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A TWO-DIMENSIONAL HAAR-LIKE TRANSFORM

1. INTRODUCTION

In two-dimensional image processing, use of the two-dimensional Fourier transform is being supplemented by the Walsh-Haramard transform (1). Haar functions (2-4) are closely related to Walsh functions (5) and have properties that suggest they may be useful in applications such as multiplexing, data transmission, and pattern recognition (6). Recently, a set of two-dimensional functions with properties analogous to Haar functions has been introduced (7). These two-dimensional functions may be well suited to image coding and related applications.

Haar Functions are briefly reviewed in Section 2. In Section 3 we describe the two-dimensional functions. Analog and digital transforms are discussed in Section 4. Computational and memory requirements are analyzed. Applications are discussed in Section 5. Before continuing, we introduce some definitions that will be used throughout this report.

DEFINITION. *Let the interval* $\left((n - 1)/2^{n-1}, n/2^{n-1}\right)$ *be denoted by* i_n^m, *where* $m = 1, 2, \ldots, 2^{n-1}$. *Thus the interval* $[0, 1]$ *is subdivided into* 2^{n-1} *equal parts labeled* $i_n^1, i_n^2, \ldots, i_n^{2^{n-1}}$.

DEFINITION. *Let the square defined by* $(\ell - 1)/2^{n-1} < x < \ell/2^{n-1}$ *and* $(k - 1)/2^{n-1} < y < k/2^{n-1}$ *be denoted by* $\diamond_n^{k\ell}$, *where* $k, \ \ell = 1, 2, \ldots, 2^{n-1}$. *The square* $\diamond_n^{k\ell}$ *is the intersection of* i_n^ℓ *along the* x *axis and of* i_n^k *along the* y *axis. The unit square will be denoted by* $\diamond_1$. *When the boundaries of the square are to be included, one writes* $[\diamond_n^{k\ell}]$ *instead of* $\diamond_n^{k\ell}$.

DEFINITION. *If* $x \in [0, 1]$ *for some* k *and* M *satisfies* $x = k/2^M$, *where* $k = 0, 1, 2, \ldots, 2^M$, *then is called a binary rational. If* x *does not satisfy* $x = k/2^M$ *for any* M, *then it is called a binary irrational. A point* $(x, y) \in \diamond_1$ *is called a binary rational if either* x *or* y *is a binary rational. It is called a binary irrational if both* x *and* y *are binary irrationals. A binary rational line segment is a set of colinear, binary-rational points with segment endpoints that are binary rational in both* x *and* y.

DEFINITION. *Let* $f(x, y) \in L^2[\diamond_1]$; *then*

$$\bar{f}(\diamond_n^{k\ell}) \equiv 2^{2n-2} \int_{\diamond_n^{k\ell}} f(x, y)\, dx dy \tag{1}$$

is the mean value of f *over the square* $\diamond_n^{k\ell}$.

DEFINITION. *If* f, g $\in$ L^2[0, 1], *then let*

$$\langle f|g\rangle \equiv \int_0^1 f(x)g(x)\,dx. \tag{2}$$

If f, g $\in$ L^2[$\Diamond_1$], *then let*

$$\langle f|g\rangle \equiv \int_{\Diamond_1} f(x,y)g(x,y)\,dxdy. \tag{3}$$

2. BRIEF REVIEW OF HAAR FUNCTIONS

The Haar orthonormal sequence is defined on the closed interval [0, 1] and is composed of functions labeled by two indices:

$$\{\varphi_n^m\} = \varphi_0;\varphi_1^1;\varphi_2^1,\varphi_2^2;\ldots;\varphi_k^1,\ldots,\varphi_k^{2^{k-1}};\ldots. \tag{4}$$

The functions are defined as follows:

$$\varphi_0(x) = 1, \qquad \text{for } 0 \leqslant x \leqslant 1$$

$$\varphi_1^1(x) = \begin{cases} 1, & \text{for } 0 \leqslant x < 1/2 \\ -1, & \text{for } 1/2 < x \leqslant 1 \end{cases}$$

$$\varphi_2^1(x) = \begin{cases} \sqrt{2}, & \text{for } 0 \leqslant x < 1/4 \\ -\sqrt{2}, & \text{for } 1/4 < x < 1/2 \\ 0, & \text{for } 1/2 < x \leqslant 1 \end{cases}$$

$$\varphi_2^2(x) = \begin{cases} 0, & \text{for } 0 \leqslant x < 1/2 \\ \sqrt{2}, & \text{for } 1/2 < x < 3/4 \\ -\sqrt{2}, & \text{for } 3/4 < x \leqslant 1 \end{cases}$$

$$\vdots$$

$$\varphi_n^m(x) = \begin{cases} 2^{(n-1)/2}, & \text{for } \dfrac{m-1}{2^{n-1}} < x < \dfrac{m-(1/2)}{2^{n-1}} \\ -2^{(n-1)/2}, & \text{for } \dfrac{m-(1/2)}{2^{n-1}} < x < \dfrac{m}{2^{n-1}} \\ 0, & \text{for } 0 < x < \dfrac{m-1}{2^{n-1}} \text{ and } \dfrac{m}{2^{n-1}} < x < 1 \end{cases}$$

At points of discontinuity, the Haar functions are defined to be the average of the limits approached on the two sides of the discontinuity. The first few Haar functions are shown in Fig. 1. We note that $\varphi_n^m(x)$ is nonzero over the interval i_n^m. The Haar functions are a complete orthonormal basis of $L^2[0,1]$.

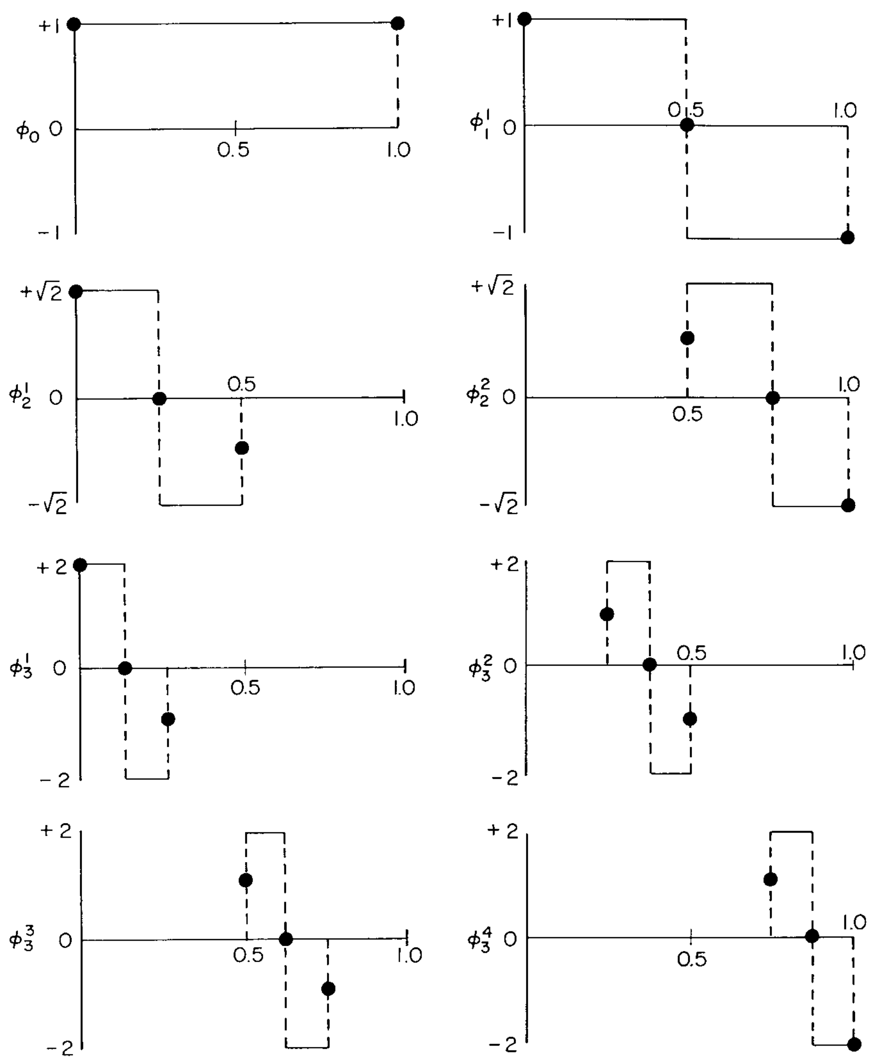

Fig. 1—The first eight Haar functions

Any function can be expressed as an infinite series in terms of Haar functions:

$$f(x) = c_0 + \sum_{n=1}^{\infty} \sum_{m=1}^{2^{n-1}} c_n^m \varphi_n^m(x), \tag{5}$$

where $c_n^m = \langle f | \varphi_n^m \rangle$. For the purposes of this report, convergence is best discussed by means of the partial sums

$$S_N(x) = c_0 + \sum_{n=1}^{N} \sum_{m=1}^{2^{n-1}} c_n^m \varphi_n^m(x), \tag{6}$$

which contain 2^N terms. Also S_{N+1} contains 2^N more terms than S_N, namely all Haar functions with the subscript $N + 1$. More general partial sums are discussed in (2) and (4).

For continuous functions, the sequence of partial sums S_N is uniformly convergent to the given function. For discontinuous functions, $\{S_N\}$ will still converge uniformly to $(1/2)[f(x+) + f(x-)]$, provided all discontinuities are at binary-rational points. This convergence property for discontinuous functions derives from the fact that all Haar-function discontinuities are at binary-rational points. For functions with discontinuities at binary-irrational points, $S_N(x)$, though no longer uniformly convergent, is still pointwise convergent everywhere except at the binary-irrational discontinuities.

Several aspects of the potential utility of Haar functions derive from an important property of the partial sum. In the expansion of $f(x)$, the Nth Haar partial sum $S_N(x)$ is a step function with 2^N equal-length steps. The value of $S_N(x)$ on each step is the mean value of $f(x)$ in the interval covered by the step. The value of $S_N(x)$ at a discontinuity between adjacent steps is halfway between the adjacent steps. Stated differently, $S_N(x)$ is the step function of 2^N steps which is the best approximation to $f(x)$ in the mean-square-error sense. The effect of additional terms is simple and intuitive, unlike the effect of additional terms when the function is expanded as a trigonometric Fourier series (or as a series in terms of other continuous bases of $L^2[0, 1]$). We note that it follows from this mean-value property that if $f(x)$ is constant in the interval covered by any step, then $S_N(x) = f(x)$ exactly on this step.

The coefficients in the Haar Series also have a simple relationship with the mean value of $f(x)$ over subintervals of $[0, 1]$. The coefficient c_ℓ^m is proportional to the difference of two adjacent steps of $S_\ell(x)$, namely the steps on either side of $x = (2m - 1)/2^\ell$. This property leads to the modest computational requirements of the Haar transform (6).

3. TWO-DIMENSIONAL HAAR-LIKE FUNCTIONS

The attractive qualities of Haar functions derive from two related properties. First, in the expansion of $f(x)$, $S_N(x)$ is a step function of 2^N equal-length steps. The value of S_N on each step is the mean value of f in the interval covered by the step. Second, the expansion coefficients are proportional to the difference in the mean value of $f(x)$ over adjacent subintervals of $[0, 1]$. As discussed in (6), these properties may be useful in applications such as data coding and pattern recognition. Whatever the application, Haar functions require a one-dimensional data stream. If the data is intrinsically two dimensional, it seems more appropriate to use a two-dimensional set of functions with analogous properties.

We therefore look for a set or orthonormal functions that have the following properties. When a function on the unit square $f(x, y)$ is expanded in terms of these functions, the Nth partial sum $P_N(x, y)$ is a step function of 2^{2N} square steps each of area $1/2^{2N}$. The value of $P_N(x, y)$ on any step is the mean value of $f(x, y)$ over the area covered by the step. The expansion coefficients are proportional to the difference in the mean value of $f(x, y)$ over adjacent subareas of the unit square.

In analogy with the double trigonometric Fourier Series, we might expect that such a set is provided by all products of the form

$$\Phi_{nk}^{m\ell}(x, y) = \varphi_n^m(x)\varphi_k^\ell(y). \tag{7}$$

The set $\{\Phi_{nk}^{m\ell}\}$ is the basis of the two-dimensional Haar transform used in Ref. 8. In fact, no subset of these products defines an adequate two-dimensional sequence, in the sense described above. Instead, we find that the desired sequence is obtained by defining three sets of functions: $S_n^{ij}(x, y)$, $V_n^{ij}(x, y)$, and $H_n^{ij}(x, y)$. The first is a subset of the products defined in Eq. (7):

$$S_0(x, y) = \varphi_0(y)\varphi_0(x) \tag{8}$$

$$S_n^{ij}(x, y) = \varphi_n^i(y)\varphi_n^j(x) \tag{9}$$

where $(x, y) \in \diamond_n^{ij}$ and $n > 0$. The first few are shown in Fig. 2. The symbol S was chosen because the functions have a saddle shape.

Introducing the block functions

$$\alpha_n^m(x) = \begin{cases} 2^{(n-1)/2}, & x \in i_n^m \\ 0, & \text{elsewhere} \end{cases} \tag{10}$$

we define

$$H_n^{ij}(x, y) = \varphi_n^i(y)\alpha_n^j(x) \tag{11}$$

$$V_n^{ij}(x, y) = \alpha_n^i(y)\varphi_n^j(x) \tag{12}$$

where $(x, y) \in \diamond_n^{ij}$ and $n > 0$. The first few H_n^{ij} are shown in Fig. 3. The symbol H was chosen because the rectangular nonzero areas are horizontally oriented. The first few V_n^{ij} are shown in Fig. 4. The symbol V was chosen because the rectangular areas are vertically oriented. The form of the definitions was chosen so that the superscripts locate the nonzero square $\diamond_n^{ij}$ according to the familiar row-column matrix notation.

On the four borders of $\diamond_n^{ij}$, all S_n^{ij}, H_n^{ij}, and V_n^{ij} are defined as the average of the limits approached on the two sides of the border. For example, if $\diamond_n^{ij}$ has no borders in common with the unit square $\diamond_1$, then the functions take one of the values 0, $\pm 2^{n-2}$. On a common border of $\diamond_n^{ij}$ and $\diamond_1$ they take one of the values 0, $\pm 2^{n-1}$.

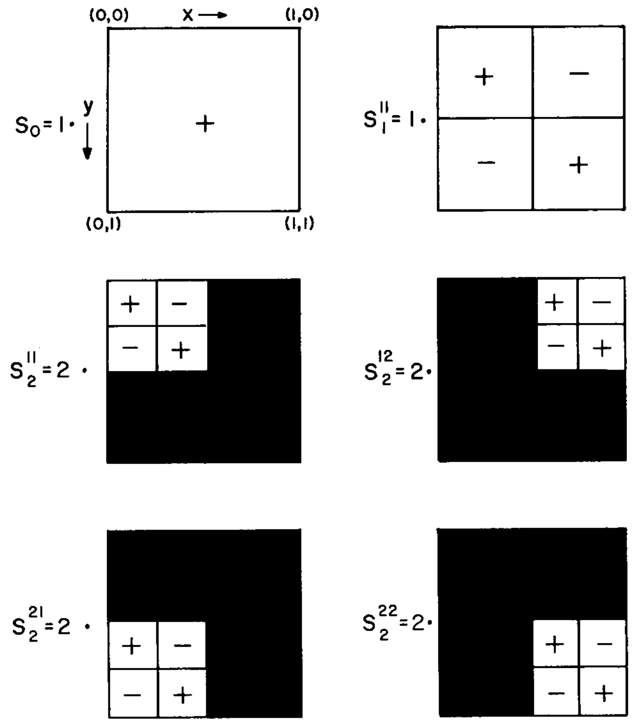

Fig. 2—The first few saddle functions $S_N^{ij}(x, y)$. The shaded areas have the value zero. Other areas have values of plus or minus the constant written outside the square.

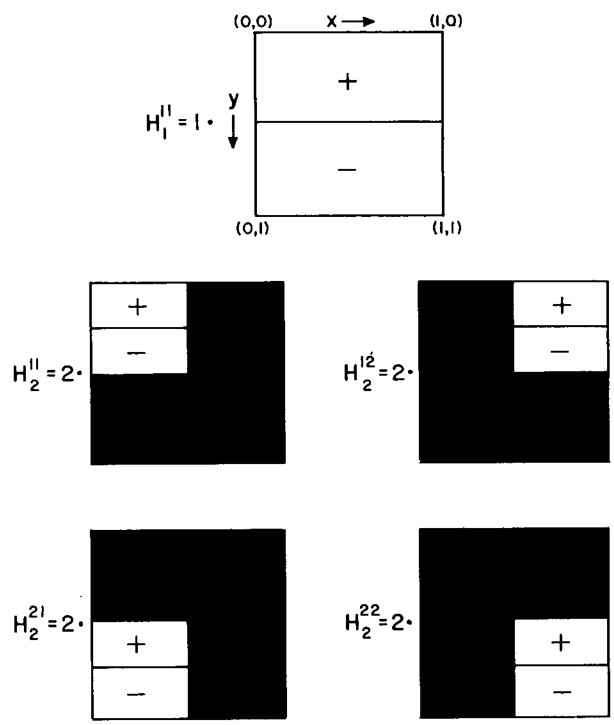

Fig. 3—The first few horizontal functions $H_N^{ij}(x, y)$

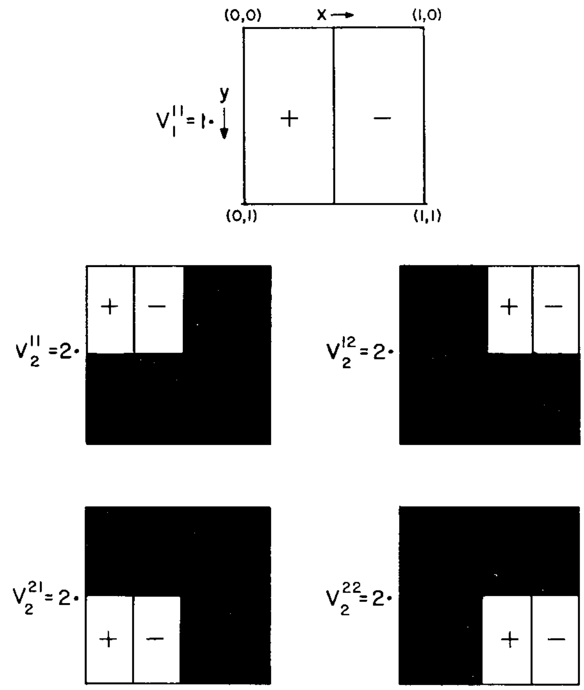

Fig. 4—The first few vertical functions $V_N^{ij}(x, y)$

As shown in Ref. 7, the set $\{S_n^{ij}(x, y), H_n^{ij}, V_n^{ij}(x, y)\}$ is orthonormal and complete in $L^2[\diamond_1]$. Series expansions in terms of this set have convergence properties analogous to those of one-dimensional Haar series. The Nth partial sum is given by

$$P_N(x, y) = a_0 S_0(x, y) + \sum_{n=1}^{N} \sum_{i=1}^{2^{n-1}} \sum_{j=1}^{2^{n-1}} \left[a_n^{ij} S_n^{ij}(x, y) + b_n^{ij} H_n^{ij}(x, y) + c_n^{ij} V_n^{ij}(x, y)\right], \tag{13}$$

where

$$a_0 = \langle f | S_0 \rangle$$

$$a_n^{ij} = \langle f | S_n^{ij} \rangle$$

$$b_n^{ij} = \langle f | H_n^{ij} \rangle$$

$$c_n^{ij} = \langle f | V_n^{ij} \rangle$$

$P_N(x, y)$ is a step function of 2^{2N} square steps each of area $1/2^{2N}$. The value of $P_N(x, y)$ on any step is the mean value of $f(x, y)$ over the area covered by the step. If $f(x, y)$ is continuous or has a finite number of discontinuities along binary-rational line segments, then $P_N(x, y)$ converges uniformly. If $f(x, y)$ has a finite number of discontinuities along binary-irrational line segments, then $P_N(x, y)$ converges pointwise, except along the discontinuities. The accuracy of the estimate $P_N(x, y)$ in the neighborhood of some point is proportional to the gradient at that point. Also the accuracy of the estimate P_{N+1} is bounded by a value which is half that which bounds the estimate P_N. The functions S_n^{ij}, H_n^{ij}, and V_n^{ij} are nonzero only over the square $\square_n^{ij}$. Each of the coefficients a_n^{ij}, b_n^{ij}, and c_n^{ij} can be written in terms of the mean value of $f(x, y)$ over the four quadrants of $\square_n^{ij}$ (Fig. 5):

$$a_\ell^{nm} = \frac{1}{2^{\ell+1}}\left[\bar{f}(\square_{\ell+1}^{2n-1\ 2m-1}) - \bar{f}(\square_{\ell+1}^{2n-1\ 2m}) - \bar{f}(\square_{\ell+1}^{2n\ 2m-1}) + \bar{f}(\square_{\ell+1}^{2n\ 2m})\right] \tag{14a}$$

$$b_\ell^{nm} = \frac{1}{2^{\ell+1}}\left[\bar{f}(\square_{\ell+1}^{2n-1\ 2m-1}) + \bar{f}(\square_{\ell+1}^{2n-1\ 2m}) - \bar{f}(\square_{\ell+1}^{2n\ 2m-1}) - \bar{f}(\square_{\ell+1}^{2n\ 2m})\right] \tag{14b}$$

$$c_\ell^{nm} = \frac{1}{2^{\ell+1}}\left[\bar{f}(\square_{\ell+1}^{2n-1\ 2m-1}) - \bar{f}(\square_{\ell+1}^{2n-1\ 2m}) + \bar{f}(\square_{\ell+1}^{2n\ 2m-1}) - \bar{f}(\square_{\ell+1}^{2n\ 2m})\right] \tag{14c}$$

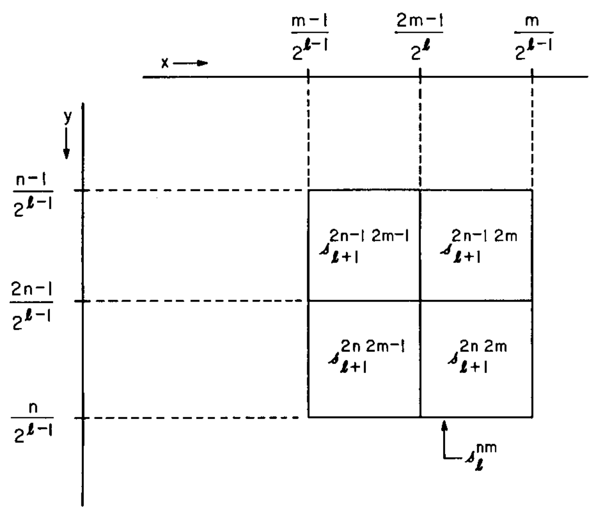

Fig. 5—The square $\square_\ell^{nm}$ and its four quarters

4. TRANSFORMS

4.1 Analog

A straightforward means of obtaining the coefficients a_ℓ^{nm}, b_ℓ^{nm}, and c_ℓ^{nm} is suggested by Eqs. (14a)-(14c), as well as by Figs. 2 - 4. In the transform of some function $f(x, y)$, each coefficient is expressed as sums and differences of the mean value of f over different areas. If we let $I(\square_n^{ij})$ represent the integral of $f(x, y)$ over the square $\square_n^{ij}$, then Eq. (14a) may be rewritten as

$$a_\ell^{nm} = 2^{\ell-1}\left[I(\square_{\ell+1}^{2n-1\ 2m-1}) - I(\square_{\ell+1}^{2n-1\ 2m}) - I(\square_{\ell+1}^{2n\ 2m-1}) + I(\square_{\ell+1}^{2n\ 2m})\right]. \tag{15}$$

Equations (14b) and (14c) may be rewritten similarly.

Dropping for the moment the factor $2^{\ell-1}$, each coefficient in the transform of a photograph can be obtained as follows. A light beam is passed through the negative and then split. Each of the two resulting beams is interrupted by a mask. Light not blocked by the mask is collected at a suitable integrating detector, perhaps a photomultiplier. The two detector outputs are subtracted, and the result, one coefficient of the transform, is converted to digital form. If required, the factor $2^{\ell-1}$ may be inserted by means of an $(\ell - 1)$-bit left shift. As an example, masks for obtaining a_2^{11}, b_2^{11}, and c_2^{11} are shown in Fig. 6.

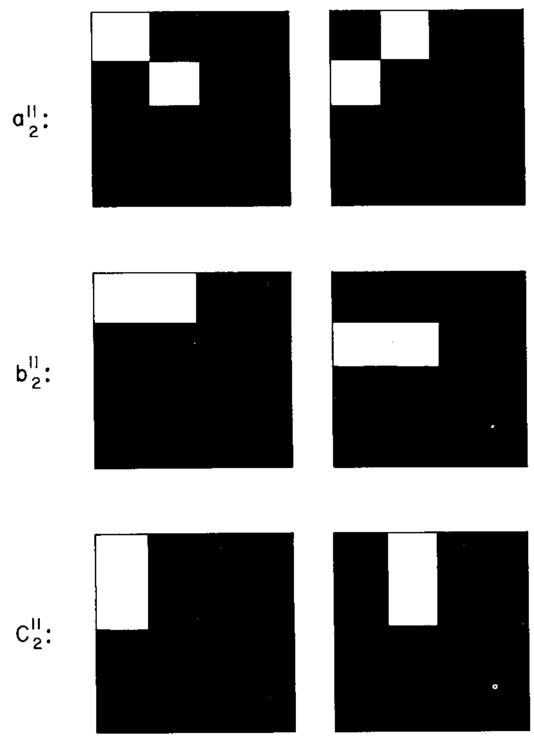

Fig. 6—Masks for analog calculation of a_2^{11}, b_2^{11}, and c_2^{11}

An alternative method uses only one beam and detector: the negative is sequentially exposed to the two masks, the detector outputs are stored digitally, and the subtraction is performed digitally. Although slower, this method would be cheaper and would avoid balance problems.

4.2 Digital

We now consider the case where the photograph has been encoded as $n = 2^{2N} = 4^N$ samples and the transform is to be obtained digitally. This data might be obtained by digitizing the output of a flying-spot scanner. Bearing in mind that each sample is usually the average value over the smallest resolution cell, and in analogy with the modified Haar transform discussed in Ref. 6, we define a modified transform as follows:

$$\hat{a}_\ell^{nm} = 2^{2N-\ell+1} a_\ell^{nm} \tag{16a}$$

$$\hat{b}_\ell^{nm} = 2^{2N-\ell+1} b_\ell^{nm} \tag{16b}$$

$$\hat{c}_\ell^{nm} = 2^{2N-\ell+1} c_\ell^{nm} \tag{16c}$$

When $\ell = N$, Eq. (16a) corresponds to dropping the leading factor in Eq. (14a). When $\ell < N$, it corresponds to replacing the leading factor in Eq. (15) with the constant 2^{2N}, which is needed to compensate for the fact that the initial samples are averages over the smallest area of resolution $1/2^{2N}$.

For illustration we discuss the case $N = 2$. We take the $n = 4^2 = 16$-point transform of data given by a four-by-four matrix x_{ij}. The brute-force calculation then proceeds as follows:

$$\begin{aligned}
\hat{a}_0 &= x_{11} + x_{12} + x_{13} + x_{14} + x_{21} + x_{22} + x_{23} + x_{24} \\
&\quad + x_{31} + x_{32} + x_{33} + x_{34} + x_{41} + x_{42} + x_{43} + x_{44} \\
\hat{a}_1^{11} &= x_{11} + x_{12} - x_{13} - x_{14} + x_{21} + x_{22} - x_{23} - x_{24} \\
&\quad - x_{31} - x_{32} + x_{33} + x_{34} - x_{41} - x_{42} + x_{43} + x_{44} \\
\hat{b}_1^{11} &= x_{11} + x_{12} + x_{13} + x_{14} + x_{21} + x_{22} + x_{23} + x_{24} \\
&\quad - x_{31} - x_{32} - x_{33} - x_{34} - x_{41} - x_{42} - x_{43} - x_{44}
\end{aligned}$$

etc. This calculation requires 112 additions. In general, $3n \log_4 n\ (1 + 3 \log_4)$ additions are required for an $n = 4^N$-point transform.

As with the Fourier, Walsh, and Haar transforms, a fast transform results from the proper grouping of terms. As in the fast modified Haar transform, sums and differences are calculated at each stage:

$$\begin{array}{llll}
u_1 = x_{11} + x_{12} & v_1 = x_{11} - x_{12} & u_2 = x_{13} + x_{14} & v_2 = x_{13} - x_{14} \\
u_3 = x_{21} + x_{22} & v_3 = x_{21} - x_{22} & u_4 = x_{23} + x_{24} & v_4 = x_{23} - x_{24} \\
u_5 = x_{31} + x_{32} & v_5 = x_{31} - x_{32} & u_6 = x_{33} + x_{34} & v_6 = x_{33} - x_{34} \\
u_7 = x_{51} + x_{42} & v_7 = x_{41} - x_{42} & u_8 = x_{43} + x_{44} & v_8 = x_{43} - x_{44}
\end{array}$$

$$\begin{array}{llll}
\hat{a}_2^{11} = v_1 - v_3 & \hat{b}_2^{11} = u_1 - u_3 & \hat{c}_2^{11} \quad v_1 + v_3 & w_1 = u_1 + u_3 \\
\hat{a}_2^{12} = v_2 - v_4 & \hat{b}_2^{12} = u_2 - u_4 & \hat{c}_2^{12} = v_2 + v_4 & w_2 = u_2 + u_4 \\
\hat{a}_2^{21} = v_5 - v_7 & \hat{b}_2^{21} = u_6 - u_7 & \hat{c}_2^{21} = v_5 + v_7 & w_3 = u_5 + u_7 \\
\hat{a}_2^{22} = v_6 - v_8 & \hat{b}_2^{22} = u_6 - u_8 & \hat{c}_2^{22} = v_6 + v_8 & w_4 = u_6 + u_8
\end{array}$$

$$y_1 = w_1 + w_2 \qquad z_1 = w_1 - w_2$$
$$y_2 = w_3 + w_4 \qquad z_2 = w_3 - w_4$$

$$\hat{a}_1^{11} = z_1 - z_2 \qquad \hat{b}_1^{11} = y_1 - y_2 \qquad \hat{c}_1^{11} = z_1 + z_2 \qquad \hat{a}_0 = y_1 + y_2$$

This fast transform requires only 40 additions. The general requirement is $(8/3)(n - 1)$.

Since it is similar to the modified Haar transform, the modified transform just presented is appropriate for applications such as pattern recognition. No information is lost by modifying the leading factors in Eqs. (14a)-(14c) and (15), since the correct factor, which is given by the identity of the coefficient, can always be reinserted. We should be careful, however, in using the modified transform in applications that involve transmitting it over a noisy channel. Depending on the coding technique and on the nature of the channel, use of the modified transform can result in unequal expected error rates for different coefficients. This is because unequal energies may be used to transmit different coefficients.

The complete two-dimensional transform is obtained by multiplying the results of the modified transform by the correct factor $2^{\ell-2N-1}$ (see Eqs. (16a)-(16c). Assuming a binary representation, this can be done by a $(2N - \ell + 1)$-bit right shift. One multibit shift is needed for each coefficient, so that n multibit shifts are needed in the $n = 4^N$-point transform. The computational requirements for the modified, modified fast, and complete fast two-dimensional transforms are summarized in Table 1.

Table 1
Computational Requirements for $n = 4^N$-point, Two-Dimensional Transforms

Transform	Additions	Multibit Shifts
Modified	$3n \log_4 n$	-
Modified Fast	$\frac{8}{3}(n-1)$	-
Complete Fast	$\frac{8}{3}(n-1)$	n

In the fast Fourier and Walsh transforms, the average number of operations per point increases as $\log_2 n$, so that the arithmetic speed requirement is a function of both the data rate and the transform size. In the fast Haar transform, the average number of operations per point is independent of the transform size. The same is true of the two-dimensional Haar-like transform just introduced. The arithmetic speed requirement is a function of the data rate alone. The only limitation on transform size is that imposed by the amount of available storage.

If the $n = 4^N$ samples are located in memory prior to the transform, this is sufficient space in which to complete the transform. If the samples are accepted one at a time from an external source and if transformed coefficients can be put out immediately after calculation, then the memory requirement is reduced. The extent of this reduction depends on the order in which the samples are made available for computation. If the samples derive from a raster scan pattern in which a full row of data is supplied before flyback, then $2\sqrt{n} - 2$ memory locations are required for the transform. If an optimum scan pattern is used, the memory requirement drops to $3 \log_4 n$ locations. Optimum scan patterns encode successively larger squares, starting at one corner of the image. For example, the following is an optimum scan pattern for the 16-point transform discussed previously:

$$x_{11}, x_{12}, x_{21}, x_{22}, x_{13}, x_{14}, x_{23}, x_{24}, x_{31}, x_{32}, x_{41}, x_{42}, x_{33}, x_{34}, x_{43}, x_{44}.$$

The calculation then takes place in the following order:

$$u_1, v_1, u_3, \hat{b}_2^{11}, w_1, v_3, \hat{a}_2^{11}, \hat{c}_2^{11},$$

$$u_2, v_2, u_4, \hat{b}_2^{12}, w_2, v_4, \hat{a}_2^{12}, \hat{c}_2^{12},$$

$$u_5, v_5, u_7, \hat{b}_2^{21}, w_3, v_7, \hat{a}_2^{21}, \hat{c}_2^{21},$$

$$u_6, v_6, u_8, \hat{b}_2^{21}, w_4, v_8, \hat{a}_2^{22}, \hat{c}_2^{22},$$

$$y_1, y_2, \hat{b}_1^{11}, \hat{a}_0, z_1, z_2, \hat{a}_1^{11}, \hat{c}_1^{11}.$$

Here six storage locations are needed, since at one stage of the calculation w_1, w_2, w_3, u_6, v_6, and u_8 must be retained.

5. APPLICATIONS

The application of the two-dimensional Haar-like functions to two-dimensional data is analogous to the application of Haar functions to one-dimensional data.

One example is image coding. Information contained in an image is often transmitted by means of the coefficients of an expansion in terms of some set of basis functions. If convergence is rapid or if many coefficients are zero, we can often reduce the transmission bandwidth from that required to send the image itself (1).

In the two-dimensional Haar-like transform, a given coefficient a_m^{ij}, b_m^{ij}, or c_m^{ij} contains information only from the square $\diamond_m^{ij}$. The full set of 4^N coefficients may be said to contain a mixture of local and global information. All points contribute to a_0, a_1^{11}, b_1^{11}, and c_1^{11}; one quarter of the points contribute to a_2^{11}; etc. In general, each point contributes to $3N$ of the 4^N coefficients. One result of this is that there is less immunity to channel errors than in the case of a transform in which all points contribute to every coefficient, such as the Fourier transform. On the other hand, greater bandwidth reduction may be possible if only the significantly nonzero coefficients are transmitted. For example, assuming a 4^N point transform, if a function is constant over the square $\diamond_m^{ij}$, where $M \leqslant N$, then

$$3\sum_{k=0}^{N-M} 4^k = 4^{N-M+1} - 1$$

coefficients are identically zero.

The potential for bandwidth reduction is summarized by the statement that coefficients of the transform are proportional to the differences in the mean value of $f(x, y)$ over adjacent squares of area $1/4^k$, where $k = 0, 1, 2, \ldots, N$ (see Eqs. (14a)-(14c)). The two-dimensional Haar-like transform should be particularly appropriate for the transmission of images that have relatively large areas of constant or slowly changing tone.

Equations (14a)-(14c) also suggest that the transform may be useful for edge detection in two-dimensional images. The coefficients b_m^{ij} are sensitive to horizontal edges; the coefficients c_m^{ij} are sensitive to vertical edges; and the coefficients a_m^{ij} are sensitive to saddle points. Stated differently, $\{H_m^{ij}\}$ is a set of horizontal edge detectors; $\{V_m^{ij}\}$ is a set of vertical edge detectors; and $\{S_m^{ij}\}$ is a set of saddle detectors. Because edge detection by differencing is sensitive to noise, if the $\{S_m^{ij}, H_m^{ij}, V_m^{ij}\}$ set is useful, it is likely to be so only as one part of an edge-detection process (9, 10).

REFERENCES

1. W.K. Pratt, et al., "Hadamard Transform Image Coding," Proc. IEEE, **57**, No. 1, 58 (Jan 1969).

2. A. Haar, "Zur Theorie der orthogonalen Funktionensysteme," Mathematische Annalen **69**, 331 (1910).

3. G. Von Faber, "Uber die Orthogonalfunktionen des Herrn Haar," Jahresbericht der deutschen Mathematiker-Vereinigung **19**, 104 (1910).

4. J.E. Shore and R.L. Berkowitz, "Convergence Properties of Haar Series," NRL Report 7470, Jan. 1973.

5. J.L. Walsh, "A Closed Set of Normal Orthogonal Functions," Amer. J. Math **55**, 5 (1923).

6. J.E. Shore, "On the Application of Haar Functions," NRL Report 7467, Jan. 1973, also to be published in *IEEE Transactions on Communications*, March 1973.

7. J.E. Shore, "A Complete, Orthonormal Set of Two-Dimensional, Haar-Like Functions," NRL Report 7471, Jan. 1973.

8. H.C. Andrews and W.K. Pratt, "Digital Image Transform Processing," in *Proceedings of the Applications of Walsh Functions Symposium and Workshop, March 31-April 3, 1970*, AD-707 431, Naval Research Laboratory, Washington, D.C.

9. A. Rosenfeld, "A nonlinear edge detection technique," Proc. IEEE (Lett.) **58**, 814 (1970).

10. A Rosenfeld and M. Thurston, "Edge and Curve Detection for Visual Scene Analysis," IEEE Transactions on Computers, **C-20**, No. 5, 562 (1971).