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The Action of Dither in Digital Matched Filters

J. J. FREEMAN

Security Systems Branch
Communications Sciences Division

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NAVAL RESEARCH LABORATORY
Washington, D.C.

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CONTENTS

THE ACTION OF DITHER IN DIGITAL MATCHED FILTERS

INTRODUCTION

This report aims at a detailed examination of how dither, a large, suitably fluctuating component added to the input of a digital correlator or matched filter can eliminate capture of a weak signal by an unwanted stronger one.

Although there exists a large literature [1-3] on the use of dither with digital correlators, to the author's best knowledge the only article dealing with the use of dither in matched filters in the context of communications is the one by Cahn [4] which applies game theory to determine the optimum distribution of dither against unknown interference. This report analyzes the behavior of various correlators with specific dithers in certain specific environments.

In the next section the classical time-sampled correlator and the polarity coincidence detector with and without dither will be described, and the action of dither will be qualitatively explained. The output signal-to-noise ratio $(\mathrm{Sig/Noi})_{\mathrm{out}}$ for the classical correlator and for the polarity coincidence detector with and without dither will be analyzed in the third section for three types of interference: Gaussian noise, *CW*, and rectangular pulses. Two types of dither will be considered: a random voltage with an amplitude uniformly distributed between the plus-and-minus peak values and a sinusoidal voltage. We will see that if the baseband signal consists of a sequence of bipolar pulses, and if the baseband jamming has the same waveform but a larger peak amplitude, then the output from a polarity coincidence detector will consist of all interference and no signal. However, if a dither voltage is added whose peak exceeds that of the interference, then the complete capture of the receiver by the jamming is avoided. On the other hand, if the dither voltage is too large, then $(\mathrm{Sig/Noi})_{\mathrm{out}}$ will become degraded. Accordingly, in the fourth section we analyze the situation in which the interference power is continuously measured and the peak dither voltage is continuously adjusted so as to exceed the peak interference by a given factor, say g, which at least exceeds unity. Graphs of $(\mathrm{Sig/Noi})_{\mathrm{out}}$ resulting from this procedure are given as a function of g for the different types of interference, allowing a determination of the optimum value of g for each type of dither and a comparison of the stability of these optimum operating points. In general we will find that uniformly distributed random dither is superior to the sinusoidal type, for the jamming environments considered. For uniform dither at the optimum value of g, the value of $(\mathrm{Sig/Noi})_{\mathrm{out}}$ for the worst-case interference was 4.3 dB below the value which would have resulted from the optimum classical correlator.

BACKGROUND

Consider a signal $s(t)$ (Fig. 1) consisting of a sequence of N bipolar pulses, each of width Δ:

$$s(t) = S \sum_{j=1}^{N} \mu_j g(t-t_j) , \tag{1}$$

where S = signal amplitude, $\mu_j = \pm 1$, $g(t) = 0$, $t < 0$ and $t > \Delta$, and $g(t) = 1$, $0 < t < \Delta$.

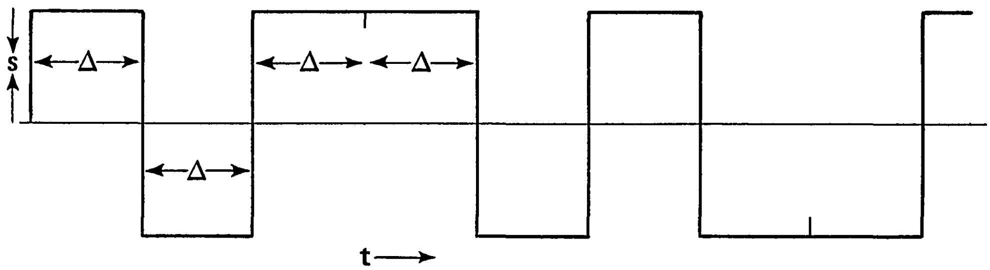


Fig. 1—Signal waveform $s(t)$

Let $n(t)$ designate the input noise or interference (sinusoidal, normal, or whatever), and let $s'(t)$ designate a stored replica of $s(t)$ with unit amplitude, so that $s(t) = Ss'(t)$. Then a functional diagram of a matched filter or correlation detector which corresponds to coherent detection is as shown in Fig. 2.

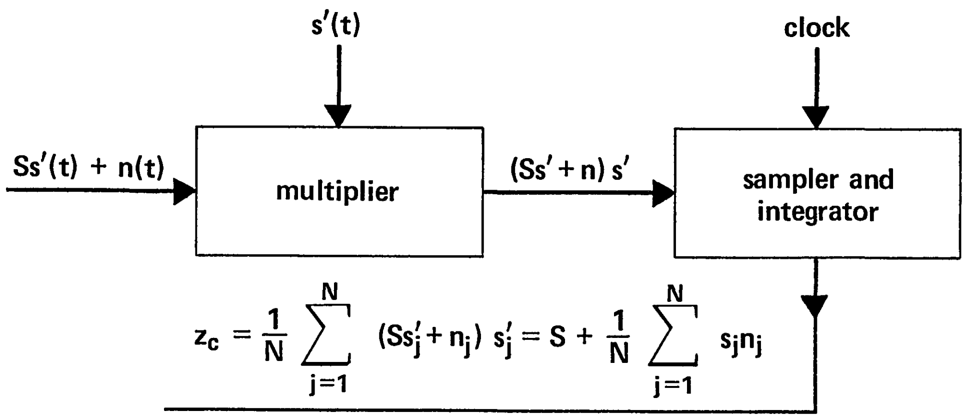


Fig. 2—Classical time-sampled correlation detector. The signal $s(t) = Ss'(t)$ is given by Eq. (1), and Eq. (2) gives the detection statistic z_c.

Let $n_j = n(t_j)$ and $s_j = s(t_j)$; then the detection statistic for the correlation detector is

$$z_c = \frac{1}{N} \sum_{j=1}^{N} (Ss_j' + n_j) s_j' . \tag{2}$$

Although the correlation detector is optimum (for white Gaussian noise), its implementation leads to many practical difficulties. An approximation to the classical correlation detector known as the polarity coincidence detector (Fig. 3), or as a digital matched filter, is adaptable to implementation by microelectronic techniques with the advantages of increased reliability and decreased weight, size, and cost.

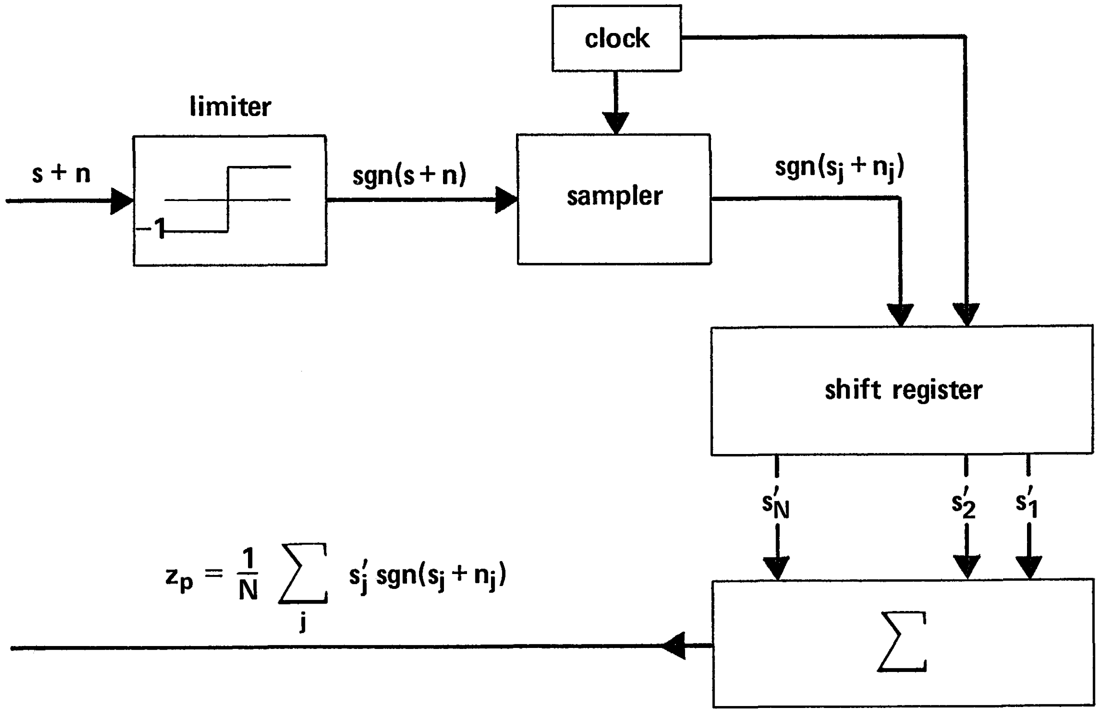


Fig. 3—Polarity coincidence detector

The statistic z_p obtained by a polarity coincidence detector is given by

$$z_p = \frac{1}{N} \sum_{j=1}^{N} \operatorname{sgn} s_j \operatorname{sgn}(s_j + n_j) , \tag{3}$$

where

$$\begin{aligned} \operatorname{sgn} x &= 1 , \quad x > 0 , \\ &= 0 , \quad x = 0 , \\ &= -1 , \quad x < 0 . \end{aligned}$$

Since Eq. (3) indicates that the coincidence detector takes the correlation of the 1-bit approximation of the input waveform with the stored replica of the signal, one expects that its behavior will be less efficient than that of the classical correlator. However, this may be more than offset by the capability of realizing a much larger storage capacity (N bits) for a coincidence detector than for a correlator.

Unfortunately the coincidence detector suffers from the defect that a rectangular jamming signal can completely capture the receiver, so that no signal output whatever emerges. This is illustrated in Fig. 4, which depicts an input signal together with a rectangular jamming signal of larger amplitude. Since $|n_j| > |s_j|$, sgn $(s_j + n_j)$ = sgn (n_j), and no signal information emerges. However, this defect can be remedied by introducing

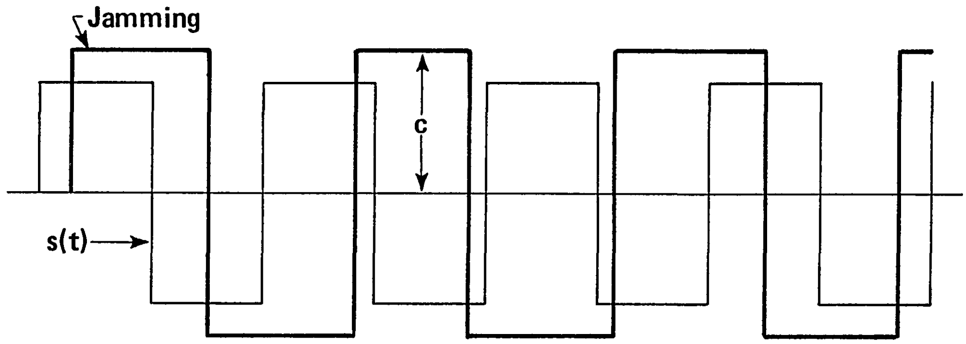


Fig. 4—Signal with rectangular jamming

a suitable noise or dither, in series with the input, as in Fig. 5. If we designate the dither by $d(t)$, then the statistic for the detector in Fig. 5 is given by

$$z_d = \frac{1}{N} \sum_{j=1}^{N} \operatorname{sgn} s_j \operatorname{sgn}(s_j+n_j+d_j) , \tag{4}$$

where $d_j = d(t_j)$.

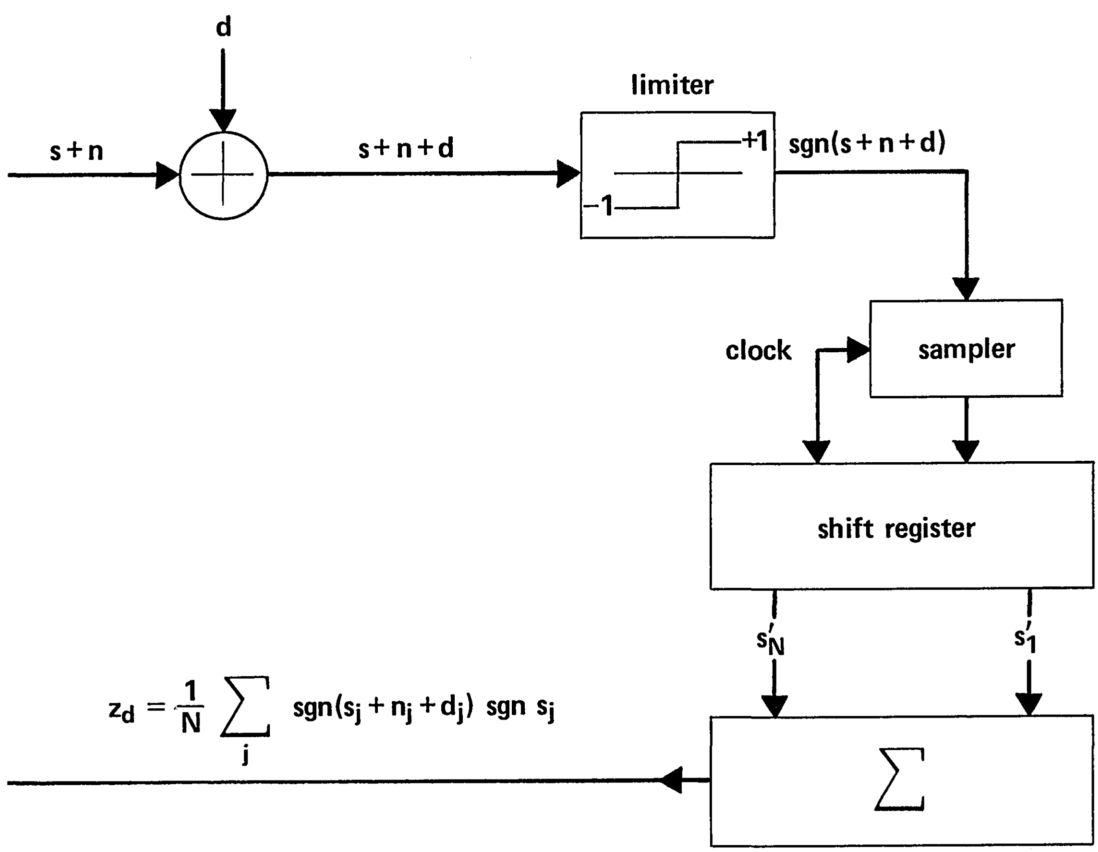


Fig. 5—Polarity coincidence detector plus dither (CW dither in the case shown)

We will use the output signal-to-noise ratio defined by

$$(\mathrm{Sig/Noi})_{\mathrm{out}} = \bar{z}^2 \Big/ \overline{(z-\bar{z})^2}$$

as the criterion of performance and compute $(\mathrm{Sig/Noi})_{\mathrm{out}}$ for the case that $d(t) = A \sin \omega_0 t$ and for the case that $d(t)$ is uniformly distributed over the range $-d_0 \leqslant d(t) \leqslant d_0$; in the latter case we assume that the values of d_j and d_k are statistically independent for $j \neq k$.

First we will give an intuitive explanation of how dither works. We assume there is a rectangular jamming noise whose shifts of polarity occur randomly and whose amplitude exceeds that of the signal (Fig. 4). Only two possibilities exist at each sampling instant t_j: either the jamming n_j and the signal s_j have opposite polarities (Fig. 6a) or they have the same polarity (Fig. 6c). In the first case an error inevitably results in the absence of dither. Suppose, however, one has a dither voltage d_j which can take on any value from $-B$ to B, where $B = |s_j| + |n_j|$. Then if d_j lies in the dotted portion of Fig. 6b, that is, if $s_j + d_j > -n_j$, then $\mathrm{sgn}(s_j + n_j + d_j) = \mathrm{sgn}\, s_j$ and what otherwise would have been an error would be corrected by the dither. Suppose now that n_j and s_j have the same polarity (Fig. 6c). In the absence of dither the receiver would read correctly. However the presence of dither could cause an error if $d_j = -B$, at which time $\mathrm{sgn}(s_j + n_j + d_j) = 0$. However the probability of this event occurring is zero. Accordingly, for the particular value

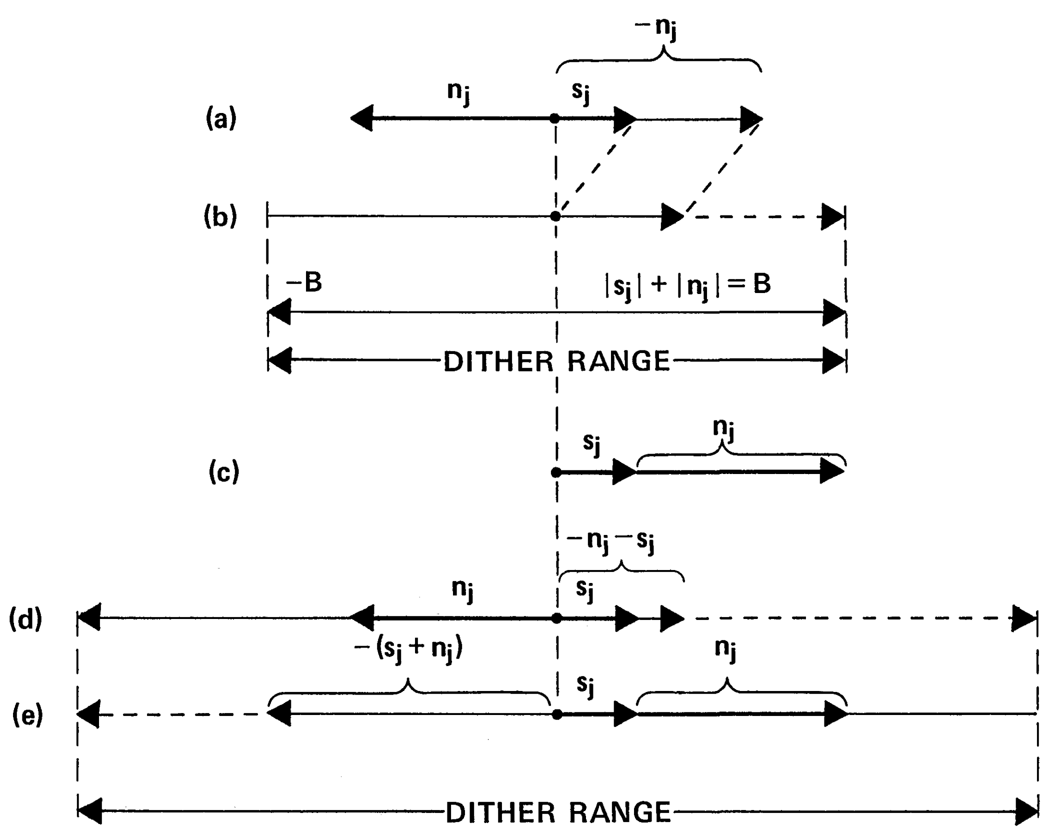


Fig. 6—Diagram (explained in the text) of how dither reduces capture

of B chosen, namely $B = |s_j| + |n_j|$, dither sometimes corrects an error and never causes one. Suppose now that the range of the dither voltage is increased, as in Fig. 6d and 6e. In Fig. 6d, where n_j and s_j have opposite polarities, so long as d_j lies in the dotted area, what would have been an error without the dither will be corrected. In Fig. 6e, where s_j and n_j are parallel, what would have been a correct indication without dither will become a false indication if the dither falls within the dotted region. Obviously the net result of the dither is helpful, but not to the extent it was in Fig. 6b. Subtracting the dotted length in Fig. 6e from that in Fig. 6d will result in a net length equal to the dotted length in Fig. 6b. However the net probability of error reduction is less than in Fig. 6b, since to obtain probabilities one must divide the resultant dotted length by the overall length, which is $2B$ in Fig. 6b and $>2B$ in Figs. 6d and 6e. Accordingly the optimum range of dither should be $|d| = |n| + |s|$. This implies that the peak values of $|s| + |n|$ should be measured, and $d(t)$ adjusted accordingly.

In the following section we will compute $(\mathrm{Sig/Noi})_{\mathrm{out}}$ for the three detectors, namely, the classical correlation detector, the polarity coincidence detector, and the polarity coincidence detector with dither, for three types of jamming: CW or sinusoidal interference, Gaussian noise, and rectangular randomly crossing pulses.

ANALYSIS

Case 1: Classical Correlation Detector

From Eq. (2) the statistic for a classical correlation detector is

$$z_c = \frac{1}{NS} \sum_{j=1}^{N} (s_j + n_j)\, s_j \, .$$

Further

$$\overline{z}_c = \frac{1}{NS} \sum_{j=1}^{N} s_j^2 = S \, . \tag{5}$$

and

$$\overline{(z_c - \overline{z}_c)^2} = \frac{1}{N^2 S^2} \sum_{j=1}^{N} \sum_{k=1}^{N} s_j\, s_k\, n_j\, n_k \, . \tag{6}$$

We are interested in evaluating the relative effectiveness of different detectors over a whole class of signals, not just for a single signal. Accordingly we consider the s_j values as random independent parameters which take on specific values for a given message, but we will assume that over the entire set of possible messages the s_j values will take on the two possible values $\pm S$ with equal probability. Accordingly $\overline{s_j} = 0$ and $\overline{s_j s_k} = \overline{s_j}\,\overline{s_k} = 0$,

$j \neq k$. Thus, since $\overline{s_j^2} = S^2$, $\overline{s_j s_k} = \delta_k{}^j S^2$, where $\delta_k{}^j = 0, j \neq k$, and $\delta_k{}^j = 1, j = k$. Averaging Eq. (6) over the s_j yields

$$\overline{(z_c - \bar{z}_c)^2} = \frac{1}{N^2 S^2} \sum_{j=1}^{N} \sum_{k=1}^{N} \overline{s_j s_k}\ \overline{n_j n_k} = \frac{1}{N^2} \sum_{j=1}^{N} \sum_{k=1}^{N} \delta_k{}^j \overline{n_j n_k} = \frac{\overline{n^2}}{N}. \tag{7}$$

From Eqs. (5) and (7) one has for any type of noise or interference

$$(\mathrm{Sig/Noi})_{\mathrm{out}} = \bar{z}_c{}^2 \Big/ \overline{(z_c - z_c)^2} = NS^2/\overline{n^2} = N(\mathrm{Sig/Noi})_{\mathrm{in}} , \tag{8}$$

where $(\mathrm{Sig/Noi})_{\mathrm{in}}$ indicates the signal-to-noise power ratio at the input.

Case 2: Polarity Coincidence Detector

From Eq. (3) the statistic for a polarity coincidence detector is

$$z_p = \frac{1}{N} \sum_{j=1}^{N} \operatorname{sgn} s_j \operatorname{sgn}(s_j + n_j) . \tag{9}$$

Averaging Eq. (9) over the independent random variables s_j, which take on the values $\pm S$ each with probability 1/2,

$$\bar{z}_p{}^{s_j} = \frac{1}{2N} \sum_{j=1}^{N} [\operatorname{sgn}(S + n_j) - \operatorname{sgn}(-S + n_j)] . \tag{10}$$

Averaging now over the n_j (which all have the same distribution),

$$\bar{z}_p = \frac{1}{2} [\mathrm{Prob}(S+n>0) - \mathrm{Prob}(S+n<0) - \mathrm{Prob}(-S+n>0) + \mathrm{Prob}(-S+n<0)] . \tag{11}$$

We assume that the probability density of the noise is symmetric about the origin, so that

$$\mathrm{Prob}(n > -S) = \mathrm{Prob}(n < S) \tag{12a}$$

and

$$\mathrm{Prob}(n < -S) = \mathrm{Prob}(n > S) . \tag{12b}$$

Use of Eqs. (12) reduces Eq. (11) to

$$\bar{z}_p = \mathrm{Prob}(n < S) - \mathrm{Prob}(n > S) = 1 - 2\,\mathrm{Prob}(n > S) . \tag{13}$$

Consider now the square of Eq. (9), namely

$$z_p^2 = \frac{1}{N^2} \sum_{j=1}^{N} \sum_{k=1}^{N} \operatorname{sgn} s_j \operatorname{sgn} s_k \operatorname{sgn}(s_j + n_j) \operatorname{sgn}(s_k + n_k) \tag{14}$$

or

$$z_p^2 = \frac{1}{N^2} \sum_{j=1}^{N} \operatorname{sgn}^2 s_j \operatorname{sgn}^2(s_j + n_j)$$

$$+ \frac{1}{N^2} \sum_{\substack{j=1 \\ (j \neq k)}}^{N} \sum_{k=1}^{N} \operatorname{sgn} s_j \operatorname{sgn} s_k \operatorname{sgn}(s_j + n_j) \operatorname{sgn}(s_k + n_k) . \tag{15}$$

When z_p^2 is averaged over s_j, and s_k.

$$\overline{z_p^2}^{s_j} = \frac{1}{N} + \frac{1}{4N^2} \sum_{\substack{j=1 \\ (j \neq k)}}^{N} \sum_{k=1}^{N} [\operatorname{sgn}(S + n_j)$$

$$- \operatorname{sgn}(-S + n_j)] [\operatorname{sgn}(S + n_k) - \operatorname{sgn}(-S + n_k)] . \tag{16}$$

For $S = 0$ (and for S large) the double sum in Eq. (16) vanishes. Thus for small input signal-to-noise ratios we may approximate

$$\overline{z_p^2} = 1/N , \tag{17}$$

regardless of the distribution of the noise. Accordingly, for small signal power compared to the interference,

$$(\text{Sig/Noi})_{\text{out}} = \frac{\bar{z}_p^2}{\overline{(z_p - \bar{z_p})^2}} \simeq \frac{\bar{z}_p^2}{\overline{z_p^2}} = N[1 - 2 \operatorname{Prob}(n > S)]^2 . \tag{18}$$

We proceed to evalûate Eq. (18) for Gaussian-noise, rectangular-pulse, and *CW* interference.

Gaussian Noise Interference—For $n(t)$ Gaussian $(\overline{n^2} = \sigma_n^2)$,

$$\operatorname{Prob}(n > S) = \frac{1}{\sqrt{2\pi}\, \sigma_n} \int_S^{\infty} e^{-x^2/2\sigma_n^2} \, dx = \frac{1}{2} - \frac{1}{2} \operatorname{erf}(S/\sqrt{2}\, \sigma_n) , \tag{19}$$

where

$$\operatorname{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2}\, dt\,. \tag{20}$$

Substituting Eq. (19) into Eq. (18), we have

$$(\text{Sig/Noi})_{\text{out}} = N \operatorname{erf}^2(S/\sqrt{2}\,\sigma_n)\,. \tag{21}$$

Since

$$\operatorname{erf} x \approx \frac{2x}{\sqrt{\pi}} \quad \text{for} \quad x < 0.3\,,$$

$$(\text{Sig/Noi})_{\text{out}} = \frac{2N}{\pi}\, S^2\Big/\sigma_n{}^2 = \frac{2N}{\pi}\, (\text{Sig/Noi})_{\text{in}}\,. \tag{22}$$

For Gaussian noise, $(\text{Sig/Noi})_{\text{out}}$ is 1.95 dB below that of a classical correlator, given by Eq. (8).

Rectangular Pulse Interference—If $n(t)$ is a sequence of random pulses with random changes of polarity and whose amplitude always exceeds that of $s(t)$, then $\text{Prob}(n > S) = 1/2$ and $\text{Prob}(n < -S) = 1/2$. Substituting the value 1/2 into Eq. (18), we have

$$(\text{Sig/Noi})_{\text{out}} = 0 \tag{23}$$

for rectangular pulse interference whose amplitude exceeds the signal. However, if the interference amplitude is less than the signal, the interference produces no effect.

CW Interference—*CW* interference is given by

$$n(t) = J \sin(\omega t + \varphi)\,, \tag{24}$$

where φ is uniformly and randomly distributed over the interval $[-\pi \leqslant \varphi \leqslant \pi]$. We let $\omega t_j + \varphi = \psi$, and we let $Z = J \sin \psi$. Then the probability density of Z is given by

$$p(Z) = \frac{1}{\pi(J^2 - Z^2)^{1/2}}\,, \qquad |Z| < J\,, \tag{25a}$$

$$= 0 \qquad\qquad , \qquad |Z| > J\,. \tag{25b}$$

Therefore

$$\text{Prob}(n > S) = \frac{1}{\pi} \int_S^J \frac{dZ}{(J^2 - Z^2)^{1/2}} = \frac{1}{2} - \frac{1}{\pi} \arcsin S/J\,. \tag{26}$$

Substituting Eq. (26) into Eq. (18), we obtain

$$(\text{Sig/Noi})_{\text{out}} = \frac{4}{\pi^2} N \arcsin^2(S/J) . \tag{27}$$

For S/J small, arcsin $(S/J) \approx S/J$, so that

$$(\text{Sig/Noi})_{\text{out}} = \frac{2}{\pi^2} N \left(S^2\Big/\frac{1}{2} J^2\right) = \frac{2}{\pi^2} N(\text{Sig/Noi})_{\text{in}} . \tag{28}$$

Comparing Eq. (28) with Eq. (8), we can determine that for *CW* interference the polarity coincidence detector has a $(\text{Sig/Noi})_{\text{out}}$ which is 6.9 dB below that of a classical correlator.

Case 3: Polarity Correlation Modified by Dither

In considering polarity correlation modified by dither we consider first uniformly distributed dither and then sinusoidal dither. We let $d(t)$ be a random variable distributed over the range $-d_0 \leqslant d(t) \leqslant d_0$ and let $d_j = d(t_j)$ be independent of d_k, for $k \neq j$. The detection statistic z_d for this case is from Eq. (4)

$$z_d = \frac{1}{N} \sum_{j=1}^{N} s_j' \operatorname{sgn}(s_j + n_j + d_j) , \tag{29}$$

where $s_j' = \operatorname{sgn} s_j$. We let $n_j + d_j = H_j$. Comparing Eq. (29) with Eq. (9), we see that the expressions become identical if $H(t)$ replaces $n(t)$ in Eq. (9). Making this substitution into Eq. (18), we find that the general expression for $(\text{Sig/Noi})_{\text{out}}$ is given by

$$(\text{Sig/Noi})_{\text{out}} = N[1-2\ \text{Prob}(H>S)]^2 . \tag{30}$$

Since H is assumed to be symmetrically distributed about the origin,

$$\text{Prob}(H>S) = \text{Prob}(H>0) - \text{Prob}(0<H<S) = \frac{1}{2} - \text{Prob}(0<H<S) .$$

Thus Eq. (30) becomes

$$(\text{Sig/Noi})_{\text{out}} = 4N[\text{Prob}(0<n+d<S)]^2 . \tag{31}$$

If $p_H(z)$ designates the probability density of H, then the characteristic function of p_H is defined to be

$$\int_{-\infty}^{\infty} e^{iz\xi}\, p_H(z)\, dz = \overline{e^{in\xi}}\ \overline{e^{id\xi}} , \tag{32}$$

since the dither and interference are independent. Therefore, by the Fourier integral theorem,

$$p_H(z) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-i\xi z} \, \overline{e^{i\xi d}} \, \overline{e^{i\xi n}} \, d\xi \, . \tag{33}$$

Therefore

$$\text{Prob}(0 < H < S) = \int_0^S p_H(z) \, dz = \frac{1}{2\pi} \int_{-\infty}^{\infty} \overline{e^{i\xi n}} \, \overline{e^{i\xi d}} \left(\frac{e^{-i\xi S} - 1}{-i\xi} \right) d\xi \, . \tag{34}$$

Since $\overline{e^{i\xi d}}$ and $\overline{e^{i\xi n}}$ are even in ξ (because n and d are symmetric about zero), and since $e^{-i\xi S} = \cos \xi S - i \sin \xi S$ with $\cos \xi S$ being even in ξ,

$$\text{Prob}(0 < H < S) = \frac{S}{\pi} \int_0^{\infty} \overline{e^{i\xi n}} \, \overline{e^{i\xi d}} \, \frac{\sin \xi S}{\xi S} \, d\xi \, . \tag{35}$$

Uniformly Distributed Dither—In the case of uniformly distributed dither

$$\overline{e^{i\xi d}} = \frac{1}{2d_0} \int_{-d_0}^{d_0} e^{i\xi\alpha} \, d\alpha = \frac{\sin \xi d_0}{\xi d_0} \, , \tag{36}$$

and Eq. (35) becomes

$$\text{Prob}(0 < H < S) = \frac{S}{\pi} \int_0^{\infty} \overline{e^{i\xi n}} \, \frac{\sin \xi d_0}{\xi d_0} \, \frac{\sin \xi S}{\xi S} \, d\xi \, . \tag{37}$$

Substituting $u = \xi d_0$ into Eq. (37), we obtain

$$\text{Prob}(0 < H < S) = \frac{S}{\pi d_0} \int_0^{\infty} \overline{e^{iun/d_0}} \, \frac{\sin u}{u} \, \frac{\sin(uS/d_0)}{uS/d_0} \, du \, . \tag{38}$$

Since we are interested in small input signals, for which $S/d_0 << 1$, Eq. (38) reduces to

$$\text{Prob}(0 < H < S) = \frac{S}{\pi d_0} \int_0^{\infty} \overline{e^{iun/d_0}} \, \frac{\sin u}{u} \, du \, . \tag{39}$$

We proceed to evaluate Eq. (39) for the cases when $n(t)$ is Gaussian, random-rectangular-pulse, and CW interference.

For Gaussian interference

$$\overline{e^{iun/d_0}} = \frac{1}{\sqrt{2\pi} \, \sigma_n} \int_{-\infty}^{\infty} e^{iun/d_0} \, e^{-n^2/2\sigma_n^2} \, dn = e^{-u^2\sigma_n^2/2d_0^2} \, . \tag{40}$$

Substituting Eq. (40) into Eq. (39), we obtain

$$\mathrm{Prob}(0 < H < S) = \frac{S}{\pi d_0} \int_0^\infty e^{-u^2\sigma_n^2/2d_0^2} \frac{\sin u}{u}\, du\ . \tag{41}$$

Integration of formula 508 in Peirce [5] gives

$$\int_0^\infty e^{-a^2x^2} \frac{\sin bx}{x}\, dx = \frac{\pi}{2} \operatorname{erf}(b/2a)\ , \tag{42}$$

which reduces Eq. (41) to

$$\mathrm{Prob}(0 < H < S) = \frac{S}{2d_0} \operatorname{erf}(d_0/\sigma_n\sqrt{2})\ . \tag{43}$$

Hence from Eq. (31)

$$(\mathrm{Sig/Noi})_{\mathrm{out}} = N\left(\frac{S}{d_0}\right)^2 \operatorname{erf}^2(d_0/\sigma_n\sqrt{2})\ . \tag{44}$$

For rectangular pulse interference (Fig. 4)

$$p_n(x) = \frac{1}{2}\, \delta(x-C) + \frac{1}{2}\, \delta(x+C)\ , \tag{45}$$

where C represents the pulse height and δ is the Dirac delta function. Thus

$$\overline{e^{iun/d_0}} = \frac{1}{2}\left[e^{iuC/d_0} + e^{-iuC/d_0}\right] = \cos(uC/d_0)\ . \tag{46}$$

Substituting Eq. (46) into Eq. (39) and simplifying, we obtain

$$\mathrm{Prob}(0 < H < S) = \frac{S}{2\pi d_0} \int_0^\infty \left[\frac{\sin(1+C/d_0)u}{u} + \frac{\sin(1-C/d_0)u}{u}\right] du\ . \tag{47}$$

From Peirce, formula 484,

$$\int_0^\infty \frac{\sin au\, du}{u} = \pi/2\ , \quad a > 0\ , \tag{48a}$$

$$= 0\ , \quad a = 0\ , \tag{48b}$$

$$= -\pi/2\ , \quad a < 0\ . \tag{48c}$$

Substituting Eqs. (48) into Eq. (47) and the results into Eq. (31), we obtain

$$(\text{Sig/Noi})_{\text{out}} = N(S/d_0)^2 \quad , \quad C/d_0 < 1 \, , \tag{49a}$$

$$= N(S/2d_0)^2 \, , \quad C/d_0 = 1 \, , \tag{49b}$$

$$= 0 \quad , \quad C/d_0 > 1 \, . \tag{49c}$$

Equations (49) point out what we discussed qualitatively. Equation (49a) indicates that if the maximum excursion of the dither exceeds the interfering pulse amplitude, then capture is prevented. (Actually, the dither peak must exceed the signal magnitude plus the interfering pulse magnitude, but in the approximations made we neglect the signal with respect to the interference.) Equation (49c) indicates that if the maximum excursion of the dither is less than the interfering pulse amplitude, capture occurs. The singularity of Eq. (49b) has zero probability of occurring and drops out of the subsequent analysis because we will be considering d_0 to just barely exceed C rather than to equal C.

For CW interference

$$n(t) = J \sin(\omega t + \varphi) \, , \tag{50}$$

$$p_n(z) = \frac{1}{\pi\sqrt{J^2 - z^2}}, \quad |z| < J \, , \tag{51a}$$

$$= 0 \quad , \quad |z| > J \, , \tag{51b}$$

and, from Magnus and Oberhettinger [6], p. 117,

$$\overline{e^{iun/d_0}} = \frac{1}{\pi} \int_{-J}^{J} \frac{e^{iuz/d_0}}{\sqrt{J^2 - z^2}} \, dz = J_0(Ju/d_0) \, . \tag{52}$$

Substituting Eq. (52) into Eq. (39), we obtain

$$\text{Prob}(0 < H < S) = \frac{S}{\pi d_0} \int_0^\infty J_0(Ju/d_0) \, \frac{\sin u}{u} \, du \, . \tag{53}$$

From Magnus and Oberhettinger, p. 36,

$$\int_0^\infty J_0(ax) \, \frac{\sin bx}{x} \, dx = \pi/2 \quad , \quad a < b \, , \tag{54a}$$

$$= \arcsin b/a \, , \quad a > b \, . \tag{54b}$$

Accordingly

$$\text{Prob}(0 < H < S) = \frac{S}{2d_0} \quad , \quad J < d_0 \, , \tag{55a}$$

$$= \frac{S}{\pi d_0} \arcsin \frac{d_0}{J} \, , \quad J > d_0 \, . \tag{55b}$$

Substituting Eqs. (55) into Eq. (31), we obtain

$$(\text{Sig/Noi})_{\text{out}} = N(S/d_0)^2 \qquad , \quad J < d_0 \; , \tag{56a}$$

$$= 4N\left(\frac{S}{\pi d_0}\right)^2 \arcsin^2(d_0/J) \; , \quad J > d_0 \; . \tag{56b}$$

Sinusoidal Dither—In the case of sinusoidal dither

$$d(t) = A \sin[\omega_0 t + \varphi(t)] \; ,$$

where φ is constant over the time interval covering a single observation but varies randomly and uniformly within $[-\pi, \pi]$ from observation to observation.

Using the same reasoning already employed in Eqs. (50) through (52), we have

$$\overline{e^{i\xi d}} = \frac{1}{\pi}\int_{-A}^{A} \frac{e^{i\xi z}}{\sqrt{A^2 - z^2}}\, dz = J_0(A\xi) \; . \tag{57}$$

Substituting Eq. (57) into Eq. (35), we obtain

$$\text{Prob}(0 < H < S) = \frac{S}{\pi}\int_0^\infty J_0(A\xi)\, \overline{e^{i\xi n}}\, \frac{\sin \xi S}{\xi S}\, d\xi \; . \tag{58}$$

Letting $u = A\xi$ in Eq. (58), we have

$$\text{Prob}(0 < H < S) = \frac{S}{\pi A}\int_0^\infty J_0(u)\, \overline{e^{iun/A}}\, \frac{\sin(uS/A)}{uS/A}\, du \; . \tag{59}$$

Similar to our treatment of Eq. (38), expanding Eq. (59) to the first term in the small quantity S/A, we reduce Eq. (59) to

$$\text{Prob}(0 < H < S) = \frac{S}{\pi A}\int_0^\infty J_0(u)\, \overline{e^{iun/A}}\, du \; , \tag{60}$$

which we proceed to evaluate for the three types of interference.

For Gaussian noise, from Eq. (40),

$$\overline{e^{iun/A}} = e^{-u^2\sigma_n^2/2A^2} \; . \tag{61}$$

Substituting Eq. (61) into Eq. (60), we get

$$\text{Prob}(0 < H < S) = \frac{S}{\pi A}\int_0^\infty J_0(u)\, e^{-u^2\sigma_n^2/2A^2}\, du \; . \tag{62}$$

From Magnus and Oberhettinger [6], p. 35,

$$\int_0^\infty J_0(u)\, e^{-p^2u^2}\, du = \frac{\sqrt{\pi}}{2p}\, {}_1F_1\left(\frac{1}{2}, 1, -\frac{1}{4p^2}\right),$$

where, from p. 87 of the same reference,

$${}_1F_1(1/2, 1, 2z) = e^z I_0(z),$$

so that

$$\int_0^\infty J_0(u)\, e^{-p^2u^2}\, du = \frac{\sqrt{\pi}}{2p}\, e^{-1/8p^2}\, I_0\left(\frac{1}{8p^2}\right), \tag{63}$$

and, from Eq. (62),

$$\text{Prob}(0 < H < S) = \frac{S e^{-A^2/4\sigma_n^2}}{\sqrt{2\pi}\,\sigma_n}\, I_0\left(A^2/4\sigma_n^2\right). \tag{64}$$

Substituting Eq. (64) into Eq. (31), we obtain

$$(\text{Sig/Noi})_{\text{out}} = \frac{2NS^2}{\pi\sigma_n^2}\, e^{-A^2/2\sigma_n^2}\, I_0^2\left(A^2/4\sigma_n^2\right). \tag{65}$$

For rectangular pulse interference, from Eq. (46)

$$\overline{e^{iun/A}} = \cos uC/A. \tag{66}$$

Substituting Eq. (66) into Eq. (60), we get

$$\text{Prob}(0 < H < S) = \frac{S}{\pi A} \int_0^\infty J_0(u) \cos uC/A\, du. \tag{67}$$

From Magnus and Oberhettinger, p. 37,

$$\int_0^\infty J_0(u) \cos uC/A\, du = 1, \quad C/A < 1, \tag{68a}$$

$$= 0, \quad C/A > 1. \tag{68b}$$

Accordingly, from Eq. (31),

$$(\text{Sig/Noi})_{\text{out}} = \frac{4NS^2}{\pi^2 A^2}, \quad C/A < 1 \tag{69a}$$

$$= 0 \quad , \quad C/A > 1. \tag{69b}$$

Thus we corroborate that *CW* dither whose maximum exceeds the interfering pulse magnitude prevents capture by the interference, although with a $(\mathrm{Sig/Noi})_{\mathrm{out}}$ which is 4 dB below that for uniform dither.

For *CW* interference, with $n(t) = J\sin(\omega t + \varphi)$, we have from Eq. (52)

$$\overline{e^{iun/A}} = J_0(uJ/A)\,. \tag{70}$$

Substituting Eq. (70) into Eq. (60), we have for S/A small

$$\mathrm{Prob}(0 < H < S) = \frac{S}{\pi A}\int_0^\infty J_0(u)J_0(uJ/A)\,du\,, \quad J + S < A\,, \tag{71a}$$

$$= \frac{S}{\pi J}\int_0^\infty J_0(u)J_0(uA/J)\,du\,, \quad A + S < J\,. \tag{71b}$$

From Gröbner and Hofreiter [7], p. 202, formula 2(a), and Magnus and Oberhettinger [6], p. 106,

$$\int_0^\infty J_0(u)J_0(bu)\,du = F(1/2, 1/2, 1, b^2) = \frac{2}{\pi}K(b)\,, \quad 0 < b < 1\,, \tag{72}$$

where F designates the hypergeometric function and $K(b)$ is the complete elliptic integral of the first kind:

$$K(b) = \int_0^{\pi/2}\frac{d\varphi}{\sqrt{1 - b^2\sin^2\varphi}}\,. \tag{73}$$

From Eqs. (72) and (73)

$$\mathrm{Prob}(0 < H < S) = \frac{2S}{\pi^2 A}K(J/A)\,, \quad S + J < A\,, \tag{74a}$$

$$= \frac{2S}{\pi^2 J}K(A/J)\,, \quad S + A < J\,. \tag{74b}$$

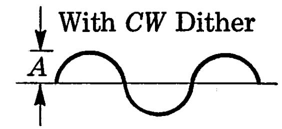
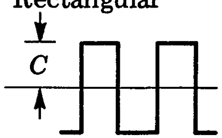
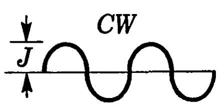
From Eq. (31)

$$(\mathrm{Sig/Noi})_{\mathrm{out}} = 4N\left[\frac{2S}{\pi^2 A}K(J/A)\right]^2\,, \quad S + J < A\,,$$

$$= 4N\left[\frac{2S}{\pi^2 J}K(A/J)\right]^2\,, \quad S + A < J\,.$$

The $(\mathrm{Sig/Noi})_{\mathrm{out}}$ formulas for the various cases have been collected in Table 1.

Table 1
Signal-to-Noise Ratios $(\text{Sig/Noi})_{\text{out}} = \bar{z}^2/\sigma_z^2$ for
Different Detectors Under Different Jamming Conditions

Jamming Noise	$(\text{Sig/Noi})_{\text{out}}$ for a Signal Having N Pulses With Amplitude $\pm S$			
	Correlation Detector	Polarity Coincidence Detector		
		Without Dither	With Uniform Dither $-d_0 \leqslant d(t) \leqslant d_0$	With *CW* Dither A 
Gaussian	$N(S^2/\sigma_n^2)$	$\frac{2}{\pi} NS^2/\sigma_n^2$	$\frac{NS^2}{2\sigma_n^2} \operatorname{erf}^2\left(\frac{d_0}{\sqrt{2}\,\sigma_n}\right) \Big/ \left(\frac{d_0}{\sqrt{2}\,\sigma_n}\right)^2$	$\frac{2NS^2}{\pi\sigma_n^2} e^{-A^2/2\sigma_n^2} I_0^2\left(\frac{A^2}{4\sigma_n^2}\right)$
Rectangular C 	$N(S^2/C^2)$	$0,\ C > S$	$NS^2/d_0^2, \quad C < d_0$	$\frac{4}{\pi^2} N(S^2/A^2), \quad C < A$
		$\infty, \quad C < S$	$0, \quad C > d_0$	$0, \quad C > A$
J *CW* 	$N\left(S^2 \Big/ \frac{1}{2} J^2\right)$	$\frac{2}{\pi^2} N\left(S^2 \Big/ \frac{1}{2} J^2\right)$	$NS^2/d_0^2, \quad J < d_0$	$\frac{16NS^2}{\pi^4 A^2} K^2\left(\frac{J}{A}\right), \quad J + S < A$
$J \sin \omega t$			$\frac{4NS^2}{\pi^2 d_0^2} \arcsin^2\left(\frac{d_0}{J}\right), \quad J > d_0$	$\frac{16NS^2}{\pi^4 J^2} K^2\left(\frac{A}{J}\right), \quad A + S < J$

COMPARISON OF $(\text{Sig/Noi})_{\text{out}}$ FOR VARYING DITHER AMPLITUDES

Table 1 shows that rectangular wave interference whose peak amplitude exceeds that of the signal will completely suppress or "capture" the signal of a polarity coincidence detector with no dither. However the addition of dither whose peak exceeds that of the interference eliminates capture. On the other hand values of peak dither exceeding the necessary threshold by too much will decrease $(\text{Sig/Noi})_{\text{out}}$. For the dither to be large enough to avoid capture and yet not so large that the $(\text{Sig/Noi})_{\text{out}}$ is unnecessarily degraded, it is necessary for the receiver to continuously monitor the interference power and to adjust the peak dither voltage to be a prescribed factor, say g, of the interference peak, the factor at least exceeding unity. Thus, if P is the measured interference power, to avoid capture we attribute it to rectangular pulse jamming and infer that $P = C^2$, where C is the pulse height (Fig. 4). Thus we choose $d_0 = gC = g\sqrt{P}$, $g > 1$, for the case of uniform dither, and we choose $A = g\sqrt{P}$ for the case of sinusoidal dither. Since it is apparent from Table 1 that for large enough dither $(\text{Sig/Noi})_{\text{out}}$ decreases as the dither increases, for each type of interference and each type of dither there is a value of g which maximizes $(\text{Sig/Noi})_{\text{out}}$. Since we are interested in insuring that the system performance always meets a minimum specification, we define the optimum value of g as that value which maximizes the minimum $(\text{Sig/Noi})_{\text{out}}$.

To determine the $(\text{Sig/Noi})_{\text{out}}$ values which would occur by choosing the peak values of dither according to $d_0 = g\sqrt{P}$, $A = g\sqrt{P}$, we substitute these values, as well as $\sigma_n^2 = P$, $J^2 = 2P$, into the formulas of Table 1, obtaining Table 2. Figures 7 and 8 graph $(\text{Sig/Noi})_{\text{out}}/(NS^2/P)$, the signal-to-noise ratio divided by that of the classical correlation detector, as a function of g for the different types of interference for uniform dither and for sinusoidal dither respectively.

Figure 7 shows that with uniform dither the minimum $(\text{Sig/Noi})_{\text{out}}$ is maximized by choosing $g = 1.36$, where the minimum $(\text{Sig/Noi})_{\text{out}}$ equals $0.37\ NS^2/P$, which is 4.3 dB below the theoretical optimum. Figure 8 shows that with CW dither the minimum $(\text{Sig/Noi})_{\text{out}}$ is maximized at $g = 1.12$, where $(\text{Sig/Noi})_{\text{out}} = 0.32\ NS^2/P$, 0.65 dB below the $(\text{Sig/Noi})_{\text{out}}$ value at the optimum point for uniform dither. However, for uniform dither the $(\text{Sig/Noi})_{\text{out}}$ curves for both rectangular waveform and Gaussian interference are above those for CW dither, and most important the optimum value of g for CW dither is fairly close to the capture point ($g = 1$), in contrast to the situation for uniform dither. Accordingly one must conclude on grounds of stability of performance as well as superior $(\text{Sig/Noi})_{\text{out}}$ that uniform dither is superior to CW dither.

Table 2
Values Obtained From Table 1 Under the Condition That $d_0 = A = g\sqrt{P}$, $\sigma_n^2 = P$, $C^2 = P$, $J^2 = 2P$, Where P is the Measured Jamming Power (Assumed to be C^2)

Jamming Noise	(Sig/Noi)$_{out}$ Values (and Comparison With Correlation Detector Values)			
	Correlation Detector	Polarity Coincidence Detector		
		Without Dither	With Uniform Dither	With CW Dither
Gaussian	NS^2/P	$\frac{2}{\pi} NS^2/P = 0.637 NS^2/P$	$\frac{NS^2}{Pg^2} \operatorname{erf}^2\left(\frac{g}{\sqrt{2}}\right), g > 1$	$\frac{2NS^2}{\pi P}\left[e^{-g^2/4} I_0(g^2/4)\right]^2, g > 1$
Rectangular	NS^2/P	$0, C > S$ $\infty, C < S$	$NS^2/Pg^2, 1 < g$ $0, 1 > g$	$4NS^2/\pi^2 g^2 P, g > 1$ $0, g < 1$
CW	NS^2/P	$\frac{2}{\pi^2} NS^2/P = 0.203 NS^2/P$	$NS^2/g^2P, g > \sqrt{2}$ $\frac{NS^2}{g^2P}\left[\frac{2}{\pi} \arcsin (g/\sqrt{2})\right]^2, g < \sqrt{2}$	$\frac{16NS^2}{\pi^4 g^2 P} K^2(\sqrt{2}/g), g > \sqrt{2}$ $\frac{8NS^2 K^2(g/\sqrt{2})}{\pi^4 P}, 1 < g < \sqrt{2}$

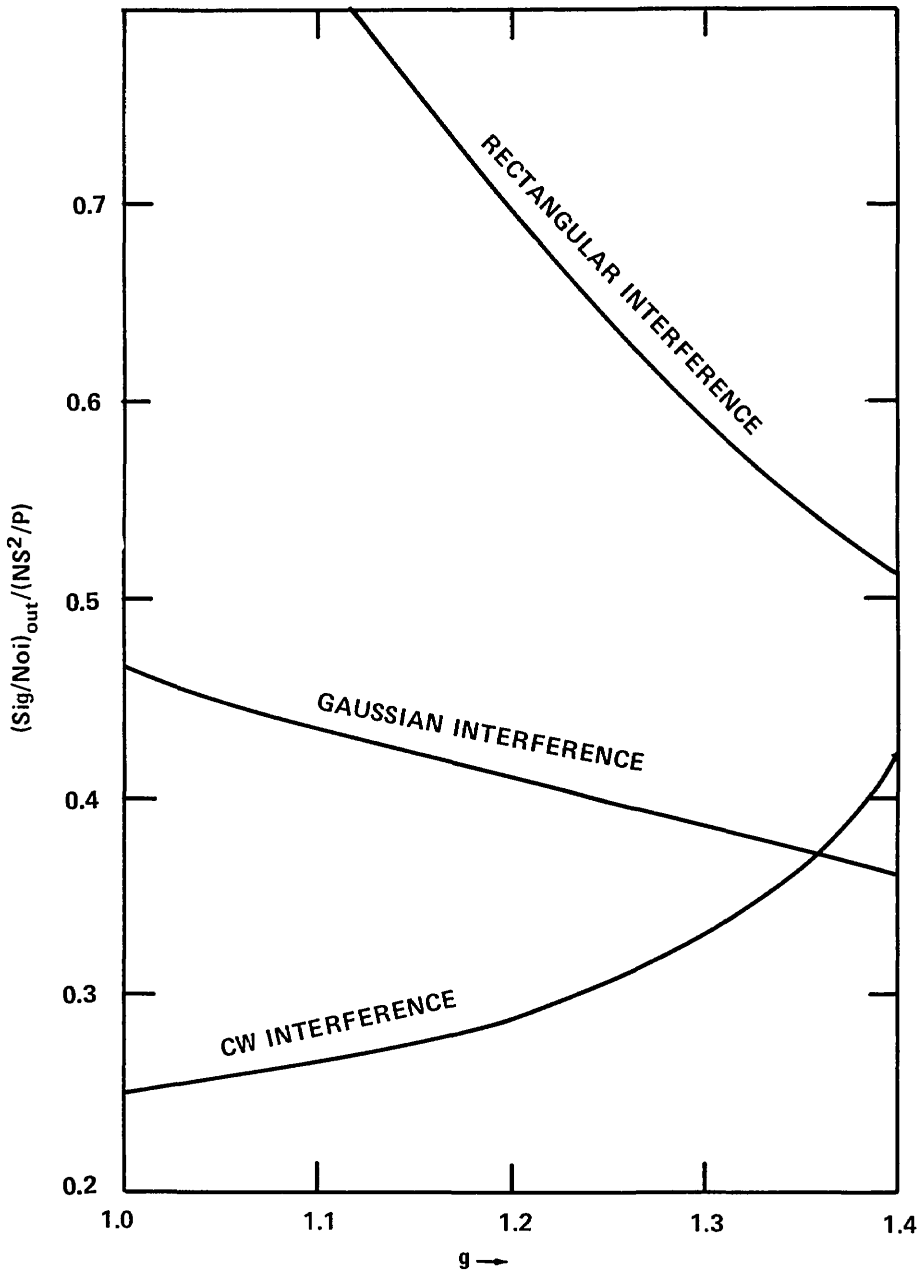


Fig. 7 — Plot to determine the value of g at which the minimum $(\mathrm{Sig/Noi})_{out}$ of a polarity coincidence detector with uniform dither divided by the output signal-to-noise ratio of a classical correlator is a maximum

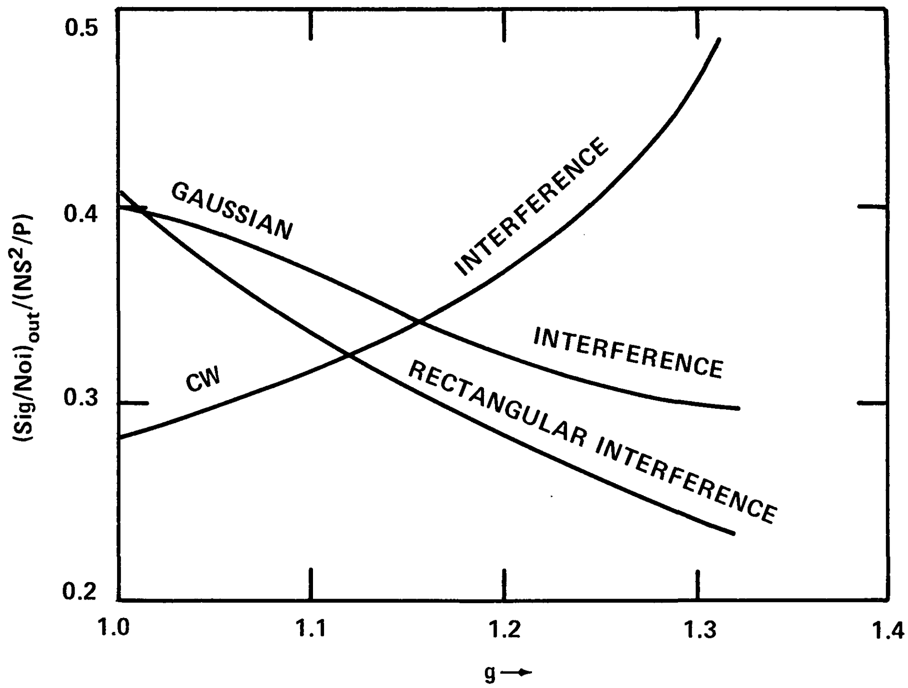


Fig. 8 — Plot to determine the value of g at which the minimum $(\text{Sig/Noi})_{out}$ of a polarity coincidence detector with *CW* dither divided by the output signal-to-noise ratio of a classical correlator is a maximum

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13. ABSTRACT

An analysis has been made of how dither, a large suitably fluctuating component added to the input of a digital matched filter (digital correlator), eliminates capture of a weak signal by an unwanted signal. The signal-to-noise ratio of various correlators with two specific dithers in certain specific jamming environments was analyzed. The correlators analyzed, in comparison with a classical ideal correlator, were a polarity coincidence detector without dither, a polarity coincidence detector with dither of random amplitude uniformly distributed between the chosen plus-and-minus peak values, and a polarity coincidence detector with *CW* dither, each being analyzed for the cases of Gaussian noise, rectangular-pulse, and *CW* interference. The choice of dither amplitude based on the measured value of the noise power was studied. In general a uniformly distributed dither was found to be superior to a *CW* dither.

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