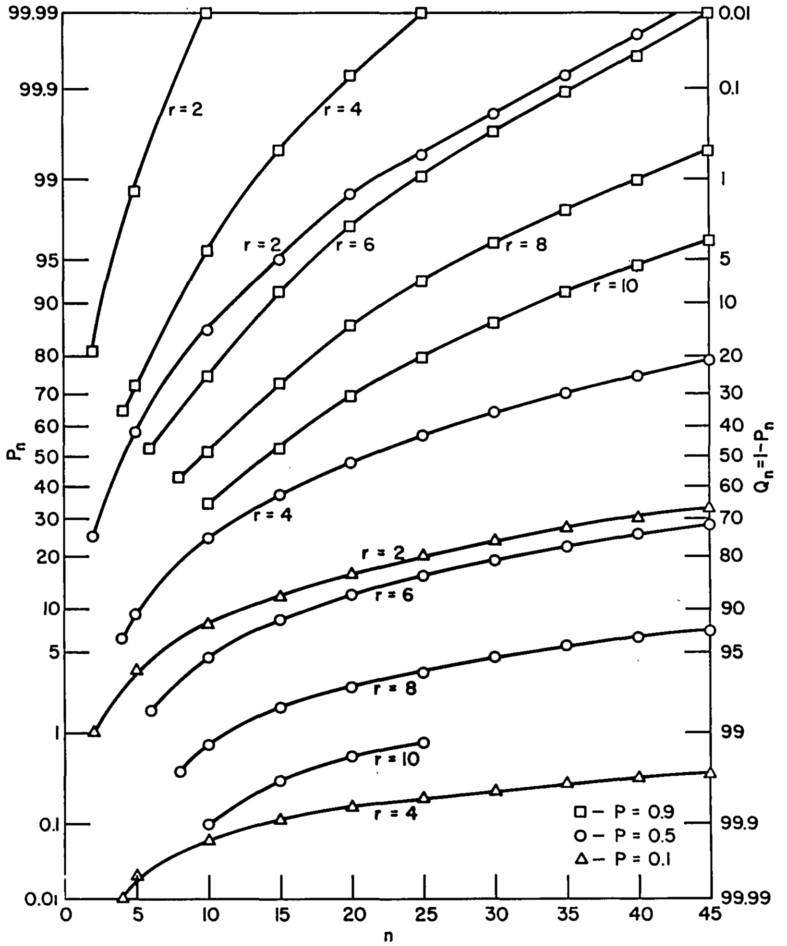
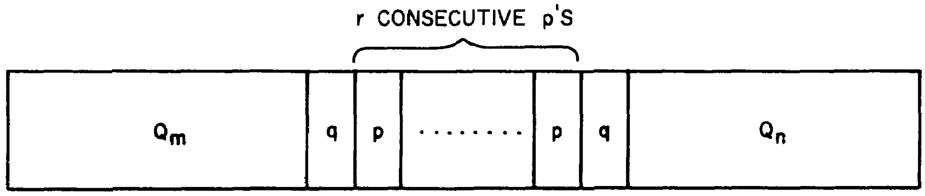


Fig. 2—Probability P_n of at least one success run of length r or greater in n trials with single-trial probability of success p

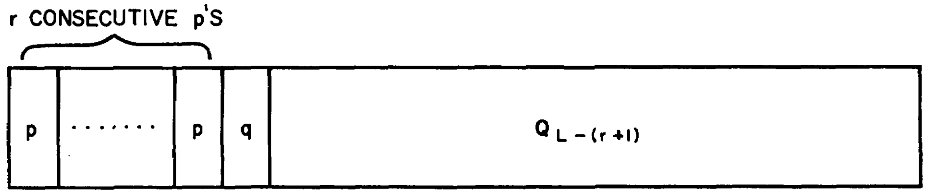
$$T = \frac{1 - p^r}{qp^r} \tag{17}$$

and is equal to the mean recurrent time of success runs.

The probability of exactly one run of length r and no other runs of length r or greater can be simply derived. This probability includes various distributions with runs of length less than r. We assume a distribution of L trials as illustrated in Fig. 3. The distribution of trials in Fig. 3a consists of m trials with no runs of length r or greater, a sequence of trials of length $r + 2$ including a success run of length r bounded by a failure on each side, followed by $L - m - (r + 2)$ trials with no success run of length r or greater. Thus this represents a series of trials of length L with exactly one success run of length r. When the end runs as shown in Fig. 3b are also taken into account, the probability of exactly one run of length r and no other runs of length greater than r in L trials $P_{LE}(1)$ is given by



(a) Control run ($L = m + n + r + 2$)



(b) End run

Fig. 3—Exactly one run of length r in L trials

$$P_{LE}(1) = \sum_{m=0}^{L-(r+2)} q^2 p^r Q_m Q_{L-(r+2)-m} + 2qp^r Q_{L-(r+1)} . \tag{18}$$

MARKOV TRIALS

When the trials in a sequence are correlated, the results of the previous section are modified. We consider the sequence of trials to be Markov dependent; that is, the probabilities for each trial depend only on the outcome of the previous trial [4-6]. This dependence is described by a transition probability matrix,

$$R = \begin{array}{c|cc l} & S & F & \text{trial } (j+1) \\ \hline S & a & \bar{a} & \\ F & b & \bar{b} & \\ \text{trial } j & & & \end{array} , \tag{19}$$

where the rows describe the probability of having a success S or failure F at the $(j + 1)$th trial having had a success or failure in the jth trial. The total probability of obtaining a success at the jth trial is p and the probability of a failure is $q = 1 - p$. It can be shown [3,4] that the correlation coefficient c between the $(j + 1)$th and jth trials is given by

$$c = a - b . \tag{20}$$

As before, we define q_n as the probability that no run of length r or greater has occurred in n trials. Now we define two new probabilities q_n' and q_n'' as [4]

q'_n – the probability that there is no run of length r or greater in n trials with the nth trial ending in a success,

q''_n – the probability that there is no run of length r or greater in n trials with the nth trial ending in a failure.

With these definitions it is clear that

$$q_n = q'_n + q''_n . \tag{21}$$

We further set up a matrix of trials similar to Fig. 1. If the nth trial ends in failure with no success runs, we can write

$$q''_n = \bar{a}q'_{n-1} + \bar{b}q''_{n-1} . \tag{22}$$

If the nth trial ends in a success with no runs, we can then write

$$q'_n = aq'_{n-1} + bq''_{n-1} - ba^{r-1}q''_{n-r} . \tag{23}$$

The last term on the right excludes the possibility that the last $r - 1$ trials leading to the $(n-1)$th trial were all successes followed by the rth consecutive success at the nth trial. This case must be excluded, since the possibility of $r - 1$ consecutive successes is included in the probability q_{n-1}.

Equations (22) and (23) are the recursive equations which must be solved to obtain q_n. Adding (22) and (23), we obtain

$$q_n = q_{n-1} - ba^{r-1}q''_{n-r} . \tag{24}$$

We must now establish the initial conditions. Two cases occur, resulting in different values of q'_n and q''_n for $n > r$. The first case assumes that the first trial in the sequence is the first trial to have occurred (there is no zeroth trial). Thus the probability of a success at the first trial is p and the probability of a failure is q. The probability of a success at the second trial is

$$p_2 = ap + bq = p \tag{25a}$$

and the probability of a failure is

$$q_2 = \bar{a}p + \bar{b}q = q . \tag{25b}$$

Furthermore the probability of no success run at the $n(<r)$th trial is p if the nth trial ends in a success and is q if the nth trial ends in a failure, since Eqs. (25a) and (25b) are identical to (22) and (23) for $n < r$. Thus we state that

$$q'_n = p , \quad n < r , \tag{26a}$$

$$q''_n = q , \quad n < r . \tag{26b}$$

Since $q_{n-r} = q''_0$ for $n = r$, we must arbitrarily assign a value to q''_0. This is given as $q''_0 = p/b$. This assumption is equivalent to the first trial having a probability of success = p

instead of b, since there is no zeroth trial. Thus Eq. (23) for $n = r$ becomes $q'_r = aq'_{r-1} + bq''_{r-1} - pa^{r-1}$. Under this assumption it can be shown that for zero correlation the Markov success run reduces to the Feller success run as developed earlier (Eq. (14)).

The second case of interest is when the sequence of n trials starts at some arbitrary point in a longer sequence of trials. In this case the zeroth trial ends in either a success or a failure. Consider that the zeroth trial has ended in a success and that the zeroth trial success does not count toward a run. Then for the first trial we have the conditional probabilities

$$q'_1 = a , \tag{27a}$$

$$q''_1 = \bar{a} . \tag{27b}$$

Using Eqs. (22) and (23) we obtain

$$q'_2 = aq'_1 + bq''_1 , \tag{28a}$$

$$q''_2 = \bar{a}q'_1 + \bar{b}q''_1 . \tag{28b}$$

Continuing and substituting Eq. (20), we obtain

$$q'_n = p + qc^n , \quad n < r \text{ and } q_0 = q'_0 , \tag{29a}$$

$$q''_n = q - qc^n , \quad n < r \text{ and } q_0 = q'_0 . \tag{29b}$$

When the zeroth trial ends in a failure, the conditional probabilities for the first trial become

$$q'_1 = b , \tag{30a}$$

$$q''_1 = \bar{b} . \tag{30b}$$

Continuing as before, we obtain

$$q'_n = p - pc^n , \quad n < r \text{ and } q_0 = q''_0 , \tag{31a}$$

$$q''_n = q + pc^n , \quad n < r \text{ and } q_0 = q''_0 . \tag{31b}$$

The generating functions for the probability of no success runs in Markov trials is obtained as in the case of Bernoulli trials (Appendix A). The generating functions are determined for $f'(z)$ and $f''(z)$, corresponding to q'_n and q''_n. The partial functions are summed to provide the conditional functions for the case of a success in the zeroth trial,

$$F(z)|(q_0 = q'_0) = \frac{1 - cz}{(1-z)(1-cz) + \bar{a}ba^{r-1}z^{r+1}} , \tag{32}$$

and for the case of a failure in the zeroth trial,

$$F(z) \mid (q_0 = q_0'') = \frac{1\ cz - ba^{r-1}z^r}{(1-z)(1-cz) + b\bar{a}a^{r-1}z^{r+1}} . \tag{33}$$

The complete generating function is

$$F(z) = p[F(z) \mid (q_0 = q_0')] + q[F(z) \mid (q_0 = q_0'')] . \tag{34}$$

Substitution of Eqs. (32) and (33) into (34) yields

$$F_1(z) = \frac{1 - cz - qba^{r-1}z^r}{(1-z)(1-cz) + \bar{a}ba^{r-1}z^{r+1}} . \tag{35}$$

The generating function ignoring the zeroth trial is obtained from Eqs. (22), (23), and (26) and results in

$$F_2(z) = \frac{(1-cz)(1-pa^{r-1}z^r) - qba^{r-1}cz^{r+1}}{(1-z)(1-cz) + \bar{a}ba^{r-1}z^{r+1}} . \tag{36}$$

The mean time to the first success run of length r is obtained from $F(z)$ by setting $z = 1$. Thus we have for the general case, from Eq. (35),

$$T_1 = F_1(1) = \frac{1 - c - qba^{r-1}}{\bar{a}ba^{r-1}} . \tag{37}$$

Since it can be shown that

$$a = p + cq ,$$

$$\bar{a} = q - cq ,$$

$$b = p - pc ,$$

$$\bar{b} = q + qc ,$$

we have

$$T_1 = \frac{1 - pq(p+cq)^{r-1}}{pq(1-c)(p+cq)^{r-1}} . \tag{38}$$

From Eq. (36), ignoring the zeroth trial, we obtain a mean time to a success run of

$$T_2 = F_2(1) = \frac{1 - p(p+cq)^{r-1}(1+qc)}{pq(1-c)(p+cq)^{r-1}} . \tag{39}$$

The difference in mean time to success runs between the choice of initial conditions is

$$T_1 - T_2 = \Delta T = \frac{p + qc}{q - qc} . \tag{40}$$

Thus ignoring the zeroth trial provides a shorter mean time to first success run than assuming that the zeroth trial can be a success or failure.

When the correlation coefficient $c = 0$, we have the Bernoulli-trial case. The mean time T_2 ignoring the zeroth trial becomes

$$T_2 = \frac{1 - p^r}{qp^r}, \tag{41}$$

and the mean time T_1 assuming a random zeroth trial becomes

$$T_1 = \frac{1 - qp^r}{qp^r}. \tag{42}$$

The first case is equivalent to the Bernoulli-trial solution given earlier and by Feller. However the solution in Eq. (42) is often more useful, since it models a real-world detection system operating prior to the arbitrary start of a trial count.

APPLICATIONS USING BERNOULLI TRIALS

Detection Runs

The success-run equation can be used to estimate design requirements for detection systems whose performance is specified by the probability of obtaining a sequence of consecutive detections over an observation span. Assume that the single-look probability of detection is $p = 0.9$ and that we require a run of at least r detections out of n looks to occur with a probability P. It is assumed that the range of the sonar is 200 n.mi., the target speed is 10 knots, the target is in view of the sonar for 100 n.mi., and the single-look duration is 10 minutes. This will result in 60 looks. Thus Eq. (15) becomes

$$P_n = P_{60} = P_{59} + (0.9)^r(0.1)(1 - P_{59-r}). \tag{43}$$

Plots of P_n vs r and p are shown in Fig. 4 for $n = 60$ and $n = 30$. A run-success probability of 0.90 is obtained for a longest run of at least 13 detections when $n = 60$ and $p = 0.9$ and for a longest run of at least 9 detections when $n = 30$ and $p = 0.9$.

Distributed Barriers

In detecting targets transiting multiline barriers the direction of transit can be obtained if at least two barrier lines detect the target. The tracking performance can be computed using the standard binomial expansion to give the probability of making at least two detections out of m attempts. For this model it is assumed that there are m barriers and a target will be detected once at each barrier with a probability p. Thus we have

$$P_2 = 1 - q^m - mpq^{m-1}. \tag{44}$$

If we introduce a further constraint and assume that tracking is achieved only when two *consecutive* barrier lines detect the target, we obtain

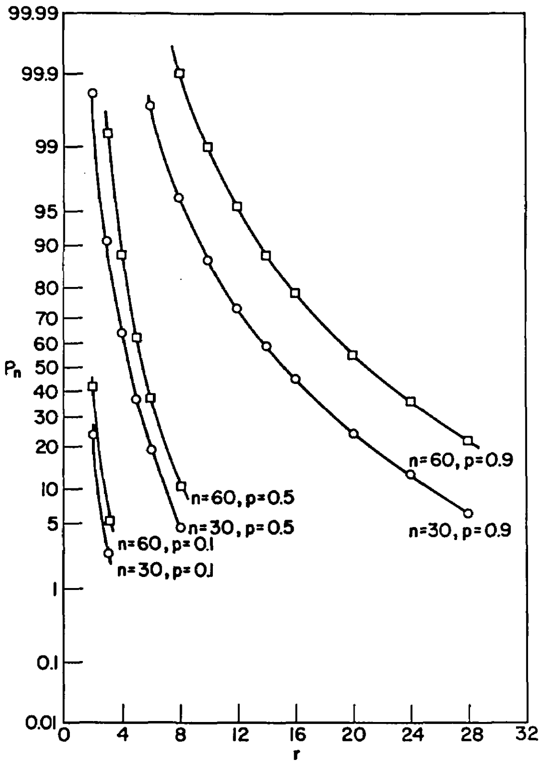


Fig. 4—Probability in n trials, P_n of at least one success run of length r or greater

$$P_n = P_{n-1} + qp^2(1 - P_{n-3}) . \tag{45}$$

If we set $p = 0.5$, then a value of $P_n = 0.90$ is obtained when $n = 12$. For the less restrictive binomial tracking, $n = 7$.

Reliability

A solution was developed to compute the probability that there would be no adjacent failures in a distributed barrier of sensors. All sensors are connected in parallel, so that a failure does not affect other sensors. Each sensor fails according to the exponential law $1 - e^{-t/T}$, where T is the mean time between failures (MTBF), in standard reliability theory terminology. A "success" run is now defined as a run of failures. Equation (15) holds directly. A typical system might consist of 25 sensors with a sensor MTBF of 10 years. We define a system failure to have occurred when two adjacent sensors fail. We would like to know the probability of system failure after 1 year. Thus

$$P_{25} = P_{24} + (1 - e^{-0.1})^2(e^{0.1})(1 - P_{22}) . \tag{46}$$

Now $p = 1 - e^{-0.1} = 0.095$, and we find that $P_{25} = 0.82$, indicating a high probability of system failure even though the sensor probability of failure is small. Reliability performances can readily be computed for other MTBF values and for barriers or parallel systems of any size.

Classification

The results of the success-run theory can be used to provide a method of classifying targets (distinguishing targets from nontarget sources) in regions of high false-alarm backgrounds. For example, consider that the single-look probability of detection is 0.90 and that a system looks at a target for $n = 30$ and $n = 60$ independent trials. From Fig. 4 we tabulate (Table 1) the probability P_n of a success run of length at least r. For each success run against a target we show the probability of false-alarm runs in which the single-look probabilities of false alarm are 0.01 and 0.1.

If we consider a classification criterion based on the probability of detecting a success run of length at least r in n trials, we will obtain a set of results as will be exemplified below. Let us set the single-trial probability detection p to be 0.9, the single-trial probability of false alarm p_{fa} to be 0.1, and set $n = 30$ and $r = 6$. Then we will obtain the success run $P_D = 0.9968$ and $P_{fa} = 2 \times 10^{-5}$. The mean time in detecting a target, to the first success run of 6 will be 9 trials and the mean recurrence time will be 9 trials, so that we may expect an average of 3 success runs in 30 trials. Furthermore the mean time to the first false alarm is 10^6 trials.

The effect of a success-run criterion is equivalent to extending the integration time of a processor operating on a continuous passive signal. Extending the integration time by n looks provides a processing gain of 5 log n. In the example cited this is equivalent to a gain of 7.5 dB. The probabilities of detection and false alarm, using the success-run criterion of $r = 6$, are equivalent to a signal-to-noise-ratio increase of 7.0 dB above that required for the single-look probabilities. In addition, since the expected number of looks to arrive at a "detection" decision is 9, the use of success runs may fit into a sequential detection process.

Markov-Dependent Trials

Application of the Markov-dependent Eqs. (22) and (23) to detection problems will be described in a subsequent report.

ACKNOWLEDGMENT

The program for calculating the success run probability was written by C. R. Rollins. Appreciation is extended to Dr. J. C. Knight for reviewing the manuscript and offering many useful suggestions.

Table 1
Probability P_n of a Success Run of Length r for Targets and False Alarms

N	p	P_n					
		$r = 2$	$r = 4$	$r = 6$	$r = 8$	$r = 10$	$r = 12$
30	0.9 (target)	1.0000	1.0000	0.9968	0.9630	0.8697	0.7310
30	0.01 (false alarm)	0.0029	2.57×10^{-7}	$<10^{-9}$	0	0	0
30	0.1 (false alarm)	0.2350	2.44×10^{-3}	2.26×10^{-5}	2.08×10^{-7}	1.90×10^{-9}	--
60	0.9 (target)	1.0000	1.0000	1.0000	0.9993	0.9901	0.9545
60	0.01 (false alarm)	0.0058	5.64×10^{-7}	$<10^{-9}$	0	0	0
60	0.1 (false alarm)	0.4197	5.13×10^{-3}	4.96×10^{-5}	4.78×10^{-7}	4.60×10^{-9}	--

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Appendix A

EQUIVALENCE OF EQ. (11) AND FELLER'S EQUATION 7.11

The probability of no runs of length r or greater in n independent trials is, from Eq. (11),

$$Q_n = q \sum_{i=1}^{r} p^{i-1} Q_{n-i} . \tag{A1}$$

If we assume the n independent trials to be a discrete process, then we can obtain the transform function $F(z)$ from

$$F(z) = \sum_{n=0}^{\infty} Q_n z^{-n} . \tag{A2}$$

Equation (A1) can be rewritten as

$$Q_{n+r} = q \sum_{i=1}^{r} p^{i-1} Q_{n+r-i} . \tag{A3}$$

The transform function for Q_{n+r} is*

$$G(z) = z^{-r}[F(z) - Q_0 - Q_1 z - \ldots - Q_{r-1} z^{r-1}] . \tag{A4}$$

Substituting this into Eq. (A3), we have

$$z^{-r}[F(z) - Q_0 - Q_1 z - \ldots - Q_{r-1} z^{r-1}]$$

$$= q \sum_{i=1}^{r} p^{i-1} z^{-(r-i)} [F(z) - Q_0 - Q_1 z - \ldots - Q_{r-1-i} z^{r-1-i}] . \tag{A5}$$

Expanding, collecting terms, and solving for $F(z)$, we obtain

$$F(z) = \frac{Q_0 + (Q_1 - Q_0 q)z + \ldots + (Q_{r-1} - Q_{r-2}q - \ldots - Q_0 q^{r-2}) z^{r-1}}{1 - qz - qpz^2 - qp^2 z^3 - \ldots - qp^{r-1} z^r} . \tag{A6}$$

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*DeRusso, Roy, and Close "State Variables for Engineers," Wiley, New York, 1965.

Now $Q_0 = Q_1 = \ldots = Q_{r-1} = 1$, and Eq. (A6) reduces to

$$F(z) = \frac{1 + \rho z + pz^2 + \ldots + \rho^{r-1}z^{r-1}}{1 - qz - qpz^2 - \ldots - qp^{r-1}z^r}$$

$$= \frac{1 - \rho^r z^r}{(1-\rho z)(1 - qz - qpz^2 - \ldots - qp^{r-1}z^r)}. \tag{A7}$$

This results in the transform for the probability of no success runs of length r in N trials:

$$F(z) = \frac{1 - p^r z^r}{1 - z + qp^r z^{r+1}}. \tag{A8}$$

Using the solution method developed in Feller, we can obtain the probability Q_n from

$$Q_n \approx \frac{\rho_1}{z_1^{n+1}} \tag{A9}$$

where ρ_1 is the coefficient of the $(n+1)$th term in the expansion of $F(z)$ or

$$\rho_1 = -\frac{U(z_1)}{V'(z_1)} \tag{A10}$$

and z_1 is the smallest root of $V(z_1) = 0$. Solving (A10) and substituting in (A9), we obtain

$$\rho_1 = \frac{p^r {z_1}^r - 1}{-1 + (r+1)qp^r {z_1}^r}, \tag{A11}$$

$$qp^r {z_1}^r = \frac{z_1 - 1}{z_1}, \tag{A12}$$

and

$$Q_n \approx \frac{\dfrac{z_1 - 1}{z_1 q} - 1}{-1 + (r+1)q\left(\dfrac{z_1 - 1}{z_1 q}\right)} \frac{1}{z_1^{n+1}} = \frac{1 - pz_1}{(r + 1 - rz_1)q} \frac{1}{z_1^{n+1}}, \tag{A13}$$

which is Feller's equation 7.11.

Appendix B

EXACT SOLUTION FOR PROBABILITY OF NO SUCCESS RUNS OF LENGTH $r = 2$

The transform for the case $r = 2$ is, from Eq. (A8),

$$F(z) = \frac{1 - p^2z^2}{1 - z + qp^2z^3} = \frac{1 + pz}{1 - qz - qpz^2}. \tag{B1}$$

Factoring the denominator, we obtain

$$F(z) = \frac{1 + pz}{\left(1 + \dfrac{2p}{1 - \sqrt{\cdot}\, z}\right)\left(1 + \dfrac{2p}{1 + \sqrt{\cdot}\, z}\right)}, \tag{B2}$$

where

$$\sqrt{\cdot} = \sqrt{1 + \frac{4p}{q}}.$$

This is simplified to

$$F(z) = \frac{A}{1 + \dfrac{2pz}{1 - \sqrt{\cdot}}} + \frac{B}{1 + \dfrac{2pz}{1 + \sqrt{\cdot}}} = \frac{A}{1 + \dfrac{z}{z_1}} + \frac{B}{1 + \dfrac{z}{z_2}}, \tag{B3}$$

where

$$B = \frac{p}{q\sqrt{\cdot}}\left(\frac{1 - \sqrt{\cdot}}{1 + \sqrt{\cdot}}\right),$$

$$A = 1 - B.$$

The probability of no success runs of length $r = 2$ or greater in n trials is computed from the nth coefficient of the expansion of $F(z)$ with $z = -z_1$ and $z = -z_Z$. This results in

$$Q_n = \left[1 + \frac{p}{q\sqrt{\cdot}}\left(\frac{\sqrt{\cdot} - 1}{\sqrt{\cdot} + 1}\right)\right]\left(\frac{2p}{\sqrt{\cdot} - 1}\right)^n - \frac{p}{q\sqrt{\cdot}}\left(\frac{\sqrt{\cdot} - 1}{\sqrt{\cdot} + 1}\right)\left(\frac{-2p}{\sqrt{\cdot} + 1}\right)^n. \tag{B4}$$

This can be slightly simplified to the form

$$Q_n = \left(\frac{2p}{\sqrt{\cdot} - 1}\right)^n + \frac{p}{q\sqrt{\cdot}}\left(\frac{\sqrt{\cdot} - 1}{\sqrt{\cdot} + 1}\right)\left(\frac{q}{2}\right)^n [(\sqrt{\cdot} + 1)^n - (-1)^n(\sqrt{\cdot} - 1)^n], \tag{B5}$$

with $Q_0 = Q_1 = 1$.

The probability of one or more success runs of length 2 or greater is

$$P_n = 1 - Q_n . \tag{B6}$$

Appendix C

MEAN TIME TO THE FIRST SUCCESS RUN

The mean time to the first success run of length r in n independent trials is evaluated from $F(z)$, the generation function of the probability of no success runs of length r. The mean time is the mean number of trials to the first success run.

We have defined Q_n as the probability that there are no success runs of length r in n trials. This also means that Q_n is the probability that the first success run of length r will occur at least at the $(n+1)$th trial. Thus, if f_n is the probability of the first success run at the nth trial, we have

$$Q_n = f_{n+1} + f_{n+2} + \cdots \qquad \text{(C1)}$$

The mean time to the first success run is given by

$$T = \sum_{k=0}^{\infty} kf_k \,, \qquad \text{(C2)}$$

where f_k is the probability of the first success run at the kth trial. It can be shown,* using equation (C1), that

$$T = \sum_{k=0}^{\infty} kf_k = \sum_{k=0}^{\infty} Q_k \,. \qquad \text{(C3)}$$

Since the generating function for Q_n is, using Eqs. (A2) and (A8) of Appendix A,

$$F(z) = \sum_{n=0}^{\infty} Q_n z^{-n} = \frac{1 - p^r z^r}{1 - z + qp^r z^{r+1}} \,, \qquad \text{(C4)}$$

we have, setting $z = 1$,

$$F(1) = \sum_{n=0}^{\infty} Q_n = T \,. \qquad \text{(C5)}$$

Thus

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*W. Feller, *An Introduction to Probability Theory and Its Applications,* Vol. 1, 2nd edition, Wiley, New York 1957.

$$T = \frac{1 - p^r}{qp^r}. \tag{C6}$$

The generating function of the probability f_n that the first success run occurs at the nth trial is

$$H(z) = \sum_{n=1}^{\infty} f_n z^{-n}. \tag{C7}$$

As developed by Feller the generating functions (C7) and (C4) are related by the expression

$$H(z) = 1 - (1-z)F(z) = \frac{(1-pz)p^r z^r}{1 - z + qp^r z^{r+1}}. \tag{C8}$$

Differentiating $H(z)$ results in

$$\frac{dH(z)}{dz} = (1-z)\frac{dF(z)}{dz} + F(z). \tag{C9}$$

Setting $z = 1$, we have

$$\left.\frac{dH(z)}{dz}\right|_{z=1} = F(z)\Big|_{z=1} = T. \tag{C10}$$

Thus the mean time to the first success run can be developed from the generating functions of either the probability of the first success run or the probability of no success run. The mean time to the first success run is the same as Feller's mean of the recurrence times of runs of length r (Feller's equation 7.7).

Typical results are shown in Table C1 for different values of p, the single-look probability of detection.

Table C1
Mean Number T of Independent Trials to First Success Run of Length r

r	T		
	$p = 0.9$	$p = 0.5$	$p = 0.1$
1	1.11	2	10
2	2.34	6	110
3	3.72	14	1110
4	5.22	30	111,110
6	8.85	126	1,111,110

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