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On the Entire Solutions of a Certain Class of Nonlinear Differential Equations

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On the Entire Solutions of a Certain Class of Nonlinear Differential Equations

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Abstract: By using the fundamental theorems of Nevanlinna theory for meromorphic functions, one can determine whether the following type of nonlinear differential equation

$$P_n(z)f^n(z) + P_{n-1}(z)f^{n-1}(z) + \ldots + P_1(z)f(z) + P_0(z) + f'(z)$$

$$= Q_1(z)e^{mp(z)} + Q_2(z)e^{kp(z)}$$

has entire solutions or not, where $p(z)$, $P_i(z)$ $(i = 0,1,2,\ldots n)$ and $Q_j(z)$ $(j = 1,2)$ are polynomials, and m and k are integers.

Some problems on the distribution of values of meromorphic functions eventually lead to the problem of whether certain differential polynomials (see Hayman [1]) in a given function $f(z)$ necessarily have zeros.

In this note we shall show how to use the Nevanlinna fundamental theorems of meromorphic functions to determine whether a certain class of nonlinear differential equations has entire solutions or not. Here and in the sequel it is assumed that the reader is familiar with the Nevanlinna functionals $T(r,f)$, $m(r,f)$, $S(r,f)$, *etc.* We begin with the following:

THEOREM 1. *Let $p(z)$, $Q(z)$ be polynomials. Then the nonlinear differential equation*

$$f^3 - f' = p(z)e^{3z} - Q(z)e^z \tag{1}$$

has an entire solution if and only if $[p(z)]^{1/3}$ is a polynomial and

$$Q(z) = c\left[p^{1/3}(z) + \frac{1}{3}p^{-2/3}(z)p'(z)\right],$$

where c is a cubic root of unity. The solution, if it exists, is unique, i.e., $f = cp^{1/3}(z)e^z$.

We shall need the following Lemma.

LEMMA (CLUNIE, SEE HAYMAN [1]). *Suppose that $f(z)$ is meromorphic and transcendental in the plane and that $f^n(z)p(f) = Q(f)$ holds, where $p(f)$, $Q(f)$ are differential polynomials in f and the degree of $Q(f)$ is at most n. Then $m\{r,p(f)\} = S(r,f)$ as $r \to +\infty$.*

Proof of the Theorem. By differentiating both sides of (1) we have

$$3f^2f' - f'' = H(z)e^{3z} - K(z)e^z, \tag{2}$$

where $H(z) \equiv p'(z) + 3p(z)$, $K(z) \equiv Q'(z) + Q(z)$.

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From (1) and (2) we obtain

$$e^{3z} = \frac{K(z)(f^3 - f') - Q(z)\ (3f^2f' - f'')}{T(z)}$$

and

$$e^{z} = \frac{H(z)(f^3 - f') - p(z)\ (3f^2f' - f'')}{T(z)},$$

where

$$T(z) = p(z)K(z) - H(z)Q(z) \not\equiv 0.$$

Eliminating e^{3z} and e^z, we have

$$\begin{aligned}[K(z)\ (f^3 - f') - Q(z)\ (3f^2f' - f'')]T(z) \\ = [H(z)\ (f^3 - f') - p(z)\ (3f^2f' - f'')]^3 \\ = [f^2(H(z)f - 3p(z)f') + pf'' - Hf']^3,\end{aligned}$$

so

$$\begin{aligned}f^6(Hf - 3pf')^3 + 3f^4(Hf - 3pf')^2(pf'' - Hf') \\ + 3f^2(Hf - 3pf')(pf'' - Hf')^2 + (pf'' - Hf')^3 \\ = K(z)f^3 - 3QT^2f^2f' + QT^2f'' - Kf' = p_3(f).\end{aligned}$$

Thus

$$\begin{aligned}f^3\left(f^3(Hf - 3pf')^3 + 3f(Hf - 3pf')^2(pf'' - Hf') + 3(pf'' - Hf')^2 \cdot \frac{(Hf - 3pf')}{f}\right) \\ = p_3(f) - (pf'' - Hf')^3 \equiv Q_3(f).\end{aligned}$$

We note $m\left(r, \dfrac{Hf - 3pf'}{f}\right) = S(r,f)$ and follow the argument of the proof of Clunie's Lemma to conclude

$$m(r,a(z)) = S(r,f), \tag{3}$$

where

$$a(z) = f^3(Hf - 3pf')^3 + 3f(Hf - 3pf')^2(pf'' - Hf') + 3\,\frac{(Hf - 3pf')}{f} \cdot (pf'' - Hf')^2.$$

Thus

$$\begin{aligned}f^3\left[(Hf - 3pf')^3 + 3\,\frac{Hf - 3pf'}{f}\,\frac{Hf - 3pf'}{f}\,(pf'' - Hf')\right] \\ = a(z) - 3\,(pf'' - Hf')^2\,\frac{Hf - 3pf'}{f}.\end{aligned} \tag{4}$$

By the same argument we can conclude from (4) that

$$m(r,b(z)) = S(r,f),$$

where $b(z) = (Hf - 3pf')^3 + 3(Hf - 3pf') \cdot \dfrac{Hf - 3pf'}{f} \cdot \dfrac{(pf'' - Hf')}{f}$.

Again by using Clunie's Lemma with respect to the function $Hf - 3pf'$ and noting that $m\left(r, \dfrac{pf'' - Hf'}{f}\right) = S(r,f)$, we conclude finally

$$m(r, Hf - 3pf') = S(r,f).$$

By noting that any possible solution f of (1) is entire, we also have

$$T(r, Hf - 3pf') = S(r,f).$$

Now if $Hf - 3pf' \not\equiv 0$, then from (3) we would have

$$\begin{aligned} m(r, f^3(Hf - 3pf')^3) &= m\Big\{r, a(z) - 3f(Hf - 3pf')^2(pf'' - Hf') \\ &\quad + 3\frac{Hf - 3pf'}{f} \cdot (pf'' - Hf')^2\Big\} \\ &= m\left\{r, a(z) - 3f(pf'' - Hf')\ (Hf - 3pf')^2 \frac{Hf - 3pf'}{f} \cdot \frac{pf'' - Hf'}{f}\right\} \\ &\leqq (2 + o(1))\ T(r,f), \end{aligned}$$

except on a set of r values of finite length.

Thus

$$T(r, f^3(Hf - 3pf')^3) \leqq (2 + o(1))\, T(r,f)$$

except on a set of r values of finite length. This is impossible since

$$\begin{aligned} T(r, f^3(Hf - 3pf')^3) &\geqq T(r, f^3) - T(r, (Hf - 3pf')^3) \\ &= (3 - o(1))\ T(r,f). \end{aligned}$$ Hence

$$Hf - 3pf' \equiv 0. \tag{5}$$

From (5) and the definition of $H(z)$ we get $f(z) = cp^{1/3}(z)e^z$ where c is a constant and is not equal to zero. By substituting this expression into Eq. (1) it is easy to verify that

$$Q(z) = c\left[p^{1/3}(z) + \frac{1}{3}p^{-2/3}(z)p'(z)\right] \quad \text{and} \quad c^3 = 1.$$

Hence the theorem is proved.

Remark. Observing the above argument, one can show that the nonlinear differential equation $R(z)f^n(z) - f'(z) = T(z)e^{Kz} - S(z)e^{Lz}$ has no meromorphic function solution, where $R(z)$, $T(z)$, $S(z)$ are rational functions and K,L are integers, unless $K/L=n$ or $L/K=n$, and if the solution exists, it must be of the form $f(z)=u(z)e^{mz}$, where $u(z)$ is a rational function and m is an integer.

The identical argument can also be used to show:

THEOREM 2. *The following nonlinear differential equation*

$$p_1(z)f^3+p_2(z)f+p_3(z)f'=Q_1(z)e^{3z}+Q_2(z)e^z,$$

where $p_1(z)$, $p_2(z)$, $p_3(z)$, $Q_1(z)$, $Q_2(z)$ *are polynomials, has no meromorphic function solution other than one of the form* $p(z)e^z$ *where* $p(z)$ *is a rational function.*

Remark. The argument of this paper does seem to work for the following more general class of nonlinear differential equations.

$$p_n(z)f^n(z)+p_{n-2}(z)f^{n-2}(z)+\ldots+\ldots+p_o(z)+f'(z)$$
$$=Q_1(z)e^{mp(z)}+Q_2(z)e^{kp(z)},$$

where $p(z)$ is a polynomial, m,k are integers and $p_i(z)$ $(i=0,1,2,\ldots n-2,n)$, $Q_j(z)$ $(j=1,2)$ are rational functions.

More generally, by putting

$$f(z)=g(z)-\frac{p_{n-1}(z)}{p_n(z)},$$

one also can determine whether the following type of differential equation has entire solutions or not:

$$p_n(z)f^n(z)+p_{n-1}(z)f^{n-1}(z)+Q_{n-2}(f)=Q_1(z)e^{mp(z)}+Q_2(z)e^{kp(z)},$$

where $Q_{n-2}(f)$ denotes a differential polynomial in f of degree at most $n-2$ with polynomials as coefficients. If the solution exists, it must have the form $f(z)=A(z)+B(z)e^{tp(z)}$, where $A(z)$, $B(z)$ are rational functions and t is a rational number.

THEOREM 3. *The following class of nonlinear differential equations*

$$p_o(z)f^n(z)+p_1(z)f^{n-1}(z)+\ldots+p_n(z)$$
$$=f^{n-1}(z)+\pi_{n-2}(f)+\pi_{n-3}(f)+\ldots+\pi_1(f)+p(z) \tag{6}$$

has no transcendental entire solutions, where $p(z)$, $p_i(z)$ $(i=0,1,2,\ldots n)$ *are rational functions and* $\pi_m(f)$ *denotes a homogeneous differential polynomial of degree* m *with rational functions as coefficients.*

Proof of the Theorem. Assume $f(z)$ is a transcendental entire solution of Eq. (6). Then, by a result of Varliron [2] we have

$$m(r,p_o(z)f^n(z)+p_1(z)f^{n-1}(z)+\ldots+p(z))$$
$$=T(r,p_o(z)f^n(z)+p_1(z)f^{n-1}(z)+\ldots+p(z))$$
$$\sim nT(r,f)+O\log r \text{ as } r\to+\infty. \tag{7}$$

But

$$m(r,f^{n-1}(z) + \pi_{n-2}(f) + \pi_{n-3}(f) + \ldots + \pi_1(f) + p(z))$$

$$\leqq m\left(r,f\left(f^{n-2} + \frac{\pi_{n-2}(f)}{f} + \ldots + \frac{\pi_1(f)}{f}\right)\right) + O \log r$$

$$\leqq m(r,f) + m\left(r,f^{n-2} + \frac{\pi_{n-2}(f)}{f} + \ldots + \frac{\pi_2(f)}{f}\right) + S(r,f)$$

$$= m(r,f) + m\left(r,f\left(f^{n-3} + \frac{\pi_{n-2}(f)}{f^2} + \ldots + \frac{\pi_2(f)}{f^2}\right)\right) + S(r,f)$$

$$\leqq m(r,f) + m(r,f) + m\left(r,f^{n-3} + \frac{\pi_{n-2}(f)}{f^2} + \ldots + \frac{\pi_3(f)}{f^2}\right) + S(r,f).$$

(Here we use the fact that $m\left(r, \frac{\pi_k(f)}{f^k}\right) = S(r,f)$.) By repeating the argument we can deduce that

$$m(r,f^{n-1}(z) + \pi_{n-2}(f) + \ldots + \pi_1(f) + p(z)) \leqq (n-1)\, m(r,f) + S(r,f). \tag{8}$$

Expressions (6), (7), and (8) will lead to a contradiction. Hence (6) has no transcendental entire solutions.

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2. Varliron, G., "Sur La dérivér des fonctions algébröides," *Bull. Soc. Math. France*, **59**:17-39 (1929)

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