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Internal Wave Generation Caused by the Growth and Collapse of a Mixed Region

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CONTENTS

ABSTRACT

We formulate a mathematical model for the mean disturbance created in a linearly stratified fluid by the growth and collapse of a localized region of turbulent, mixed fluid. Linearized differential equations for the disturbance are obtained from the Navier-Stokes equations under the Boussinesq approximation after various intuitive and empirical considerations. The divergence of the mean turbulent density flux enters these equations as the sole forcing function for the disturbance; this differs from previous theoretical models in which an initial, perturbed density field acts as the forcing function.

We replace the original expression we find for the forcing function by an approximation devised to simplify the numerical evaluation of the disturbance. We then obtain a solution for the disturbance in a fluid of constant depth and compare the theoretical results with measured data. The comparison indicates that the theory provides satisfactory results at large distances from the mixed region but has deficiencies near the mixed region. Specific proposals are made for correcting the deficiencies of the model at small distances from the mixed region.

PROBLEM STATUS

This is a final report on one phase of a continuing NRL problem.

AUTHORIZATION

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SYMBOLS

A	constant of proportionality
a_n	auxiliary quantity
b_n	auxiliary quantity
C_n	auxiliary quantity
D	total depth of fluid
$\mathfrak{D}$	Fourier's transform of $-\nabla \cdot \mathbf{D}$
dT^*/dZ	temperature gradient prior to mixing
$\mathbf{D}$	ensemble average density flux due to turbulent motion, $\langle \rho' \mathbf{u}' \rangle$
F_n	auxiliary quantity
g, $\mathbf{g}$	acceleration of gravity
$H(x)$	Heaviside unit step function
i	$\sqrt{-1}$
I_n	auxiliary quantity
2ℓ	dimension of square used to represent mixed region
N	Brunt-Väisälä frequency
p	pressure
p'	fluctuating component of pressure
P	ensemble average pressure
P^*	pressure in undisturbed fluid
$r(t)$	radius of mixed region
r_0	effective radius of mixer
R	Reynolds stresses, $-\rho_0 \langle \mathbf{u}'\mathbf{u}' \rangle$
S	viscous stress tensor
T	ensemble average temperature
T^*	temperature prior to mixing
t	time
$t_b(r)$	time at which mixed region has radius r
$t_b(x,y)$	time for mixed region to reach point (x,y)
t_{BV}	Brunt-Väisälä period
t_e	excitation time; interval during which mixer agitates the fluid
t_m	duration of mixing phase
t_0	reference time, virtual origin of mixed region
$\mathbf{u}$	fluid velocity
$\mathbf{u}'$	fluctuating component of velocity
$\mathbf{v}$	ensemble average velocity
w	vertical mean velocity component
$\widetilde{w}$	Fourier transform of w
w_-	vertical velocity component associated with frequencies less than N
w_+	vertical velocity component associated with frequencies greater than N
x	horizontal coordinate
y	vertical coordinate referred to center of mixed region, $z - z_0$
z	vertical coordinate

z_0	depth to center of mixer
α_n	auxiliary quantity
β	degree of mixing
β_n	auxiliary quantity
ΔH	average temperature duration
ΔT	ensemble average temperature change
$\delta(x)$	Dirac delta function
$\delta\rho$	density change produced by mixing
η	particle displacement in vertical direction caused by mean motion
μ	coefficient of viscosity
υ	kinematic viscosity
ρ	density
ρ_0	mean density of the fluid
$\langle\rho\rangle$	ensemble average density
ρ'	fluctuating component of density
ρ^*	density prior to mixing
σ	rms temperature deviation from the average
ϕ_n	set of orthogonal functions
ω	frequency, rad/s

INTERNAL WAVE GENERATION CAUSED BY THE GROWTH AND COLLAPSE OF A MIXED REGION

INTRODUCTION

The phenomenon of internal wave generation by the turbulent wake of a self-propelled body moving through a density-stratified fluid was first demonstrated by Schooley and Stewart (1) in 1963. Their procedure involved an experimental study of the disturbance produced by a small-scale, self-propelled body in a laboratory tank under carefully controlled conditions.

Schooley and Stewart noted that the horizontal and vertical dimensions of the turbulent wake initially increased, but eventually the vertical dimension reached a maximum and afterward decreased in value. They attributed this collapse of the wake to the return of mixed fluid in the wake to an appropriate equilibrium level under the action of gravity. They also argued that for bodies moving at moderate speeds the collapse process should be approximately a two-dimensional process. This concept of collapse and the resulting internal wave generation as essentially a two-dimensional process led to further experimental work (2-6) in which the mechanism of inducing turbulence was by means of various cylindrical mixers, chosen so as to produce motions whose average characteristics could be regarded as two dimensional.

These later experiments fall into two distinct categories. The first category is represented by Wu's experiment (2), in which a portion of the fluid is kept separate from the surrounding fluid by a rigid partition as it is being mixed; once turbulence has subsided in the mixed fluid the rigid partition is removed and collapse ensues. Mei (7) presented a theoretical analysis of the collapse phase, and Wessel (8) developed a numerical method for studying the process based on the Navier-Stokes equations.

The second category provides a more realistic representation of the interaction between the turbulent wake of a self-propelled body and the ambient stratified fluid than does the first category; it consists of those experiments in which the mixed, turbulent fluid is always in contact with, and interacting with, the surrounding unmixed fluid. Most experimental investigations of two-dimensional collapse fall into this category (see Refs. 3-6). A characteristic feature of these experiments, not present in Wu's experiment, is that the mixed region initially expands in extent until some characteristic time is reached when collapse of the region takes place.

Recently, Schooley and Hughes (9) developed a theoretical model for wave generation by collapse and made some comparison with the results of an experiment of the second category. They found rough agreement with measurements made at large distances from the mixer; however, they took their origin of time as arbitrary and determined it by requiring as good a match as possible between their results and the measured data. An undesirable feature of their model, which explains their need for an arbitrary time origin, is the neglect of the initial growth stage of the mixed region and its role in the wave generation process. They assume, in fact, that wave generation begins essentially at the same time as does collapse of the mixed region. This assumption is not in accord with the experimental results of Williams (6) who made measurements near the mixer and found that wave generation begins as soon as the mixer is activated.

In this report we develop a theoretical model for internal wave generation which takes approximate account of the entire history of the mixed region. We do this by specifying on empirical and intuitive grounds only those properties of the mixed region which we feel are important for the process of wave generation. Several different models could be developed by this procedure, and we have selected one of the simplest as a candidate for study.

We base our model on a linearized version of the Navier-Stokes equations in the form applicable to the ensemble average motion. The linearization is motivated by practical considerations rather than by experimental evidence: Linearizing the Navier-Stokes equations produces a system of equations amenable to mathematical analysis whose applicability to the physical process of interest can then be examined.

Our linearized equations contain the mean turbulent density flux and the Reynolds stresses as forcing terms for the disturbance. None of these quantities have been measured in the experiments cited above, and little appears to be known about their behavior in a stratified fluid. On intuitive grounds, we take the mean turbulent density flux as the primary forcing term and obtain the information about this flux required to complete the mathematical model through empirical considerations. We then solve the differential equations of the model for the particular case of a linearly stratified fluid of constant depth. The analytical solution is compared to experimental results of Schooley and Hughes (9) which apply to points at some distance from the mixer. We also compare the theory with measurements made by Williams (6) near the mixed region. In this case, interpretation of the comparison is difficult because experimental conditions do not agree with requirements of the theory. We do find that similar trends occur in both the theoretical results and the measured data. However, the main implication of our comparison with the Williams data seems to be that the present model is deficient for points near the mixed region. Finally, we suggest modifications of the model which may remove the deficiencies in the solution for points near the mixer.

THE MATHEMATICAL MODEL

Approximate Linearized Equations

Our main objective is the development of a mathematical model to describe the internal wave system created by the growth and collapse of a localized region of turbulence in a linearly stratified fluid.† We assume that the stratified fluid has a finite depth D but is effectively infinite in horizontal extent. A cylindrical mechanical device called a mixer with an effective radius r_0 and located at some depth z_0 creates turbulence in the fluid when oscillated or rotated back-and-forth around its axis at a sufficient rate. Figure 1 shows the geometry of the flow. We assume that the mixer acts for a finite interval of time and produces a flow whose ensemble average characteristics are two dimensional, dependent only on x, z, and t. Figure 1 shows the coordinate system used to describe

†Our approach can be easily extended to include more general stratifications, but the numerical problem of evaluating the disturbance becomes complicated.

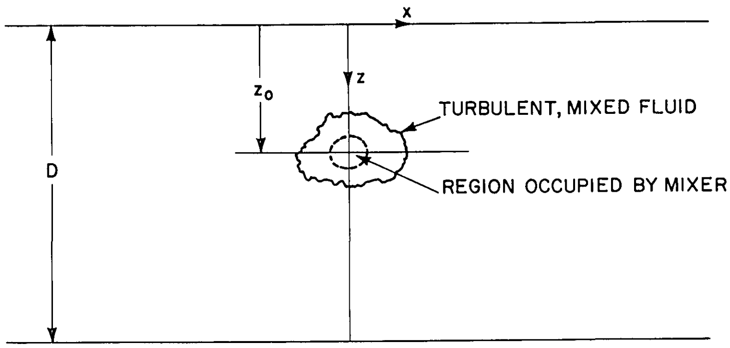


Fig. 1—Coordinate system for description of flow; r_0 designates the effective radius of the mixer and z_0 the depth of the center of the mixer

the flow. It will be further assumed that the structural details of the mixer can be ignored to a first approximation. (Inclusion of the mixer in the mathematical problem as a rigid boundary to the flow generally leads to an intractable problem.) In this section we formulate approximate equations for the description of internal wave generation caused by the mixing of the fluid.

The basic equations describing the flow are the Navier-Stokes equations modified by the Boussinesq approximation (see Ref. 10). These equations consist of the incompressibility equation

$$\frac{d\rho}{dt} = 0 \tag{1}$$

for the density ρ, the solenoidal condition

$$\nabla \cdot \mathbf{u} = 0 \tag{2}$$

for the fluid velocity $\mathbf{u}$, and the momentum equation

$$\rho_0 \frac{d\mathbf{u}}{dt} = -\nabla p + \nabla \cdot S + \rho \mathbf{g},$$

in which ρ_0 denotes the mean density of the fluid, p the pressure, S the viscous stress tenor, and $\mathbf{g}$ the acceleration of gravity. For those cases of interest to us the coefficient of viscosity μ can be regarded as constant and this allows the momentum equation to be written as

$$\frac{d\mathbf{u}}{dt} = -\frac{1}{\rho_0}\nabla p + \frac{\rho}{\rho_0}\mathbf{g} + \nu\nabla^2\mathbf{u}, \tag{3}$$

with the kinematic viscosity $\nu = \mu/\rho_0$.

For turbulent flows it is not possible to describe mathematically particular realizations of the flow; instead we must use a statistical description based on the average flow characteristics. In our problem we can define the average, or mean, flow as the ensemble average taken over many individual realizations of the flow, each of which possesses the same values for controllable parameters. We require then differential equations applicable to the mean flow.

Each flow quantity can be written as the sum of its ensemble average and a fluctuating term whose mean value vanishes. Thus, we write

$$\rho = \langle\rho\rangle + \rho'$$

$$\mathbf{u} = \mathbf{v} + \mathbf{u}'$$

$$p = P + p', \tag{4}$$

in which the prime quantities denote fluctuating terms. We note that the mean flow quantities generally depend on time. Differential equations for the mean flow result from applying the ensemble average to Eqs. (1)-(3). Equation (1) provides

$$\frac{\partial}{\partial t}\langle\rho\rangle + \mathbf{v} \cdot \nabla\langle\rho\rangle = -\nabla \cdot \mathbf{D}, \tag{5}$$

in which $\mathbf{D} = \langle\rho'\mathbf{u}'\rangle$ is the average density flux due to the fluctuating motion. From Eq. (2) we obtain

$$\nabla \cdot \mathbf{v} = 0, \tag{6}$$

which expresses the fact that the mean flow is solenoidal. Equation (3) leads to

$$\frac{\partial \mathbf{v}}{\partial t} + \mathbf{v} \cdot \nabla\mathbf{v} = -\frac{1}{\rho_0}\nabla P + \frac{\langle\rho\rangle}{\rho_0}\mathbf{g} + \upsilon\nabla^2\mathbf{v} + \frac{1}{\rho_0}\nabla \cdot R \tag{7}$$

in which $R = -\rho_0\langle\mathbf{u}'\mathbf{u}'\rangle$ designates the Reynolds stresses acting on the mean flow as a consequence of the fluctuating velocity field. We note that measurements of the Reynolds stresses and the average density flux do not exist for those situations of interest to us.

To simplify Eqs. (5)-(7), we can use the fact that the Reynolds stresses in Eq. (7) can be written as a pressure term plus a term containing shear stresses alone. We can combine the pressure term with P to essentially redefine P. As for the shear stresses, we assume they are unimportant for describing the wave generation process. We note that for locally isotropic turbulence these shear stresses vanish. However, in the present problem they probably are important at least in the vicinity of the mixer. We also assume that the viscous stresses do not play a crucial role for the wave generation process and drop them from Eq. (7).

We write

$$\langle\rho\rangle = \rho^*(z) + \hat{\rho}$$

$$P = P^*(z) + \hat{p}, \tag{8}$$

in which ρ^* and P^* refer to the equilibrium state of the stratified fluid prior to mixing and satisfy

$$\frac{dP^*}{dz} = \rho^* g. \tag{9}$$

We always have the condition that $\hat{\rho}$ is small in comparison with $\rho^*(z)$. However we shall assume that $\nabla\hat{\rho}$ can be neglected in comparison with $\nabla\rho^*$ and that the mean flow velocity $\mathbf{v}$ can be regarded as small. These assumptions do not stem from experimental evidence. Rather, they are made for the sole purpose of obtaining an analytically tractable system of equations whose applicability to the physical process of interest can then be studied. We should point out that, even though these assumptions may turn out to be untenable, the simplified equations they produce may still provide an approximate description of some features of the physical process.

Substituting Eqs. (8) and (9) into Eqs. (5)-(7) and neglecting those terms which are of second order in small quantities, we obtain the following linear equations for the mean disturbance:

$$\nabla \cdot \mathbf{v} = 0 \tag{10}$$

and

$$\frac{\partial\hat{\rho}}{\partial t} + w\frac{d\rho^*}{dz} = -\nabla \cdot \mathbf{D}, \tag{11}$$

in which w is the vertical velocity component, and

$$\frac{\partial\mathbf{v}}{\partial t} = -\frac{1}{\rho_0}\nabla\hat{p} + \frac{\hat{\rho}}{\rho_0}\mathbf{g}. \tag{12}$$

Equations (10)-(12) represent the basic differential equations we shall employ in describing the internal wave system produced by the mixer. They have been established mainly on intuitive grounds because of the lack of detailed experimental measurements.

An auxiliary quantity of use in describing the disturbance is the displacement η defined by

$$\frac{\partial\eta}{\partial t} = w$$

and

$$\eta(x, z, 0) = 0. \tag{13}$$

The function $\eta(x, z, t)$ gives the displacement in the vertical direction of the particle initially (that is, prior to mixing) at (x, z) produced by the mean motion. In those regions of the flow where mixing or other diffusive mechanisms do not occur, the temperature of a fluid particle is conserved throughout its motion; this property leads to a relation between the temperature change at a point $\Delta T = T - T^*$ (T^* being the temperature prior to the start of mixing) and the displacement η given by

$$\Delta T = -\eta\frac{dT^*}{dz}. \tag{14}$$

This relationship is quite important for comparing theoretical results with experimental measurements because internal waves are often measured through the time-dependent temperature change they produce at a point.

The form of Eqs. (10)-(12) indicates that the mean density flux **D** acts as the forcing function for the disturbance. This can be made more explicit by integrating Eq. (11) and using Eq. (13) as well as the initial condition that no disturbance is present at $t = 0$ to get

$$\hat{\rho} = -\eta \frac{d\rho^*}{dz} + \int_0^t (-\nabla \cdot \mathbf{D})dt. \tag{15}$$

This equation states that the density variation at a point is the sum of two effects. The first effect is the mean convection of the isopycnals represented by the first term on the right-hand side of Eq. (15). The second effect leading to a change in density is the presence of turbulence or mixing, which changes the mean value of the density. This effect is represented by the second term on the right side of Eq. (15). Since the buoyancy force is given by $\hat{\rho}\mathbf{g}$ we can also consider the buoyancy force as the sum of two components. One component is the reaction of the stratified fluid to displacement of the isopycnals by the mean flow, which is always in a direction to restore equilibrium. The second component of the buoyancy force is

$$\mathbf{g} \int_0^t (-\nabla \cdot \mathbf{D})dt \tag{16}$$

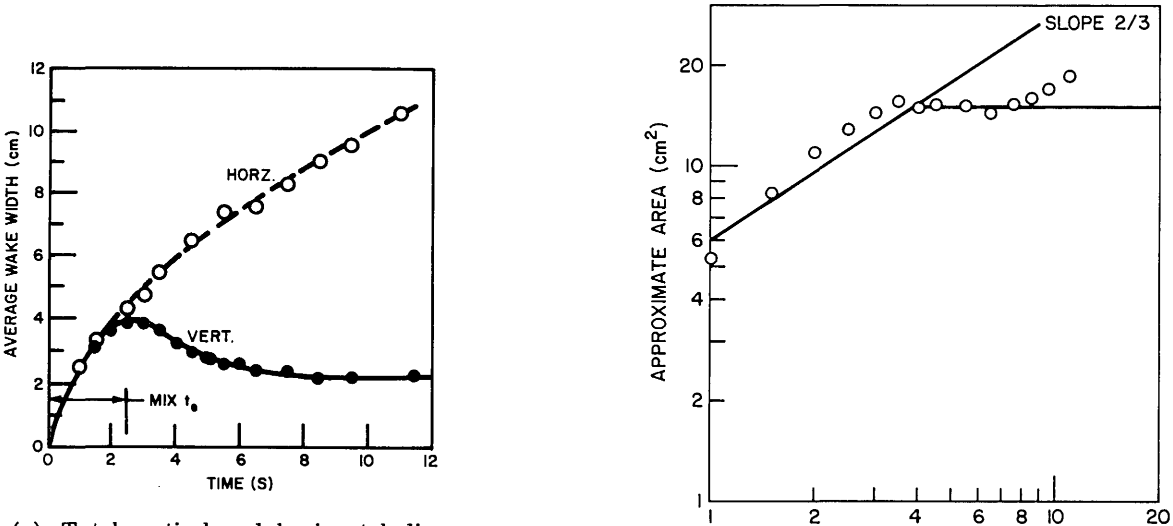
and depends entirely on the nature of the turbulence. Immediately after the start of mixing this second component of the buoyancy force begins to act on the fluid and creates a disturbance which ultimately propagates throughout the entire fluid.

To complete the statement of the mathematical problem we must specify $\nabla \cdot \mathbf{D}$ as well as the boundary conditions and initial conditions. Measurements of $\nabla \cdot \mathbf{D}$ do not exist for the situations of interest to us, and we are forced to rely on empirical considerations in order to define a functional form for $\nabla \cdot \mathbf{D}$.

Specification of the Forcing Function

We find a functional form for the quantity $\nabla \cdot \mathbf{D}$ which describes the net mixing of the fluid by use of experimental results and intuitive arguments. A few properties of this function are obvious: outside the mixed region it has the value zero; once the generation of turbulence by the mixer has ceased there is a finite interval of time during which the turbulence loses energy by working against the stratification and, at the end of this time interval, the net mixing is zero, for all practical purposes. We designate by t_e, called the excitation time, the time interval during which the mixer generates turbulence and by t_m the duration of the mixing phase, which is the time interval from the moment the mixer is activated to the time when net mixing vanishes. Turbulence may still be present after the end of the mixing phase, but it will produce no net mixing of the fluid.

The quantity $\nabla \cdot \mathbf{D}$ will therefore be nonzero inside the boundary of the mixed region over the time interval $0 \leqslant t \leqslant t_m$. Figure 2a shows a representative history of the horizontal and vertical dimensions of the mixed region. The dramatic decrease of the vertical dimension while the horizontal dimension continues to increase is clearly indicated. Figure 2b gives the time variation of the area of the mixed region; the area was calculated by assuming an elliptical cross section for the mixed region and using the results in Fig. 2a. The important features of the behavior are the rapid initial increase and the attainment of a constant value. The increase in area at later times is difficult to explain but apparently may be due to the action of boundary layers on the side walls of the tank (the side walls were about 1 cm apart). We do not consider this increase in area at the latest times as playing a crucial role in the process of wave generation. Thus, we idealize the behavior by interpreting it as an initial increase to a final, constant value.



(a) Total vertical and horizontal dimensions of the region

(b) Area of turbulent region determined by use of (a) and the assumption of an elliptical cross section for the region (based on Schooley (5))

Fig. 2—Representative time history of turbulent region. Conditions are Brunt-Väisälä period $t_{BV} = 7$ s, mixer radius $r_0 = 0.64$ cm, excitation time $t_e = 2.5$ s

A constant value for the area of the mixed fluid indicates that entrainment of fluid has ceased; presumably, the turbulent motion no longer has sufficient energy to mix additional fluid. We therefore assume that mixing is complete when the area of the mixed region no longer increases with time. This offers a means for determining t_m, the time interval during which mixing occurs. We take t_m to depend on the excitation time t_e and the Brunt-Väisälä period according to the simple law

$$t_m = \text{constant} \times t_{BV} + t_e .$$

From Fig. 2b the value of the constant is 0.2, which gives

$$t_m = 0.2 t_{BV} + t_e . \tag{17}$$

Intuitively we would expect a relationship of the form given by Eq. (17), since it states that mixing occurs during the excitation interval plus an interval of time dependent on the Brunt-Väisälä period. Equation (17) also seems to be in accord with the measurements of van de Watering (3).

We now know the period of time over which $\nabla \cdot \mathbf{D}$ is nonzero. We further require knowledge of the spatial region in which this quantity is nonvanishing. To satisfy this requirement we assume that a boundary can be defined for the mixed region which has a circular form over the interval $0 \leqslant t \leqslant t_m$ with a time-dependent radius $r(t)$. This approximately agrees with measured results (see Fig. 2a). For analytical convenience we assume that

$$r(t) = Ar_0\sqrt[3]{t + t_0} \qquad \text{for } 0 \leqslant t \leqslant t_m, \tag{18}$$

in which $t = -t_0$ gives the virtual origin of the mixed region. The quantities A and t_0 can be evaluated by requiring that the area $\pi r^2(t)$ should agree with measured data. For those cases of interest to us t_0 is about 0.1 s and quite small compared to the Brunt-Väisälä period. Thus, we shall shift our time origin by t_0 and regard the mixed region as expanding from a point. This allows us to write

$$r(t) = Ar_0\sqrt[3]{t} \qquad \text{for } 0 \leqslant t \leqslant t_m, \tag{19}$$

In Fig. 2b it is shown that a 2/3 law provides an adequate fit to the data.

We expect that the quantity A appearing in Eqs. (18) and (19) depends on the effective frequency of oscillation of the mixer and on the geometrical structure of the mixer. Measured data do not permit this dependence to be determined. We shall treat A as a constant to which a value will be assigned on the basis of measured results for each particular experiment.

The main assumption we use to obtain a value for $\nabla \cdot \mathbf{D}$ is that mixing, in the absence of mean motion, produces a constant degree of mixing β inside the mixed region. The degree of mixing β is defined as

$$\beta = \frac{\delta\rho}{\rho^*(z_0) - \rho^*(z)}, \tag{20}$$

in which $\delta\rho$ designates the density change produced by mixing. Since $\rho^*(z)$ has a linear variation we can write Eq. (20) as

$$\delta\rho = \beta\frac{d\rho^*}{dz}(z_0 - z) \tag{21}$$

inside the mixed region. Figure 3 shows the density variation along a vertical line through the center of the mixed region.

Using Eqs. (15) and (19), and the assumption of a constant degree of mixing in the absence of mean motion, we find that the mean turbulent density flux must satisfy

$$-\nabla \cdot \mathbf{D} = \beta\frac{d\rho^*}{dz}(z_0 - z)\delta(t - t_b(r))H(t_m - t) \qquad \text{for } t \geqslant 0, \tag{22}$$

in which $\delta(x)$ designates the delta function and $H(x)$ the unit step function, with the properties

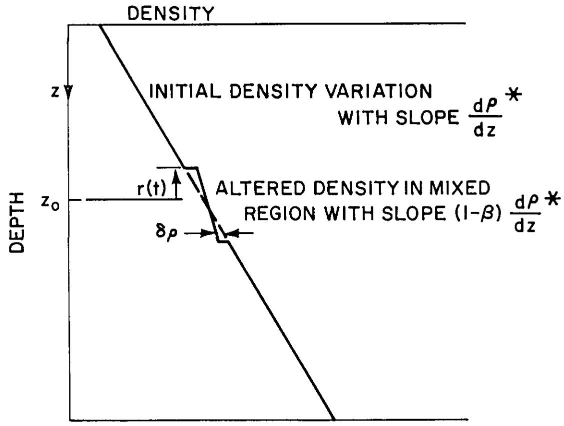


Fig. 3—Density variation along line through center of mixed region; z_0 gives depth of mixed region, $r(t)$ the radius of mixed region at time t, β the degree of mixing, and $\delta\rho$ the density change produced by mixing in the absence of mean motion

$$H(x) = 1 \quad \text{for } x \geqslant 0,$$

$$H(x) = 0 \quad \text{for } x < 0.$$

The quantity $t_b(r)$ gives the time at which the boundary of the mixed region reaches a point at a distance r from the center of the mixed region; analytically,

$$t_b(r) = \left(\frac{r}{Ar_0}\right)^3 .$$

We now have an explicit functional expression for the forcing function. However, when the differential equations are solved using this expression, we obtain a solution for the disturbance in which four integrations appear. This leads to a complicated numerical problem which we prefer to avoid. Thus, instead of employing the form of $\nabla \cdot \mathbf{D}$ given by Eq. (22), we shall use an alternative expression for $\nabla \cdot \mathbf{D}$ which approximately produces the same effects as the expression in Eq. (22). We take this approach for the sake of mathematical convenience and note that several approximate forms for $\nabla \cdot \mathbf{D}$ could be developed. It is difficult to select the best approximate form to employ without actually solving the complete mathematical problem and comparing the results to measurements. In this report, we consider the simplest approximate form.

To obtain the approximate form we shall use, we first evaluate the contribution to the buoyancy force due to mixing with $\nabla \cdot \mathbf{D}$ given by Eq. (22). From Eq. (16) the contribution to the buoyancy force is

$$\beta \frac{d\rho^*}{dz}(z_0 - z)\mathbf{g} \ \text{for} \ \begin{cases} 0 \leqslant r \leqslant r(t), & 0 \leqslant t \leqslant t_m \\ 0 \leqslant r \leqslant r(t_m), & t \geqslant t_m \end{cases} \tag{23}$$

and zero elsewhere.

Figure 4 shows the distribution of this force. The net downward force acting on the upper half of the region in Fig. 4 is

$$\frac{2}{3}\beta g \frac{d\rho^*}{dz} r^3(t) = \frac{2}{3}\beta g \frac{d\rho^*}{dz} A^3 r_0^3 t \qquad \text{for } 0 \leqslant t \leqslant t_m$$

$$\frac{2}{3}\beta g \frac{d\rho^*}{dz} r^3(t_m) = \frac{2}{3}\beta g \frac{d\rho^*}{dz} A^3 r_0^3 t_m \qquad \text{for } t \geqslant t_m . \tag{24}$$

By symmetry, a force equal in magnitude acts upward on the lower half of the region.

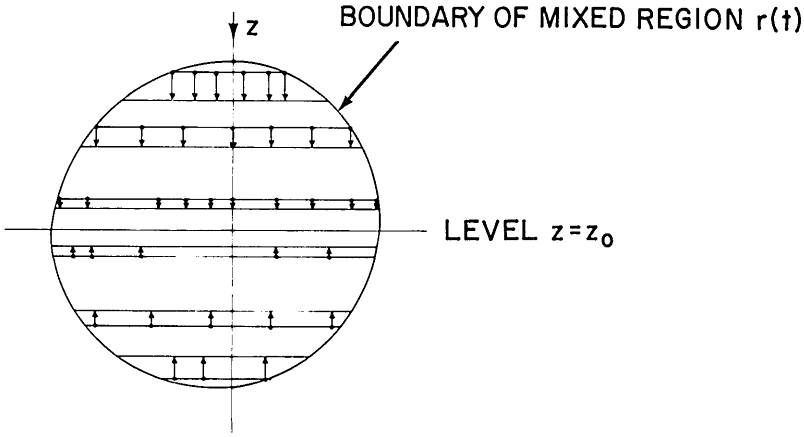


Fig. 4—Form of contribution to the buoyancy force due to mixing. Arrows indicate relative magnitude and direction of force. The force is constant along constant-z lines inside the region but changes with vertical distance from $z = z_0$; outside the region the force is zero.

We now develop an alternate form of $\nabla \cdot \mathbf{D}$ that produces the same net force. First we take $\nabla \cdot \mathbf{D}$ to be nonzero over a square region of dimension 2ℓ; a square region is used for mathematical convenience in later calculations. Within this square we take

$$-\nabla \cdot \mathbf{D} = \beta \frac{d\rho^*}{dz} (z_0 - z) \frac{1}{t_m} \qquad \text{for } 0 \leqslant t \leqslant t_m , \tag{25}$$

and zero otherwise. This leads to a contribution to the buoyancy force from mixing of the form

$$\beta \frac{d\rho^*}{dz} (z_0 - z) \frac{t}{t_m} \mathbf{g} \qquad \text{for } 0 \leqslant t \leqslant t_m$$

and

$$\beta \frac{d\rho^*}{dz} (z_0 - z) \mathbf{g} \qquad \text{for } t \geqslant t_m \tag{26}$$

inside the square region. The net buoyancy force due to mixing acting on the upper half of the square region is

$$\beta g \frac{d\rho^*}{dz} \ell^3 \frac{t}{t_m} \qquad \text{for } 0 \leqslant t \leqslant t_m$$

$$\beta g \frac{d\rho^*}{dz} \ell^3 \qquad \text{for } t \geqslant t_m . \tag{27}$$

Comparing Eq. (27) with Eq. (24), we see that the *net* force will agree at each instant of time if we take

$$\ell = \sqrt[3]{\frac{2}{3}}\, r(t_m) = 0.874 A r_0 \sqrt[3]{t_m} . \tag{28}$$

We note that for $t \geqslant t_m$ the detailed force distribution given by Eq. (26) has precisely the same behavior as in Eq. (23), although the force now acts on a slightly different area. Equations (25), (17), and (28) specify the forcing function we shall use in our evaluation of the internal waves produced by the growth and collapse of a mixed region.

Solution for Fluid with Constant N and Constant Depth

In obtaining a mathematical solution to the problem we shall first solve Eqs. (10), (11), and (12) in terms of the divergence of the mean turbulent density flux $\nabla \cdot \mathbf{D}$; afterward the solution will be made specific for the mean density flux described by Eq. (25). This procedure reveals the necessity of employing simple analytical forms for the divergence of the mean density flux if the numerical problem of evaluating the disturbance is to be practicable.

We assume at the outset that $\nabla \cdot \mathbf{D}$ vanishes everywhere outside a square region of side 2ℓ centered on the z axis at a depth z_0; furthermore, $\nabla \cdot \mathbf{D}$ is taken to be an even function of x. We consider the fluid to have a constant depth D with constant Brunt-Väisälä frequency N given by

$$N^2 = \frac{g}{\rho_0} \frac{d\rho^*}{dz} .$$

The horizontal extent of the fluid is taken as infinite (if vertical boundaries are present they are regarded as nonreflecting). As already mentioned, we deal with situations in which the mean motion is two dimensional, dependent on x, z, and t. Figure 5 shows the geometric situation. The initial conditions are that

$$\mathbf{v} = \hat{\rho} = 0 \text{ at } t = 0, \tag{29}$$

and the boundary conditions are

$$\left.\begin{array}{c} w = 0 \text{ on } z = 0, D \\ w(x, z, t) \text{ remains bounded as } x \to \pm\infty . \end{array}\right\} \tag{30}$$

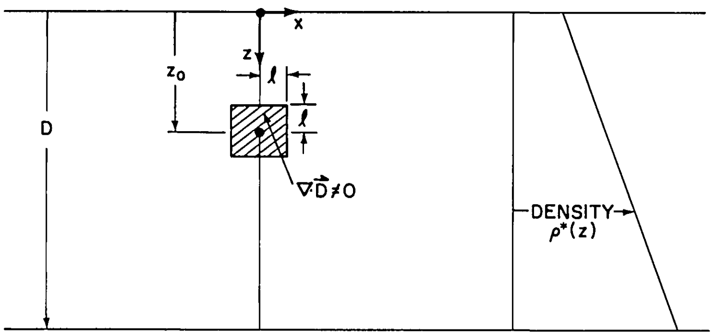


Fig. 5—Location of region in which mean turbulent density flux **D** is nonzero. The density $\rho^*(z)$ is the initial, linear stratification.

We introduce the Fourier transform in time defined by

$$\tilde{w}(x, z, \omega) = \int_0^\infty w(x, z, t)e^{-i\omega t}d\omega, \tag{31}$$

with similar expressions for the other dependent variables. The transformed quantities will be analytic functions of ω in some region $Im(\omega) \leqslant \omega_0$ (ω_0 positive) of the complex ω plane. From the inversion theorem we get

$$w(x, z, t) = \frac{1}{2\pi} \int_{-\infty-i\epsilon}^{\infty-i\epsilon} \tilde{w}(x, z, \omega)e^{i\omega t}d\omega, \tag{32}$$

with ϵ chosen so that the path of integration lies below the singularities of $\tilde{w}$.

Applying the transform to Eqs. (10)-(12) and using the initial conditions in Eq. (29), we find after some elimination that

$$\tilde{w}_{zz} + \left(1 - \frac{N^2}{\omega^2}\right)\tilde{w}_{xx} = -\frac{1}{\omega^2}\frac{g}{\rho_0}\mathfrak{D}_{xx}, \tag{33}$$

in which subscripts denote differentiation, and

$$\mathfrak{D} = \int_0^\infty (-\nabla \cdot \mathbf{D})e^{-i\omega t}d\omega. \tag{34}$$

The boundary conditions for $\tilde{w}$ follow from Eq. (30) and can be written as

$$\tilde{w} = 0 \text{ on } z = 0, D$$

$$\tilde{w} \text{ remains bounded as } x \to \pm\infty. \tag{35}$$

We now want to find $\tilde{w}$ (x, z, ω).

Considering $\tilde{w}$ (x, z, ω) as a function of z, we can expand it in a Fourier series

$$\tilde{w}(x, z, \omega) = \sum_{n=1}^{\infty} \phi_n(z) F_n(x, \omega), \tag{36}$$

in which

$$\phi_n(z) = \sin\left(\frac{n\pi z}{D}\right) \tag{37}$$

and the F_n coefficients depend on x and ω. The motivation for using the orthogonal functions sin $(n\pi z/D)$ rather than some other set of orthogonal functions follows from the fact that the trigonometric functions are eigenfunctions of the differential operator appearing in Eq. (33).

Substitution of the expansion given by Eq. (36) into Eq. (33) and use of Eq. (37) and the orthogonal property of ϕ_n produces for the determination of F_n

$$\frac{d^2}{dx^2} F_n + \alpha_n^2 F_n = \frac{1}{N^2 - \omega^2} \frac{2g}{\rho_0 D} \int_0^D \phi_n \mathfrak{D}_{xx} dz, \tag{38}$$

in which

$$\alpha_n^2 = \left(\frac{n\pi}{D}\right)^2 \frac{\omega^2}{\mathrm{N}^2 - \omega^2}. \tag{39}$$

We define

$$I_n(x, \omega) = \frac{1}{N^2 - \omega^2} \frac{2g}{\rho_0 D} \int_0^D \phi_n \mathfrak{D} dz, \tag{40}$$

so that Eq. (38) can be written in the form

$$\frac{d^2}{dx^2} F_n + \alpha_n^2 F_n = \frac{d^2 I_n}{dx^2}; \tag{41}$$

the boundary conditions for F_n are that it remain bounded as $x \to \pm \infty$.

We now wish to find the solution F_n (x, ω) to Eq. (41). The dependence of F_n on ω is affected by α_n which also depends on ω. From Eq. (39), which gives $\alpha_n^2(\omega)$ we define a single-valued root as

$$\alpha_n(\omega) = \frac{n\pi}{D} \frac{\omega}{\sqrt{N^2 - \omega^2}}, \tag{42}$$

in which we use the principal branch for $\sqrt{N^2 - \omega^2}$. In the complex ω plane, then, branch cuts occur for Im $(\omega) = 0$, $|\omega| \geqslant N$; $\alpha_n(\omega)$ is not defined on these branch cuts, but it possesses values through out the remainder of the plane as given by Eq. (42).

Some properties of α_n needed in the construction of the solution for F_n are that the imaginary part of α_n is negative for Im $(\omega) < 0$ and equal to zero on the line segment Im $(\omega) = 0$, $|\omega| < N$.

The solution to Eq. (41) follows from the variation-of-parameters technique, in which we set

$$F_n(x) = a_n(x)e^{i\alpha_n x} + b_n(x)e^{-i\alpha_n x}$$

and use Eq. (41) with its boundary conditions to find $a_n(x)$ and $b_n(x)$. This process yields

$$F_n(x) = I_n - \frac{i\alpha_n}{2}\left[e^{i\alpha_n x}\int_x^{\infty} e^{-i\alpha_n \eta}I_n(\eta)d\eta + e^{-i\alpha_n x}\int_{-\infty}^{x} e^{i\alpha_n \eta}I_n(\eta)d\eta\right]. \qquad (43)$$

It is easy to show that since $\mathfrak{D}(x, z, \omega)$ is an even function of x, then F_n possesses the same property.

We now have the general form of the solution for the vertical velocity component as

$$w(x, z, t) = \sum_{n=1}^{\infty} \sin\left(\frac{n\pi z}{D}\right)\frac{1}{2\pi}\int_{-\infty-i\epsilon}^{\infty-i\epsilon} F_n(x, \omega)e^{i\omega t}d\omega, \qquad (44)$$

with F_n given by Eq. (43) and the path of integration located below the singularities of F_n. Since F_n is an even function of x, we conclude that the vertical velocity component—in fact the complete disturbance—is an even function of x, as we would expect from the physics of the problem. We can therefore confine our attention to the region $x \geqslant 0$.

Equation (44) gives the form of the solution when the only assumptions on $\mathfrak{D}$ are that it be an even function of x vanishing outside some finite region. Four integrations occur in the solution: one to obtain the function $\mathfrak{D}$, a second for I_n, a third for F_n, and the fourth for the vertical velocity component w as exhibited explicitly in Eq. (44). For the sake of convenience in performing the numerical evaluation of the disturbance, it is clearly desirable to be able to analytically perform as many of these integrations as possible. This was our motivation for selecting the function given in Eq. (25) in place of that given in Eq. (22).

We now develop the specific form of the solution when the mean turbulent density flux satisfies Eq. (25). From Eq. (34) we find that

$$\mathfrak{D} = \beta\frac{d\rho^*}{dz}(z_0 - z)\frac{1 - \exp(-i\omega t_m)}{i\omega t_m}H(\ell - x)H(\ell - y)H(\ell + y), \qquad (45)$$

where we restrict our attention to the region $x \geqslant 0$; Eq. (28) specifies the quantity ℓ. This result yields for I_n defined in Eq. (40),

$$I_n = 4\beta D \left(\frac{1}{n\pi}\right)^2 \cos\left(\frac{n\pi z_0}{D}\right) \left[\left(\frac{n\pi \ell}{D}\right) \cos\left(\frac{n\pi \ell}{D}\right) - \sin\left(\frac{n\pi \ell}{D}\right)\right]$$

$$\times \frac{N^2}{N^2 - \omega^2} \frac{1 - \exp(-i\omega t_m)}{i\omega t_m} H(\ell - x). \tag{46}$$

We write I_n as

$$I_n = C_n H(\ell - x) \tag{47}$$

in which C_n can be read off from Eq. (46). C_n is independent of x, and Eq. (47) will allow F_n to be written in a condensed form.

From Eqs. (43) and (47) we find that

$$F_n = C_n \left\{\cos(\alpha_n x) e^{-i\alpha_n \ell} H(\ell - x) - i e^{-i\alpha_n x} \sin(\alpha_n \ell) H(x - \ell)\right\}, \tag{48}$$

with α_n defined by Eq. (42). As a function of ω, F_n has singularities at $\omega = \pm N$, and the branch cuts for α_n leave F_n undefined on these cuts. However, F_n is an analytic function of ω below the real axis. In Eq. (44), then, ϵ can be taken as any positive number. Application of Cauchy's theorem permits us to transfer the integration path in Eq. (44) to an integration along the real axis. In carrying out this process we must take proper account of the singular points and also define F_n along the branch cuts through a limiting process.

Substitution of Eq. (48) into Eq. (44) and use of the procedure just outlined yields the specific form of the vertical velocity component w for the model under consideration. Different expressions exist for $0 \leqslant x < \ell$ and $x > \ell$. For both regions we find that we can write

$$w(x, z, t) = w_-(x, z, t) + w_+(x, z, t), \tag{49}$$

in which w_- involves an integration over frequencies $0 \leqslant \omega \leqslant N$ and w_+ an integration over frequencies $\omega \geqslant N$. These two functions have the following explicit forms. For $0 \leqslant x < \ell$,

$$w_-(x, z, t) = \frac{4}{\pi^3} \beta D \sum_{n=1}^{\infty} \frac{1}{n^2} \sin\left(\frac{n\pi z}{D}\right) \cos\left(\frac{n\pi z_0}{D}\right) \left[\left(\frac{n\pi \ell}{D}\right) \cos\left(\frac{n\pi \ell}{D}\right)\right.$$

$$\left. - \sin\left(\frac{n\pi \ell}{D}\right)\right] \int_0^N \frac{N^2}{N^2 - \omega^2} \cos(\alpha_n x)$$

$$\times \left[\frac{\sin(\omega t - \alpha_n \ell) - \sin(\omega t - \omega t_m - \alpha_n \ell)}{\omega t_m}\right] d\omega, \tag{50}$$

in which

$$\alpha_n = \frac{n\pi}{D} \frac{\omega}{\sqrt{N^2 - \omega^2}}; \tag{51}$$

and

$$w_+(x, z, t) = \frac{4}{\pi^3} \beta D \sum_{n=1}^{\infty} \frac{1}{n^2} \sin\left(\frac{n\pi z}{D}\right) \cos\left(\frac{n\pi z_0}{D}\right) \left[\left(\frac{n\pi \ell}{D}\right) \cos\left(\frac{n\pi \ell}{D}\right) - \sin\left(\frac{n\pi \ell}{D}\right)\right]$$

$$\times \int_N^{\infty} \frac{N^2}{N^2 - \omega^2} \exp(-\beta_n \ell) \cosh(\beta_n x) \left[\frac{\sin \omega t - \sin \omega(t - t_m)}{\omega t_m}\right] d\omega, \tag{52}$$

in which

$$\beta_n = \frac{n\pi}{D} \frac{\omega}{\sqrt{\omega^2 - N^2}}. \tag{53}$$

For the region $x > \ell$, we have

$$w_-(x, z, t) = \frac{4}{\pi^3} \beta D \sum_{n=1}^{\infty} \frac{1}{n^2} \sin\left(\frac{n\pi z}{D}\right) \cos\left(\frac{n\pi z_0}{D}\right) \left[\left(\frac{n\pi \ell}{D}\right) \cos\left(\frac{n\pi \ell}{D}\right) - \sin\left(\frac{n\pi \ell}{D}\right)\right]$$

$$\times \int_0^{N} \frac{N^2}{N^2 - \omega^2} \sin(\alpha_n \ell) \left[\frac{\cos(\omega t - \omega t_m - \alpha_n x) - \cos(\omega t - \alpha_n x)}{\omega t_m}\right]$$

$$\times \, d\omega, \tag{54}$$

in which α_n is as given in Eq. (51), and

$$w_+(x, z, t) = \frac{4}{\pi^3} \beta D \sum_{n=1}^{\infty} \frac{1}{n^2} \sin\left(\frac{n\pi z}{D}\right) \cos\left(\frac{n\pi z_0}{D}\right) \left[\left(\frac{n\pi \ell}{D}\right) \cos\left(\frac{n\pi \ell}{D}\right) - \sin\left(\frac{n\pi \ell}{D}\right)\right]$$

$$\times \int_N^{\infty} \frac{N^2}{N^2 - \omega^2} \exp(-\beta_n x) \sinh(\beta_n \ell) \left[\frac{\sin \omega(t - t_m) - \sin \omega t}{\omega t_m}\right] d\omega, \tag{55}$$

in which β_n is as in Eq. (53).

The above expressions provide the vertical velocity component throughout the fluid as a function of time. We can easily determine the remaining quantities describing the disturbance from Eqs. (10)-(13). In particular the displacement η defined by Eq. (13) follows from an integration over time of the expressions in Eqs. (50), (52), (54), and (55); since the results can be written immediately from the cited equations, we shall not write out the explicit forms. In our later comparison to experimental results we use the temperature change ΔT at a point outside the mixed region, which can be expressed in terms of the displacement by use of Eq. (14).

We note that all quantities describing the disturbance can be written as the sum of two terms; the first one contains an integration over frequencies $0 \leqslant \omega \leqslant N$, while the second has an integration over frequencies $\omega \geqslant N$. The term involving frequencies less than the Brunt-Väisälä frequency N represents a propagating system of internal waves. The term with frequencies greater than N does not represent propagating waves; rather, it is a forced disturbance maintained by the mean turbulent density flux. This forced disturbance decays exponentially with horizontal distance from the boundary of the mixed region and decays at sufficiently large times as well.

COMPARISON WITH EXPERIMENTAL RESULTS

Before proceeding with our comparison, we make some comments on the degree of mixing β. Direct measurements of β do not exist. We used $\beta = 0.5$ in our calculations and cite some indirect experimental evidence supporting this choice. In Fig. 2a, if we assume that at the time of maximum vertical expansion a constant degree of mixing β exists in the mixed region, then we have

$$\beta = \frac{\text{final vertical dimension}}{\text{maximum vertical dimension}} = 0.55.$$

Similarly, Schooley and Stewart (1) present data that imply a value $\beta = 0.60$ at the time of maximum vertical extent; in this case the mixing device is a self-propelled body that moves through the fluid. These approximate values for β apply to the mixed region at the time of its maximum vertical extent. However, we take these values as indicating that various mixing devices produce approximately the same degree of mixing and that a constant value of $\beta = 0.5$ is acceptable as a first approximation.

Comparison for Large Horizontal Distances from the Mixed Region

Schooley and Hughes (9) present measurements of the temperature change produced at points near the top surface of a stratified fluid as a consequence of localized mixing in the interior of the fluid. Figure 6 shows their measured data for two stratifications, which can be approximated by constant values of the Brunt-Väisälä frequency N. At least two sets of measured data were obtained for each stratification. Pertinent values of the parameters are listed below.

N (s^{-1})	dT^*/dz, (°C/cm)	z (cm)	t_e(s)
1.19	11.1	0.250	2.0
0.68	3.81	0.300	2.0

$D = 7.3$ cm, $z_0 = 4.5$ cm, $r_0 = 0.65$ cm

The quantity z gives the depth at which measurements were made; for each stratification temperature as a function of time was measured at the three horizontal distances $x = 5.72$, 8.26, and 12.7 cm.

The solid curves in Fig. 6 are from the theory of Schooley and Hughes (9). Their theory assumes that wave generation does not begin until the mixed region has attained its maximum vertical dimension; furthermore, the mixing is assumed complete so that

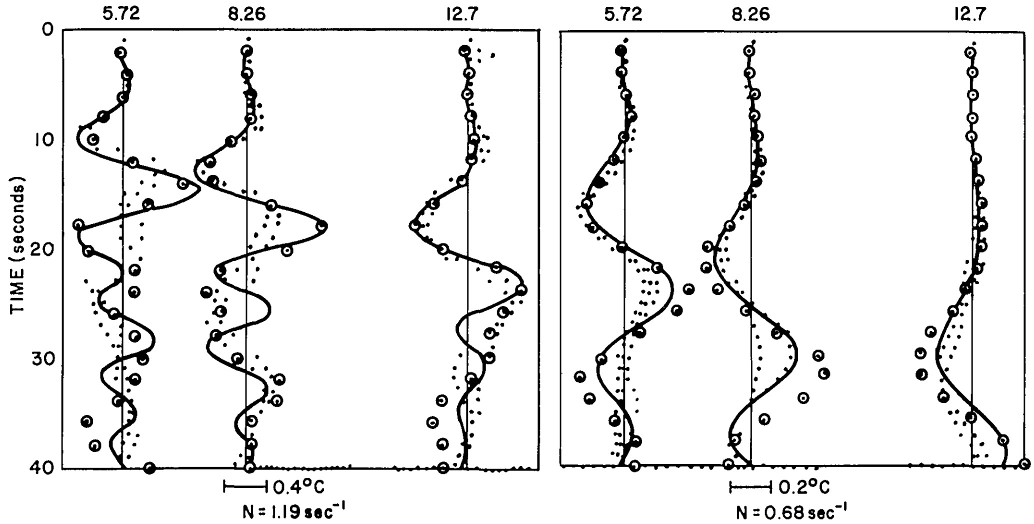


Fig. 6—Measured and theoretical values of temperature changes produced by growth and collapse of a mixed region.

. . . . . .measured data of Schooley and Hughes (9),———— theory of Schooley and Hughes (9), ⊙ according to theory developed in this report. Values at the top of each curve give the horizontal distance x in centimeters at which measurements and calculations were made. (based on Schooley and Hughes (9))

$\beta = 1$. Thus, their mathematical model represents wave generation as the result of collapse of an initial density perturbation in a stratified fluid initially at rest. They take their time origin at the moment of collapse. In comparing their theoretical values with the measured values they shifted their curves until the best fit was obtained; this was considered permissible since the actual moment of collapse was unknown. Figure 6 shows that their theory yields fair agreement with amplitudes but appears to indicate more oscillations than are observed.

Our theory assumes that the mixed region expands from a point according to the rule given by Eq. (19). A value is needed for the quantity A which appears in Eqs. (19) and (28). An approximate value of $A = 2.06\ s^{-1/3}$ was obtained from the measured radii of the mixed region at the time of collapse by setting the measured radius equal to $Ar_0(t_m)^{1/3}$ for each stratification and then averaging the resulting values of A. Values of the measured radii are given by Schooley and Hughes (9).

The circled points in Fig. 6 were calculated with the theory developed in this report. In evaluating the modal sum we replaced the sum over all the modes by a partial sum containing a sufficient number of modes to provide a value for the displacement η stable to $\pm$ 0.001 cm. For both stratifications the number of modes required increased with time at each location. A minimum number of four modes was calculated for each point. The maximum number of modes required decreased with horizontal distance from the mixed region. Thus for $N = 1.19\ s^{-1}$ the maximum number of modes was 30 at 5.72 cm, 21 at 8.26 cm, and 14 at 12.7 cm. Approximately 2/3 as many modes were required at the same locations for $N = 0.68\ s^{-1}$.

On the whole, our theoretical values show a fair agreement with the measured amplitudes and phases. We should point out that Schooley and Hughes made their measurements very close to a rigid boundary and that they included in their theory a correction for the viscous layer at the boundary. It seems that if our theory were modified to account for the boundary layer effect, we would improve our agreement with the measured data because our calculated amplitudes would be reduced at later times.

Although our theoretical values also show some agreement with the theory of Schooley and Hughes, the basic premises of the two theories are quite different. Our model describes the disturbance as originating at the start of mixing instead of at the time of maximum vertical expansion; thus there is no uncertainty about the meaning of our time origin, and the model accounts for the entire growth phase of the mixed region. Furthermore, we use a constant degree of mixing, $\beta = 1/2$, which seems partially supported by experimental evidence, while the calculations of Schooley and Hughes assume that $\beta = 1$.

Comparison Near the Mixed Region

We now consider the experimental results of Williams (6). He made temperature measurements at a point just outside the mixed region at its maximum vertical extent. These measurements describe the near-field behavior of the disturbance caused by mixing in contrast to the far-field behavior measured by Schooley and Hughes (9). Moreover, Williams studied the effect of excitation time t_e (the time interval during which the mixer agitates the fluid) on the disturbance.

He used an axisymmetric mixer consisting of a rod wound with wire having a total effective radius $r_0 = 0.95$ cm. The mixer was placed at a depth $z_0 = 9.5$ cm in fluid of total depth $D = 25.4$ cm. Measurements were made at $x = 1.35$ cm, $z = 7.5$ cm. The stratification of the fluid consisted of two approximately homogeneous layers at the top and bottom with an intermediate layer having an approximately linear density variation. Excitation time t_e varied from 1 s to 32 s.

Before proceeding with a comparison of our theory with the measurements we mention several reasons why our theory is not entirely applicable to the experiment.

1. Our theory assumes that a small disturbance exists in the fluid, and this assumption may fail to hold near the mixer.

2. We used a simplified expression for the divergence of the turbulent density flux $\nabla \cdot \mathbf{D}$ in order to evaluate several integrals that appear in the theory. This simplified expression may produce a poor approximation to the near-field behavior but a good approximation to the far-field behavior.

3. The presence of the solid mixer requires the fluid to flow around it, and this was not accounted for in our boundary conditions.

4. Stratification of the fluid used in the experiment does not follow the linear behavior (constant N) assumed by the theory.

5. The measurements apply to a particular realization of the flow. On the other hand, the theory deals with the ensemble average disturbance. It is not clear to what extent particular realizations deviate from the ensemble average outside the mixed region. This matter will eventually have to be settled by experiment. Several of these effects seem quite important for our comparison because the near-field behavior intimately

depends on the detailed flow processes occurring in the mixed region. Despite these qualifications we are motivated to make a comparison by the belief that some of the gross features contained in the measured data should also appear in the theoretical results.

In order to apply our theory to the experiment we must determine a value for the parameter A entering Eq. (18). This parameter is related to the size of the mixed region. Rather than assume a virtual origin for the mixed region, we shall use Eq. (18) in which $t = 0$ denotes the start of mixing. Thus at $t = 0$ we require that the boundary of the mixed region coincide with the effective radius of the mixer; this gives

$$r(0) = r_0 = A r_0 \sqrt[3]{t_0} \; ,$$

which establishes a relation between A and t_0. The solid curves in Fig. 7 show the measured temperature oscillation at a distance 2.42 cm from the center of the mixer. Turbulent fluctuations do not appear. However, for the greatest excitation time of 32 s there are high-frequency oscillations which we take as an indication that the boundary of the mixed region is in the vicinity of the measured point. We find from Eq. (18) that $A \approx 0.7\ \mathrm{s}^{-1/3}$ by requiring the maximum radius of the mixed region $r(t_m)$ to equal the distance of the measurement point from the center of the mixer. This is the approximate value we employed in our calculations.

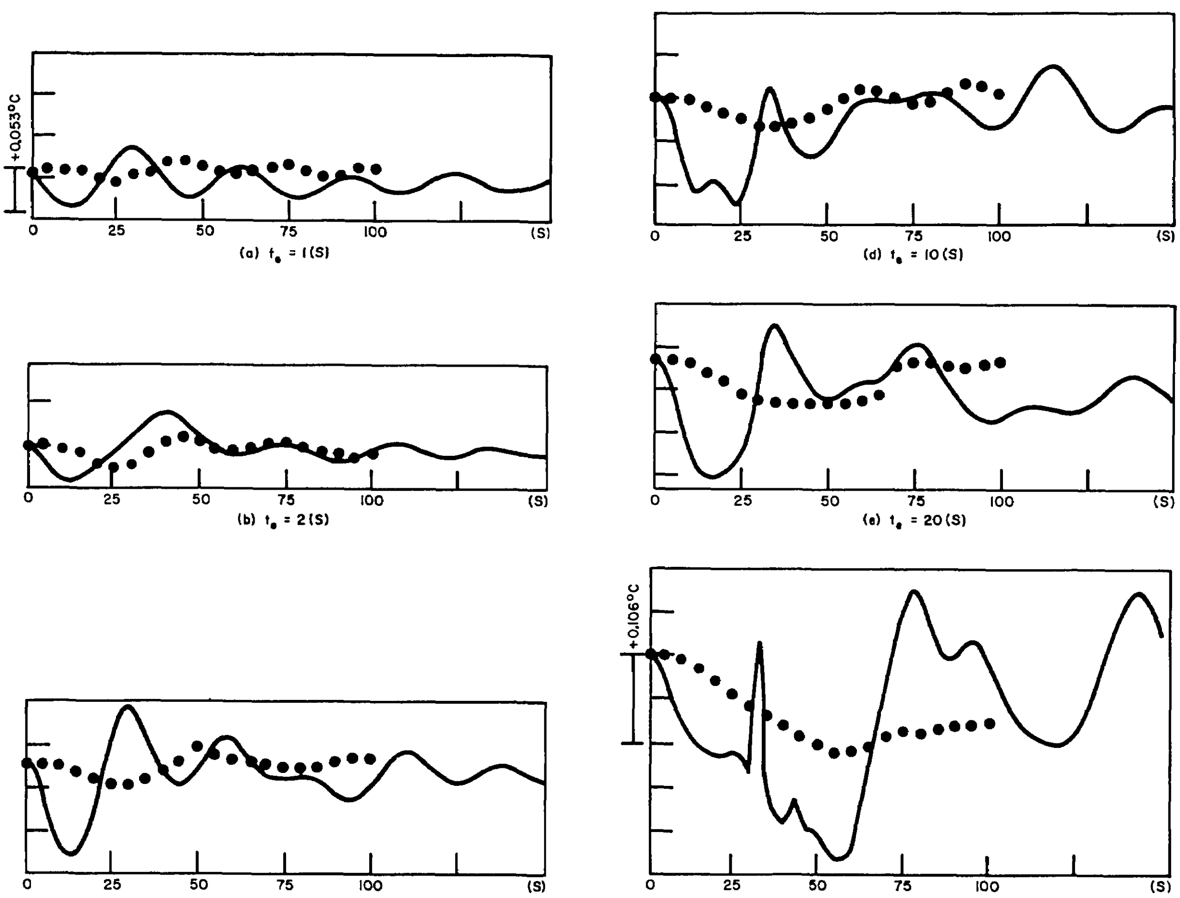


Fig. 7—Temperature oscillation at a point for various excitation times. Conditions are D = 25.4 cm, z_0 = 9.5 cm, r_0 = 0.95 cm, x = 1.35 cm, z = 7.5 cm. The Brunt-Väisälä period increases from t_{BV} = 26.6 s in (a) to 31 s in (f)

________ measurements of Williams (6), ...theory

We account for the presence of the mixer in the following indirect fashion. We consider the mixed fluid to lie between the outside radius given by Eq. (18) and the surface of the mixer, which has an effective radius r_0. In this region the turbulent density flux is assumed to satisfy Eq. (22) in which $t = 0$ now designates the start of mixing. Using the procedure following Eq. (22) we again are led to the approximation for the turbulent density flux given by Eqs. (25) and (28). This scheme partially accounts for the fact that mixed fluid does not fill the region occupied by the mixer.

Figure 7 shows a comparison between measured temperature changes (solid curves) and temperature changes according to our theory for 100 s. (We note that for times in excess of 120 s there may be interference in the measured data due to wave reflections from the walls of the tank.) While the comparison shows no agreement in detail, some general properties agree. Our theory gives an increase in temperature as soon as mixing begins, indicative of a downward motion of the fluid. This agrees qualitatively with the measurements, but our calculated temperature does not demonstrate the rapid initial increase found in the measurements.

A further point of qualitative agreement arises when we compare the behavior of average properties of the temperature fluctuation. The properties we compare are the average temperature fluctuation ΔH and the root-mean-square (rms) temperature deviation σ from the average. For the measured data these averages were calculated for the time range $0 \leqslant t \leqslant 120$ s by using values of the temperature fluctuation at 4-s intervals. For the theoretical curves the averages were calculated for the time interval $0 \leqslant t \leqslant 100$ s at 5-s intervals. Thus, ΔH and σ are defined by

$$\Delta H = \frac{1}{n} \sum \Delta T_n$$

and

$$\sigma = \sqrt{\frac{1}{n} \sum (\Delta T_n - \Delta H)^2} ,$$

in which ΔT_n is the temperature change for the interval n. Figure 8 shows how these two quantities vary with excitation time according to the measured data and according to theory. All the curves demonstrate a trend to increase with excitation time, which roughly indicates that the disturbance grows in magnitude with an increase of excitation time.

Figure 9 shows the displacement as a function of time at various locations in the fluid for an excitation time $t_e = 5$ s. These curves were calculated using the theory to demonstrate how the features of the signal change with distance from the mixer and with depth in the fluid. An interesting feature shown by the curves is that the temperature fluctuation has a more complex form in the interior of the fluid (Fig. 7a and 7b) than it has near the surface (Fig. 7c).

The comparison seems to indicate that our theory is inadequate near the mixed region. The major reason for this inadequacy may be that our theory does not lead to a disturbance increasing in amplitude at a sufficient rate during the initial phases of the motion, as demonstrated in Fig. 7. We base this remark on the expectation that, if the first maximum temperature deviation were larger, then the amplitudes of later temperature extremes would be correspondingly increased. The fact that our theory does not

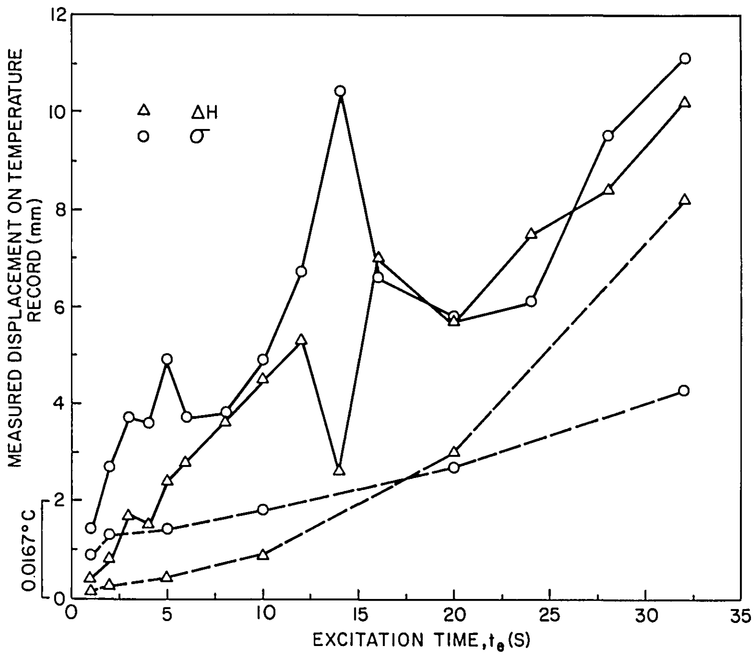


Fig. 8—Average characteristics of the temperature oscillations as a function of excitation time

——— measurements of Williams (6), ...theory

produce a sufficiently large initial disturbance must of course be associated with our forcing function $\nabla \cdot \mathbf{D}$.

We recall that we replaced the expression for $\nabla \cdot \mathbf{D}$ in Eq. (22) by the expression in Eq. (25) on the grounds of mathematical convenience. Actually, $\nabla \cdot \mathbf{D}$ given by Eq. (25) corresponds to smearing out $\nabla \cdot \mathbf{D}$ in Eq. (22) over a prescribed area. The buoyancy force due to mixing is also spread out over the prescribed area but such as to preserve the net force acting on the fluid. What this means is that at most points in the region of mixing we weakened the buoyancy force due to mixing. It seems reasonable to expect that the disturbance at a point near the mixed region will depend more on the actual distribution of the buoyancy force due to mixing than on its intergral properties. Thus, it may be that in smearing out the buoyancy force we weakened the disturbance in the near field.

We give some suggestions later as to how the present theory may be improved by the use of an expression for $\nabla \cdot \mathbf{D}$ which should approximate $\nabla \cdot \mathbf{D}$ in Eq. (22) in such a way as to avoid the smearing out process, and, consequently, the weakening of the near-field disturbance.

CONCLUSION

We developed in this report an approximate mathematical model to describe the internal wave disturbance caused by the growth and collapse of a localized, two-dimensional region of turbulence in a linearly stratified fluid. Our theory differs from earlier work to the extent that we incorporated the growth phase of the turbulent, mixed region in the model. We found adequate agreement with measured data describing the disturbance at

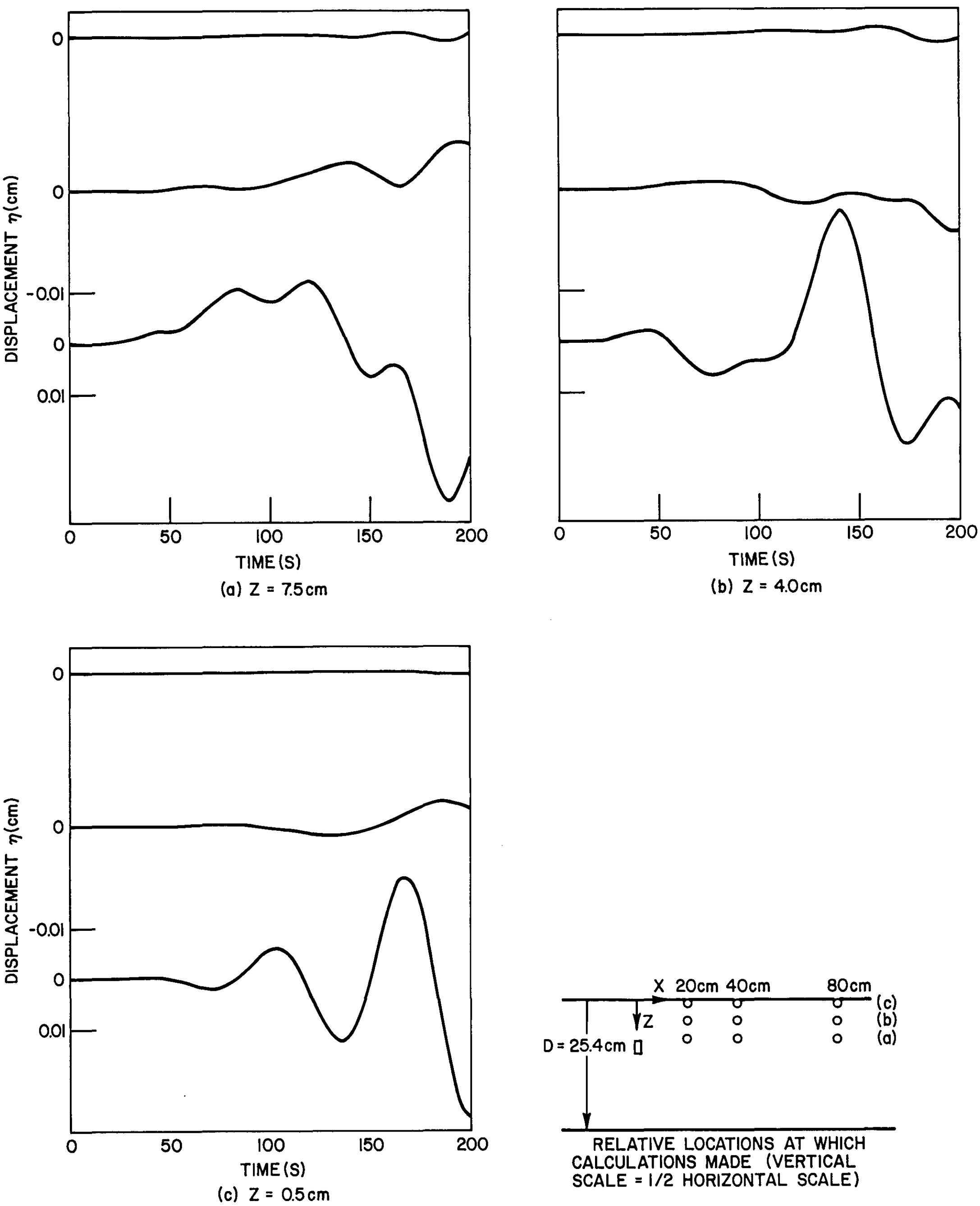


Fig. 9—Displacement as a function of time at various positions in the fluid (according to theory). The conditions are z_0 = 9.5 cm, t_e = 5 sec, dT^*/dz = .3510°C/cm, t_{BV} = 27.7 sec; in (a), (b) and (c) the curves from bottom to top are for lateral distances x = 20, 40, and 80 cm respectively.

large distances from the mixed region; however, comparison with data describing the disturbance near the mixed region indicated that our model does not provide an entirely faithful representation close to the mixed region. We present further details of the comparison after we summarize the essential features of the model.

Our model applies to the mean disturbance, determined as an ensemble average of the motion. We linearized the Navier-Stokes equations in the form applicable to the mean disturbance to obtain equations amenable to mathematical analysis whose applicability to the physical process could then be studied. The forcing function for the disturbance in the linearized system of differential equations is the divergence of the mean density flux $\nabla \cdot \mathbf{D}$ associated with the turbulent motion. This density flux describes how the mean density in the turbulent region changes as a consequence of mixing. Specification of $\nabla \cdot \mathbf{D}$ in functional form constituted a basic problem of the model.

Intuitive and empirical considerations provided an analytical expression for $\nabla \cdot \mathbf{D}$ for mixed regions which can be described by a constant degree of mixing β. Further assumptions that entered the argument establishing the expression for $\nabla \cdot \mathbf{D}$ were

1. Mixing in the fluid occurs during a finite time interval of length $t_m = 0.2t_{BV} + t_e$ in which t_{BV} denotes the Brunt-Väisälä period and t_e the time interval during which the mixer device agitates the fluid.

2. The mixed region can be regarded as circular during the time interval $0 \leqslant t \leqslant t_m$ ($t = 0$ represents the virtual origin of the mixed region) with a radius increasing according to a $t^{1/3}$ law.

3. $\beta = 1/2$ represents a reasonable value for the degree of mixing produced by mixer devices.

We cited limited experimental support for some of these assumptions.

It was pointed out that the above model leads to a rather complicated numerical problem due to the presence of several integrals in the solution. For the sake of simplifying the numerical problem we replaced the analytical expression for the flux term $\nabla \cdot \mathbf{D}$ by an alternate expression which allowed several of the intergrals to be evaluated explicitly. Our alternate expression for $\nabla \cdot \mathbf{D}$ is equivalent to the original expression in the following sense. At any instant of time the two expressions lead to the same net buoyancy force associated with mixing for the upper (or lower) half of the mixed region.

Using this alternate expression for the forcing term, we obtained a solution for the vertical velocity component in the case of a linearly stratified fluid of constant depth. The other quantities describing the disturbance can be immediately evaluated in terms of this velocity component. In particular, the vertical displacement of a particle following the mean motion is the time integral of the vertical velocity component. The importance of the vertical displacement stems from the fact that outside the region of mixing it is directly proportional to the temperature change produced by the disturbance, which often is the quantity measured in experimental investigations of internal waves.

We compared our theoretical solution with results of two experimental investigations. The first case pertained to the nature of the disturbance at large distances from the mixed region. Figure 6 shows the comparison of our theory with measured data from Schooley and Hughes (9) for two values of the Brunt-Väisälä frequency, and with the results of their theory. Our theory demonstrates a fair agreement with the measured data. We note that the measurements were made near a rigid boundary so that viscous boundary-layer effects were probably of some importance. If corrections were included in our theory to account for the viscous layer at the boundary the agreement with the data would probably be improved because the disturbance would tend to be diminished, especially at later times.

The comparison of our calculations with calculations given by Schooley and Hughes (9), shown in Fig. 6, indicated a rather close agreement during the early stages of the motion. However, at later times, our theory appears to yield fewer oscillations—in agreement with the measured data—than does the theory of Schooley and Hughes, even though our theory employs premises quite different from those used by Schooley and Hughes. For example, we account for the initial growth stage of the mixed region, whereas Schooley and Hughes assume that the initial growth stage can be ignored; we assume a degree of mixing $\beta = 1/2$, whereas Schooley and Hughes take the degree of mixing $\beta = 1$; in our theory, internal wave generation begins at the same time as does mixing, whereas Schooley and Hughes assume that wave generation does not begin until the mixed region has attained its maximum vertical extent and collapse has begun.

The second case pertained to measurements made at a point very close to the mixed region (Fig. 7). While agreement in detail is poor, we must qualify this comparison with the remark that our theory is not directly applicable to near-field measurements for specific reasons given in the text.

Thus Fig. 7 shows that the amplitudes and periods are significantly different. Since the calculated values cannot really be expected to agree in detail with near-field measurements, a more valid comparison can be made on the basis of observed trends in the data. In Fig. 8, we show how the average temperature change taken over a finite time interval and the rms temperature change from its average over the same time interval vary with excitation time. The trend demonstrated by the measured data, is also exhibited by the theoretical values, although the rates of increase do not match very well.

We concluded from this comparison that our model is deficient in its description of the disturbance near the mixed region. A partial cause of this deficiency must stem from our use of a modified expression for the flux term $\nabla \cdot \mathbf{D}$. The modified expression was used in order to simplify some of the mathematical expressions appearing in the theory. However, it does not preserve all of the characteristics of the original expression we found for $\nabla \cdot \mathbf{D}$. Since the disturbance near the mixed region probably depends on the detailed nature of the forcing term, we must expect that our modified expression for $\nabla \cdot \mathbf{D}$ does not give a good description of the disturbance near the mixed region.

Some remarks follow on how this deficiency might be corrected, but in such a manner as to retain mathematical simplicity of the theory.

SUGGESTIONS FOR FURTHER WORK

We pointed out that our approximation to the forcing term $\nabla \cdot \mathbf{D}$ given in Eq. (22) by the expression in Eq. (25) is probably responsible for weakening the disturbance in the near field. We now outline a procedure for correcting this deficiency.

We retain the basic premises which led us to Eq. (22). The problem is to devise an approximate form for $\nabla \cdot \mathbf{D}$ which retains the essential features of the expression in Eq. (22) and at the same time allows as many as possible of the integrals appearing in the theory to be evaluated analytically.

Since $\nabla \cdot \mathbf{D}$ vanishes for times $t > t_m$ we only have to consider the mixed region for the interval $0 \leqslant t \leqslant t_m$. In this time interval we suggest that (a) the boundary† $r(t) = Ar_0\sqrt[3]{t}$ of the mixed region be replaced by a square region of the same area, as in Fig. 10a, (b) the variation of the square boundary with time then be approximated by a series of linear laws, as in Fig. 10b, and (c) the divergence of the mean turbulent density flux be approximated by

$$-\nabla \cdot \mathbf{D} = \beta \frac{d\rho^*}{dz} y\delta(t - t_b(x, y)), \tag{56}$$

in which $y = z_0 - z$, and $t_b(x, y)$ designates the time at which the boundary reaches the point (x, y). This expression for $\nabla \cdot \mathbf{D}$ leads to a density change caused by mixing of the form

$$\delta\rho = \beta \frac{d\rho^*}{dz}(z_0 - z)$$

inside the square region. Thus we would have a constant degree of mixing inside the square region.

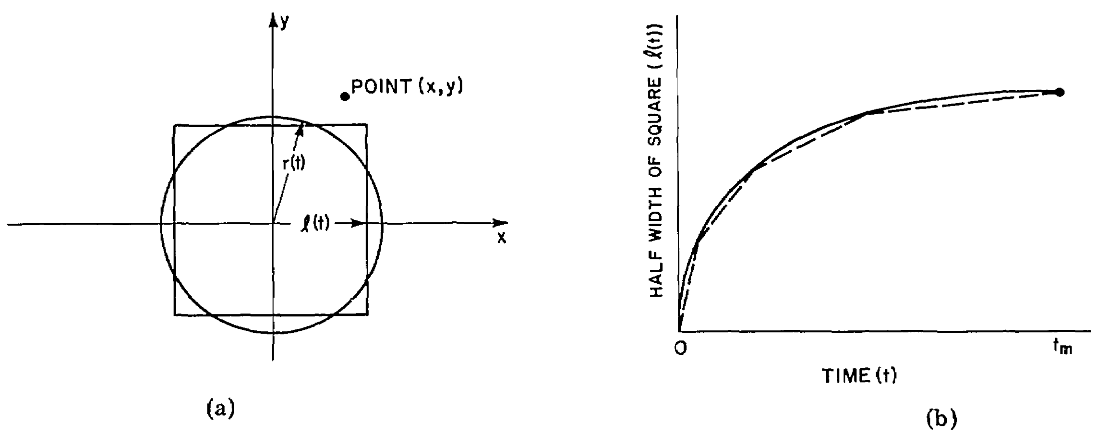


Fig. 10—(a) Square of side $2\ell(t)$ used to replace circular region, (b)———— $\ell(t)$ proportional to $t^{1/3}$ ---------series of linear growth laws for $\ell(t)$ to approximate the solid curve.

($t = 0$ is the virtual origin of the mixed region)

In Eq. (56) the quantity $t_b(x, y)$ depends linearly on x and y because a linear growth rate is assumed for the boundary. This property of $t_b(x, y)$ should enable the integrals appearing in the theory, apart from the final integration over frequency, to be evaluated in analytic terms since the integrands will be of trigonometric type.

† the procedure outlined can be applied to any growth law for the radius of the mixed region.

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13. ABSTRACT

We formulate a mathematical model for the mean disturbance created in a linearly stratified fluid by the growth and collapse of a localized region of turbulent, mixed fluid. Linearized differential equations for the disturbance are obtained from the Navier-Stokes equations under the Boussinesq approximation after various intuitive and empirical considerations. The divergence of the mean turbulent density flux enters these equations as the sole forcing function for the disturbance; this differs from previous theoretical models in which an initial, perturbed density field acts as the forcing function.

We replace the original expression we find for the forcing function by an approximation devised to simplify the numerical evaluation of the disturbance. We then obtain a solution for the disturbance in a fluid of constant depth and compare the theoretical results with measured data. The comparison indicates that the theory provides satisfactory results at large distances from the mixed region but has deficiencies near the mixed region. Specific proposals are made for correcting the deficiences of the model at small distances from the mixed region.

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