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Gain Adjustment of an Alpha-Beta Filter With Random Updates

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ABSTRACT

A procedure for adjusting the gains in an α - β filter used in tracking air targets by search radars is given for the case in which the track updates appear randomly in time. The filter gains are given by $\alpha = 1 - e^{-2\xi\omega_0 T}$ and $\beta = 1 + e^{-2\xi\omega_0 T} - 2e^{-\xi\omega_0 T} \cos \omega_d T$ where ξ , ω_0 , and ω_d are constants and T is the randomly varying time between updates. Using this gain adjustment procedure, we found that the tracking errors are smaller than when the gains α and β are held constant for tracks which are randomly updated.

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GAIN ADJUSTMENT OF AN ALPHA-BETA FILTER WITH RANDOM UPDATES

INTRODUCTION

Search radars sometimes track air targets by use of an α - β filter [1]. The filter computes the target's velocity from the measured position, smoothes both the position and velocity, and finally predicts the position the target will have at the next look of the radar. In most of these cases a uniform update time can be assumed and such a system can be analyzed with standard techniques involving sampled-data systems [2-4].

Let us consider merging the target reports from a number of radars. The radars are assumed to rotate at different speeds, and the detection capability of any given radar on a target depends on many environmental and radar factors. For these reasons the track updates for a given target might be thought of as appearing randomly in time. Under some circumstances a phased-array radar could also be modeled with a random update time. In this report we analyze the α - β filter by using the postulated random updates, and we consider a means of adjusting the filter's gain so as to improve its system response.

REVIEW OF THE α - β FILTER

The review of the α - β filter with constant update times includes an examination of the transfer function, its properties, and the response of the system in both the mean and variance. We begin by defining the α - β filter [1]:

$$x_s(k) = x_p(k) + \alpha[x_m(k) - x_p(k)] , \quad (1)$$

$$v_s(k) = v_s(k-1) + (\beta/T)[x_m(k) - x_p(k)] , \quad (2)$$

$$x_p(k+1) = x_s(k) + v_s(k) T , \quad (3)$$

where

$x_s(k)$ = smoothed position,

$v_s(k)$ = smoothed velocity,

$x_p(k)$ = predicted position,

$x_m(k)$ = measured position,

T = sampling period (constant until specified otherwise),

α, β = system gains.

We obtain a convenient form of the filter equations by substituting (3) into (1) and (2), obtaining

$$\begin{bmatrix} x_s(k) \\ v_s(k) \end{bmatrix} = \begin{bmatrix} (1-\alpha) & (1-\alpha)T \\ -\beta/T & (1-\beta) \end{bmatrix} \begin{bmatrix} x_s(k-1) \\ v_s(k-1) \end{bmatrix} + \begin{bmatrix} \alpha \\ \beta/T \end{bmatrix} [x_m(k)] \quad (4)$$

and

$$x_p(k+1) = [1 \ T] \begin{bmatrix} x_s(k) \\ v_s(k) \end{bmatrix}. \quad (5)$$

Applying the z transform [2, 3] to (4) and (5), we find that the transfer functions of the system are

$$H_x = x_s(z)/x_m(z) = \frac{\alpha z [z + (\beta - \alpha)/\alpha]}{z^2 - z(2 - \alpha - \beta) + (1 - \alpha)}, \quad (6)$$

$$H_v = v_s(z)/x_m(z) = \frac{(\beta/T) z(z-1)}{z^2 - z(2 - \alpha - \beta) + (1 - \alpha)}, \quad (7)$$

$$H_p = x_p(z)/x_m(z) = \frac{(\alpha + \beta) z [z - \{\alpha/(\alpha + \beta)\}]}{z^2 - z(2 - \alpha - \beta) + (1 - \alpha)}. \quad (8)$$

By setting $z = e^{j\omega T}$, we find the frequency responses for a typical system (Fig. 1). The smoothed position is obtained by passing the measured position through a low-pass filter, and the smoothed velocity is obtained by differentiating the measured position. The sampler itself acts as a low-pass filter, and any excitation whose frequency range is above $1/T$ is simply folded into the frequency range from zero to $1/T$. The normally used values of α and β are shown in Fig. 2. This is obtained by examining the pole and zero locations of the transfer functions (6), (7), and (8). The transfer functions (6), (7), (8) are placed into standard notation for a second-order system:

$$H(\cdot) = \frac{(\cdot)}{z^2 - 2z e^{-\xi\omega_0 T} \cos \omega_d T + e^{-2\xi\omega_0 T}}. \quad (9)$$

Equating terms in the denominator of (9) with the denominators of (6), (7), and (8), we obtain

$$\alpha = 1 - e^{-2\xi\omega_0 T} \quad (10)$$

and

$$\beta = 1 + e^{-2\xi\omega_0 T} - 2e^{-\xi\omega_0 T} \cos \omega_d T, \quad (11)$$

where ξ is the damping coefficient of the second-order system, ω_0 is the natural frequency, and ω_d is the damped natural frequency. Equations (10) and (11) will be used later.

Before observing the system response, we consider incorporating a constant-rate high-speed sampler in the system and defining $x_m(k)$ as

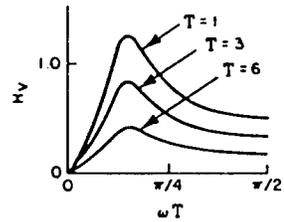
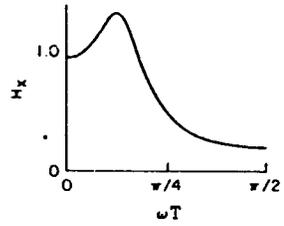


Fig. 1—Frequency response of an α - β filter, which is given by Eqs. (6), (7), and (8) with $z = e^{j\omega T}$, for $\alpha = 0.529$ and $\beta = 0.579$

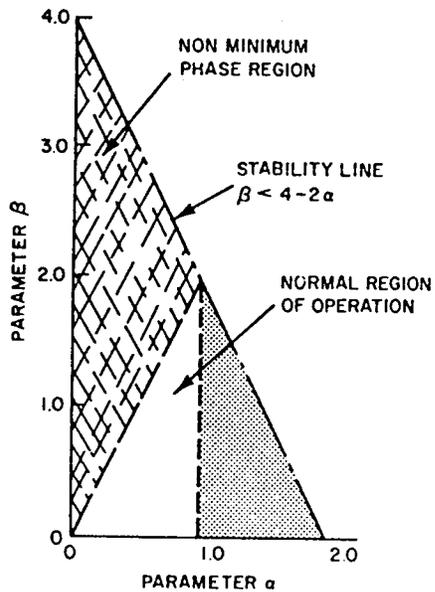
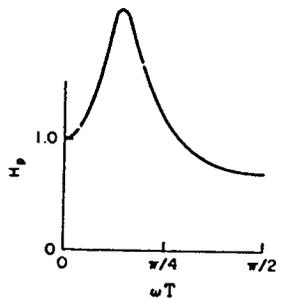


Fig. 2—Permissible values of α and β

$$x_m(k) = u(t_k) + w(t_k), \quad (12)$$

with $w(t_k)$ assumed to be a white stationary Gaussian noise having zero mean and having variance σ_w^2 representing measurement error of the radar and with $u(t_k)$ assumed to be samples from a deterministic target trajectory. A block diagram of the filter is shown in Fig. 3. We define the matrices in Eqs. (4) and (5) as

$$A(T) = \begin{bmatrix} (1 - \alpha) & (1 - \alpha)T \\ -\beta/T & (1 - \beta) \end{bmatrix},$$

$$B(T) = \begin{bmatrix} \alpha \\ \beta/T \end{bmatrix},$$

$$X(k) = \begin{bmatrix} x_s(k) \\ v_s(k) \end{bmatrix},$$

$$C(i) = [1 \quad i\Delta],$$

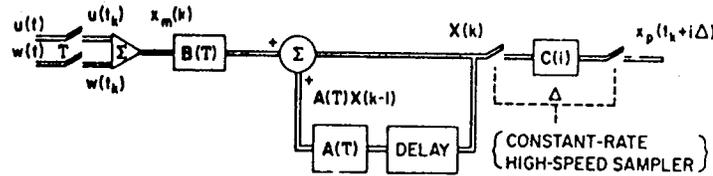


Fig. 3—An α - β filter incorporating a constant-rate high-speed sampler

where Δ is the time between samples of the high-speed sampler. We then write the filter equations (4) and (5) as

$$X(k) = A(T) X(k-1) + B(T) [u(t_k) + w(t_k)] \quad (13)$$

and

$$x_p(kT + i\Delta) = C(i) X(k), \quad (14)$$

where $i = 1, 2, 3, \dots, T/\Delta$, with T/Δ being an integer.

The response of the filter is obtained in terms of means and covariances. The equations describing the covariances are of the form [4].

$$P(k+1) = A(T) P(k) A'(T) + B(T) \sigma_w^2 B'(T), \quad (15)$$

where

$$P(k) = \begin{bmatrix} P_{x_s x_s}(k) & P_{x_s v_s}(k) \\ P_{v_s x_s}(k) & P_{v_s v_s}(k) \end{bmatrix},$$

in which

$$P_{x_s x_s}(k) = \text{cov}[x_s(k) \quad x_s(k)] .$$

The covariance equations for the α - β filter are derived using (13), (14), and (15):

$$\begin{bmatrix} P_{x_s x_s}(k) \\ P_{x_s v_s}(k) \\ P_{v_s v_s}(k) \end{bmatrix} = \begin{bmatrix} (1-\alpha)^2 & 2(1-\alpha)^2 T & (1-\alpha)^2 T^2 \\ -\beta(1-\alpha)/T & (1-\alpha)(1-2\beta) & (1-\alpha)(1-\beta)T \\ (\beta/T)^2 & -2\beta(1-\beta)/T & (1-\beta)^2 \end{bmatrix} \begin{bmatrix} P_{x_s x_s}(k-1) \\ P_{x_s v_s}(k-1) \\ P_{v_s v_s}(k-1) \end{bmatrix} + \begin{bmatrix} \alpha^2 \\ 2\beta/T \\ (\beta/T)^2 \end{bmatrix} \sigma_w^2 , \quad (16)$$

$$P_{x_p x_p}(kT + i\Delta) = P_{x_s x_s}(k) + 2i\Delta P_{x_s v_s}(k) + (i\Delta)^2 P_{v_s v_s}(k) , \quad (17)$$

for $i = 1, 2, \dots, T/\Delta$. The steady-state solution of (15) is obtained by setting $P(k+1)$ and $P(k)$ and solving the resulting algebraic equation. The results are

$$\dot{P}_{x_s x_s}(k)/\sigma_w^2 = \frac{2\beta - 3\alpha\beta + 2\alpha^2}{\alpha(4 - 2\alpha - \beta)} , \quad (18)$$

$$P_{x_s v_s}(k)/\sigma_w^2 = \frac{\beta(2\alpha - \beta)}{\alpha(4 - 2\alpha - \beta)} , \quad (19)$$

$$P_{v_s v_s}(k)/\sigma_w^2 = \frac{\beta(2\alpha^2 - \alpha^3 + 2\beta - \alpha\beta)}{\alpha(4 - 2\alpha - \beta)} . \quad (20)$$

We define $\sigma_0^2(i)$ to be

$$\sigma_0^2(i) = P_{x_p x_p}(kT + i\Delta)/\sigma_w^2 . \quad (21)$$

Combining (17) through (21), we plot the normalized variance of the predicted position $\sigma_0^2(i = T/\Delta)$ at the update time of the filter as a function of α and β (Fig. 4). In Fig. 5 we plot $\sigma_0^2(i)$ for a given α and β to show the intrasample ripple in the variance of the predicted position.

We now consider the mean response of the filter. A target is flown in a circle at a large distance from the radar. (This trajectory represents a turning target, and the full circle is used such that a steady state is obtained.) The range variation as a function of time is [5]

$$u(t) = R_0 + (v^2/a_n) \cos(a_n/v)t , \quad (22)$$

where R_0 is the range to the center of circle (ft), a_n is the normal acceleration (ft/s²), and v is the velocity of target (ft/s). When the filter is excited by samples of $u(t)$ taken

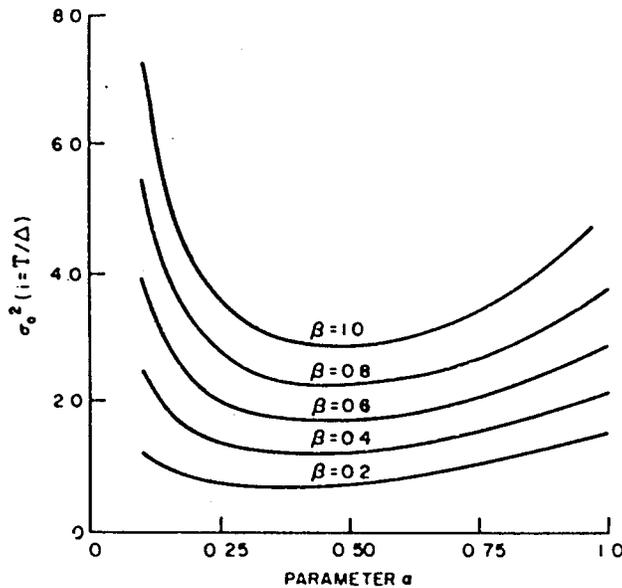


Fig. 4—Normalized variance of the predicted position at sample instants k as a function of α and β

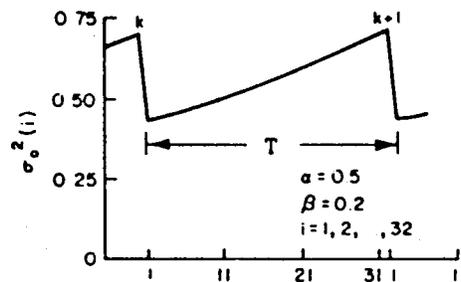


Fig. 5—Normalized variance of the predicted position, showing intrasample ripple

at uniform instants in time (constant T), the response of the filter $x_s(k)$, $v_s(k)$, and $x_p(kT + T)$ are sinusoids of the same frequency but different amplitudes and phases at the sampling instants. This result is well known [2,3], and is easily obtained by recursively solving the filter equations using the sampled values $u(t_k)$. The intrasample ripple is obtained by using the constant-rate high-speed sampler $i = 1, 2, \dots, T/\Delta$. A system response including the intrasample ripple is shown in Fig. 6. This filter was mis-designed purposely so that the error could be shown easily.

The various ways of designing the filter basically attempt to reject the influence of $w(t)$ while maintaining $x_p(k)$ as close as possible to $u(t_k)$ [1,6-10]. We now consider the filter's operation when it is randomly updated.

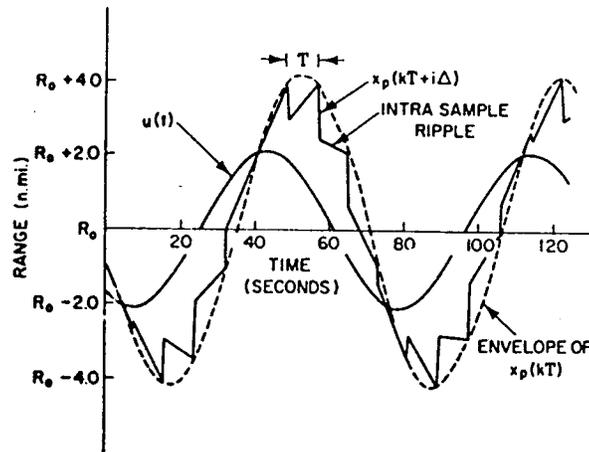


Fig. 6—Mean response of a filter excited by $u(t)$ with $T = 8$ s, $\alpha = 0.798$, $\beta = 0.534$, $a_n = 96.6$ ft/s², and $v = 1100$ ft/s

GAIN ADJUSTMENT WITH RANDOM SAMPLING

We begin by computing the means and covariances of the filter under random sampling, considering only the excitation $w(t_k)$ for a fixed-gain and a variable-gain α - β filter. We then compute the mean and covariances of the α - β filter considering only the excitations $u(t_k)$ and $X(0)$. The total mean and covariance response can be found by superimposing the two solutions. A procedure for showing that superposition is permissible is shown in Appendix A.

Fixed-Gain α - β Filter

The α - β filter with random updates can be placed in the form

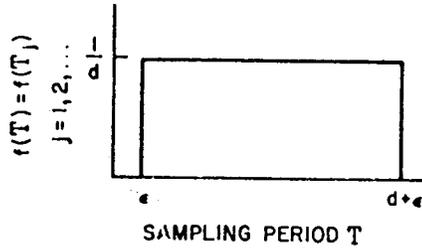
$$X(k) = A(T_k) X(k-1) + B(T_k) [u(t_k) + w(t_k)], \quad (23)$$

$$x_p(t_{k+1}) = C(T_{k+1}) X(k), \quad (24)$$

where T_k is a random variable representing the time between the $(k-1)$ th and k th sample, $t_k = T_1 + T_2 + \dots + T_k$ is the time of occurrence of the k th sample, and $A(T_k)$, $B(T_k)$, and $C(T_k)$ are matrices whose elements depend on T_k . The joint probability density $f(\cdot)$ of the process is

$$f(T_1) f(T_2) \dots f(T_k) f[X(0)] f[w(t_1)] f[w(t_2)] \dots f[w(t_k)] f[u(t_1), u(t_2), \dots, u(t_k)]. \quad (25)$$

Random samples t_k from a continuous deterministic process $u(t)$ are not independent. The time interval T_j between samples is assumed to be identically distributed for all j from the uniform distribution shown in Fig. 7. The subscript on T_k will be dropped for convenience.

Fig. 7—Probability density of a sampling interval T

The covariances of the system states are next considered. Considering the response only due to $w(t_k)$, we find $\overline{X(k)} = 0$, where the bar denotes expected value. As shown in Appendix A, the recursive equation

$$P(k) = \overline{A(T) P(k-1) A'(T)} + \overline{B(T) \sigma_w^2 B'(T)} \quad (26)$$

can be used to find the covariances $P(k)$ due to $w(t_k)$. The covariance equations for the α - β filter are

$$\begin{bmatrix} P_{x_s x_s}(k) \\ P_{x_s v_s}(k) \\ P_{v_s v_s}(k) \end{bmatrix} = \begin{bmatrix} \overline{(1-\alpha)^2} & \overline{2(1-\alpha)^2 T} & \overline{(1-\alpha)^2 T^2} \\ \overline{-\beta(1-\alpha)/T} & \overline{(1-\alpha)(1-2\beta)} & \overline{(1-\alpha)(1-\beta)T} \\ \overline{(\beta/T)^2} & \overline{-2\beta(1-\beta)/T} & \overline{(1-\beta)2} \end{bmatrix} \begin{bmatrix} P_{x_s x_s}(k-1) \\ P_{x_s v_s}(k-1) \\ P_{v_s v_s}(k-1) \end{bmatrix} + \begin{bmatrix} \overline{\alpha^2} \\ \overline{\alpha\beta/T} \\ \overline{(\beta/T)^2} \end{bmatrix} \sigma_w^2, \quad (27)$$

$$P_{x_p x_p}(t_{k+1}) = P_{x_s x_s}(k) + 2\bar{T} P_{x_s v_s}(k) + \bar{T}^2 P_{v_s v_s}(k). \quad (28)$$

Since α and β are constant, the coefficients in Eqs. (27) and (28) are

$$\bar{T} = (1/d) \int_{\epsilon}^{d+\epsilon} T dT = (d/2) + \epsilon, \quad (29)$$

$$\bar{T}^2 = (d^2/3) + d\epsilon + \epsilon^2, \quad (30)$$

$$\overline{1/T} = [\ln(d+\epsilon) - \ln \epsilon] / d, \quad (31)$$

$$\overline{1/T^2} = 1/[\epsilon(T+\epsilon)]. \quad (32)$$

As the results of using the coefficients (29) through (32) in (27) and (28), the normalized variance of the predicted position

$$\sigma_0^2 = P_{x_p x_p}(t_{k+1})/\sigma_w^2 \quad (33)$$

is shown in Fig. 8 as a function of ϵ for steady-state conditions. These results were also computed using Monte Carlo techniques. Observing Fig. 8, we find that σ_0^2 increases rapidly for small values of ϵ . This can be explained as follows: In a uniform update system, if the update time becomes short, the variance in the velocity increases rapidly. However, this effect is directly canceled in the variance of the predicted position, because one needs to predict only over this same small interval of time. In the random update system the time between updates can first be short, creating a large error in velocity. This can then be followed by a long time interval in which the target's position must be estimated using the poor velocity estimate. Therefore σ_0^2 increases under these conditions. A method of avoiding these large errors is next considered.

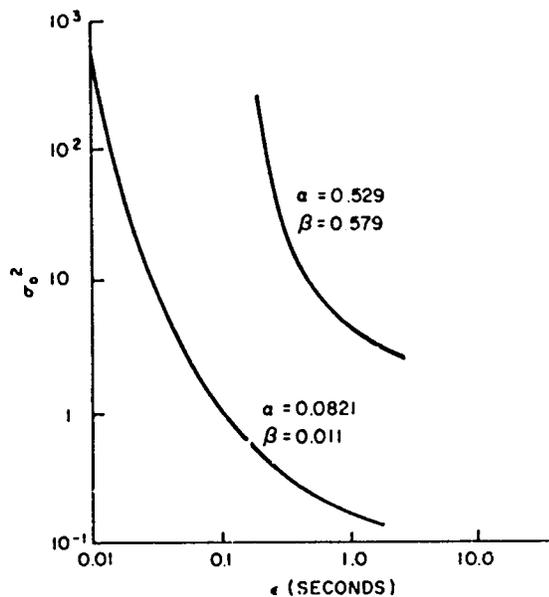


Fig. 8—Normalized predicted position variance as a function of ϵ with $d = 6$ seconds for a fixed-gain α - β filter

Variable-Gain α - β Filter

The method of avoiding large errors in the random update system adjusts the system gains α and β according to (10) and (11), where T is the random update time. The covariance equations considering only $w(t_k)$ as an excitation is again given by Eqs. (27) and (28). The coefficients are computed by numerical integration with the use of the uniform probability density given in Fig. 7. The normalized variance of the predicted position (33) is shown in Fig. 9 as a function of ϵ . We find that σ_0^2 is not a strong function of ϵ , as was the case in Fig. 8. This can be explained as follows: When the time interval between samples becomes short, α and β approach 0, thus smoothing the data heavily and negating the rise in velocity errors. As the time between samples becomes long, α and β approach 1 and in effect no smoothing is used. The gain adjustments

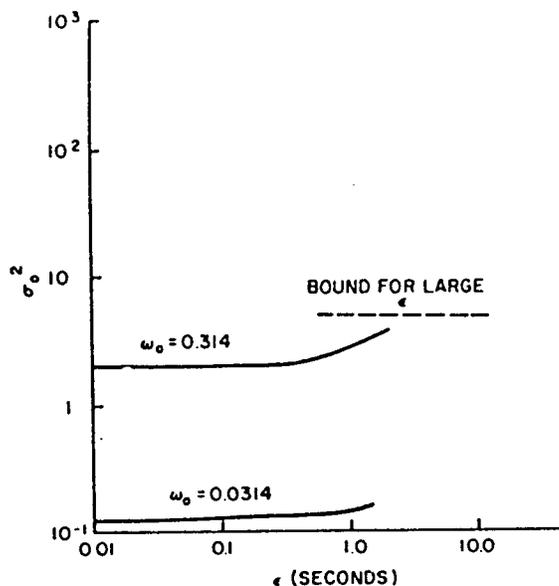


Fig. 9—Normalized predicted position variance as a function of ϵ with $\xi = 0.4$ and $d = 6$ seconds for a variable-gain α - β filter

(10) and (11) appear to maintain a rather constant σ_0^2 under the random sampling distribution proposed.

We now investigate the effects of the excitation $u(t_k)$ and $X(0)$ on the system response using the gains (10) and (11). We will find the effect of $u(t_k)$ and $X(0)$ on the randomly updated filter by using simulation procedures. (Because $E\{[u(t_j) - \overline{u(t_j)}][u(t_l) - \overline{u(t_l)}]\}$ is not 0 for all $l \neq j$, an analytic formulation appears to be very difficult.) An approach to the problem would be to find the mean and covariance of the predicted position at the k th sample. However the k th sample appears randomly in time. A more meaningful calculation would be to find these quantities at a given instant of time. Consider the constant-rate high-speed sampler defined in Fig. 3. Under any randomly sampled excitation the filter response can be found at a given sample of the high-speed constant-rate sampler which corresponds to a given instant of time. The quantity Δ is made smaller than ϵ such that the effect of each random sample can be easily seen. The adjustable-gain filter is excited by taking random samples, distributed as shown in Fig. 7, from a sinusoid, given by Eq. (22). Over many trials the mean and variance of x_p is computed at each switch closing of the constant-rate high-speed sampler. The results are shown in Figs. 10 and 11. Even though $u(t)$ is a deterministic process, under random sampling the filter output is a random variable. One finds that if the filter has a sufficient bandwidth, the filter with adjustable gains (10) and (11) follows $u(t)$ quite well (Fig. 11).

The reason x_p follows $u(t)$ fairly well with the adjustable-gain filter may be explained as follows: As the time between samples becomes short, the system smoothes heavily, which counteracts the increase in bandwidth due to the rapid sampling. Conversely, as the time between samples becomes large, the system lightly smoothes, counteracting the sluggishness induced by the long time between samples. In fact the system attempts to maintain a constant bandwidth, and if this bandwidth is wider than the frequency content of the signal, then x_p follows $u(t)$ fairly closely. This system behaves like a continuous system of fixed bandwidth being excited with inputs appearing randomly in time.

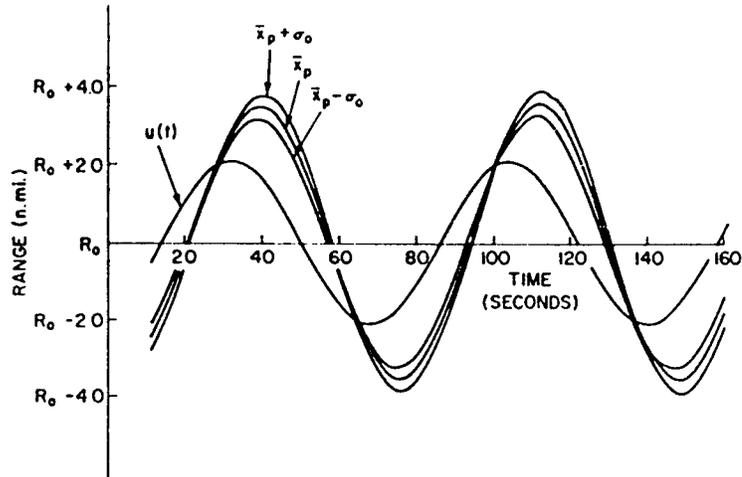


Fig. 10—Mean (and mean plus and minus a standard deviation) of x_p vs time for $\xi = 0.4$, $\omega_0 = 0.1$ rad/s, $d = 8$ s, and $\epsilon = 0.1$ s

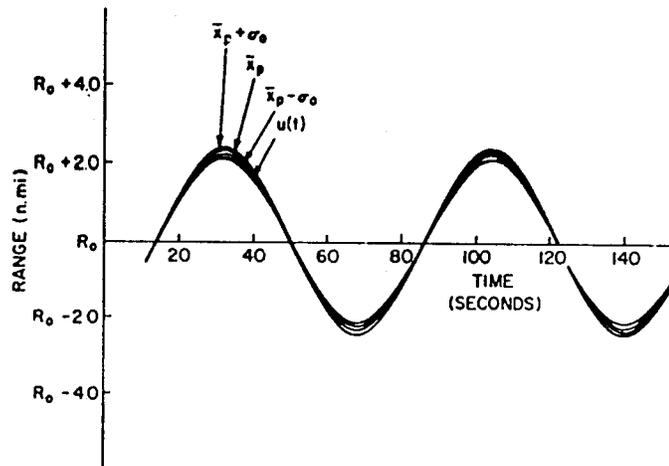


Fig. 11—Mean (and mean plus and minus a standard deviation) of x_p vs time for $\xi = 0.4$, $\omega_0 = 0.314$ rad/s, $d = 6$ s, and $\epsilon = 0.1$ s

The total filter response in both the mean and covariance can be obtained by superimposing the effects of $w(t)$ and $[X(0), u(t)]$. The filter design would adjust ξ and ω_0 such that \bar{x}_p followed $u(t)$ as closely as possible while minimizing $P_{x_p x_p}$, which is composed of the variance $P_{x_p x_p}$ due to $w(t)$ plus the variance $P_{x_p x_p}$ due to $u(t)$.

SUMMARY

The constant-coefficient α - β filter when randomly updated was found to have large errors in the variance under certain conditions, namely, a short time between updates followed by a long time between updates. To circumvent this problem, variable gains $\alpha = 1 - e^{-2\xi\omega_0 T}$ and $\beta = 1 + e^{-2\xi\omega_0 T} - 2e^{-\xi\omega_0 T} \cos \omega_d T$, where ξ , ω_0 , and ω_d are constants and T is the randomly varying time between updates, were postulated. Using these gains, one found that the variance in the predicted position remained at reasonably low values under the same conditions. Also it was shown that the variable-gain filter's response to a given trajectory could achieve a reasonable small error.

Although the gain adjustments were simply postulated and shown to work well with the examples cited, no exact justification for their use was given. However arguments were given as to why the system seemed to work well.

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Appendix A

FORMULATIONS OF MEAN AND COVARIANCE EQUATIONS

The purpose of this appendix is to outline a procedure for showing that superposition can be used in computing the means and covariances of (23) and to outline a method of obtaining (26). We begin by writing the system equations (23) using a convenient form.

$$X(k) = A X(k-1) + B[u(t_k) + w(t_k)] , \quad (A1)$$

where

A = random variable which is identically distributed and independent from sample to sample,

B = random variable which is identically distributed and independent from sample to sample,

$u(t_k)$ = random samples from an arbitrary function,

$X(0)$ = initial condition,

$w(t_k)$ = zero-mean white Gaussian noise with variance σ_w^2 .

Recursively solving (A1), we can obtain the solution in the form

$$X(k) = g[A, B, X(0), u(t_k), w(t_k)] . \quad (A2)$$

The mean $M(k)$ and covariance $P(k)$ of $X(k)$ are computed using the probability density (25). The means $M'(k)$ and $M''(k)$ and the covariances $P'(k)$ and $P''(k)$ are computed in the same manner by first using $X(0)$ and $u(t_k)$ as excitations and then using $w(t_k)$ as an excitation. Superposition is then shown to hold by noting

$$M(k) = M'(k) + M''(k) , \quad (A3)$$

$$P(k) = P'(k) + P''(k) . \quad (A4)$$

The computation is straightforward but quite lengthy.

We compute (26) by noting the mean value of $X(k)$ excited with only $w(t_k)$ is 0 and forming the covariance:

$$\begin{aligned} \overline{X(k) X'(k)} &= \overline{A X(k-1) X'(k-1) A'} + \overline{B w(t_k) w(t_k) B'} \\ &+ \overline{A X(k-1) w(t_k) B'} + \overline{B w(t_k) X(k-1) A'} , \end{aligned} \quad (A5)$$

where the bar denotes expected value and the prime denotes transpose. Because of the independence assumed, (A5) becomes

$$\begin{aligned} \overline{X(k) X'(k)} &= \overline{A X(k-1) X'(k-1) A'} + \overline{B w(t_k) w(t_k) B'} \\ &+ \overline{A X(k-1) w(t_k) B'} + \overline{B w(t_k) X(k-1) A'}. \end{aligned} \quad (A6)$$

Defining $P(k) = \overline{X(k) X'(k)}$ and noting $\overline{X(k-1) w(t_k)} = 0$, we obtain

$$P(k) = \overline{A P(k-1) A'} + \overline{B \sigma_w^2 B'}. \quad (A7)$$