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Low-Frequency Propagation in the Composite Region of a Wedge and Two Parallel Plates

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ABSTRACT

Wave propagation in the composite region of a wedge which leads into a duct was considered where mixed boundary conditions were imposed. The problem was reformulated as an integral equation where, instead of the free-space Green's function being used as the Kernel function, the Green's function for the wedge region was used. This permitted considerable simplification, and a Fredholm integral equation of the second kind was obtained where the region of integration is merely a plane and is valid for any value of the wave number. For small values of the wave number the boundary integral equation may be solved iteratively, and the solution is given as a Neumann series.

PROBLEM STATUS

This is an interim report on the problem; work is continuing.

AUTHORIZATION

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LOW-FREQUENCY SCATTERING IN THE COMPOSITE REGION OF A WEDGE AND TWO PARALLEL PLATES

INTRODUCTION

The problem of wave diffraction in the region bounded by a wedge was considered in 1896 by Sommerfeld (1, 2), who obtained the two Green's functions for a wedge of angle $n/m\pi$ (n, m integers) using a Riemann surface of n sheets. In 1915 MacDonald (3) was the first to obtain expressions for the two Green's functions for a wedge of any angle by using the hard analysis characteristic of nineteenth-century mathematical physics. In 1954 Oberhettinger (4) obtained the two Green's functions for a wedge of any angle by using more conventional analysis. Williams (5, 6) and Lauwerier (7) used integral representations and reduced the problem to the solution of a potential problem. For a more extensive review of the literature published on this topic since its first solutions, see Oberhettinger (4, 8), Bouwkamp (9), and Williams (10).

An outgrowth of this problem is the problem of propagation in the composite region of a wedge leading into a duct; this is solved in this report for mixed boundary conditions (i.e., Dirichlet on one portion of the surface and Neumann on the remainder). Although this problem is important both in radar theory and underwater linear acoustics, except for reports based on experiments, very little has been written on this problem. Kearsley (11) and Wait (12) give some treatment of the subject; however, the problem is still far from being satisfactorily resolved. The approach used in this report is first to solve Helmholtz's equation for the region inside a wedge with mixed boundary conditions. This is done in the next section by following the procedure that Oberhettinger (4) used to obtain the two Green's functions to the Dirichlet and the Neumann problems. In the third section, the composite problems for both a line source and a point source are reformulated as integral equations using the Green's identities. Instead, however, of using the traditional free-space Green's function as the kernel function, the corresponding Green's functions for the wedge are used. Considerable simplification results, whereby a Fredholm integral equation of the second kind is obtained for each problem. In the fourth section, both boundary integral equations are solved by direct iteration, and the solution in each case is given as a Neumann series which converges for small, but non-zero, values of the wave number.

THE GREEN'S FUNCTION FOR A WEDGE

In this section the Green's functions for a wedge with mixed boundary conditions are obtained for both line and point sources. The procedure used is the same as that employed by Oberhettinger (4) to obtain the two Green's functions corresponding to Neumann and Dirichlet boundary conditions.

A cylindrical polar-coordinate system (r, φ, z) is chosen such that the z axis is along the edge of the wedge and the wedge occupies the region

$$0 \leqslant \phi \leqslant \Omega,$$

where Ω is any angle. Denote the region inside the wedge by W; i.e.,

$$W = \{(r, \phi, z) | 0 < \phi < \Omega\}.$$

Let P be a point with coordinates (r, ϕ, z). The distance between two points P and P_0 will be denoted by $R(P, P_0)$ or simply R and

$$R(P, P_0) = \{r^2 + r_0^2 - 2rr_0 \cos(\phi - \phi_0) + (z - z_0)^2\}^{1/2}. \tag{1}$$

Line Source

Consider first the two-dimensional case of a cylindrical incident field which radiates outward from a line source passing through the point P_0 and is parallel to the edge of the wedge. We wish to solve the following problem for the Green's function g:

$$g(P, P_0) = g^i(P, P_0) + g^s(P, P_0) \tag{2}$$

$$(\nabla^2 + k^2) g^s(P, P_0) = 0 \text{ for } P \epsilon W$$

$$g(P, P_0) = 0 \text{ for } \phi = \Omega$$

$$\frac{\partial}{\partial n} g(P, P_0) = 0 \text{ for } \phi = 0$$

$$\lim_{r \to \infty} r^{1/2} \left(\frac{\partial}{\partial r} - ik \right) g^s = 0,$$

where $g^i(P, P_0) = H_0^{(1)}(kR)$. $H_0^{(1)}(kR)$ is a Hankel function of the first kind of order zero and has the following integral representation Ref. 13, p. 106:

$$H_0^{(1)}(kR) = \frac{-4i}{\pi^2} \int_0^\infty K_{ix}(\gamma r) K_{ix}(\gamma r_0) \cosh[x(\pi - |\phi - \phi_0|)] \, dx, \tag{3}$$

where $\gamma = -ik$ and $K_\nu(\zeta)$ is a modified Hankel function defined by

$$K_\nu(\zeta) = \frac{1}{2} i\pi e^{i\frac{\pi}{2}\nu} H_\nu^{(1)}(i\zeta). \tag{4}$$

Assume the following representation for g^s:

$$g^s(P, P_0) = \frac{-4i}{\pi^2} \int_0^\infty K_{ix}(\gamma r) K_{ix}(\gamma r_0) [f_1(x) e^{\phi x} + f_2(x) e^{-\phi x}] \, dx. \tag{5}$$

The functions $f_1(x)$ and $f_2(x)$ may be determined from the boundary conditions. If $\phi = 0$, then

$$\frac{\partial}{\partial \phi} g(P, P_0) = 0$$

and

$$f_1(x) - f_2(x) = - \sinh [\pi - \phi_0)]. \tag{6}$$

If $\phi = \Omega$, then $g(P, P_0) = 0$ and

$$e^{\Omega x} f_1(x) + e^{-\Omega x} f_2(x) = - \cosh [x(\pi - \Omega + \phi_0)]. \tag{7}$$

Solving Eqs. (6) and (7) for f_1 and f_2, after some simplification we obtain from Eqs. (3) and (5)

$$g(P, P_0) = \frac{-4i}{\pi^2} \int_0^\infty K_{ix}(\gamma r) K_{ix}(\gamma r_0) \frac{\sinh \pi x}{\cosh \Omega x} \{\sinh [x(\Omega - |\phi - \phi_0|)]$$

$$+ \sinh [x(\Omega - |\phi + \phi_0|)]\} \, dx. \tag{8}$$

To obtain a more convenient representation for g we convert the one-sided integral in Eq. (8) to a two-sided integral which can be evaluated by residue theory. From Ref. 13, p. 66 we have

$$K_\nu(\zeta) = \frac{\pi}{2} [\sin \pi\nu]^{-1} [I_{-\nu}(\zeta) - I_\nu(\zeta)]. \tag{9}$$

Substituting Eq. (9) into Eq. (8) and observing that $i \sinh \pi x / \sin \pi i x = 1$, we obtain

$$g(P, P_0) = -\frac{2}{\pi} \int_0^\infty K_{ix}(\gamma r) I_{-ix}(\gamma r_0) \left\{ \frac{\sinh [x(\Omega - |\phi - \phi_0|)] + \sinh [x(\Omega - |\phi + \phi_0|)]}{\cosh \Omega x} \right\} dx$$

$$+ \frac{2}{\pi} \int_0^\infty K_{ix}(\gamma r) I_{ix} (\gamma r_0) \left\{ \frac{\sinh [x(\Omega - |\phi - \phi_0|)] + \sinh [x(\Omega - |\phi + \phi_0|)]}{\cosh \Omega x} \right\} dx \tag{10}$$

for $r_0 \leqslant r$ and the same expression with r and r_0 interchanged if $r_0 > r$. In the last integral in Eq. (10), replace x by $-x$ and obtain

$$\int_0^{-\infty} K_{-ix}(\gamma r) I_{-ix}(\gamma r_0) \left\{ \frac{\sinh [x(\Omega - |\phi - \phi_0|)] + \sinh [x(\Omega - |\phi + \phi_0|)]}{\cosh \Omega x} \right\} dx$$

$$= - \int_{-\infty}^0 K_{ix}(\gamma r) I_{-ix}(\gamma r_0) \left\{ \frac{\sinh [x(\Omega - |\phi - \phi_0|)] + \sinh [x(\Omega - |\phi + \phi_0|)]}{\cosh \Omega x} \right\} dx, \tag{11}$$

where from Ref. 13, p. 67 we have

$$K_{ix}(\gamma r) = K_{-ix}(\gamma r). \tag{12}$$

Substituting Eq. (11) into Eq. (10), we obtain

$$g(P, P_0) = -\frac{2}{\pi}\int_{-\infty}^{\infty} K_{ix}(\gamma r) I_{-ix}(\gamma r_0)\left\{\frac{\sinh[x(\Omega - |\phi - \phi_0|)] + \sinh[x(\Omega - |\phi + \phi_0|)]}{\cosh \Omega x}\right\} dx. \tag{13}$$

We can obtain an infinite series representation for g from Eq. (13) by residue theory. Close the path of integration in Eq. (13) in the upper half-plane by a half-circle with radius $(n + 1)\pi/\Omega$, apply the residue theorem, and then let n tend to infinity through nonnegative integers. The only poles of the integrand of Eq. (13) within the closed contour are simple and occur at

$$x_n = \left(n + \frac{1}{2}\right)\frac{\pi}{\Omega}\, i \qquad n = 0,1,2,.... \tag{14}$$

It can be shown that the residue of the integrand at $x = x_n$ is

$$\frac{1}{\Omega} K_{[(n+1/2)\pi]/\Omega}(\gamma r) I_{[(n+1/2)\pi]/\Omega}(\gamma r_0)\left\{\cos\left[\frac{(n + 1/2)\pi}{\Omega}|\phi - \phi_0|\right]\right.$$
$$\left. + \cos\left[\frac{(n + 1/2)\pi}{\Omega}|\phi + \phi_0|\right]\right\} \tag{15}$$

and consequently

$$g(P, P_0) = -\frac{8i}{\Omega}\sum_{n=0}^{\infty} K_{[(n+1/2)\pi]/\Omega}(\gamma r) I_{[(n+1/2)\pi]/\Omega}(\gamma r_0)$$
$$\times \cos\frac{(n + 1/2)\pi}{\Omega}\phi \cos\frac{(n + 1/2)\pi}{\Omega}\phi_0. \tag{16}$$

From Ref. 13, p. 66

$$I_\nu(\zeta) = e^{-i\frac{\pi}{2}\nu} J_\nu(i\zeta). \tag{17}$$

If we replace γ by $-ik$ and let $\tau_n = (n + 1/2)\pi/\Omega$, it follows from Eqs. (4), (16), and (17) that

$$g(P, P_0) = \frac{4\pi}{\Omega}\sum_{n=0}^{\infty} \cos\tau_n\phi \cos\tau_n\phi_0 H_{\tau_n}^{(1)}(kr_>) J_{\tau_n}(kr_<). \tag{18}$$

From Ref. 14, p. 265 we have that

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$$H^{(1)}_{\tau_n}(kr_>)J_{\tau_n}(kr_<) = \sum_{m=0}^{\infty} \frac{\left(\frac{1}{2}\, krr_0\right)^{2m+\tau_n}}{m!\Gamma(m+\tau_n+1)} \frac{H^{(1)}_{2m+\tau_n}(k\sqrt{r^2+r_0^2})}{(\sqrt{r^2+r_0^2})^{2m+\tau_n}} . \tag{19}$$

Thus

$$g(P, P_0) = \frac{4\pi}{\Omega} \sum_{n=0}^{\infty} \cos \tau_n\phi \cos \tau_n\phi_0 \sum_{m=0}^{\infty} \frac{\left(\frac{1}{2}\, krr_0\right)^{2m+\tau_n}}{m!\Gamma(m+\tau_n+1)} \frac{H^{(1)}_{2m+\tau_n}(k\sqrt{r^2+r_0^2})}{(\sqrt{r^2+r_0^2})^{2m+\tau_n}} . \tag{20}$$

Point Source. Consider the three-dimensional case of an incident spherical wave emitted from a point source at P_0. The problem we wish to solve is the same as the one in Eq. (2) except that here the incident field G^i is represented by e^{ikR}/R, and the scattered field G^s satisfies a three-dimensional radiation condition; i.e., in terms of spherical coordinates

$$\lim_{\rho\to\infty} \rho\left(\frac{\partial}{\partial\rho} - ik\right)G^s = 0, \qquad 0 \leqslant \phi \leqslant \Omega,$$

where $\rho = \sqrt{r^2 + z^2}$. From Ref. 13, p. 487 and Ref. 15, p. 827 we have

$$\frac{e^{ikR}}{R} = \frac{i}{2}\int_{-\infty}^{\infty} H_0^{(1)}(\xi\sqrt{k^2 - \alpha^2})e^{-i\alpha|z-z_0|}d\alpha, \tag{21}$$

where $\xi = [r^2 + r_0^2 - 2rr_0 \cos(\phi - \phi_0)]^{1/2}$. Thus it is seen that the incident field resulting from a point source in a cylindrical geometry is obtained from an incident field for a line source by replacing k by $\sqrt{k^2 - \alpha^2}$, multiplying by $(i/2\, exp\, (-\, i\alpha|z - z_0|)$ and integrating with respect to α from $-\infty$ to ∞. Similarly, the total field resulting from a point source can be obtained from the total field resulting from a line source and from Eq. (18); thus,

$$G(P, P_0) = \frac{2\pi i}{\Omega} \sum_{n=0}^{\infty} \cos \tau_n\phi \cos \tau_n\phi_0 S_{\tau_n}, \tag{22}$$

where

$$S_{\tau_n} = \int_{-\infty}^{\infty} J_{\tau_n}(r_<\sqrt{k^2-\alpha^2})H^{(1)}_{\tau_n}(r_>\sqrt{k^2-\alpha^2})e^{-i\alpha|z-z_0|}d\alpha. \tag{23}$$

From Ref. 14, p. 270 it can be shown that

$$S_{\tau_n} = 2k \sum_{m=0}^{\infty} \frac{\left(\frac{1}{2}\, krr_0\right)^{2m+\tau_n}}{m!\Gamma(m+\tau_n+1)} \frac{h^{(1)}_{2m+\tau_n}(k\sqrt{r^2+r_0^2+(z-z_0)^2})}{(\sqrt{r^2+r_0^2+(z-z_0)^2})2m+\tau_n} , \tag{24}$$

where $h_\nu^{(1)}$ is a spherical Hankel function. Substituting Eq. (24) into Eq. (22), we get

$$G(P, P_0) = \frac{4\pi ki}{\Omega} \sum_{n=0}^{\infty} \cos \tau_n\phi \cos \tau_n\phi_0$$

$$\times \sum_{m=0}^{\infty} \frac{\left(\frac{1}{2}\, krr_0\right)^{2m+\tau_n}}{m!\Gamma(m+\tau_n+1)} \frac{h^{(1)}_{2m+\tau_n m}\left(k\sqrt{r^2+r_0^2+(z-z_0)^2}\right)}{\left(\sqrt{r^2+r_0^2+(z-z_0)^2}\right)^{2m+\tau_n}} \,. \tag{25}$$

COMPOSITE REGION

In this section two scattering problems are considered, one corresponding to a line source and the other corresponding to a point source. We first set forth the notation to be used. We next formally state the two problems, and finally we obtain integral representations for them.

Consider the composite region formed by the intersection of a wedge of angle $\Omega < \pi/2$ and two parallel plates, where one plate coincides with one surface of the wedge and the second plate intersects the other face of the wedge. Choose a cylindrical coordinate system (r, ϕ, z) so that the apex of the wedge is the z axis and its faces are at $\phi = 0$ and $\phi = \Omega$, where the surface corresponding to $\phi = \Omega$ is the one which coincides with one of the plates. Suppose the plates are a distance a apart. This situation is shown in Fig. 1. Denote the region formed by the intersection of the wedge and the plates by V, and let S_1 denote the surface corresponding to $\phi = \Omega$; let S_2 denote the surface corresponding to $\phi = 0$; and let S_3 denote the surface corresponding to $a = r \sin(\Omega - \phi)$, $0 \leqslant \phi \leqslant \Omega$. Let V_e denote the region exterior to V and its boundary.

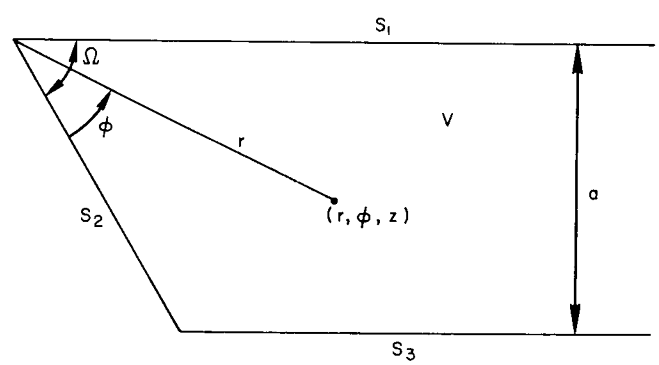

Fig. 1—The composite Region

In the two problems considered in this section, we wish to find the total field $u(P, P_1)$ where P_1 is a point source such that

$$u(P, P_1) = u^i(P, P_1) + u^s(P, P_1)$$

$$(\nabla^2 + k^2)u^s(P, P_1) = 0 \text{ for } P \epsilon V$$

$$u(P, P_1) = 0 \text{ for } P \epsilon S_1 \tag{26}$$

$$\frac{\partial}{\partial n}\, u(P, P_1) = 0 \text{ for } P\epsilon S_2 \cup S_3, \tag{26}$$
(Continued)

and $u^s(P, P_1)$ satisfies a radiation condition at infinity. The function u^i is a known incident field. For one of the problems the incident field results from a line source through the point $P_1 \epsilon V$ and parallel to the z axis. For the other problem the incident field is emitted from a point source located at $P_1 \epsilon V$. For notational convenience and to avoid unnecessary confusion, let $u(P, P_1)$ represent the total resulting from the line source and let $u^i(P, P_1)$ and $u^s(P, P_1)$ denote the incident and scattered fields, respectively. Let $U(P, P_1)$ denote the total field resulting from the point source and let $U^i(P, P_1)$ and $U^s(P, P_1)$ denote its incident and scattered fields, respectively. Thus $u^i(P, P_1) = H_0^{(1)}(kR)$ and $U^i(P, P_1) = e^{ikR}/R$. The radiation condition that u^s satisfies is

$$\lim_{x\to\infty} \left(\frac{\partial}{\partial x} - ik\right) u^s = 0, \tag{27}$$

where $x = r \cos(\Omega - \phi)$; and if $\rho = \sqrt{x^2 + z^2}$, U^s satisfies

$$\lim_{\rho\to\infty} \rho^{1/2}\left(\frac{\partial}{\partial \rho} - ik\right) U^s = 0. \tag{28}$$

We now reformulate the problem in Eq. (26) as an integral representation both for u and U. Since the argument is essentially the same in both instances, we shall give only the derivation for U and merely state the corresponding results for u. From its construction the function $G(P, P_0)$ given in Eq. (25) satisfies

$$(\nabla^2 + k^2)G - 4\pi\delta(P - P_0) \tag{29}$$

for $(P, P_0 \epsilon W)$, where W denotes the wedge region and δ is the Dirac delta function. We see that Eq. (29) is valid in any subregion of W and in particular for $(P, P_0 \epsilon V)$ since $V \subset W$. Also,

$$(\nabla^2 + k^2)U = -4\pi\delta(P - P_1) \tag{30}$$

for $(P, P_1 \epsilon V)$. The singularity of G for P near P_0 is like $e^{ik|P-P_0|}/|P - P_0|$ and the singularity of U for P near P_1 is like $e^{ik|P-P_1|}/|P - P_1|$. From Green's identities (e.g., Ref. 16, p. 256), we have for $P_1 \epsilon V$

$$-\frac{1}{4\pi}\int_{S\cup S_\infty}\left\{U(P, P_1)\frac{\partial}{\partial n_P}G(P, P_0) - G(P, P_0)\frac{\partial}{\partial n_P}U(P, P_1)\right\} dSp$$

$$= \begin{cases} U(P_0, P_1) - G(P_1, P_0) & P_0 \epsilon V \\ \dfrac{\sigma(P_0)}{4\pi}\, U(P_0, P_1) - G(P_1, P_0) & P_0 \epsilon S \\ -G(P_1, P_0) & P_0 \epsilon V_e \end{cases} \tag{31}$$

where

$$S = \overset{3}{\underset{i=1}{U}} S_i$$

and S_∞ is that portion of the surface of a sphere of infinite radius centered at the origin which intersects V. The unit normal $\hat{n}$ is directed into V_e and is not uniquely defined at all points of S; however,

$$S = \overset{3}{\underset{i=1}{U}} S_i$$

and a unique normal is defined on each S_i. It will become apparent later why we chose G as our kernel function rather than the more traditional free-space Green's function. The function $\sigma(P_0)$ is defined by

$$\sigma(P_0) = -\lim_{\epsilon \to 0} \int_{\Sigma_\epsilon} \frac{\partial}{\partial n} \frac{1}{R}\, dS, \tag{32}$$

where $\Sigma_\epsilon = \partial(B_\epsilon(P_0) \cap V)$; that is, the boundary of that part of a ball of radius ϵ and center P_0 lying in V. $\sigma(P_0)$ is a measure of solid angle (Ref. 17), and when P_0 is on a smooth portion of S, $\sigma(P_0) = 2\pi$; when P_0 is on the z axis, $\sigma(P_0) = 2\Omega$; and when P_0 is along the other edge (i.e., when $r = a \csc \Omega$, $\phi = 0$), $\sigma(P_0) = 2(\pi - \Omega)$. The total field U satisfies a radiation condition, since both the scattered field and the incident field satisfy this condition. Also G satisfies a radiation condition, and it follows that the boundary integral in Eq. (31) over S_∞ vanishes.

It is seen from Eq. (31) that if the total field is known on S, it is known everywhere in V. Those portions of S which are not smooth (i.e., the two edges) are of measure zero and consequently do not contribute in the determination of the total field in V. For this reason we shall consider only the boundary integral where the field point is on a smooth portion of S. From Eq. (31) we have

$$-\frac{1}{4\pi} \int_S \left\{ U(P, P_1) \frac{\partial}{\partial n} G(P, P_0) - G(P, P_0) \frac{\partial}{\partial n} U(P, P_1) \right\} dS = \frac{1}{2} U(P_0, P_1) - G(P_1, P_0). \tag{33}$$

Now $G(P, P_0)$ satisfies the boundary conditions in Eq. (2), thus,

$$G(P, P_0) = 0 \qquad P \epsilon S_1 \text{ and } \frac{\partial}{\partial n} G(P, P_0) = 0 \qquad P \epsilon S_2. \tag{34}$$

Also, $U(P, P_1)$ satisfies the boundary conditions in Eq. (26),

$$U(P, P_1) = 0 \qquad P \epsilon S_1 \text{ and } \frac{\partial}{\partial n} U(P, P_1) = 0 \qquad P \epsilon S_2 \cup S_3. \tag{35}$$

From Eqs. (33), (34), and (35) we have

$$U(P_0, P_1) + \frac{1}{2\pi} \int_{S_3} U(P, P_1) \frac{\partial}{\partial n_p} G(P, P_0) dS_p = 2G(P_0, P_1). \tag{36}$$

The boundary integral equation in Eq. (36) is a Fredholm integral equation of the second kind where the region of integration is only a half-plane. It is now clear why we chose G as our Kernel function instead of the free-space Green's function. If the latter were chosen, then our boundary integral in Eq. (33) would involve both $U(P, P_1)$ and its normal derivative, which is a considerably more difficult integral equation to solve.

An integral representation for $u(P, P_1)$ similar to the one in Eq. (31) can be obtained by using Green's identities and choosing $g(P, P_0)$ defined in Eq. (20) as the Kernel function. Let Λ denote a plane which is perpendicular to the z axis and intersects the line source at the point P_1. Let $A = V \cap \Lambda$, $A_e = V_e \cap \Lambda$, $C_i = S_i \cap \Lambda$, and $i = 1,2,3,\infty$. It can be shown that

$$\frac{1}{4i} \int_{C \cup C_\infty} \left\{ u(P, P_1) \frac{\partial}{\partial n} g(P, P_0) - g(P, P_0) \frac{\partial}{\partial n} u(P, P_1) \right\} dS_P$$

$$= \begin{cases} u(P_0, P_1) - g(P_1, P_0) & P_0 \epsilon A \\ \dfrac{\mu(P_0)}{2\pi} u(P_0, P_1) - g(P_1, P_0) & P_0 \epsilon C \\ - g(P_1, P_0) & P_0 \epsilon A_e, \end{cases} \tag{37}$$

where

$$\mu(P_0) = \lim_{\epsilon \to 0} \int_{\Sigma_\epsilon} \frac{\partial}{\partial n} \ell n R dS. \tag{38}$$

In Eq. (38), $\Sigma_\epsilon = \partial(B_\epsilon \cap A)$; that is, the boundary of that part of a circle of radius ϵ and center P_0 lying in A. The boundary integral over C_∞ vanishes as a result of both u and g satisfying radiation conditions at infinity. It also may be argued that we need only consider the integral equation for P_0 on a smooth portion of

$$C = \bigcup_{i=1}^{3} C_i$$

to obtain the total field u in A. For P_0 on a smooth portion of C, $\mu(P_0) = \pi$, and we have

$$\frac{1}{4i} \int_C \left\{ u(P, P_1) \frac{\partial}{\partial n} g(P, P_0) - g(P, P_0) \frac{\partial}{\partial n} u(P, P_1) \right\} dS_P = \frac{1}{2} u(P_0, P_1) - g(P_1, P_0). \tag{39}$$

This is analogous to the integral equation in Eq. (33) for U. From Eq. (2) we have that

$$g(P, P_0) = 0, \quad P\epsilon C_1; \quad \frac{\partial}{\partial n} g(P, P_0) = 0, \quad P\epsilon C_2 \tag{40}$$

and from Eq. (26) we have

$$u(P, P_1) = 0 \quad P\epsilon C_1 \text{ and } \frac{\partial}{\partial n} u(P, P_1) = 0 \quad P\epsilon C_2 \cup C_3. \tag{41}$$

Substituting Eqs. (40) and (41) into Eq. (39), we obtain

$$u(P_0, P_1) - \frac{1}{2i} \int_{C_3} u(P, P_1) \frac{\partial}{\partial n} g(P, P_0) dS = 2g(P_0, P_1). \tag{42}$$

If

$$LU = -\frac{1}{2\pi} \int_{S_3} U(P, P_1) \frac{\partial}{\partial n_p} G(P, P_0) dS_p \tag{43}$$

and

$$Mu = \frac{1}{2i} \int_{C_3} u(P, P_1) \frac{\partial}{\partial n} g(P, P_0) dS_P, \tag{44}$$

the integral equations in Eqs. (36) and (42) may be written

$$(I - L)U = 2G \tag{45}$$

and

$$(I - M)u = 2g, \tag{46}$$

where I is the identity operator.

OBTAINING THE SOLUTION TO THE PROPAGATION PROBLEMS AS A NEUMANN SERIES

With L and M as defined in Eqs. (43) and (44), we show in this section that

$$U = 2 \sum_{n=0}^{\infty} L^n G \text{ and } u = 2 \sum_{n=0}^{\infty} M^n g$$

are the solutions to the two scattering problems formulated in the previous section everywhere on S except at the two vertices of the composite region. Once the solutions are known for this portion of S, from the integral representations in Eqs. (31) and (37), they are also known in V.

It is known (e.g., Ref. 18, p. 173) that if L and M are linear operators on a Banach space and if the above Neumann series converge in the norm of the Banach space, then the series converge to the unique solution of Eqs. (45) and (46). To discuss the convergence of the two series we must define the function spaces, show that L and M are operators in the space, and establish a suitable norm.

Let T denote the set of points on S which are not on either of the two vertices of S. It is noted that the integral equations in Eqs. (45) and (46) are valid only for $P_0 \epsilon T$. First let us discuss the problem corresponding to the point source. Let $C(S)$ and $C(T)$ denote the spaces of complex continuous functions, bounded and continuous as P_0 tends to infinity, which are defined on S and T, respectively. Since we desire the infinite series

$$U = 2 \sum_{n=0}^{\infty} L^n G$$

to converge pointwise to the boundary values at every point on T, we must establish convergence in the sup norm. Unfortunately, since T does not contain all of its limit points,

$$||U|| = \sup_{\substack{U \epsilon C(T) \\ P \epsilon T}} |U(P)| \tag{47}$$

may not exist for every function $U(P) \epsilon C(T)$. On the other hand, S does contain all of its limit points, the relationship

$$||U|| = \sup_{\substack{U \epsilon C(S) \\ P \epsilon S}} |U(P)| \tag{48}$$

does exist, and $C(S)$ is complete with respect to this norm. Now $G(P_0, P_1) \epsilon C(S)$, since $P_1 \epsilon V$. Later it will be proven that $U(P_0) \epsilon C(S)$ implies that $(LU)(P_0) \epsilon C(S)$. Note that since we are interested only in the functional behavior of $U(P_0, P_1)$ at P_0, and not at the fixed source point P_1, we have omitted the P_1 variable. For small values of k it will also be shown that $||L|| < 1$ with respect to the norm in Eq. (48). Thus for these values of k, the Neumann series

$$U = 2 \sum_{n=0}^{\infty} L^n G$$

converges pointwise to the solution of the boundary value problem for $P_0 \epsilon T$.

Now we discuss the problem corresponding to the line source. Let Λ be a plane perpendicular to the z axis and intersecting the line source at the point P_1. Let $C(S \cap \Lambda)$ denote the space of complex continuous functions, bounded and continuous as P_0 tends to infinity, which is defined on $S \cap \Lambda$. We will show that the Neumann series

$$u = 2 \sum_{n=0}^{\infty} M^n g$$

converges pointwise to the boundary value problem for $P_0 \epsilon T$ by first demonstrating that $u(P_0) \epsilon C(S \cap \Lambda)$ implies that $(Mu)(P_0) \epsilon C(S \cap \Lambda)$ and then by showing that, for small values of k, $||M|| < 1$ with respect to the norm,

$$\|u\| = \sup_{\substack{u \epsilon C(S \cap \Lambda) \\ P \epsilon S \cap \Lambda}} |u(P)|. \tag{49}$$

We note that $g(P_0, P_1) \epsilon C(S \cap \Lambda)$ and that $C(S \cap \Lambda)$ is complete with respect to the norm in Eq. (49).

We now prove the theorem that $U(P_0) \epsilon C(S)$ implies that $(LU)(P_0 \epsilon C(S)$. To do this we first establish two lemmas. The analogous result for u can be proven in a similar manner, and hence after each result we shall merely state the corresponding one for u. Consider the following.

Lemma 1. $\lim\limits_{\rho \to \infty} \frac{\partial}{\partial n} G(P, P_0) = 0\left(\frac{1}{\rho^2}\right)$, where $\rho = \sqrt{r^2 + z^2}$ and $P \epsilon S_3$.

Now $\nabla G = \left(\frac{\partial}{\partial r} G, \quad \frac{1}{r} \frac{\partial}{\partial \phi} G, \quad \frac{\partial}{\partial z} G\right)$ and it can be shown that

$$\hat{\mathrm{n}} = \sin(\Omega - \phi)\hat{\mathrm{r}} - \cos(\Omega - \phi)\hat{\phi}. \tag{50}$$

Thus

$$\frac{\partial}{\partial n} G = \sin(\Omega - \phi) \frac{\partial}{\partial r} G - \cos(\Omega - \phi) \frac{1}{r} \frac{\partial}{\partial \phi} G. \tag{51}$$

Since $P \epsilon S_3$, $r \sin(\Omega - \phi) = a$. It follows from Eq. (25) that

$$\sin(\Omega - \phi) \frac{\partial}{\partial r} G = \frac{4\pi k a i}{\Omega} \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{\cos \tau_n \phi \cos \tau_n \phi_0 \left(\frac{1}{2} k r_0\right)^s}{m! \Gamma(m + \tau_n + 1)}$$
$$\times \left\{ \frac{s r^{s-2} h_s^{(1)}(k\xi)}{\xi^s} + h_s^{(1)}(k\xi)' k \frac{r^s}{\xi^{s+1}} - s \frac{r^s}{\xi^{s+2}} h_s^{(1)}(k\xi) \right\} \tag{52}$$

and

$$\frac{\cos(\Omega - \phi)}{r} \frac{\partial}{\partial \phi} G = \frac{-4\pi k i}{\Omega} \cos(\Omega - \phi) \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{\tau_n \sin \tau_n \phi \cos \tau_n \phi_0}{m! \Gamma(m + \tau_n + 1)} \frac{\left(\frac{1}{2} k r_0 r\right)^s h_s^{(1)}(k\xi)}{r \xi^s}, \tag{53}$$

where

$$\tau_n = \frac{(n + 1/2)\pi}{\Omega}$$

$$s = 2m + \tau_n$$

$$\xi = \sqrt{r^2 + r_0^2 + (z - z_0)^2}.$$

From Ref. 13, p. 139, we have that for large ξ

$$h_s^{(1)}(\xi) \cong \frac{exp\left\{i\left[\xi - \frac{\pi}{2}(s+1)\right]\right\}}{\xi}. \tag{54}$$

After Eq. (54) is substituted into Eqs. (52) and (53), it follows from Eq. (51) that

$$\frac{\partial}{\partial n} G = 0\left(\frac{1}{\rho^2}\right) . \tag{55}$$

Now we give the corresponding result for g.

Lemma 2. $\lim_{r\to\infty} \frac{\partial}{\partial n} g(P, P_0) = 0\left(\frac{1}{r^{3/2}}\right)$, where $P \epsilon S_3 \cap \Lambda$.

The proof is similar to the one for Lemma 1. We use the representation for $g(P, P_0)$ given in Eq. (20). From Ref. 13, p. 139, for large values of ξ,

$$H_s^{(1)}(\xi) = \sqrt{\frac{2}{\pi}}\, \frac{exp[i(\xi - (\pi s/2) - \pi/4)]}{\sqrt{\xi}} . \tag{56}$$

The proof of Lemma 4.2 follows from Eq. (56).

Next let us prove Lemma 3.

Lemma 3. $\lim_{\rho_0\to\infty} \int_{S_3} \left|\frac{\partial}{\partial n} G\right| dS = 0\left(\frac{1}{\rho_0}\right).$

From Lemma 1, it follows that for $\epsilon>0$ there exists a compact set $\widetilde{S}_3(\epsilon)\subset S_3$ such that

$$\left|\int_{S_3} \left|\frac{\partial}{\partial n} G\right| dS - \int_{\widetilde{S}_3} \left|\frac{\partial}{\partial n} G\right| dS\right| < \epsilon \tag{57}$$

where $\widetilde{S}_3 \subset S_3$ is any compact set containing $\widetilde{S}_3(\epsilon)$. Thus

$$\int_{S_3} \left|\frac{\partial}{\partial n} G\right| dS < \epsilon + \int_{\widetilde{S}_3} \left|\frac{\partial}{\partial n} G\right| dS. \tag{58}$$

From Eqs. (51), (52), and (53), we have

$$\frac{\partial}{\partial n} G = \frac{1}{\sqrt{r_0^2 + z_0^2}} f(P, P_0), \tag{59}$$

where $f(P, P_0)$ is a function, continuous in P_0, which satisfies the following property

$$\lim_{\rho_0 \to \infty} f(P, P_0) = 0(1). \tag{60}$$

The important feature of $f(P, P_0)$ is not its explicit form, but that it satisfies the property in Eq. (60). Hence,

$$\lim_{\rho_0 \to \infty} \int_{\tilde{S}_3} |f(P, P_0)| dS = 0(1), \tag{61}$$

and from Eqs. (58), (59), and (61) the proof of the lemma follows.

The corresponding result for g is Lemma 4.4.

Lemma 4. $\lim_{r_0 \to \infty} \int_{C_3} \left|\frac{\partial}{\partial n} g\right| ds = 0\left(\frac{1}{\sqrt{r_0}}\right)$

Now we establish the following theorem.

Theorem 1. $U(P_0) \epsilon C(S) \Rightarrow (LU)(P_0) \epsilon C(S)$

In view of Lemma 3, we need to consider only the case when ρ_0 is finite. Suppose $P_0 \epsilon S_1 \cup S_2$. Then $(\partial/\partial n)G(P, P_0) \epsilon C(S)$, and from Lemma 1 it follows that $(LU)(P_0) \epsilon C(S)$.

Suppose $P_0 \epsilon S_3$. From Lemma 1, it follows that for $\epsilon > 0$ there exists a compact set $\tilde{S}_3(\epsilon) \subset S_3$ such that

$$\left| \int_{S_3} \left|\frac{\partial}{\partial n} G\right| dS - \int_{\tilde{S}_3} \left|\frac{\partial}{\partial n} G\right| dS \right| < \epsilon, \tag{62}$$

where $\tilde{S}_3 \subset S_3$ is any compact set containing $\tilde{S}_3(\epsilon)$. G has a singularity of the type $1/R$ at P_0 and may be expressed as

$$G(P, P_0) = \frac{1}{R} + V(P, P_0), \tag{63}$$

where $V(P, P_0)$ and its derivative are continuous on S_3. Let $\epsilon > 0$ be given. Consider

$$\begin{aligned} |(LU)(P_0) - (LU)(P')| = \frac{1}{2\pi} \Bigg| &\int_{\tilde{S}_3} U(P) \frac{\partial}{\partial n_P} \left(\frac{1}{|P - P_0|} - \frac{1}{|P - P'|} \right) dS \\ &+ \int_{\tilde{S}_3} U(P) \frac{\partial}{\partial n_P} [V(P, P_0) - V(P, P')] dS \\ &+ \int_{S_3 - \tilde{S}_3} U(P) \frac{\partial}{\partial n_P} [G(P, P_0) - G(P, P')] dS \Bigg|, \end{aligned} \tag{64}$$

where $\tilde{S}_3 \subset S_3$ is a compact set which satisfies

$$\left| \frac{1}{2\pi} \int_{S_3 - \tilde{S}_3} U(P) \frac{\partial}{\partial n} (G(P, P_0) - G(P, P'))dS \right| < \frac{\epsilon}{3} . \tag{65}$$

It follows immediately from Eq. (62) that such a set $\tilde{S}_3$ exists, and we may assume that $\tilde{S}_3$ contains a neighborhood of P_0, for if not, a set $\tilde{S}_3^*$ containing $\tilde{S}_3$ and some neighborhood of P_0 may be chosen. Since V is continuous, it follows that there exists a $\delta_1 > 0$ such that $|P_0 - P'| < \delta_1$ implies that

$$\left| \frac{1}{2\pi} \int_{\tilde{S}_3} U(P) \frac{\partial}{\partial n} [V(P, P_0) - V(P, P')] dS \right| < \frac{\epsilon}{3} . \tag{66}$$

It is well known (e.g., Ref. 19, p. 157) that the kernel of integrals of the form

$$LU = - \frac{1}{2\pi} \int_{\tilde{S}_3} U(P) \frac{\partial}{\partial n_P} \frac{1}{R} dS \tag{67}$$

has weak singularity, i.e., $|(\partial/\partial n)(1/R)| \leqslant (C/R^{2-\alpha})$, where $0 < \alpha \leqslant 1$ and C is some constant. It is also known (Ref. 19, p. 115) that an operator with a weakly singular kernel whose domain of integration is a compact set in m-dimensional Euclidean space is completely continuous and hence continuous. Thus there is a $\delta_2 > 0$ such that $|P_0 - P'| < \delta_2$ implies

$$\left| \frac{1}{2\pi} \int_{\tilde{S}_3} U(P) \frac{\partial}{\partial n} \left(\frac{1}{|P - P_0|} - \frac{1}{|P - P'|} \right) dS \right| < \frac{\epsilon}{3} . \tag{68}$$

Take $\delta = \min (\delta_1, \delta_2)$. It follows from Eqs. (64), (65), (66), and (68) that for $|P_0 - P'| < \delta$,

$$|(LU)(P_0) - (LU)(P')| < \epsilon,$$

and hence $(LU)(P_0) \epsilon C(S)$.

Similarly, we may prove the following theorem.

Theorem 2.2. $u(P_0) \epsilon C(S \cap \Lambda) \Rightarrow (Mu)(P_0) \epsilon C(S \cap \Lambda)$.

It remains to be demonstrated that for small values of k, $||L|| < 1$ with respect to the norm in Eq. (48) and that $||M|| < 1$ with respect to the norm in Eq. (49). Since the argument is the same in both cases, we shall give only the one for L and merely state the corresponding results for M. From Ref. 13, pp. 66 and 80, we have

$$h^{(1)}_{2m+\tau_n}(k\xi) = \left(\frac{1}{\frac{2\pi}{k\xi}}\right)^{1/2} \left\{ \left(\frac{k\xi}{2}\right)^{2m+\tau_n+1/2} \sum_{s=0}^{\infty} \frac{(-1)^s \left(\frac{k\xi}{2}\right)^{2s}}{s!\Gamma(2m+\tau_n+3/2)} \right. \tag{69}$$

$$+ \frac{2i}{\Gamma(2m+\tau_n+1)\sqrt{\pi}} \left(\frac{k\xi}{2}\right)^{2m+\tau_n+1/2} \left[\int_0^1 (1-t^2)^{2m+\tau_n} \sin(k\xi t)dt \right.$$

$$\left.\left. - \int_0^{\infty} e^{-k\xi t}(1+t^2)^{2m+\tau_n}dt \right] \right\}.$$

We see that the spherical Hankel function has a factor of $k^{2m+\tau_n}$; i.e.,

$$h^{(1)}_{2m+\tau_n}(k\xi) = k^{2m+\tau_n}A(\xi), \tag{70}$$

where $A(\xi)$ is some function with terms of k of order $\geqslant 0$. Now $\tau_n = (n + 1/2)\pi/\Omega$, and from Eq. (69) we see that $k^{\pi/2\Omega}$ is a factor of each $h^{(1)}_{2m+\tau_n}(k\xi)$. Thus from Eq. (25), we see that G has a factor of $k^{1+\pi/2\Omega}$, and L may be expressed as

$$LU = k^{1+\pi/2\Omega} L^*U, \tag{71}$$

where L^* is an operator whose kernel involves terms of k of order greater than or equal to zero. Thus $||L|| < 1$ is valid for small values of k, provided it can be shown that L is a bounded operator, which is demonstrated below. Similarly, by using the representation for g in Eq. (20) we can show that

$$Mu = k^{\frac{\pi}{2\Omega}} M^*u, \tag{72}$$

and for small values of k, $||M|| < 1$ with respect to the norm in Eq. (49).

Here we demonstrate that L is a bounded operator, and we could use similar argument to show that M is bounded. From Eq. (43) it follows that

$$\left|(LU)(P_0)\right| \leqslant \frac{||U||}{2\pi} \int_{S_3} \left|\frac{\partial}{\partial n} G\right| dS. \tag{73}$$

Consider the integral $\int_{S_3} \left|\frac{\partial}{\partial n} G\right| dS$. Suppose $P_0 \epsilon S_1 \cup S_2$. Then $\frac{\partial}{\partial n} G$ is continuous, hence $\int_{S_3} \left|\frac{\partial}{\partial n} G\right| dS$ is continuous, and from Lemma 3 this integral is equal to zero as P_0 tends to infinity. It follows that the integral may be bounded by some constant C; i.e.,

$$\int_{S_3} \left|\frac{\partial}{\partial n} G\right| dS < C \quad \text{for } P_0 \epsilon S_1 \cup S_2. \tag{74}$$

Suppose $P_0 \epsilon S_3$. From Lemma 3 we have that $\lim_{\rho \to \infty} \int_{S_3} \left| \frac{\partial}{\partial n} G \right| dS = 0$. Thus suppose $\rho_0 < \infty$. As in Theorem 1 it may be argued that for $\epsilon > 0$ there exists a compact set $\tilde{S}_3(\epsilon) \subset S_3$ such that

$$\int_{S_3 - \tilde{S}_3} \left| \frac{\partial}{\partial n} G \right| dS < \epsilon \tag{75}$$

for all compact sets $\tilde{S}_3$ in S_3 containing $\tilde{S}_3(\epsilon)$. Further, we may suppose that $\tilde{S}_3$ contains a neighborhood of P_0. If we express G as

$$G = \frac{1}{|P - P_0|} + V(P, P_0), \tag{76}$$

where $V(P, P_0)$ and its derivative are continuous on S_3, it follows that $\int_{S_3} \left| \frac{\partial}{\partial n} V \right| dS$ is bounded by some constant C_1. Consider the integral

$$\int_{\tilde{S}_3} \left| \frac{\partial}{\partial n} \frac{1}{|P - P_0|} \right| dS. \tag{77}$$

From the convexity of $\tilde{S}_3$ it can be shown that

$$\left| \frac{\partial}{\partial n} \frac{1}{|P - P_0|} \right| = - \frac{\partial}{\partial n} \frac{1}{|P - P_0|} . \tag{78}$$

Thus the integral in Eq. (77) is of the type in Eq. (67). Such operators are completely continuous, and it is known (e.g. Ref. 19, p. 117) that they are bounded. Thus it is seen that for $P_0 \epsilon S_3$, $\int_{\tilde{S}_3} \left| \frac{\partial}{\partial n} G \right| dS$ is bounded and from Eq. (75), since ϵ does not depend on P_0 it follows that $\int_{S_3} \left| \frac{\partial}{\partial n} G \right| dS$ is uniformly bounded by some constant C. Thus for and P_0 in S, the integral in Eq. (73) is bounded by a constant, and from Eq. (73) it follows that

$$|(LU)(P_0)| \leqslant C \| U \|. \tag{79}$$

Since this is true for any $P_0 \epsilon S$, it follows that

$$\| LU \| \leqslant C \| U \|,$$

and hence L is bounded.

SUMMARY AND CONCLUSIONS

In this report the mixed boundary value problem for the Helmholtz equation, formally stated in Eq. (26), is considered in the composite region formed by the intersection of a wedge and two parallel plates. Both cases, a line source and a point source, are discussed for the incident field. The problem is reformulated via Green's identities, where instead of the traditional choice of the free-space Green's function as the kernel function, the Green's function for the corresponding wedge region is used. If the free-space Green's function were used as the kernel function, the following boundary integral equation would be obtained via Green's identities:

$$\frac{1}{2} U(P_0, P_1) - G_0(P_1, P_0) = \frac{1}{4\pi} \int_{S_1} G_0(P, P_0) \frac{\partial}{\partial n_P} U(P, P_1) dS_P - \frac{1}{4\pi} \int_{S_2 \cup S_3} U(P, P_1) \frac{\partial}{\partial n_P} G_0(P, P_0) dS_P, \tag{80}$$

where $G_0(P_1, P_0)$ is the three-dimensional, free-space Green's function; $P_1 \epsilon V$; and P_0 is on a smooth portion of S; i.e., P_0 is not on one of the vertices of S. If the Green's function $G(P, P_0)$ for the wedge region is used as the kernel function, the following integral equation is obtained through Green's identities;

$$\frac{1}{2} U(P_0, P_1) - G(P_1, P_0) = \frac{1}{4\pi} \int_{S_3} U(P, P_1) \frac{\partial}{\partial n_P} G(P, P_0) dS_P, \tag{81}$$

where G is given explicitly in Eq. (25).

Although the kernel function in the integral equation in Eq. (80) is extremely simple in form, nevertheless it is doubtful that the integral equation can be solved either by matrix inversion or by iteration. On the other hand, it is proven in this report that the solution to the integral equation in Eq. (81) may be expressed as a Neumann series, which converges for sufficiently small values of k; and in fact, at least theoretically, the solution may be obtained to any degree of accuracy desired. From a comparison of the regions of integration of Eqs. (80) and (81), we see that the one in Eq. (80) is considerably more simple. Also, the second integral equation is in a form which lends itself to matrix inversion. Once the total field $U(P_0, P_1)$ is known for $P_0 \epsilon S_3$, it follows from the integral representation in Eq. (31) that it is known everywhere in V. Certainly the kernel function $G(P, P_0)$ is somewhat complicated and presents numerical problems; however, for small values of the wave number, it can be reasonably approximated.

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13. ABSTRACT

Wave propagation in the composite region of a wedge which leads into a duct was considered where mixed boundary conditions were imposed. The problem was reformulated as an integral equation where, instead of the free-space Green's function being used as the Kernel function, the Green's function for the wedge region was used. This permitted considerable simplification, and a Fredholm integral equation of the second kind was obtained where the region of integration is merely a plane and is valid for any value of the wave number. For small values of the wave number the boundary integral equation may be solved iteratively, and the solution is given as a Neumann series.

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