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The Geometry of the Singularities of Harmonic Functions

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The objective of this investigation is to shed some light on the problem of finding for harmonic functions in three complex variables an analogue to the Mittag-Leffler representation for meromorphic functions in one variable, given their poles.

We shall consider the geometry of certain sets of possible singularities for harmonic functions which we shall call singularity sets.

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R. P. Gilbert has shown that these singularity sets are developable surfaces. In the present work it is shown that these singularity sets are either tangent surfaces or cones. A tangent surface may be completely characterized by its edge of regression. It is found that this edge of regression is characterized by the fact that it is an isotropic curve. It is also characterized by the fact that its projection onto the real space is a minimal surface. We also find that any minimal surface will in this way characterize a singularity set.

In analogy to the Mittag-Leffler representation for meromorphic functions in one variable, we give a general representation for harmonic functions in terms of their singularity sets, characterized by minimal surfaces whose parametric equations are given.

Particular examples of this representation are given. The minimal surfaces considered in these examples are the catenoid and the right helicoid.

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THE GEOMETRY OF THE SINGULARITIES OF HARMONIC FUNCTIONS

I. INTRODUCTION

1. Abstract

The objective of this investigation is to shed some light on the problem of finding for harmonic functions in three variables an analogue to the Mittag-Leffler representation for meromorphic functions in one variable given their poles.

Mittag-Leffler Representation: Let $\{b_\nu\}$ be a sequence of complex numbers with $\lim_{\nu\to\infty} b_\nu = \infty$, and let $P_\nu(\zeta)$ be polynomials without constant term. Then there are functions which are meromorphic in the whole plane with poles at the points b_ν and the corresponding singular parts $P_\nu(1/(z-b_\nu))$. Moreover, the most general meromorphic function of this kind can be written in the form

$$f(z) = \sum_\nu \left[P_\nu\left(\frac{1}{z-b_\nu}\right) - p_\nu(z)\right] + g(z) \qquad \text{(I-1.1)}$$

where the $p_\nu(z)$ are suitably chosen fixed polynomials and $g(z)$ is analytic in the whole plane.

A meromorphic function may also be represented in terms of its poles by utilizing the Weierstrass representation which expresses an entire function in terms of its zero's.

Weierstrass Representation: There exists a meromorphic function with arbitrarily prescribed zeros a_n provided that, in the case of infinitely many zeros, $a_n \to \infty$. Every

<u>entire function with these and no other zeros can be written in the form</u>

$$f(z) = z^m e^{g(z)} \prod_{n=1}^{\infty} \left(1 - \frac{z}{a_n}\right) e^{\frac{z}{a_n} + \frac{1}{2}\left(\frac{z}{a_n}\right)^2 + \cdots + \frac{1}{m_n}\left(\frac{z}{a_n}\right)^{m_n}} \tag{I-1.2}$$

<u>where the product is taken over all</u> $a_n \neq 0$, <u>the</u> m_n <u>are certain integers, and</u> $g(z)$ <u>is an entire function.</u>

Since every meromorphic function is the quotient of two entire functions, the Weierstrass representation may be used to express a meromorphic function in terms of its poles and its zeros.

The problem of extending these results to analytic functions of several complex variables was posed by Cousin [C.2] in 1895. His "First Problem" was to find the analogue of the Mittag-Leffler representation and his "Second Problem" was to find the analogue of the Weierstrass representation.

<u>Cousin's First Problem</u>: <u>Suppose that to every point</u> P <u>of the space</u> C^n <u>there corresponds a neighborhood</u> V_P <u>of</u> P <u>and a function</u> f_P <u>meromorphic in that neighborhood. Suppose also that if two such neighborhoods</u> V_P <u>and</u> V_Q <u>of the points</u> P <u>and</u> Q <u>have a common portion, then the function</u> $f_P - f_Q$ <u>is holomorphic in</u> $V_P \cap V_Q$. <u>Find a function</u> F <u>meromorphic at all points of the space</u> C^n <u>and such that</u> $F - f_P$ <u>is holomorphic in</u> V_P .

Cousin's Second Problem: Suppose that to every point P of the space C^n there corresponds a neighborhood V_P of P and a function f_P meromorphic in that neighborhood. Suppose also that if two such neighborhoods V_P and V_Q of the points P and Q have a common portion, then the function f_P/f_Q is holomorphic and different from zero in $V_P \cap V_Q$. Find a function F meromorphic at all points of the space C^n and such that F/f_P is holomorphic and different from zero in V_P.

One way in which harmonic functions differ from meromorphic functions is that they form a linear space rather than an algebra. Both the class of meromorphic functions and the class of harmonic functions are closed under addition and subtraction. This allows for the possibility of finding for harmonic functions an analogue to the Mittag-Leffler representation and Cousin's first problem. Unlike meromorphic functions, however, the class of harmonic functions are not closed under multiplication and division. As a result one would not expect to find for harmonic functions an analogue to the Weierstrass representation or to Cousin's second problem.

As a simple example of a harmonic function and its singularities consider

$$H(\underset{\sim}{z}) = \frac{1}{r} \qquad \underset{\sim}{z} \in C^3 \qquad \text{(I-1.3)}$$

where, following the convention of summing on i from 1 to 3,

$$r = [(z_i-c_i)(z_i-c_i)]^{\frac{1}{2}}. \qquad \text{(I-1.4)}$$

The singularity points of this function are the solutions of

$$(\underset{\sim}{z}-\underset{\sim}{c})^2 = 0 \; . \tag{I-1.5}$$

This is the equation of a cone, in the complex space C^3, with its vertex at the point $\underset{\sim}{z} = \underset{\sim}{c}$, and with isotropic lines as its generators. We will express $\underset{\sim}{z}$ and $\underset{\sim}{c}$ in terms of their real and imaginary parts as

$$\underset{\sim}{z} = \underset{\sim}{x} + i\underset{\sim}{y} \tag{I-1.6}$$

and

$$\underset{\sim}{c} = \underset{\sim}{a} + i\underset{\sim}{b} \; . \tag{I-1.7}$$

When the vertex of the cone is a real point, i.e.

$$\underset{\sim}{c} = \underset{\sim}{a} \tag{I-1.8}$$

and we restrict our attention to the real space R^3, we have an isolated singular point

$$\underset{\sim}{x} = \underset{\sim}{a} \; . \tag{I-1.9}$$

To find the set of singularities in R^3 when $\underset{\sim}{c}$ is not a real point we take the real and imaginary parts of equation (I-1.5) and set $\underset{\sim}{y} = 0$.

$$(\underset{\sim}{x}-\underset{\sim}{a})^2 - \underset{\sim}{b}^2 = 0 \tag{I-1.10}$$

$$\underset{\sim}{b}\cdot(\underset{\sim}{x}-\underset{\sim}{a}) = 0 \tag{I-1.11}$$

Equation (I-1.10) represents a sphere with its center at $\underset{\sim}{x} = \underset{\sim}{a}$ and with a radius $R = |\underset{\sim}{b}|$. Equation (I-1.11) represents a plane with a normal vector $\underset{\sim}{b}$ and passing through the point $\underset{\sim}{x} = \underset{\sim}{a}$. We obtain the set of singularities in R^3 by intersecting Equations (I-1.10) and (I-1.11) which results in a circle with its center at $\underset{\sim}{x} = \underset{\sim}{a}$ and with the vector $\underset{\sim}{b}$ normal to its plane.

We know that to specify the singularities of a meromorphic function in one variable we should specify a discrete set of points. Likewise, if we wish to specify the singularities of a harmonic function, we should know what sort of geometric objects may be used. The present work endeavors to answer this question. Having determined the geometrical nature of the singularities, we then proceed to determine how we may represent a harmonic function in terms of its given singularities.

We shall consider the geometry of certain sets of possible singularities for harmonic functions defined over C^3 which we shall call singularity sets. The result will be a characterization of the singularity sets by simpler geometric forms which may be used to categorize the singularity sets and their corresponding harmonic functions.

R. P. Gilbert [G.7, p. 70] has shown that the singularity sets are developable surfaces in C^3. In the geometry of the real space R^3 a developable surface is either a plane, a cylinder, a cone, a tangent surface, or a composition of these [K.1, p. 185]. We may extend the definitions of these surfaces in a natural way to the complex space C^3. It is shown in Sections (I-3) and (III-3) that the singularity sets we are considering are either tangent surfaces or cones in C^3.

A tangent surface may be completely characterized by its edge of regression. It is found in Section (III-1) that the edge of regression of a singularity set is characterized by the fact that it is an isotropic curve. In Section (III-2) it is shown that the edge of regression is also characterized by the fact that its projection onto either the real or the imaginary space is a minimal surface. We also find in Section (III-2) that any minimal surface will in this way characterize a singularity set.

If the singularity set is a cone, we find in Section (III-3) that it may be characterized by its vertex along with the fact that its generators are isotropic lines.

In analogy to the Mittag-Leffler representation for meromorphic functions in one variable, in Section (V-1) we give a general representation for harmonic functions in terms of their singularity sets, characterized by minimal surfaces whose parametric equations are given. In Sections (V-2) and (V-3) particular examples of this representation are given. The minimal surfaces considered in these examples are the catenoid and the right helicoid.

2. Previous Results

One approach to the study of partial differential equations is the generation of solutions by means of integral operators applied to analytic functions. As a simple example of this idea we may consider Laplace's equation in two dimensions.

The solutions, which are harmonic functions, may be generated by taking the real parts of analytic functions. Although this is not a true integral operator we will proceed to give some typical examples of integral operators.

Elliptic Equations in Two Variables

Consider the elliptic partial differential equation

$$\underset{\sim}{e}[u] \equiv \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + a(x,y)\frac{\partial u}{\partial x} + b(x,y)\frac{\partial u}{\partial y} + c(x,y)u = 0 \quad (\text{I-2.1})$$

where a, b, and c are entire functions in a bicylinder $\mathfrak{S}^{(2)}$. Bergman's integral operator of the first kind [B.5,6] will associate in a one-to-one manner the solutions of this equation with analytic functions of a complex variable. If we transform variables according to the equations

$$z = x + iy \quad (\text{I-2.2})$$

$$z^* = x - iy \quad (\text{I-2.3})$$

we obtain an equation of the form

$$\underset{\sim}{E}(U) = U_{zz^*} + A(z,z^*)U_z + B(z,z^*)U_{z^*} + C(z,z^*)U = 0 \,. \quad (\text{I-2.4})$$

The solutions of this equation may be represented by the integral operator

$$U(z,z^*) = \underset{\sim}{b}_2 f \equiv \int_{\mathfrak{L}} E(z,z^*,t)\, f\left(\frac{z}{2}\,[1-t^2]\right) \frac{dt}{(1-t^2)^{1/2}} \,. \quad (\text{I-2.5})$$

$E(z,z^*,t)$ is the generating function for Equation (I-2.4).

It is given as

$$E(z,z^*,t) \equiv \tilde{E}(z,z^*,t) \exp\left\{-\int_0^{z^*} A(z,\zeta^*)d\zeta^* + n(z)\right\} \qquad (I-2.6)$$

where $n(z)$ is an arbitrary analytic function of z. $\tilde{E}(z,z^*,t)$ satisfies the partial differential equation

$$(1-t^2)\tilde{E}_{z^*t} - t^{-1}\tilde{E}_{z^*} + 2tz(\tilde{E}_{zz^*} + D\,\tilde{E}_{z^*} + F\,\tilde{E}) = 0\,. \qquad (I-2.7)$$

Vekua [V.1] has also obtained an integral operator representation for solutions to Equation (I-2.1). The class of real solutions which are analytic in $\mathfrak{S}^{(2)}$ are given by

$$u(x,y) = \mathrm{Re}\left\{H_0(z,\overline{z})\varphi(z) + \int_{\mathfrak{L}} H(z,\overline{z},t)\varphi(t)dt\right\} \qquad (I-2.8)$$

where

$$H_0(z,\overline{z}) \equiv R(z,0;z,\overline{z})\,, \qquad (I-2.9)$$

$$H(z,\overline{z},t) \equiv -\frac{\partial}{\partial t}R(t,0;z,\overline{z}) + B(t,0)\,R(t,0;z,\overline{z})\,. \qquad (I-2.1$$

$\varphi(z)$ is an arbitrary holomorphic function and $\overline{z}$ is the restriction of z^* to real values of x and y.

The function $R(\zeta,\zeta^*;z,z^*)$ is a complex Riemann function for Equation (I-2.4). It is analytic for $(\zeta,\zeta^*) \in \mathfrak{S}^{(2)}$ and satisfies the equation

$$\frac{\partial^2}{\partial\zeta\partial\zeta^*}R - \frac{\partial}{\partial\zeta}(AR) - \frac{\partial}{\partial\zeta^*}(BR) + CR = 0 \qquad (I-2.11)$$

and the conditions

$$R(z,\zeta^*;z,z^*) = \exp \int_z^{\zeta^*} A(z,\eta)d\eta \ , \qquad \text{(I-2.12)}$$

$$R(\zeta,z^*;z,z^*) = \exp \int_z^{\zeta} B(\xi,z^*)d\xi \quad . \qquad \text{(I-2.13)}$$

<u>The Generalized Biaxially Symmetric Helmholtz Equation</u>

For the following equation we consider solutions which are of class $C^{(2)}$ in some neighborhood of the origin and are even functions in x and in y.

$$\underset{\sim}{h}_{\mu\nu}[u] = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{2\mu}{x}\frac{\partial u}{\partial x} + \frac{2\nu}{y}\frac{\partial u}{\partial y} + k^2 u = 0 \quad (\mu,\nu>0) \qquad \text{(I-2.14)}$$

Henrici [H.2] has given an integral representation for solutions of this equation which was used by Gilbert and Howard [G.H.6,7] to study the singularity structure of analytic solutions. This is represented by the integral operator

$$u(x,y) = \underset{\sim}{H}_{\mu\nu} f \equiv ax^{-\mu} \int_{\mathcal{L}+1}^{-1} f(k\sigma)\sigma^{\mu}\left(\zeta - \frac{1}{\zeta}\right)^{2\nu-1} \Phi_2(\mu,1-\mu,\nu;\xi,\eta)\frac{d\zeta}{\zeta} \ , \qquad \text{(I-2.15)}$$

where

$$\sigma = x + \frac{iy}{2}\left(\zeta + \frac{1}{\zeta}\right) \ , \qquad \text{(I-2.16)}$$

$$\xi = \frac{y^2(\zeta - \frac{1}{\zeta})}{16x\sigma} \ , \qquad \text{(I-2.17)}$$

$$\eta = \frac{k^2y^2(\zeta - \frac{1}{\zeta})}{16} \ , \qquad \text{(I-2.18)}$$

$$\mathcal{L} = \{\zeta \mid \zeta = e^{i\varphi},\ 0 \le \varphi \le \pi\} \ , \qquad \text{(I-2.19)}$$

with

$$a = \frac{2}{(2i)^{2\nu}\Gamma(\nu)\Gamma(\frac{1}{2})} \tag{I-2.20}$$

Φ_2 is a confluent hypergeometric function of two variables [E.3, Vol. I, p. 225] defined by the double power series

$$\Phi_2(\alpha,\beta,\gamma;x,y) \equiv \sum_{m,n=0}^{\infty} \frac{(\alpha)_m(\beta)_m}{(\gamma)_{m+n}\, m!\, n!} x^m y^n \;. \tag{I-2.21}$$

Generalized Axially Symmetric Helmholtz Equation

This is a special case of the biaxially symmetric equation where $\mu = 0$. It is represented as

$$\underset{\sim}{H}_{0\nu} u \equiv \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{2\nu}{y}\frac{\partial u}{\partial y} + k^2 u = 0 \;, \qquad (\nu > 0) \;. \tag{I-2.22}$$

Gilbert and Howard [G.H.2] have given the following integral operator representation for solutions to this equation:

$$\underset{\sim}{H}_{\nu} f \equiv C_{\nu k} \int_{\substack{+1 \\ |\zeta|=1}}^{-1} J_{\nu-1}\left(\frac{ky[\zeta - 1/\zeta]}{2i}\right) f(k\sigma)(\zeta - \frac{1}{\zeta})^{\nu} \frac{d\zeta}{\zeta} \tag{I-2.23}$$

where

$$\sigma = x + \frac{1}{2} iy[\zeta + \frac{1}{\zeta}] \tag{I-2.24}$$

$$C_{\nu k} \equiv \frac{-(iky)^{1-\nu}}{2} \frac{\Gamma(\nu + 1/2)}{\Gamma(\frac{1}{2})} \tag{I-2.25}$$

The path of integration is the upper semicircle from $+1$ to -1 . $J_{\nu-1}$ is a Bessel function. Colton [C.1] has obtained uniqueness theorems for Equation (I-2.22) for the case $\nu < 0$. Previously, only the range of $\nu \geq 0$ had been known [H.3],[P.1].

Generalized Biaxially Symmetric Potential Equation

This is a special case of the Generalized Biaxially Symmetric Helmholtz Equation where $k = 0$. It is represented as

$$L_{\mu\nu}[\phi] \equiv \frac{\partial^2\phi}{\partial x^2} + \frac{2\mu}{x}\frac{\partial\phi}{\partial x} + \frac{\partial^2\phi}{\partial y^2} + \frac{2\nu}{y}\frac{\partial\phi}{\partial y} = 0\ , \quad \mu,\nu > 0\ . \quad \text{(I-2.26)}$$

The work of Gilbert [G.6] gives the following integral operator for solutions of this equation:

$$\phi(x,y) = G_{\mu\nu}f \equiv \frac{1}{2\pi i}\int_{|\zeta|=\varepsilon} f(\tau)\left[1+\frac{ix}{y}\zeta\right]^{\nu-\frac{1}{2}}\left[1+\frac{iy}{x}\zeta\right]^{\mu-\frac{1}{2}} d\zeta \quad \text{(I-2.27)}$$

where

$$\tau = x^2 - y^2 + ixy[\zeta + \frac{1}{\zeta}] \quad \text{(I-2.28)}$$

Generalized Axially Symmetric Potential Equation

This is a special case of both the generalized axially symmetric Helmholtz equation and the generalized biaxially symmetric potential equation. It is represented as

$$L_{\mu}[\phi] \equiv \frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} + \frac{2\mu}{y}\frac{\partial\phi}{\partial y} = 0 \quad \text{(I-2.29)}$$

Gilbert [G.5] (see also Henrici [H.1,2]) has given the following integral operator representation for solutions to this equation:

$$\Phi = \underset{\sim}{A}_\mu f = \frac{4\Gamma(2\mu)}{(4i)^{2\mu}\Gamma(\mu)^2} \int_{\mathcal{L}} f(\sigma)(\zeta-\zeta^{-1})^{2\mu-1} \frac{d\zeta}{\zeta} , \qquad \text{(I-2.30)}$$

where

$$\mathcal{L} = \{\zeta | \zeta = e^{i\varphi};\ 0 \leq \varphi \leq \pi\} . \qquad \text{(I-2.31)}$$

The Elliptic Operator T_{p+2}

We will now consider the following class of elliptic partial differential equations:

$$\underset{\sim}{T}_{p+2}[\Phi] \equiv \frac{\partial^2\Phi}{\partial x_\mu \partial x_\mu} + A(r^2)x_\mu \frac{\partial\Phi}{\partial x_\mu} + C(r^2)\Phi = 0 , \qquad \text{(I-2.32)}$$

where $A(r^2)$ and $C(r^2)$ are analytic functions of r^2. Bergman [B.3,4,8] has given an operator which generates solutions to this equation for $p = 0,1$. Gilbert and Howard [G.H.1] have generalized this result to include $p \geq 2$. The integral operator is given by

$$\Phi(\underset{\sim}{x}) = \underset{\sim}{\Omega}_{p+2}f \equiv \left(\frac{1}{2\pi i}\right)^p \int_{\mathcal{P}} \frac{d\zeta}{\zeta} \int_{t=-1}^{+1} E(r,t)f(v[1-t^2];\zeta)dt , \quad \text{(I-2.33)}$$

where $f(v,\zeta)$ is an analytic function of $p+1$ variables. The auxiliary variable v is given by

$$v = N_\mu x_\mu , \qquad \text{(I-2.34)}$$

where the analytic functions $N_\mu(\zeta)$ satisfy the relation,

$$N_\mu N_\mu = 0 . \qquad \text{(I-2.35)}$$

A vector satisfying this relation is called isotropic. $\mathcal{J}^p$

is a product of regular contours $\mathfrak{L}_k$ in the ζ_k-planes which do not pass through their respective origins. The function $E(r,t)$ is given by

$$E(r,t) = \exp\left\{-\frac{1}{2}\int_0^r rA\,dr\right\}\tilde{H}(r,t)\ , \tag{I-2.36}$$

where the function $\tilde{H}(r,t)$, $|t|\leq 1$ is a solution of the equation

$$(1-t^2)\tilde{H}_{rt} - t^{-1}(t^2+1)\tilde{H}_r + rt\left\{\tilde{H}_{rr} + \frac{p-1}{r}\tilde{H}_r + B\tilde{H}\right\} = 0\ , \tag{I-2.37}$$

where B is given by

$$B = -\frac{r}{2}A_r - \frac{p+2}{2}A - \frac{r^2A^2}{4} + C\ . \tag{I-2.38}$$

We, furthermore, impose the restriction that $\tilde{H}_r/rt$ be continuous at $r = t = 0$.

Laplace's Equation in Three Variables

Laplace's Equation in three variables is a special case of the elliptic operator T_{p+2}. The solutions of Laplace's equation, which are harmonic functions, may be generated by applying an integral operator to holomorphic functions in two variables. Consider a holomorphic function with a Laurent series representation

$$f(u,\zeta) = \sum_{n=0}^{\infty}\sum_{m=-n}^{n} a_{nm}u^n\zeta^m \tag{I-2.39}$$

in the region $\{(u,\zeta)\,\big|\,|u| < \rho \text{ and } 1-\delta \leq |\zeta| \leq 1+\delta\}$. Using the same coefficients a_{nm} we may represent a formal harmonic function as

$$H(\underline{z}) = \sum_{n=0}^{\infty}\sum_{m=-n}^{n}\frac{a_{nm}n!}{(n+m)!}\, i^m r^n P_n^m(\cos\theta)\, e^{im\varphi}\ , \tag{I-2.40}$$

$$\underset{\sim}{z} = (r \sin\theta \sin\varphi\ ,\ r \sin\theta \cos\varphi\ ,\ r \cos\theta)\ . \qquad \text{(I-2.41)}$$

The functions P_n^m are the Legendre functions, [E.3] . This formal harmonic function may be represented in terms of the holomorphic function, whose Laurent coefficients are also these a_{nm} , by using the Bergman-Whittaker operator [B.3,4; W.1]. The Bergman-Whittaker operator is defined as

$$H(\underset{\sim}{z}) = \underset{\sim}{B}_3 f = \frac{1}{2\pi i} \int_{\mathcal{L}} f(u,\zeta) \frac{d\zeta}{\zeta}\ ,\ \mathcal{L} = \{|\zeta| = 1\}\ . \qquad \text{(I-2.42)}$$

The auxiliary variable u depends upon ζ in the following manner,

$$u = \zeta^{-1} u_i(\zeta) z_i\ , \quad i = 1,2,3\ . \qquad \text{(I-2.43)}$$

The functions $u_i(\zeta)$ are analytic. They are also the components of an isotropic vector and thus satisfy the relation

$$u_i u_i = 0\ . \qquad \text{(I-2.44)}$$

A suitable choice for the components of the vector $u_i(\zeta)$ is

$$u_1 = \frac{1}{2}(\zeta^2 - 1)\ , \qquad \text{(I-2.45)}$$

$$u_2 = \frac{i}{2}(\zeta^2 + 1)\ , \qquad \text{(I-2.46)}$$

$$u_3 = \zeta\ . \qquad \text{(I-2.47)}$$

Another choice for the components of u_i is obtained by making the substitution

$$\zeta = e^{i\alpha}\ . \qquad \text{(I-2.48)}$$

This gives the following expressions for the u_i

$$u_1 = i\, e^{i\alpha} \sin \alpha \;, \tag{I-2.49}$$

$$u_2 = i\, e^{i\alpha} \cos \alpha \;, \tag{I-2.50}$$

$$u_3 = e^{i\alpha} \tag{I-2.51}$$

To continue the function $H(\underset{\sim}{z})$ and remain on the same branch, the path of integration $\mathcal{L}$ should be deformed, if necessary, to prevent a singularity from crossing over it. The effect of letting a pole cross over the path of integration corresponds to a jump in the value of $H(\underset{\sim}{z})$.

If the path of integration $\mathcal{L}$ is regarded as fixed, the space becomes separated into regions called <u>domains of association</u> separated by surfaces called <u>surfaces of separation</u> [B.8, p.49]. Passing through a surface of separation represents a pole crossing over the path of integration, and the value of $H(\underset{\sim}{z})$ will jump from one branch to another.

Bergmann[B.2,8] has given an integral formula which transforms a harmonic function $H(\underset{\sim}{z})$ into its B_3-associate. We first define the C_3-associate of $H(\underset{\sim}{z})$ which is expressed in terms of the variables

$$w = \frac{1}{2}\,(z_1 + iz_2) \;, \tag{I-2.52}$$

$$w^* = -\,\frac{1}{2}\,(z_1 - iz_2) \;, \tag{I-2.53}$$

and is equal to the restriction of $H(\underset{\sim}{z})$ to the (complex) <u>characteristic space</u>

$$z_i z_i = 0 \,. \tag{I-2.54}$$

The C_3-associate is given by

$$\chi(w,w^*) = H\left(w-w^*, -i(w+w^*), 2(ww^*)^{1/2}\right) \,. \tag{I-2.55}$$

The B_3-associate is expressed in terms of the C_3-associate by

$$f(u,\zeta) = 2\int_0^1 u^{1/2} \frac{\partial}{\partial u} \left\{\chi\left(\zeta ut^{\,2}, u\zeta^{-1}(1-t)^2\right)\right\} dt \,. \tag{I-2.56}$$

The B_3-associate resulting from the application of Equations (I-2.55) and (I-2.56) to $H(\underset{\sim}{z})$ is called the <u>normalized B_3-associate</u>, and is the function given by Equation (I-2.39). We may add to it a <u>null-associate</u> and the resulting holomorphic function will be mapped by the Bergmann-Whitaker operator into the same harmonic function. The null-associates are all mapped onto the zero harmonic function and are given by

$$n(u,\zeta) = \sum_{n=0} \sum_{|k|>n} a_{nk} u^n \zeta^{k-1} \tag{I-2.57}$$

The C_3-associates are holomorphic functions of two complex variables which may be used to generate harmonic functions. This is done by applying successively the operators defined by Equations (I-2.56) and (I-2.42).

Bergmann [B.1,7,8] has obtained a result regarding the singularities of a harmonic function when $f(u,\zeta)\zeta^{-1}$ is a rational function of u and ζ. In this case we may express the harmonic function as follows

$$H(\underset{\sim}{z}) = \underset{\sim}{B}_3 \frac{\zeta p}{q} = \frac{1}{2\pi i} \int_{\mathcal{L}} \frac{p(u,\zeta)}{q(u,\zeta)} d\zeta , \qquad \text{(I-2.58)}$$

where p and q are polynomials. Defining $Q(\underset{\sim}{z},\zeta)$ as

$$Q(\underset{\sim}{z},\zeta) \equiv q\Big(u(\underset{\sim}{z},\zeta),\zeta\Big) , \qquad \text{(I-2.59)}$$

it is clearly seen that $H(\underset{\sim}{z})$ becomes singular for those values of $\underset{\sim}{z}$ which satisfy simultaneously the equations

$$Q(\underset{\sim}{z},\zeta) = 0 , \qquad \text{(I-2.60)}$$

$$\frac{\partial}{\partial\zeta} Q(\underset{\sim}{z},\zeta) = 0 . \qquad \text{(I-2.61)}$$

Gilbert [G.2,3] has obtained a more general result of this type. It is as follows:

Theorem: Let the defining function for the set of singularities of $f(u,\zeta)\zeta^{-1}$ be a global defining function in C^2. Then if $n(u,\zeta) \equiv s(\underset{\sim}{z},\zeta) = 0$ is such a defining function we have that $H(\underset{\sim}{z}) = \underset{\sim}{B}_3 f$ is regular for all points $\underset{\sim}{z}$, which may be reached by continuation along a curve Γ starting at some point of definition $\underset{\sim}{z}^0$, provided $\underset{\sim}{z}$ (and hence the curve Γ) does not lie on the set

$$\mathfrak{S} = \cup_{\zeta}' \Big(\{\underset{\sim}{z} |\ s(\underset{\sim}{z},\zeta)=0\} \cap \{\underset{\sim}{z} |\ s_{\zeta}(\underset{\sim}{z},\zeta)=0\}\Big) . \qquad \text{(I-2.62)}$$

We shall refer to the set of possible singularities $\mathfrak{S}$ as a singularity set. If the function $f(u,\zeta)\zeta^{-1}$ is a rational function, every point of the singularity set $\mathfrak{S}$ is a singularity. Whether this is so for any $f(u,\zeta)\zeta^{-1}$ is an open question. The singularity set is in general a four-dimensional manifold in the six-dimensional space C^3. Its intersection with the three-dimensional real space R^3 is a one-dimensional curve.

Kreyszig [K.2] has given several examples of the function $h(u,\zeta)$ along with the resulting set of singularities of $H(\underset{\sim}{x}) = \underset{\sim}{B}_3 f$ in R^3.

If $h(u,\zeta)$ is of the form

$$h(u,\zeta) = a + (c+u)\zeta + k\zeta^2 , \tag{I-2.63}$$

then $H(\underset{\sim}{x})$ is singular at a point, if c is real, or on a circle otherwise. The point or the center of the circle is given by

$$x_1 = -\mathrm{Re}\ c , \tag{I-2.64}$$

$$x_2 = -\mathrm{Im}\ (a+k) , \tag{I-2.65}$$

$$x_3 = \mathrm{Re}\ (a-k) . \tag{I-2.66}$$

The components of a vector $\underset{\sim}{b}$ normal to the plane of the circle are given by

$$b_1 = \mathrm{Im}\ c , \tag{I-2.67}$$

$$b_2 = -\mathrm{Re}\ (a+k) , \tag{I-2.68}$$

$$b_3 = \mathrm{Im}\ (k-a) , \tag{I-2.69}$$

and the radius of the circle is given by

$$R = |\underline{b}| \; . \qquad (I-2.70)$$

If $h(u,\zeta)$ is of the form

$$h(u,\zeta) = \zeta(1+\zeta)u \; , \qquad (I-2.71)$$

then $H(\underline{x})$ is singular along the x_1- and x_3-axes.

If $h(u,\zeta)$ is of the form

$$h(u,\zeta) = \zeta(u+\zeta^2) \; , \qquad (I-2.72)$$

then $H(\underline{x})$ is singular along two plane curves. One of them lies in the x_1,x_3-plane and can be represented in the form

$$x_3^4 + (x_1^2 - 18x_1 - 27)x_3^2 - 16x_1^3 = 0 \; ; \qquad (I-2.73)$$

the other lies in the x_1, x_2-plane and can be represented in the form

$$x_2^4 + (x_1^2 + 18x_1 - 27)x_2^2 + 16x_1^3 = 0 \; . \qquad (I-2.74)$$

If $h(u,\zeta)$ is of the form

$$h(u,\zeta) = 1 + u\zeta + u\zeta^2 + \zeta^3 \qquad (I-2.75)$$

then $H(\underline{x})$ is singular on the line

$$x_2 = 0 \qquad (I-2.76)$$

and

$$x_3 = -2 \; , \qquad (I-2.77)$$

on the two curves in the x_1, x_3-plane

$$x_1 = 3 , \tag{I-2.78}$$

and

$$(x_1+1)(x_1+3)^2 + x_3^2(x_1-3) = 0 , \tag{I-2.79}$$

and on the curve given by the following pair of equations,

$$(x_1-3)\left[(x_1+1)(x_1+3)^2 + (x_1-3)x_3^2 - 12x_2^2\right] - x_2^2(x_2^2+x_3^2) = 0 , \tag{I-2.80}$$

$$(x_1-3)(x_1^2+x_3^2-9) + x_2^2(x_1-5) = 0 . \tag{I-2.81}$$

We shall consider the special case where the equation

$$h(u,\zeta) = 0 \tag{I-2.82}$$

may be solved for u . This will be the case if

$$\frac{\partial}{\partial u} h(u,\zeta) \neq 0 . \tag{I-2.83}$$

After solving for u we may express Equation (I-2.82) in terms of an analytic function $\varphi(\zeta)$ as

$$u\zeta + \varphi(\zeta) = 0 . \tag{I-2.84}$$

We henceforth define the functions $s(\underline{z},\zeta)$ and $h(u,\zeta)$ by

$$s(\underline{z},\zeta) = h(u,\zeta) \equiv u\zeta + \varphi(\zeta) . \tag{I-2.85}$$

Gilbert [G.7, p.70] has demonstrated the following theorem regarding the singularities of a harmonic function.

<u>Theorem: Let $H(\underset{\sim}{z})$ be a harmonic function given by $H(\underset{\sim}{z}) = \underset{\sim}{B}_3 f$ where the singularities of $f(u,\zeta)\zeta^{-1}$ have a global representation, $h(u,\zeta) \equiv su + \varphi(\zeta) = 0$. Then the singularity set $\mathfrak{S}$ is a developable surface in $\mathbb{C}^3$.</u>

In the special case where the components u_i are given by Equations (I-2.45) - (I-2.47) the singularity set $\mathfrak{S}$ is given by the parametric Equations [G.7, p.65]

$$z_1 = (\mu/2)[\zeta - 1/\zeta] + \varpi - \frac{1}{2}[\zeta + 1/\zeta]\varpi' , \tag{I-2.86}$$

$$z_2 = (i\mu/2)[\zeta + 1/\zeta] + i(\varpi - \frac{1}{2}[\zeta - 1/\zeta]\varpi') , \tag{I-2.87}$$

$$z_3 = \mu . \tag{I-2.88}$$

3. <u>The Edge of Regression</u>

If φ is not of the form

$$\varphi = a\zeta^2 + b\zeta + c , \tag{I-3.1}$$

Equations (I-2.86) - (I-2.88), representing the singularity set $\mathfrak{S}$, may be put into the form

$$z_i = Y_i(\zeta) + \beta Y_i'(\zeta) , \quad (\beta,\zeta) \in \mathbb{C}^2 . \tag{I-3.2}$$

A complex surface which may be represented by an equation of this form is called a <u>tangent surface</u>. The complex curve represented by

$$z_i = Y_i(\zeta) , \quad \zeta \in \mathbb{C}, \tag{I-3.3}$$

is called the <u>edge of regression</u> of the tangent surface. These are natural extensions of definitions from real geometry. Equations (I-2.86) - (I-2.88) may be put into the form represented by Equation (I-3.2) by making the substitution

$$\mu = -\varphi' + \zeta\varphi'' + \beta\zeta\varphi''' \ . \qquad \text{(I-3.4)}$$

This gives

$$z_1 = \varphi - \zeta\varphi' + \frac{1}{2}\varphi''(\zeta^2-1) + \frac{1}{2}\beta\varphi'''(\zeta^2-1) \ , \qquad \text{(I-3.5)}$$

$$z_2 = i\varphi - i\zeta\varphi' + \frac{1}{2} i\varphi''(\zeta^2+1) + \frac{i}{2}\beta\varphi'''(\zeta^2+1) \ , \qquad \text{(I-3.6)}$$

$$z_3 = -\varphi' + \zeta\varphi'' + \beta\zeta\varphi''' \ . \qquad \text{(I-3.7)}$$

It follows that the edge of regression is given by

$$z_1 = \varphi - \zeta\varphi' + \frac{1}{2}\varphi''(\zeta^2-1) \ , \qquad \text{(I-3.8)}$$

$$z_2 = i\varphi - i\zeta\varphi' + \frac{i}{2}\varphi''(\zeta^2+1) \ , \qquad \text{(I-3.9)}$$

$$z_3 = -\varphi' + \zeta\varphi'' \qquad \text{(I-3.10)}$$

The derivative with respect to ζ of the position vector $\underset{\sim}{z}$ can be represented coordinate-wise as

$$z_i' = u_i\varphi''' , \qquad \text{(I-3.11)}$$

where u_i is given by Equations (I-2.45) - (I-2.47). We shall consider, however, the more general case where u_i is taken to be <u>any</u> analytic isotropic vector.

II. THE METRIC TENSOR FOR THE EDGE OF REGRESSION

1. The Six-Dimensional Real Space $\mathcal{R}^6$

In addition to regarding the edge of regression as a complex curve over C^3 we may also regard it as a two-dimensional manifold over $\mathcal{R}^6$. The coordinates of a point (z_1, z_2, z_3) in C^3 shall be denoted in $\mathcal{R}^6$ by $(x_1, x_2, x_3, y_1, y_2, y_3)$. A point on the edge of regression

$$\zeta = v^1 + iv^2 \tag{II-1.1}$$

shall have the coordinates (v^1, v^2) on the corresponding surface.

2. The Metric Tensor in $\mathcal{R}^6$

The edge of regression is a two-dimensional surface in $\mathcal{R}^6$ whose metric tensor is given by

$$g_{\alpha\beta} = \frac{\partial x_i}{\partial v^\alpha}\frac{\partial x_i}{\partial v^\beta} + \frac{\partial y_i}{\partial v^\alpha}\frac{\partial y_i}{\partial v^\beta} \; . \tag{II-2.1}$$

We are following the convention of summing upon repeated indices. A latin index shall take on the values 1,2, and 3; while a greek index shall take on the values 1 and 2. Using equation (I-3.11) to calculate the partial derivatives $\partial x_i/\partial v^\alpha$ and $\partial y_i/\partial v^\alpha$ and substituting into Equation (II-2.1) gives the following expression for the metric tensor,

$$g_{\alpha\beta} = \sum_i |u_i \varphi^{*}|^2 \delta_{\alpha\beta} \; . \tag{II-2.2}$$

Equation (II-2.2) follows solely from the assumption that the functions u_i are analytic. The assumption that u_i is an isotropic vector has not been used. (The derivation of this result is discussed further in Appendix A.)

We see that the metric tensor is of the general form

$$g_{\alpha\beta} = \lambda(v^1,v^2)\delta_{\alpha\beta} \,, \tag{II-2.3}$$

where

$$\lambda = \sum_i |u_i\varphi'''|^2 \,. \tag{II-2.4}$$

Whenever this is the case we say that the surface parameters are <u>isothermal</u>, [E.1, p.93]. The coordinate curves form an orthogonal system. When they are spaced according to equal infinitesimal increments in the values of v^1 and v^2 they divide the surface into a network of small squares. The size of these squares may vary as one moves about the surface.

Let us proceed to evaluate the metric tensor when u_i is given by Equations (I-2.45) - (I-2.47).

$$\begin{aligned} u_i\bar{u}_i &= \frac{1}{4}(\zeta^2-1)(\bar{\zeta}^2-1) + \frac{1}{4}(\zeta^2+1)(\bar{\zeta}^2+1) + \zeta\bar{\zeta} \\ &= \frac{1}{4}(\zeta^2\bar{\zeta}^2 - \zeta^2 - \bar{\zeta}^2 + 1 + \zeta^2\bar{\zeta}^2 + \zeta^2 + \bar{\zeta}^2 + 1 + 4\zeta\bar{\zeta}) \\ &= \frac{1}{4}(2\zeta^2\bar{\zeta}^2 + 4\zeta\bar{\zeta} + 2) = \frac{1}{2}(\zeta\bar{\zeta} + 1)^2 \,. \end{aligned} \tag{II-2.5}$$

In this case the expression for the metric tensor becomes

$$g_{\alpha\beta} = \frac{1}{2}(|\zeta|^2+1)^2\,|\varphi'''|^2\,\delta_{\alpha\beta} \,. \tag{II-2.6}$$

3. The Assumption that a Vector is Isotropic

For future reference we shall derive a number of equations that follow from the assumption that a vector N_i is isotropic. Taking the real part of the equation

$$N_i N_i = 0 \qquad \text{(II-3.1)}$$

gives

$$RN_i\, RN_i - IN_i\, IN_i = 0\ . \qquad \text{(II-3.2)}$$

Taking the imaginary part of Equation (II-3.1) gives

$$RN_i IN_i = 0\ . \qquad \text{(II-3.3)}$$

Multiplying N_i by its complex conjugate gives

$$N_i \overline{N}_i = RN_i RN_i + IN_i IN_i \qquad \text{(II-3.4)}$$

Using Equation (II-3.2) we obtain

$$RN_i RN_i = \frac{1}{2} \sum_i |N_i|^2 \qquad \text{(II-3.5)}$$

and

$$IN_i IN_i = \frac{1}{2} \sum_i |N_i|^2 \qquad \text{(II-3.6)}$$

Differentiating Equation (II-3.1) gives

$$N_i N_i{}' = 0\ . \qquad \text{(II-3.7)}$$

Taking the real part of this equation yields

$$RN_i RN_i{}' - IN_i IN_i{}' = 0\ ; \qquad \text{(II-3.8)}$$

whereas, taking the imaginary part yields

$$RN_iIN_i' + IN_iRN_i' = 0 . \qquad \text{(II-3.9)}$$

Let us consider the real and imaginary parts of $N_i\bar{N}_i'$. One has

$$R(N_i\bar{N}_i') = RN_iRN_i' + IN_iIN_i' , \qquad \text{(II-3.10)}$$

and

$$I(N_i\bar{N}_i') = -RN_iIN_i' + IN_iRN_i' \qquad \text{(II-3.11)}$$

Using Equations (II-3.8) and (II-3.10)

$$RN_jRN_j' = IN_jIN_j' = \frac{1}{2} R(N_j\bar{N}_j') , \qquad \text{(II-3.12)}$$

and from Equations (II-3.9) and (II-3.11)

$$RN_jIN_j' = -IN_jRN_j' = - \frac{1}{2} I(N_j\bar{N}_j') . \qquad \text{(II-3.13)}$$

4. The Metric Tensor in $\mathbb{R}^3$

Let us consider the projection of the edge of regression onto the real space $\mathbb{R}^3$. This time it is necessary to use the assumption that u_i is an isotropic vector in order to obtain a result which closely resembles Equation (II-2.2). The metric tensor in $\mathbb{R}^3$ is given by

$$g_{\alpha\beta} = \frac{\partial x_i}{\partial v^\alpha} \frac{\partial x_i}{\partial v^\beta} \qquad \text{(II-4.1)}$$

We will denote the right hand side of Equation (I-3.11) by N_i , that is

$$z_i{}' = u_i\varphi''' \equiv N_i \ . \qquad \text{(II-4.2)}$$

Substituting the partial derivatives $\partial x_i/\partial v^\alpha$ as determined from Equation (II-4.2) into Equation (II-4.1) gives the following expression for the components of the metric tensor

$$g_{11} = RN_iRN_i \ , \qquad \text{(II-4.3)}$$

$$g_{12} = g_{21} = -RN_iIN_i \ , \qquad \text{(II-4.4)}$$

$$g_{22} = IN_iIN_i \qquad \text{(II-4.5)}$$

At this point by means of Equations (II-3.3), (II-3.5) and (II-3.6) we introduce the assumption that u_i is isotropic. This gives by (II-3.3), (II-3.5), and (II-3.6) that

$$g_{\alpha\beta} = \frac{1}{2}\sum_i |N_i|^2 \delta_{\alpha\beta} \ . \qquad \text{(II-4.6)}$$

Substituting for N_i the expression of Equation (II-4.2) gives the following result for the metric tensor

$$g_{\alpha\beta} = \frac{1}{2}\sum_i |u_i\varphi'''|^2 \delta_{\alpha\beta} \ . \qquad \text{(II-4.7)}$$

(The derivation of this result is discussed further in Appendix B.)

Equation (II-4.7) is of the form of Equation (II-2.3) where λ is given by

$$\lambda = \frac{1}{2}\sum_i |u_i\varphi'''|^2 \ . \qquad \text{(II-4.8)}$$

It follows that v^1 and v^2 are isothermal parameters for the surface under consideration. In the special case where u_i is given by Equations (I-2.45) - (I-2.47), i.e. the Bergman formulation, the metric tensor is given by

$$g_{\alpha\beta} = \frac{1}{4}\left(|\zeta|^2+1\right)^2 |\varphi'''|^2 \delta_{\alpha\beta} . \tag{II-4.9}$$

5. The Metric Tensor in I^3

Similarly, we may consider the projection of the edge of regression onto the imaginary space I^3 . The metric tensor for I^3 is given by

$$g_{\alpha\beta} = \frac{\partial y_i}{\partial v^\alpha}\frac{\partial y_i}{\partial v^\beta} . \tag{II-5.1}$$

Substituting the partial derivatives $\partial y_i/\partial v^\alpha$ as determined from Equation (II-4.2) into Equation (II-5.1) gives the following result for the metric tensor:

$$g_{11} = IN_i IN_i , \tag{II-5.2}$$

$$g_{12} = g_{21} = IN_i RN_i , \tag{II-5.3}$$

$$g_{22} = RN_i RN_i . \tag{II-5.4}$$

Using Equations (II-3.3), (II-3.5), and (II-3.6) gives:

$$g_{\alpha\beta} = \frac{1}{2}\sum_i |N_i|^2 \delta_{\alpha\beta} . \tag{II-5.5}$$

We see that in the case of the imaginary space we obtain the same metric tensor as for the real space. (The derivation of this result is discussed further in Appendix C.)

III. CHARACTERIZING THE SINGULARITY SETS

1. The Edge of Regression is an Isotropic Curve

Considering the points on the edge of regression as points in the complex space C^3, the length of an infinitesimal line segment is given by

$$dz_i dz_i = z_i' z_i' (d\zeta)^2 , \qquad \text{(III-1.1)}$$

and substituting Equation (II-4.2) into this gives

$$dz_i dz_i = u_i u_i (\omega''' d\zeta)^2 . \qquad \text{(III-1.2)}$$

Furthermore, since

$$u_i u_i = 0 \qquad \text{(III-1.3)}$$

it follows that

$$dz_i dz_i = 0 . \qquad \text{(III-1.4)}$$

A complex curve that satisfies this equation is called an isotropic curve [K.3, p.186]. In particular the edge of regression represented by Equations (I-3.8), (I-3.9) and (I-3.10) is an isotropic curve. From the following lemma we have that any isotropic analytic curve is the edge of regression of some singularity set.

Lemma: Any isotropic analytic curve may be expressed using Equations (I-3.8) - (I-3.10) .

<u>Proof</u>: Equation (III-1.4) may be written

$$z_k' z_k' d\zeta^2 = 0 \qquad \text{(III-1.5)}$$

It follows that

$$(z_1')^2 + (z_2')^2 + (z_3')^2 = 0 \qquad \text{(III-1.6)}$$

Using Equation (III-1.6) define the parameter ξ as follows,

$$\xi = \frac{z_1' - iz_2'}{z_3'} = \frac{-z_3'}{z_1' + iz_2'} \qquad \text{(III-1.7)}$$

A change of parameter given by

$$\zeta = \zeta(\xi) \qquad \text{(III-1.8)}$$

leaves the form of the second and third members of Equation (III-1.7) invariant. Equation (III-1.7) can also be written as

$$z_1' - iz_2' = \xi z_3' , \qquad \text{(III-1.9)}$$

$$\xi z_1' + i\xi z_2' = -z_3' \quad . \qquad \text{(III-1.10)}$$

Defining β''' as

$$\beta''' \equiv \frac{z_3'}{\xi} , \qquad \text{(III-1.11)}$$

and solving Equations (III-1.9) - (III-1.11) simultaneously for z_1', z_2' and z_3' yields

$$z_1' = \frac{1}{2}(\xi^2 - 1)\beta''' , \qquad \text{(III-1.12)}$$

$$z_2' = \frac{i}{2}(\xi^2 + 1)\beta''' , \qquad \text{(III-1.13)}$$

$$z_3' = \xi\beta''' \quad . \qquad \text{(III-1.14)}$$

Integrating Equations (III-1.12) - (III-1.14) gives

$$z_1 = \beta - \xi\beta' + \frac{1}{2}\beta''(\xi^2-1) + c_1 \ , \qquad \text{(III-1.15)}$$

$$z_2 = i\beta - i\xi\beta' + \frac{1}{2}i\beta''(\xi^2+1) + ic_2 \ , \qquad \text{(III-1.16)}$$

$$z_3 = -\beta' + \xi\beta'' + c_3 \ , \qquad \text{(III-1.17)}$$

where c_1 , c_2 , and c_3 are arbitrary constants of integration. $\beta(\xi)$ may be expressed in terms of a function φ as

$$\beta = \varphi + \frac{1}{2}(\xi^2-1)c_1 - \frac{1}{2}(\xi^2+1)c_2 + \xi c_3 \ . \qquad \text{(III-1.18)}$$

Substituting Equation (III-1.18) into Equations(III-1.15) - (III-1.17) will give equations which are identical to Equations (I-3.8) - (I.3.10).

2. Minimal Surfaces Characterize the Singularity Sets

Theorem: A complex curve in C^3 is an isotropic analytic curve if and only if its projection onto the real space R^3 is a minimal surface. Two isotropic analytic curves with the same projection in R^3 differ only by an imaginary translation.

Proof: First we will assume that we have an isotropic analytic curve in C^3 represented by

$$z_k = z_k(\zeta) \ , \qquad \text{(III-2.1)}$$

where

$$dz_k dz_k = 0 \ . \qquad \text{(III-2.2)}$$

z_k' can be expressed as

$$z_k' = \frac{\partial x_k}{\partial v^1} - i \frac{\partial x_k}{\partial v^2} \ . \qquad \text{(III-2.3)}$$

Substituting this expression for z_k' into Equation (III-1.6) yields

$$\left(\frac{\partial x_k}{\partial v^1} - i \frac{\partial x_k}{\partial v^2}\right)\left(\frac{\partial x_k}{\partial v^1} - i \frac{\partial x_k}{\partial v^2}\right) = 0 \ . \qquad \text{(III-2.4)}$$

Expanding the left hand side we obtain

$$\frac{\partial x_k}{\partial v^1}\frac{\partial x_k}{\partial v^1} - \frac{\partial x_k}{\partial v^2}\frac{\partial x_k}{\partial v^2} - 2i \frac{\partial x_k}{\partial v^1}\frac{\partial x_k}{\partial v^2} = 0 \ . \qquad \text{(III-2.5)}$$

Substituting Equation (II-4.1) into Equation (III-2.5) and setting the real and imaginary parts individually equal to zero gives

$$g_{11} - g_{22} = 0 \ , \qquad \text{(III-2.6)}$$

$$g_{12} = 0 \ . \qquad \text{(III-2.7)}$$

We see that v^1 and v^2 are isothermal parameters for the surface in R^3. Also, the surface is represented by the harmonic functions

$$x_k = x_k(v^1, v^2) \ . \qquad \text{(III-2.8)}$$

A surface expressed using isothermal parameters is a minimal surface <u>if</u> <u>and</u> <u>only</u> <u>if</u> each coordinate function $x_k(v^1, v^2)$ is a harmonic function [0.1, p.1099]. It follows that the surface under consideration is a minimal surface.

Let us now assume that we are given a minimal surface in R^3. Any minimal surface may be represented by Equation (III-2.8) where $x_k(v^1,v^2)$ is harmonic and v^1 and v^2 are isothermal parameters. Now introduce $y_k(v^1,v^2)$ as the harmonic functions conjugate to the x_k. The y_k are determined to within arbitrary constants. An analytic curve in C^3 whose projection onto R^3 is the given minimal surface is given by

$$z_k = x_k + iy_k \; . \qquad \text{(III-2.9)}$$

Let us proceed to calculate $dz_k dz_k$. Using the assumption that x_k and y_k are conjugate harmonic functions and repeating the calculations represented by Equations (III-1.2) and (III-2.3) - (III-2.5) we obtain

$$dz_k dz_k = \frac{\partial x_k}{\partial v^1}\frac{\partial x_k}{\partial v^1} - \frac{\partial x_k}{\partial v^2}\frac{\partial x_k}{\partial v^2} - 2i\,\frac{\partial x_k}{\partial v^1}\frac{\partial x_k}{\partial v^2} \qquad \text{(III-2.10)}$$

Substituting Equation (II-4.1) into this equation gives

$$dz_k dz_k = (g_{11}-g_{22}) - 2ig_{12} \; . \qquad \text{(III-2.11)}$$

Since we have isothermal parameters we may use Equations (III-2.6) and (III-2.7). This gives:

$$dz_k dz_k = 0 \; . \qquad \text{(III-2.12)}$$

Therefore our analytic curve is also an isotropic curve. This completes the proof.

The results of this and the previous section lead us to the following

<u>Theorem: Assume $\varphi(\zeta)$ is not of the form $a\zeta^2 + b\zeta + c$. Then a singularity set has an edge of regression which is an isotropic curve in C^3 and its projection onto R^3 is a minimal surface. Any two singularity sets that determine in this way the same minimal surface differ only by an imaginary translation. Any minimal surface will characterise in this way a singularity set.</u>

3. The Cone as a Special Case

We will now consider the case where φ is of the form

$$\varphi = a\zeta^2 + b\zeta + c . \tag{III-3.1}$$

We may perform the following change of parameter on Equations (I-2.86) - (I-2.88):

$$\mu = \sigma\zeta - \varphi' + \zeta\varphi'' . \tag{III-3.2}$$

We will then have

$$z_1 = c - a + \frac{1}{2}\sigma(\zeta^2 - 1) , \tag{III-3.3}$$

$$z_2 = i(c+a) + \frac{i}{2}\sigma(\zeta^2+1) , \tag{III-3.4}$$

$$z_3 = -b + \sigma\zeta . \tag{III-3.5}$$

These equations are of the form:

$$z_i = v_i + \sigma u_i \tag{III-3.6}$$

and represent a family of cones which differ only in the location of their vertices. The vertex of a cone is given by

$$v_1 = c - a \, , \qquad \text{(III-3.7)}$$

$$v_2 = i(c+a) \, , \qquad \text{(III-3.8)}$$

$$v_3 = -b \, . \qquad \text{(III-3.9)}$$

The family of isotropic vectors u_i determine the orientations of the generators of a cone. We see that in this case the singularity set may be characterized by a point, the vertex of the cone.

IV. THE EXTRINSIC GEOMETRY OF THE EDGE OF REGRESSION

1. The Geometry of the Projections onto $\mathfrak{R}^3$ and I^3

The extrinsic geometry of a surface may be specified by its second fundamental form $b_{\alpha\beta}$, which is defined by the formula

$$b_{\alpha\beta} = \left(\frac{\partial^2 x^i}{\partial v^\alpha \partial v^\beta} - g^{\delta\gamma} \frac{\partial x^i}{\partial v^\delta} \frac{\partial x^j}{\partial v^\gamma} \frac{\partial^2 x^j}{\partial v^\alpha \partial v^\beta}\right) \nu^i \qquad \text{(IV-1.1)}$$

where $\underset{\sim}{\nu}$ is a unit vector normal to the surface. The second fundamental form may be interpreted geometrically by the fact that the curvature κ of a geodesic passing through a point in the direction of a unit surface vector η^α is given by the formula

$$\kappa = b_{\alpha\beta}\eta^\alpha\eta^\beta. \qquad \text{(IV-1.2)}$$

The second fundamental forms and also the unit normal vectors for the projections of the edge of regression onto the spaces $\mathfrak{R}^3$ and I^3 are calculated in Appendices D, E, and F.

For both $\mathfrak{R}^3$ and I^3 the components ν^i of the unit normal are the same, and are as follows,

$$\nu^1 = \frac{2v^1}{1+(v^1)^2+(v^2)^2} \qquad \text{(IV-1.3)}$$

$$\nu^2 = \frac{-2v^2}{1+(v^1)^2+(v^2)^2} \qquad \text{(IV-1.4)}$$

$$\nu^3 = \frac{1-(v^1)^2-(v^2)^2}{1+(v^1)^2+(v^2)^2} . \qquad \text{(IV-1.5)}$$

For $\mathfrak{R}^3$ it is found that the components of the second fundamental form are

$$b_{11} = -b_{22} = P''' \qquad \text{(IV-1.6)}$$

$$b_{12} = b_{21} = -Q''' \qquad \text{(IV-1.7)}$$

where P and Q are the real and imaginary parts of φ. For I^3 the second fundamental form has the components

$$b_{11} = -b_{22} = Q''' , \qquad \text{(IV-1.8)}$$

$$b_{12} = b_{21} = P''' \qquad \text{(IV-1.9)}$$

A minimal surface may be defined as a surface with its mean curvature M identically equal to zero,

$$M = g^{\alpha\beta} b_{\alpha\beta} \equiv 0 . \qquad \text{(IV-1.10)}$$

Since the parameters are isothermal one may express this condition as

$$b_{\alpha\alpha} = 0 . \qquad \text{(IV-1.11)}$$

Observing that the set of Equations (IV-1.6) and (IV-1.7) and also the set of Equations (IV-1.8) and (IV-1.9) satisfy Equation (IV-1.11), we have an additional demonstration that the surfaces in $\mathfrak{R}^3$ and I^3 are minimal surfaces.

In Appendix G we calculate the <u>Gaussian curvature</u> for the surfaces in $\mathfrak{R}^3$ and I^3. The result, which is the same in both cases, is given by the formula

$$K = \frac{-4(u_i'\bar{u}_i'u_j\bar{u}_j - \bar{u}_i u_i' u_j \bar{u}_j')}{(u_k\bar{u}_k)^3 \varphi'' \bar{\varphi}'''} . \qquad \text{(IV-1.12)}$$

This expression for the Gaussian curvature is nowhere positive. Evaluating Equation (IV-1.12) in the special case where u_i is given by Equations (I-2.45) - (I-2.47), we obtain

$$K = \frac{-4\left[(2\zeta\bar{\zeta}+1)\,\frac{1}{2}\,(\zeta\bar{\zeta}+1)^2 - \bar{\zeta}(\zeta\bar{\zeta}+1)\zeta(\zeta\bar{\zeta}+1)\right]}{\frac{1}{8}\,(\zeta\bar{\zeta}+1)^6 \varphi''' \bar{\varphi}'''}$$

$$= \frac{-32\left[\frac{1}{2}\,(2\zeta\bar{\zeta}+1) - \zeta\bar{\zeta}\right]}{(\zeta\bar{\zeta}+1)^4 \varphi''' \bar{\varphi}'''} = \frac{-16}{(|\zeta|^2+1)^4\, |\varphi'''|^2} \,. \qquad \text{(IV-1.13)}$$

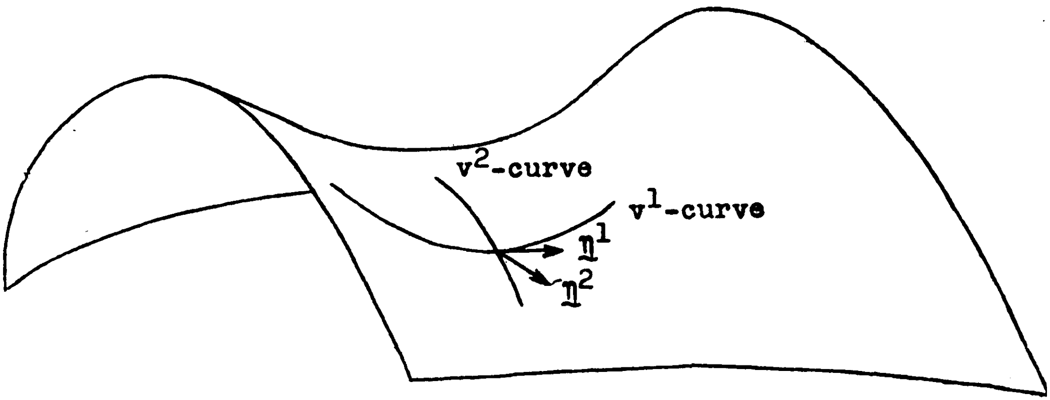
At a point determined by the parameters v^α the surfaces in R^3 and I^3 have the same normal ν^i. Since they are minimal surfaces with the same Gaussian curvature, they have the same local shape. It is saddle-like with principal curvatures of the same magnitude. By comparing their second fundamental forms, we see that the orientations of the two surface elements differ by forty-five degrees as is shown in the diagram on page 44.

2. The Geometry of the Edge of Regression in R^6

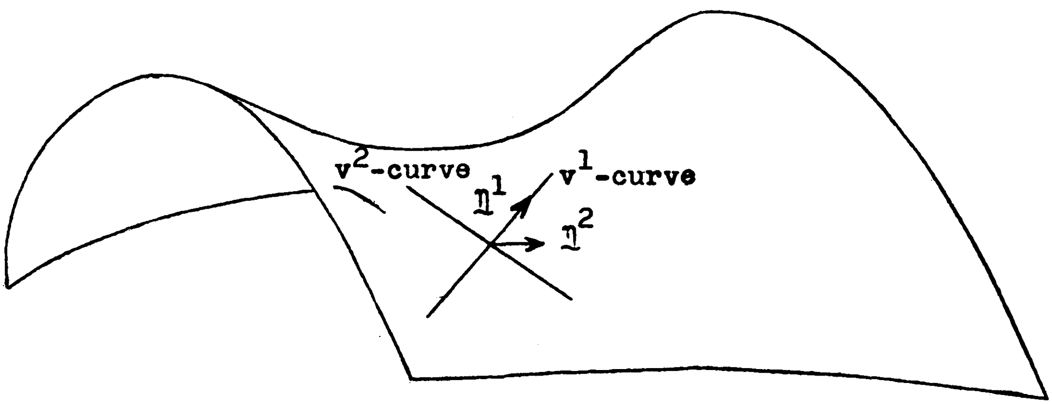
In Appendices H, I, and J we calculate the Gaussian curvature for the edge of regression in R^6. The result is given by the formula

$$K = \frac{-2(u_i' \bar{u}_i' u_j \bar{u}_j - \bar{u}_i u_i' u_j \bar{u}_j')}{(u_k \bar{u}_k)^3\, \varphi''' \bar{\varphi}'''} \,. \qquad \text{(IV-2.1)}$$

This expression for the Gaussian curvature is nowhere positive. It is one-half of the result obtained for the projections of the edge of regression onto R^3 and I^3; hence,



Surface element in $\mathfrak{R}^3$



Surface element in I^3

Diagram of surface elements in $\mathfrak{R}^3$ and I^3 for the point with surface coordinates v^α .

for $\mathcal{R}^6$ one has

$$K = \frac{-8}{(|\zeta|^2+1)^4 \; |\varphi'''|^2} \tag{IV-2.2}$$

In Appendix K we calculate the curvature κ of a geodesic passing through a point on the edge of regression in $\mathcal{R}^6$, in the direction of the unit surface vector η^α, as

$$\kappa^2 = \lambda^{-2}\left(u_i'\bar{u}_i' - \frac{\bar{u}_i u_i' u_j \bar{u}_j'}{u_k \bar{u}_k}\right) |\varphi'''|^2 , \tag{IV-2.3}$$

where λ_1 is given by Equation (II-2.4). We see that the curvature κ is independent of the direction η^α. A point with this property is called an <u>umbilical point</u>.

An isolated umbilical point on a surface in $\mathcal{R}^3$ is a point of positive Gaussian curvature. If a surface in $\mathcal{R}^3$ is not a plane, and if every point is an umbilical point, then it is a sphere. This is not so, however, for a surface in $\mathcal{R}^N$ with $N > 3$. The edge of regression, which is a surface in $\mathcal{R}^6$, gives us an example.

<u>Theorem</u>: <u>For the surface in $\mathcal{R}^6$ determined by Equations (I-3.8) - (I-3.10) the Gaussian curvature is nowhere positive and every point is an umbilical point</u>.

In order to understand this we refer to the diagram on page 44. The curvature of a geodesic in $\mathcal{R}^6$ is determined by the contributions from $\mathcal{R}^3$ and I^3. For $\underline{\eta} = \underline{\eta}^1$ there is a maximum contribution from $\mathcal{R}^3$ and no contribution from I^3. For $\underline{\eta} = \underline{\eta}^2$ we have the reverse situation. For an

$\underset{\sim}{\eta}$ located at some intermediate position between $\underset{\sim}{\eta}^1$ and $\underset{\sim}{\eta}^2$ the constant curvature in $\mathcal{R}^6$ results from a composition of the curvatures in $\mathcal{R}^3$ and I^3 .

V. EXAMPLES OF MINIMAL SURFACES

1. To Find φ Given a Minimal Surface

Suppose we are given a minimal surface which is represented by parametric equations in the form

$$x_i = x_i(w^1, w^2) \ . \tag{V-1.1}$$

The three components of a unit normal ν^i to this surface are surface scalars. In terms of the isothermal parameters v^1 and v^2 the components ν^i are given by Equations (IV-1.3) - (IV-1.5). We may use the equations

$$\nu^i(w^1, w^2) = \pm \ \nu^i(v^1, v^2) \tag{V-1.2}$$

to find the functions $w^\alpha(v^1, v^2)$. From the second fundamental form in the w^α-coordinate system, $b^*_{\delta\gamma}$, we may find the second fundamental form in the v^α-coordinate system, $b_{\alpha\beta}$, by using the equations

$$b_{\alpha\beta} = \frac{\partial w^\delta}{\partial v^\alpha} b^*_{\delta\gamma} \frac{\partial w^\gamma}{\partial v^\beta} \ . \tag{V-1.3}$$

Theorem: Given a minimal surface whose second fundamental form has components $b_{\alpha\beta}$ with respect to isothermal parameters, let

$$\varphi''' = b_{11} - i b_{12} \ . \tag{V-1.4}$$

Then, there are harmonic functions with their singularity sets characterized by the given minimal surface. Moreover, the most general harmonic function of this kind can be written in the form

$$H(\underset{\sim}{z}) = \underset{\sim}{B}_3 \left[\frac{\zeta}{u\zeta+\varphi}\right] + H_0(\underset{\sim}{z}) \tag{V-1.5}$$

<u>where $H_0(\underset{\sim}{z})$ is an arbitrary entire harmonic function</u>.

2. The Catenoid

As an example we will consider a catenoid as the given minimal surface. We may represent this surface with the set of equations

$$x_1 = \sin w^2 \cosh w^1 , \tag{V-2.1}$$

$$x_2 = w^1 , \tag{V-2.2}$$

$$x = \cos w^2 \cosh w^1 . \tag{V-2.3}$$

Taking the partial derivatives of these equations with respect to w^a and substituting the resulting expressions into Equation (II-4.1) gives the expression for the metric tensor, namely

$$g_{\alpha\beta} = (\cosh w^1)^2 \, \delta_{\alpha\beta} . \tag{V-2.4}$$

Since this is of the form of Equation (II-2.3) we have isothermal parameters with λ given by

$$\lambda = \cosh^2 w^1 . \tag{V-2.5}$$

In Appendix L we find that the components of the unit vector normal to the surface are

$$\nu^1 = - \frac{\sin w^2}{\cosh w^1} , \tag{V-2.6}$$

$$\nu^2 = \tanh w^1 . \tag{V-2.7}$$

$$\nu^3 = -\frac{\cos w^2}{\cosh w^1}\,, \tag{V-2.8}$$

and that the components of the second fundamental form are given by

$$b^*_{11} = -b^*_{22} = -1\,, \tag{V-2.9}$$

$$b^*_{12} = b^*_{21} = 0\,. \tag{V-2.10}$$

The catenoid may also be parameterized in terms of the isothermal parameters v^α. In this case the components of the unit normal vector are given by Equations (IV-1.3) - (IV-1.5). The relationship between the w^α and the v^α parameterizations is given by Equation (V-1.2) from which we have

$$\frac{\nu^1(w^1,w^2)}{\nu^3(w^1,w^2)} = \frac{\nu^1(v^1,v^2)}{\nu^3(v^1,v^2)}\,, \tag{V-2.11}$$

$$\nu^2(w^1,w^2) = \nu^2(v^1,v^2)\,. \tag{V-2.12}$$

Substituting Equations (V-2.6) - (V-2.8) and (IV-1.3) (IV-1.5) into Equations (V-2.11) and (V-2.12) gives

$$\tan w^2 = \frac{\zeta+\bar\zeta}{1-\zeta\bar\zeta}\,, \tag{V-2.13}$$

and

$$\tanh w^1 = \frac{(\zeta-\bar\zeta)i}{1+\zeta\bar\zeta} \tag{V-2.14}$$

Equation (V-2.14) may be written as

$$\tan iw^1 = \frac{(\bar\zeta-\zeta)}{1+\zeta\bar\zeta}\,. \tag{V-2.15}$$

We may express the parameters w^{α} using the complex variable

$$\eta = w^1 + iw^2 . \tag{V-2.16}$$

Substituting Equations (V-2.13) and (V-2.15) into the identity

$$\tan i\eta = \tan (iw^1 - w^2) = \frac{\tan iw^1 - \tan w^2}{1 + \tan iw^1 \tan w^2} \tag{V-2.17}$$

yields

$$\tan i\eta = \frac{\dfrac{(\bar{\zeta}-\zeta)}{1+\zeta\bar{\zeta}} - \dfrac{\zeta+\bar{\zeta}}{1-\zeta\bar{\zeta}}}{1 + \dfrac{(\bar{\zeta}-\zeta)(\bar{\zeta}+\zeta)}{1-(\zeta\bar{\zeta})^2}}$$

$$= \frac{\bar{\zeta}-\zeta\bar{\zeta}^2-\zeta+\zeta^2\bar{\zeta}-\bar{\zeta}-\zeta\bar{\zeta}^2-\zeta-\zeta^2\bar{\zeta}}{1+\bar{\zeta}^2-\zeta^2-(\zeta\bar{\zeta})^2}$$

$$= \frac{-2\zeta(1+\bar{\zeta}^2)}{(1-\zeta^2)(1+\bar{\zeta}^2)} = \frac{-2\zeta}{1-\zeta^2} , \tag{V-2.18}$$

which in standard quadratic form, becomes

$$(\tan i\eta)\zeta^2 - 2\zeta - \tan i\eta = 0 . \tag{V-2.19}$$

The solutions to this equation are given by:

$$\zeta = \frac{1 \pm \sqrt{1+\tan^2 i\eta}}{\tan i\eta} = \frac{1 \pm \sec i\eta}{\tan i\eta}$$

$$= \frac{1 \pm \dfrac{1}{\cos i\eta}}{\dfrac{\sin i\eta}{\cos i\eta}} = \frac{\cos i\eta \pm 1}{\sin i\eta} \tag{V-2.20}$$

We may simplify the two solutions as follows:

$$\zeta = \cot \frac{1}{2} i\eta \tag{V-2.21}$$

$$\zeta = - \tan \frac{1}{2} i\eta \ . \tag{V-2.22}$$

Only Equation (V-2.21) checks with Equation (V-1.2). The other solution is therefore extraneous. Solving Equation (V-2.21) for η, we have

$$\eta = -2i \cot^{-1} \zeta \ . \tag{V-2.23}$$

Differentiating Equation (V-2.23) with respect to v^1 and v^2 gives

$$\frac{\partial \eta}{\partial v^1} = \frac{2i}{1+\zeta^2} \ , \tag{V-2.24}$$

and

$$\frac{\partial \eta}{\partial v^2} = \frac{-2}{1+\zeta^2} \ . \tag{V-2.25}$$

These Equations can be written as

$$\frac{\partial w^1}{\partial v^1} = \frac{\partial w^2}{\partial v^2} = -I \left(\frac{2}{1+\zeta^2}\right) \ , \tag{V-2.26}$$

and

$$\frac{\partial w^2}{\partial v^1} = - \frac{\partial w^1}{\partial v^2} = R \left(\frac{2}{1+\zeta^2}\right) \ . \tag{V-2.27}$$

Substituting Equations (V-2.9), (V-2.10), (V-2.26) and (V-2.27) into Equation (V-1.3) gives the second fundamental form in the v^α-coordinate system, namely

$$b_{11} = -b_{22} = R\left(\frac{4}{(1+\zeta^2)^2}\right), \tag{V-2.28}$$

and

$$b_{12} = b_{21} = -I\left(\frac{4}{(1+\zeta^2)^2}\right). \tag{V-2.29}$$

The derivation of this result is discussed further in Appendix L. Using Equation (V-1.4) we obtain the function φ modulo a quadratic by

$$\varphi''' = \frac{4}{(1+\zeta^2)^2}. \tag{V-2.30}$$

3. The Right Helicoid

Another example of a minimal surface is the right helicoid. It may be represented by the set of parametric equations

$$x_1 = w^1 \cos w^2, \tag{V-3.1}$$

$$x_2 = w^1 \sin w^2, \tag{V-3.2}$$

$$x_3 = kw^2. \tag{V-3.3}$$

Taking the partial derivatives of these equations with respect to w^α and substituting the resulting expressions into Equation (II-4.1) gives the expression for the metric tensor, namely

$$g_{11} = 1, \tag{V-3.4}$$

$$g_{12} = g_{21} = 0 \; , \tag{V-3.5}$$

$$g_{22} = (w^1)^2 + k^2 \tag{V-3.6}$$

In Appendix M we find that the components of the unit vector normal to the surface are given by

$$\nu^1 = \frac{k \sin w^2}{[(w^1)^2+k^2]^{1/2}} \; , \tag{V-3.7}$$

$$\nu^2 = \frac{-k \cos w^2}{[(w^1)^2+k^2]^{1/2}} \; , \tag{V-3.8}$$

and

$$\nu^3 = \frac{w^1}{[(w^1)^2+k^2]^{1/2}} \; , \tag{V-3.9}$$

and that the components of the second fundamental form are given by

$$b^*_{11} = b^*_{22} = 0 \; , \tag{V-3.10}$$

and

$$b^*_{12} = b^*_{21} = -k\left[(w^1)^2 + k^2\right]^{-\frac{1}{2}} \tag{V-3.11}$$

We may also parameterize the right helicoid in terms of the isothermal parameters v^α . For this parameterization the components of the unit normal vector are given by Equations (IV-1.3) - (IV-1.5). The relationship between the w^α and the v^α parameterizations is given by Equation (V-1.2) from which we obtain

$$\frac{\nu^3(w^1,w^2)}{\nu^1(w^1,w^2)} = \frac{\nu^3(v^1,v^2)}{\nu^1(v^1,v^2)} \quad , \tag{V-3.12}$$

and

$$\frac{\nu^3(w^1,w^2)}{\nu^2(w^1,w^2)} = \frac{\nu^3(v^1,v^2)}{\nu^2(v^1,v^2)} \quad . \tag{V-3.13}$$

Substituting Equations (IV-1.3) - (IV-1.5) and (V-3.7) - (V-3.9) into Equations (V-2.12) and (v-2.13) gives

$$\frac{w^1}{k \sin w^2} = \frac{1-(v^1)^2-(v^2)^2}{2v^1} \quad . \tag{V-3.14}$$

and

$$\frac{w^1}{k \cos w^2} = \frac{1-(v^1)^2-(v^2)^2}{2v^2} \quad . \tag{V-3.15}$$

These equations can be expressed as

$$2v^1w^1 = k \sin w^2 \left[1 - (v^1)^2 - (v^2)^2\right] , \tag{V-3.16}$$

and

$$2v^2w^1 = k \cos w^2 \left[1 - (v^1)^2 - (v^2)^2\right] . \tag{V-3.17}$$

Eliminating w^2 we have

$$4(w^1)^2 \left[(v^1)^2 + (v^2)^2\right] = k^2 \left[1 - (v^1)^2 - (v^2)^2\right]^2 . \tag{V-3.18}$$

?

Solving for w^1 gives

$$w^1 = \frac{\pm k[1-(v^1)^2-(v^2)^2]}{2[(v^1)^2+(v^2)^2]^{1/2}} \tag{V-3.19}$$

Eliminating w^1 from Equations (V-3.16) and (V-3.17) gives

$$\tan w^2 = \frac{v^1}{v^2} , \qquad \text{(V-3.20)}$$

or inverting this

$$w^2 = \tan^{-1} \frac{v^1}{v^2} . \qquad \text{(V-3.21)}$$

Taking the partial derivative of Equation (V-3.19) with respect to v^1 and v^2 gives

$$\frac{\partial w^1}{\partial v^1} = \frac{-kv^1[1+(v^1)^2+(v^2)^2]}{2[(v^1)^2+(v^2)^2]^{3/2}} , \qquad \text{(V-3.22)}$$

and

$$\frac{\partial w^1}{\partial v^2} = \frac{-kv^2[1+(v^1)^2+(v^2)^2]}{2[(v^1)^2+(v^2)^2]^{3/2}} . \qquad \text{(V-3.23)}$$

Performing these operations on Equation (V-3.21) gives

$$\frac{\partial w^2}{\partial v^1} = \frac{1}{1 + \frac{(v^1)^2}{(v^2)^2}} \left(\frac{1}{v^2}\right) = \frac{v^2}{(v^1)^2+(v^2)^2} , \qquad \text{(V-3.24)}$$

and

$$\frac{\partial w^2}{\partial v^2} = \frac{1}{1 + \frac{(v^1)^2}{(v^2)^2}} \left[\frac{-v_1}{(v^2)^2}\right] = \frac{-v^1}{(v^1)^2+(v^2)^2} . \qquad \text{(V-3.25)}$$

Substituting Equations (V-3.19), (V-3.22) - (V-3.25), (V-3.10) and (V-3.11) into Equation (V-1.3) gives the following expression for the second fundamental form in the v^α-coordinate system,

$$b_{11} = -b_{22} = \frac{2kv^1v^2}{\left[(v^1)^2+(v^2)^2\right]^2} , \tag{V-3.26}$$

$$b_{12} = b_{21} = \frac{k[(v^2)^2-(v^1)^2]}{\left[(v^1)^2+(v^2)^2\right]^2} . \tag{V-3.27}$$

(Tne derivation of this result is discussed further in Appendix M.) Using Equation (V-1.4) we obtain

$$\varphi''' = k\left[(v^1)^2 + (v^2)^2\right]^{-2}\left\{2v^1v^2 + \left[(v^1)^2 - (v^2)^2\right] i\right\}$$

$$= \frac{ik[(v^1)^2-2(v^1)(iv^2)+(iv^2)^2]}{\left[(v^1)^2-(iv^2)^2\right]^2}$$

$$= \frac{ik[(v^1)-i(v^2)]^2}{[v^1-iv^2]^2[v^1+iv^2]^2} = \frac{ik}{(v^1+iv^2)^2} . \tag{V-3.28}$$

Using Equation (II-1.1) we may express this as

$$\varphi''' = \frac{ik}{\zeta^2} . \tag{V-3.29}$$

VI. FURTHER CONSIDERATIONS

It has been shown that a minimal surface in $\mathfrak{R}^3$ will determine a singularity set. However, a minimal surface will be determined by a boundary curve. This is known as Plateau's problem. It often but not always has a unique solution. When it does have a unique solution, a closed curve in $\mathfrak{R}^3$ is sufficient to determine the singularity set in C^3. This may also be a fruitful way in which to characterize the singularity sets.

The singularity set is a four-dimensional manifold in the six-dimensional space C^3. It will, in general, intersect the real space $\mathfrak{R}^3$ in a one-dimensional curve. In other words the singularity set for a harmonic function on $\mathfrak{R}^3$ is in general a curve. One area for future study is to categorize these curves in $\mathfrak{R}^3$.

Boundary curves of the minimal surfaces are sufficient to determine the singularity sets. <u>This raises the question as to whether a singularity set in $\mathfrak{R}^3$, which is also a curve, will determine the singularity set in C^3.</u>

Another possibility for further developments is the extension of the study of the geometry of singularities to harmonic functions in four dimensions. Gilbert [G.2,4] has introduced an operator which is a four dimensional analogue of the Bergman-Whittaker operator. It maps holomorphic functions of three complex variables $f(\tau,\eta,\xi)$ into harmonic functions in four variables. It is expressed as

$$H(\underset{\sim}{z}) = -\frac{1}{4\pi^2}\int_{|\xi|=1}\frac{d\xi}{\xi}\int_{|\eta|=1}\frac{d\eta}{\eta}f(\tau,\eta,\xi) \tag{VI-1.1}$$

$$\tau = N_\mu(\eta,\xi)z_\mu , \quad \mu = 1,2,3,4 , \tag{VI-1.2}$$

$$N_\mu N_\mu = 0 . \tag{VI-1.3}$$

Let

$$h(\tau,\eta,\xi) \equiv S(\underset{\sim}{z},\eta,\xi) = 0 \tag{VI-1.4}$$

be a global defining function in C^3 for the set of singularities of $f(\tau,\eta,\xi)\eta^{-1}\xi^{-1}$, then $H(\underset{\sim}{z})$ is regular for all points $\underset{\sim}{z}$ which do not lie on the set

$$\mathfrak{S} \equiv \cup_{\eta,\xi}\ \{\underset{\sim}{z}|\ S=0\} \cap \{\underset{\sim}{z}|\ S_\eta=0\} \cap \{\underset{\sim}{z}|\ S_\xi=0\} . \tag{VI-1.5}$$

If we consider the case where the equation

$$h(\tau,\eta,\xi) = 0 \tag{VI-1.6}$$

may be solved for τ, we may express $S(\underset{\sim}{z},\eta,\xi)$ in the form

$$S(\underset{\sim}{z},\eta,\xi) = \eta\xi\tau(\underset{\sim}{z},\eta,\xi) + \varphi(\eta,\xi) , \tag{VI-1.7}$$

where $\varphi(\eta,\xi)$ is a holomorphic function of η and ξ . A possibility for the components of the isotropic vector N_μ are

$$N_1 = 1 + \frac{1}{\eta\xi} , \tag{VI-1.8}$$

$$N_2 = i\left(1 - \frac{1}{\eta\xi}\right) , \tag{VI-1.9}$$

$$N_3 = \frac{1}{\xi} - \frac{1}{\eta}, \qquad \text{(VI-1.10)}$$

$$N_4 = i\left(\frac{1}{\xi} + \frac{1}{\eta}\right). \qquad \text{(VI-1.11)}$$

The singularity set $\mathfrak{S}$ will then be given by the parametric equations

$$z_1 = \frac{2i(1+\eta\xi)\mu+(\eta+\xi)\varphi+(1-\eta^2)\varphi_\eta+(1-\xi^2)\varphi_\xi}{-2(\eta+\xi)}, \qquad \text{(VI-1.12)}$$

$$z_2 = \frac{2i(1-\eta\xi)\mu-(\eta+\xi)\varphi+(1+\eta^2)\varphi_\eta+(1+\xi^2)\varphi_\xi}{-2i(\eta+\xi)}, \qquad \text{(VI-1.13)}$$

$$z_3 = \frac{i(\eta-\xi)\mu+\eta\varphi_\eta-\xi\varphi_\xi}{-(\eta+\xi)}, \qquad \text{(VI-1.14)}$$

$$z_4 = \mu, \qquad (\mu,\eta,\xi) \in C^3. \qquad \text{(VI-1.15)}$$

B. L. Tjong in her dissertation [T.2] has introduced an operator which maps holomorphic functions in two variables onto solutions to the following elliptic partial differential equation,

$$\frac{\partial^2 \psi}{\partial z_i \partial z_i} + F(\underset{\sim}{z})\psi = 0 \quad i = 1,2,3 \qquad \text{(VI-1.16)}$$

where $F(\underset{\sim}{z})$ is an entire function in C^3. When this equation reduces to Laplace's equation, Tjong's operator reduces to the Bergman-Whittaker operator. This suggests the possibility of generalizing the theory of singularities from harmonic functions to solutions of the above elliptic partial differential equation.

A similar development may be possible for the four-dimensional case since Colton and Gilbert [C.G.1] have introduced an operator which is a four dimensional analogue of Tjong's operator.

APPENDIX A

The Metric Tensor in $\mathcal{R}^6$

As an alternative notation we shall denote the six components of a position vector in $\mathcal{R}^6$ by x^{iA} where $i=1,2,3$ and A or any other capitalized roman index takes on either of the two values denoted by R and I. These components are defined as

$$x^{iR} = x_i \; , \tag{A-1}$$

$$x^{iI} = y_i \; . \tag{A-2}$$

For their partial derivatives we shall use the notation

$$x_\alpha^{iA} = \frac{\partial x^{iA}}{\partial v^\alpha} \; . \tag{A-3}$$

From the real and imaginary parts of Equation (I-3.11) we have

$$x_1^{iR} = Ru_i\varphi''' \; , \tag{A-4}$$

$$x_1^{iI} = Iu_i\varphi''' \; . \tag{A-5}$$

R and I are used to denote the real and imaginary parts of an expression. Applying the Cauchy-Riemann equations to Equations (A-4) and (A-5) gives

$$x_2^{iI} = Ru_i\varphi''' \; , \tag{A-6}$$

$$x_2^{iR} = -Iu_i\varphi''' \; . \tag{A-7}$$

Equations (A-4) - (A-7) may be expressed as

$$x_{\alpha}^{iA} = \begin{bmatrix} R & -I \\ I & R \end{bmatrix}_{A\alpha} u_i\varphi''' \ . \tag{A-8}$$

As one moves along a row in the above matrix the index α takes on the values 1 and 2 , and as one moves down a column the index A takes on the values R and I .

The metric tensor for the edge of regression in $\Re^6$ is given by

$$g_{\alpha\beta} = x_{\alpha}^{iA} x_{\beta}^{iA} \ . \tag{A-9}$$

Substituting Equation (A-8) into Equation (A-9) gives

$$g_{\alpha\beta} = \begin{bmatrix} R & I \\ -I & R \end{bmatrix}_{\alpha A} u_i\varphi''' \begin{bmatrix} R & -I \\ I & R \end{bmatrix}_{A\beta} u_i\varphi''' \ . \tag{A-10}$$

After performing the indicated matrix multiplication we have

$$g_{\alpha\beta} = \sum_i \begin{bmatrix} R^2(u_i\varphi''')+I^2(u_i\varphi''') & 0 \\ 0 & R^2(u_i\varphi''')+I^2(u_i\varphi''') \end{bmatrix}_{\alpha\beta} \ . \tag{A-11}$$

This result simplifies to Equation (II-2.2)

APPENDIX B

The Metric Tensor in $\Re^3$

The metric tensor for the edge of regression in $\Re^3$ is given by

$$g_{\alpha\beta} = x_\alpha^{iR} x_\beta^{iR} \quad . \tag{B-}$$

Expressing those components of Equation (A-8) where $A = R$ we have

$$x_\alpha^{iR} = [R \quad -I]_\alpha \, u_i \varphi''' \quad . \tag{B-2}$$

Using the notation indicated in Equation (II-4.2) we have

$$x_\alpha^{iR} = [R \quad -I]_\alpha \, N_i \quad . \tag{B-3}$$

Substituting this equation into Equation (B-1) yields

$$g_{\alpha\beta} = \begin{bmatrix} R \\ \\ -I \end{bmatrix}_\alpha N_i \; [R \quad -I]_\beta \; N_i = \begin{bmatrix} RN_iRN_i & -RN_iIN_i \\ & \\ -RN_iIN_i & IN_iIN_i \end{bmatrix}_{\alpha\beta} \tag{B-4}$$

Using Equations (II-3.3), (II-3.5), and (II-3.6) we introduce the assumption that u_i is isotropic. Equation (II-4.6) follows.

APPENDIX C

The Metric Tensor in I^3

The metric tensor for the edge of regression in I^3 is given by

$$g_{\alpha\beta} = x_\alpha^{iI} x_\beta^{iI} \; . \tag{C-1}$$

Expressing those components of Equation (A-8) where $A = I$ we have

$$x_\alpha^{iI} = [I \quad R]_\alpha u_i \varphi''' \quad . \tag{C-2}$$

Using the notation indicated in Equation (II-4.2) we have

$$x_\alpha^{iI} = [I \quad R]_\alpha N_i \; . \tag{C-3}$$

Substituting this equation into Equation (C-1) yields

$$g_{\alpha\beta} = \begin{bmatrix} I \\ \\ R \end{bmatrix}_\alpha N_i [I \quad R]_\beta N_i = \begin{bmatrix} IN_i IN_i & IN_i RN_i \\ \\ RN_i IN_i & RN_i RN_i \end{bmatrix}_{\alpha\beta} \; . \tag{C-4}$$

Using Equations (II-3.3), (II-3.5), and (II-3.6) we introduce the assumption that u_i is isotropic. Equation (II-5.5) follows.

APPENDIX D

The Calculation of $x^{iR}_{\alpha,\beta}$ in $\Re^3$

In studying the geometry of a surface it is useful to calculate the covariant derivatives of the quantities x^{iR}_{α}. In $\Re^3$ they are given by

$$x^{iR}_{\alpha,\beta} = x^{iR}_{\alpha\beta} - g^{\delta\gamma} x^{iR}_{\delta} x^{jR}_{\gamma} x^{jR}_{\alpha\beta} \quad . \tag{D-1}$$

The quantities $x^{jR}_{\alpha\beta}$ denote second partial derivatives

$$x^{iR}_{\alpha\beta} = \frac{\partial^2 x^{jR}}{\partial v^{\alpha} \partial v^{\beta}} \quad . \tag{D-2}$$

Using the fact that we have isothermal parameters, Equation (D-1) becomes

$$x^{iR}_{\alpha,\beta} = x^{iR}_{\alpha\beta} - \lambda^{-1} x^{iR}_{\delta} x^{jR}_{\delta} x^{jR}_{\alpha\beta} \tag{D-3}$$

where λ is given by Equation (II-4.8). From Equation (B-3) we have

$$x^{iR}_{\alpha\beta} = [R \quad -I \quad -I \quad -R]_{\alpha\beta} N'_i \quad . \tag{D-4}$$

Proceeding to evaluate the right side of Equation (D-3) we have

$$x^{jR}_{\delta} x^{jR}_{\alpha\beta} = \begin{bmatrix} \cdot\ R \\ \\ -I \end{bmatrix}_{\delta} N_i [R \quad -I \quad -I \quad -R]_{\alpha\beta} N'_i$$

$$= \begin{bmatrix} RN_j RN'_j & -RN_j IN'_j & -RN_j IN'_j & -RN_j RN'_j \\ -IN_j RN'_j & IN_j IN'_j & IN_j IN'_j & IN_j RN'_j \end{bmatrix}_{\delta,\alpha\beta} \quad . \tag{D-5}$$

Using the assumption that u_i is isotropic we substitute using Equations (II-3.12) and (II-3.13).

$$x_\delta^{jR} x_{\alpha\beta}^{jR} = \begin{bmatrix} R & I & I & -R \\ -I & R & R & I \end{bmatrix}_{\delta,\alpha\beta} \frac{1}{2} N_j \bar{N}_j' \quad . \tag{D-6}$$

Multiplying on the left by x_δ^{iR} we have

$$x_\delta^{iR} x_\delta^{jR} x_{\alpha\beta}^{jR} = [R \quad -I]_\delta \, N_i \begin{bmatrix} R & I & I & -R \\ -I & R & R & I \end{bmatrix}_{\delta,\alpha\beta} \frac{1}{2} N_j \bar{N}_j'$$

$$= \begin{bmatrix} R & -I \\ -I & -R \end{bmatrix}_{\alpha\beta} \frac{1}{2} N_i \bar{N}_j N_j' \quad . \tag{D-7}$$

Using Equations (II-4.8), (D-4) and (D-7), Equation (D-3) becomes

$$x_{\alpha,\beta}^{iR} = \begin{bmatrix} R & -I \\ -I & -R \end{bmatrix}_{\alpha\beta} \left(N_i' - \frac{N_i \bar{N}_j N_j'}{N_k \bar{N}_k}\right) . \tag{D-8}$$

Substituting Equation (II-4.2) into the preceding result gives

$$x_{\alpha,\beta}^{iR} = \begin{bmatrix} R & -I \\ -I & -R \end{bmatrix}_{\alpha\beta} \left(u_i' - \frac{u_i \bar{u}_j u_j'}{u_k \bar{u}_k}\right) \varphi'' \quad . \tag{D-9}$$

Evaluating $x_{\alpha,\beta}^{iR}$ in the particular case where u_i is given by Equations (I-2.45), (I-2.46) and (I-2.47) we have

$$x^{iR}_{\alpha,\beta} = \begin{bmatrix} R & -I \\ -I & -R \end{bmatrix}_{\alpha\beta} \begin{bmatrix} \zeta+\bar{\zeta} \\ i(\zeta-\bar{\zeta}) \\ 1-\zeta\bar{\zeta} \end{bmatrix}_i \frac{\varphi'''}{1+\zeta\bar{\zeta}} \quad . \tag{D-10}$$

We may also express this result in terms ov v^1, v^2, P and Q by substituting Equation (II-1.1) .

$$x^{iR}_{\alpha,\beta} = \left[1+(v^1)^2+(v^2)^2\right]^{-1} \begin{bmatrix} 2v^1 \\ -2v^2 \\ 1-(v^1)^2-(v^2)^2 \end{bmatrix}_i \begin{bmatrix} P''' & -Q''' \\ -Q''' & -P''' \end{bmatrix}_{\alpha\beta} \tag{D-11}$$

APPENDIX E

The Calculation of $x^{iI}_{\alpha,\beta}$ in I^3

In the imaginary space I^3, the formula for $x^{iI}_{\alpha,\beta}$ is

$$x^{iI}_{\alpha,\beta} = x^{iI}_{\alpha\beta} - \lambda^{-1} x^{iI}_{\delta} x^{jI}_{\delta} x^{jI}_{\alpha\beta} \quad . \tag{E-1}$$

From Equation (C-3) we have

$$x^{iI}_{\alpha\beta} = [I \quad R \quad R \quad -I]_{\alpha\beta} N_i' \quad . \tag{E-2}$$

Proceeding to evaluate the right side of Equation (E-1) we have

$$\begin{aligned} x^{jI}_{\delta} x^{jI}_{\alpha\beta} &= \begin{bmatrix} I \\ R \end{bmatrix}_{\delta} N_j \, [I \quad R \quad R \quad -I]_{\alpha\beta} N_j' \\ &= \begin{bmatrix} IN_jIN_j' & IN_jRN_j' & IN_jRN_j' & -IN_jIN_j' \\ RN_jIN_j' & RN_jRN_j' & RN_jRN_j' & -RN_jIN_j' \end{bmatrix}_{\delta,\alpha\beta} . \end{aligned} \tag{E-3}$$

Using the assumption that u_i is isotropic we substitute using Equations (II-3.12) and (II-3.13).

$$x^{jI}_{\delta} x^{jI}_{\alpha\beta} = \begin{bmatrix} R & I & I & -R \\ -I & R & R & I \end{bmatrix}_{\delta,\alpha\beta} \frac{1}{2} N_j \overline{N}_j' \tag{E-4}$$

Multiplying this equation by Equation (C-3) we have

$$x^{iI}_{\delta} x^{jI}_{\delta} x^{jI}_{\alpha\beta} = \begin{bmatrix} I & R \\ R & -I \end{bmatrix}_{\alpha\beta} \frac{1}{2} N_i \overline{N}_j N_j' \tag{E-5}$$

Using Equations (II-4.8), (E-2) and (E-5), Equation (E-1) becomes

$$x^{iI}_{\alpha,\beta} = \begin{bmatrix} I & R \\ R & -I \end{bmatrix}_{\alpha\beta} \left(N'_i - \frac{N_i \bar{N}_j N'_j}{N_k \bar{N}_k} \right) \tag{E-6}$$

Substituting Equation (II-4.2) into the preceding result gives

$$x^{iI}_{\alpha,\beta} = \begin{bmatrix} I & R \\ R & -I \end{bmatrix}_{\alpha\beta} \left(u'_i - \frac{u_i \bar{u}_j u'_j}{u_k \bar{u}_k} \right) \varpi''' \tag{E-7}$$

Evaluating $x^{iI}_{\alpha,\beta}$ in the particular case where u_i is given by Equations (I-2.45),(I-2.46), and (I-2.47) we have

$$x^{iI}_{\alpha,\beta} = \begin{bmatrix} I & R \\ R & -I \end{bmatrix}_{\alpha\beta} \begin{bmatrix} \zeta+\bar{\zeta} \\ i(\zeta-\bar{\zeta}) \\ 1-\zeta\bar{\zeta} \end{bmatrix}_i \frac{\varpi'''}{1+\zeta\bar{\zeta}} \tag{E-8}$$

Substituting Equation (II-1.1) we may express this result as

$$x^{iI}_{\alpha,\beta} = \left(1+(v^1)^2+(v^2)^2\right)^{-1} \begin{bmatrix} 2v^1 \\ -2v^2 \\ 1-(v^1)^2-(v^2)^2 \end{bmatrix}_i \begin{bmatrix} Q''' & P''' \\ P''' & -Q''' \end{bmatrix}_{\alpha\beta} . \tag{E-9}$$

APPENDIX F

The Second Fundamental Forms in $\mathcal{R}^3$ and I^3

The second fundamental form $b_{\cdot\alpha\beta}$ of a surface in $\mathcal{R}^3$ or I^3 is defined by the equation

$$x^{iA}_{\alpha,\beta} = b_{\alpha\beta}\nu^i \tag{F-1}$$

where ν^i is a unit vector normal to the surface. Observe that the expression

$$\nu^i = \left[1 + (v^1)^2 + (v^2)^2\right]^{-1} \begin{bmatrix} 2v^1 \\ -2v^2 \\ 1-(v^1)^2-(v^2)^2 \end{bmatrix}_i \tag{F-2}$$

which appears in Equation (D-11), is a unit vector. It follows that the second fundamental form in $\mathcal{R}^3$ is given by

$$b_{\alpha\beta} = \begin{bmatrix} P''' & -Q''' \\ -Q''' & -P''' \end{bmatrix}_{\alpha\beta} . \tag{F-3}$$

Likewise, from Equation (E-9) it follows that the second fundamental form in I^3 is given by

$$b_{\alpha\beta} = \begin{bmatrix} Q''' & P''' \\ P''' & -Q''' \end{bmatrix}_{\alpha\beta} . \tag{F-4}$$

APPENDIX G

The Gaussian Curvature in $\mathcal{R}^3$ and I^3

In $\mathcal{R}^3$ we may u... the following formula for the Riemann curvature tensor [G.1, p.155]

$$R_{\alpha\beta\lambda\mu} = x^{iR}_{\alpha,\lambda} x^{iR}_{\beta,\mu} - x^{iR}_{\alpha,\mu} x^{iR}_{\beta,\lambda} \quad . \tag{G-1}$$

Substituting Equation (D-9) into Equation (G-1) gives

$$R_{\alpha\beta\lambda\mu} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & -RR-II & RR+II & 0 \\ 0 & RR+II & -RR-II & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}_{\alpha\beta,\lambda\mu} \left(u_i' - \frac{u_i \bar{u}_j u_j'}{u_k \bar{u}_k} \right) \varphi''' \tag{G-2}$$

We are led to the following formula for the Riemann curvature tensor in $\mathcal{R}^3$:

$$R_{\alpha\beta\lambda\mu} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}_{\alpha\beta,\lambda\mu} \left(u_i' \bar{u}_i' - \frac{u_i' \bar{u}_i u_j \bar{u}_j'}{u_k \bar{u}_k} \right) \varphi''' \bar{\varphi}''' \quad . \tag{G-3}$$

In the imaginary space I^3 the Riemann curvature tensor is given by the formula

$$R_{\alpha\beta\lambda\mu} = x^{iI}_{\alpha,\lambda} x^{iI}_{\beta,\mu} - x^{iI}_{\alpha,\mu} x^{iI}_{\beta,\lambda} \tag{G-4}$$

Substituting Equation (E-7) into Equation (G-4) again leads to Equation (G-3).

The Riemann curvature scalar is given by the formula

$$R = R_{\alpha\beta\lambda\mu} g^{\alpha\mu} g^{\beta\lambda} \; . \qquad \text{(G-5)}$$

Since we have isothermal parameters, $g^{\alpha\beta}$ is given by

$$g^{\alpha\beta} = \lambda^{-1} \delta^{\alpha\beta} \; . \qquad \text{(G-6)}$$

Substituting Equation (G-6) into Equation (G-5) gives

$$R = \lambda^{-2} R_{\alpha\beta\beta\alpha} \; . \qquad \text{(G-7)}$$

We obtain a formula for the Riemann curvature scalar by substituting Equations (II-4.8) and (G-3) into Equation (G-7).

$$R = \frac{8(u_i' \bar{u}_i' u_j \bar{u}_j - \bar{u}_i u_i' u_j \bar{u}_j')}{(u_k \bar{u}_k)^3 \varpi''' \bar{\varpi}'''} \qquad \text{(G-8)}$$

The Gaussian curvature of a surface is given by the formula

$$K = -\frac{1}{2} R \; . \qquad \text{(G-9)}$$

Substituting Equation (G-8) into Equation (G-9) gives Equation (IV-1.12) as a formula for the Gaussian curvature.

APPENDIX H

The Calculation of $x^{iA}_{\alpha,\beta}$ in $\Re^6$

In $\Re^6$ the covariant derivatives of the quantities x^{iA}_{α} are given by the formula

$$x^{iA}_{\alpha,\beta} = x^{iA}_{\alpha\beta} - g^{\delta\gamma}x^{iA}_{\delta}x^{jB}_{\gamma}x^{jB}_{\alpha\beta} . \tag{H-1}$$

The quantities $x^{iA}_{\alpha\beta}$ denote second partial derivatives.

$$x^{iA}_{\alpha\beta} = \frac{\partial^2 x^{iA}}{\partial v^{\alpha}\partial v^{\beta}} \tag{H-2}$$

Using the fact that we have isothermal parameters in which case the metric tensor is given by Equation (II-2.3), Equation (H-1) becomes

$$x^{iA}_{\alpha,\beta} = x^{iA}_{\alpha\beta} - \lambda^{-1}x^{iA}_{\delta}x^{jB}_{\delta}x^{jB}_{\alpha\beta} . \tag{H-3}$$

Taking the partial derivative with respect to v^{β} of each side of Equation (A-8), we obtain

$$x^{iA}_{\alpha\beta} = \begin{bmatrix} R & -I & -I & -R \\ I & R & R & -I \end{bmatrix}_{A,\alpha\beta} N'_i \qquad \alpha\beta = 11,12,21,22 . \tag{H-4}$$

We proceed in evaluating the right side of Equation (H-3)

$$x^{jB}_{\delta}x^{jB}_{\alpha\beta} = \begin{bmatrix} R & I \\ -I & R \end{bmatrix}_{\delta B} N_j \begin{bmatrix} R & -I & -I & -R \\ I & R & R & -I \end{bmatrix}_{B,\alpha\beta} N'_j$$

$$= \begin{bmatrix} R & -I & -I & -R \\ I & R & R & -I \end{bmatrix}_{\delta,\alpha\beta} \overline{N}_j N'_j \quad . \tag{H-5}$$

Multiplying on the left by x_{δ}^{iA} we have

$$x_{\delta}^{iA} x_{\delta}^{jB} x_{\alpha\beta}^{jB} = \begin{bmatrix} R & -I \\ I & R \end{bmatrix}_{A\delta} N_i \begin{bmatrix} R & -I & -I & -R \\ I & R & R & -I \end{bmatrix}_{\delta,\alpha\beta} \overline{N}_j N_j'$$

$$= \begin{bmatrix} R & -I & -I & -R \\ I & R & R & -I \end{bmatrix}_{A,\alpha\beta} N_i \overline{N}_j N_j' \quad . \tag{H-6}$$

Using Equations (II-2.4), (H-4) and (H-6); Equation (H-3) becomes

$$x_{\alpha,\beta}^{iA} = \begin{bmatrix} R & -I & -I & -R \\ I & R & R & -I \end{bmatrix}_{A,\alpha\beta} \left(N_i' - \frac{N_i \overline{N}_j N_j'}{\overline{N}_k N_k} \right) \quad . \tag{H-7}$$

Substituting Equation (II-4.2) into the preceding equation gives the result

$$x_{\alpha,\beta}^{iA} = \begin{bmatrix} R & -I & -I & -R \\ I & R & R & -I \end{bmatrix}_{A,\alpha\beta} \left(u_i' - \frac{u_i \overline{u}_j u_j'}{\overline{u}_k u_k} \right) \varphi''' \quad . \tag{H-8}$$

This result depends only upon the assumption that the edge of regression is an analytic curve. We proceed to evaluate $x_{\alpha,\beta}^{iA}$ in the particular case where u_i is given by Equations (I-2.45), (I-2.46) and (I-2.47).

$$\left(u_i' - \frac{u_i \overline{u}_j u_j'}{\overline{u}_k u_k} \right) = \frac{1}{\zeta\overline{\zeta}+1} \begin{bmatrix} \zeta+\overline{\zeta} \\ i(\zeta-\overline{\zeta}) \\ -\zeta\overline{\zeta}+1 \end{bmatrix}_i \tag{H-9}$$

Substituting this expression into Equation (H-8) gives

$$x^{iA}_{\alpha,\beta} = \begin{bmatrix} R & -I & -I & -R \\ I & R & R & -I \end{bmatrix}_{A,\alpha\beta} \begin{bmatrix} \zeta+\bar{\zeta} \\ i(\zeta-\bar{\zeta}) \\ 1-\zeta\bar{\zeta} \end{bmatrix}_i \frac{\varphi'''}{1+\zeta\bar{\zeta}} \quad . \tag{H-10}$$

Using the notation

$$\varphi = P + iQ \tag{H-11}$$

along with Equation (II-1.1) we may express Equation (H-10) as

$$x^{iA}_{\alpha,\beta} = \left[1+(v^1)^2+(v^2)^2\right]^{-1} \begin{bmatrix} 2v^1 \\ -2v^2 \\ 1-(v^1)^2-(v^2)^2 \end{bmatrix}_i \begin{bmatrix} P''' & -Q''' & -Q''' & -P''' \\ Q''' & P''' & P''' & -Q''' \end{bmatrix}_{A,\alpha\beta} \quad . \tag{H-1}$$

APPENDIX I

The Second Fundamental Form in $\mathfrak{R}^6$

In a space of greater than three dimensions the second fundamental form of a surface is defined differently from the way it is defined in a three dimensional space. Rather than being given by Equation (F-1), it is given by [E.2, p. 166]

$$\Omega_{\alpha\beta\gamma\delta} = x^{iA}_{\alpha,\beta} x^{iA}_{\gamma,\delta} \tag{I-1}$$

By substituting Equation (H-8) into Equation (I-1) we obtain the following expression for the second fundamental form in $\mathfrak{R}^6$:

$$\Omega_{\alpha\beta\lambda\mu} = \begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ -1 & 0 & 0 & 1 \end{bmatrix}_{\alpha\beta,\lambda\mu} E\varphi'''\overline{\varphi}''' \tag{I-2}$$

where

$$E = \left| u_i' - \frac{u_i \overline{u}_\ell u_\ell'}{u_m \overline{u}_m} \right|^2 = \left(\overline{u}_i' u_i' - \frac{\overline{u}_i u_i' u_j \overline{u}_j'}{u_k \overline{u}_k} \right) . \tag{I-3}$$

We shall proceed to evaluate $\Omega_{\alpha\beta\delta\gamma}$ in the special case where u_i is given by Equations (I-2.45) - (I-2.47). We first evaluate E .

$$u_k \overline{u}_k = \frac{1}{2} (\zeta\overline{\zeta}+1)^2 \tag{I-4}$$

$$\overline{u}_i u_i' = \overline{\zeta}(\zeta\overline{\zeta}+1) \tag{I-5}$$

$$u_j\bar{u}_j' = \zeta(\zeta\bar{\zeta}+1) \tag{I-6}$$

$$u_i'\bar{u}_i' = 2\zeta\bar{\zeta}+1 \tag{I-7}$$

$$E = 2\zeta\bar{\zeta}+1 - \frac{\bar{\zeta}(\zeta\bar{\zeta}+1)\zeta(\zeta\bar{\zeta}+1)}{\frac{1}{2}(\zeta\bar{\zeta}+1)^2} = 1 \tag{I-8}$$

Substituting this value of E into Equation (I-2) gives

$$\Omega_{\alpha\beta\lambda\mu} = \begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ -1 & 0 & 0 & 1 \end{bmatrix}_{\alpha\beta,\lambda\mu} \varphi'''\bar{\varphi}''' \tag{I-9}$$

APPENDIX J

The Gaussian Curvature in $\Re^6$

The Riemann curvature tensor in $\Re^6$ is given by the formula

$$R_{\alpha\beta\lambda\mu} = \Omega_{\alpha\lambda\beta\mu} - \Omega_{\alpha\mu\beta\lambda} \qquad \text{(J-1)}$$

Substituting Equation (I-2) into this formula gives:

$$R_{\alpha\beta\lambda\mu} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & -2 & 2 & 0 \\ 0 & 2 & -2 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}_{\alpha\beta,\lambda\mu} \left(u_i'\bar{u}_i' - \frac{\bar{u}_i u_i' u_j \bar{u}_j'}{u_k \bar{u}_k} \right) \varphi'''\bar{\varphi}''' . \qquad \text{(J-2)}$$

We obtain a formula for the Riemann curvature scalar by substituting Equations (II-2.4) and (J-2) into Equation (G-7).

$$R = \frac{4(u_i'\bar{u}_i' u_j \bar{u}_j - \bar{u}_i u_i' u_j \bar{u}_j')}{(u_k \bar{u}_k)^3 \; \varphi'''\bar{\varphi}'''} \qquad \text{(J-3)}$$

Substituting Equation (J-3) into Equation (G-9) gives Equation (IV-2.1) as a formula for the Gaussian curvature.

APPENDIX K

Geodesic Curvature on the Surface in $\Re^6$

The curvature κ of a geodesic passing through a point on the edge of regression in $\Re^6$ in the direction of a unit surface vector η^α is given by the formula [2.2, p.165]

$$\kappa^2 = |\Omega_{\alpha\beta\gamma\delta}\eta^\alpha\eta^\beta\eta^\gamma\eta^\delta| \quad . \tag{K-1}$$

We express the assumption that η^α is a unit vector as

$$g_{\alpha\beta}\eta^\alpha\eta^\beta = 1 \ . \tag{K-2}$$

Since we have isothermal parameters, we may substitute for $g_{\alpha\beta}$ using Equation (II-2.3). This gives

$$\eta^\alpha\eta^\alpha = \lambda^{-1} \ . \tag{K-3}$$

Substituting Equation (I-2) into Equation (K-1) we have

$$\kappa^2 = (\eta^\alpha\eta^\alpha)^2 \left(\bar{u}_i' u_i' - \frac{\bar{u}_i u_i' u_j \bar{u}_j'}{u_k \bar{u}_k}\right) |\varpi'''|^2 \ . \tag{K-4}$$

Substituting Equation (K-3) into this equation gives Equation (IV-2.3).

APPENDIX L

The Catenoid

We may represent the partial derivatives of Equations (V-2.1) - (V-2.3) with respect to w^{α} as

$$x_{\alpha}^{iR} = \begin{bmatrix} \sin w^2 \sinh w^1 & \cos w^2 \cosh w^1 \\ 1 & 0 \\ \cos w^2 \sinh w^1 & -\sin w^2 \cosh w^1 \end{bmatrix}_{i\alpha} . \qquad \text{(L-1)}$$

Using Equation (D-3) we may calculate the covariant derivatives of the quantities given by Equation (L-1).

$$x_{\alpha,\beta}^{iR} = \begin{bmatrix} \frac{\sin w^2}{\cosh w^1} & 0 & 0 & -\frac{\sin w^2}{\cosh w^1} \\ -\tanh w^1 & 0 & 0 & \tanh w^1 \\ \frac{\cos w^2}{\cosh w^1} & 0 & 0 & -\frac{\cos w^2}{\cosh w^1} \end{bmatrix}_{\alpha\beta,i}$$

$$= \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}_{\alpha\beta} \left[-\frac{\sin w^2}{\cosh w^1} , \tanh w^1 , -\frac{\cos w^2}{\cosh w^1} \right]_i \qquad \text{(L-2)}$$

Comparing this result with Equation (F-1) we see that the unit vector normal to the surface is given by

$$\nu^i = \left[-\frac{\sin w^2}{\cosh w^1} , \tanh w^1 , -\frac{\cos w^2}{\cosh w^1} \right]_i \qquad \text{(L-3)}$$

and the second fundamental form is given by

$$b^{*}_{\alpha\beta} = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}_{\alpha\beta} \qquad \text{(L-4)}$$

Equations (V-2.26) and (V-2.27) may be expressed as

$$\frac{\partial w^{\delta}}{\partial v^{\alpha}} = \begin{bmatrix} -I & -R \\ R & -I \end{bmatrix}_{\delta\alpha} \frac{2}{1+\zeta^2} \qquad \text{(L-5)}$$

Substituting Equations (L-4) and (L-5) into Equation (V-1.3) yields

$$b_{\alpha\beta} = \begin{bmatrix} -I & R \\ -R & -I \end{bmatrix}_{\alpha\delta} \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}_{\delta\gamma} \begin{bmatrix} -I & -R \\ R & -I \end{bmatrix}_{\gamma\beta} \frac{2}{1+\zeta^2}$$

$$= \begin{bmatrix} -I & R \\ -R & -I \end{bmatrix}_{\alpha\delta} \begin{bmatrix} I & R \\ R & -I \end{bmatrix}_{\delta\beta} \frac{2}{1+\zeta^2}$$

$$= \begin{bmatrix} RR-II & -2IR \\ -2IR & -RR+II \end{bmatrix}_{\alpha\beta} \frac{2}{1+\zeta^2}$$

$$= \begin{bmatrix} R & -I \\ -I & -R \end{bmatrix}_{\alpha\beta} \frac{4}{(1+\bar{\zeta}^2)^2} \quad . \qquad \text{(L-6)}$$

From this result we have Equations (V-2.28) and (V-2.29).

APPENDIX M

The Right Helicoid

Differentiating Equations (V-3.1) - (V-3.3) with respect to w^α gives

$$x^{iR}_{\alpha} = \begin{bmatrix} \cos w^2 & -w^1 \sin w^2 \\ \sin w^2 & w^1 \cos w^2 \\ 0 & k \end{bmatrix}_{i\alpha} . \tag{M-1}$$

Substituting Equation (M-1) into Equation (B-1) gives the following result for the metric tensor

$$g_{\alpha\beta} = \begin{bmatrix} \cos w^2 & \sin w^2 & 0 \\ -w^1 \sin w^2 & w^1 \cos w^2 & k \end{bmatrix}_{\alpha i} \begin{bmatrix} \cos w^2 & -w^1 \sin w^2 \\ \sin w^2 & w^1 \cos w^2 \\ 0 & k \end{bmatrix}_{i\beta}$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & (w^1)^2 + k^2 \end{bmatrix}_{\alpha\beta} . \tag{M-2}$$

We will also need the contravariant form of the metric tensor which is given by

$$g^{\alpha\beta} = \begin{bmatrix} 1 & 0 \\ 0 & \left[(w^1)^2 + k^2 \right]^{-1} \end{bmatrix}_{\alpha\beta} . \tag{M-3}$$

We obtain the quantities $x^{iR}_{\alpha\beta}$ by differentiating Equation (M-1) with respect to w^β .

$$x^{iR}_{\alpha\beta} = \begin{bmatrix} 0 & 0 & 0 \\ -\sin w^2 & \cos w^2 & 0 \\ -\sin w^2 & \cos w^2 & 0 \\ -w^1\cos w^2 & -w^1\sin w^2 & 0 \end{bmatrix}_{\alpha\beta,i} \tag{M-4}$$

The Christoffel symbols may be calculated using the equations

$$\Gamma^{\delta}_{\alpha\beta} = \frac{1}{2} g^{\delta\gamma}\left(\frac{\partial g_{\beta\gamma}}{\partial w^{\alpha}} + \frac{\partial g_{\alpha\gamma}}{\partial w^{\beta}} - \frac{\partial g_{\alpha\beta}}{\partial v^{\gamma}}\right) . \tag{M-5}$$

Substituting Equations (M-2) and (M-3) into Equation (M-5) gives

$$\Gamma^{\delta}_{\alpha\beta} = \frac{1}{2}\begin{bmatrix} 1 & 0 \\ 0 & \left[(w^1)^2+k^2\right]^{-1} \end{bmatrix}_{\delta\nu} \begin{bmatrix} 0 & 0 & 0 & -2w^1 \\ 0 & 2w^1 & 2w^1 & 0 \end{bmatrix}_{\nu,\alpha\beta}$$

$$= \begin{bmatrix} 0 & 0 & 0 & -w^1 \\ 0 & \dfrac{w^1}{(w^1)^2+k^2} & \dfrac{w^1}{(w^1)^2+k^2} & 0 \end{bmatrix}_{\delta,\alpha\beta} . \tag{M-6}$$

The quantities $x^{iR}_{\alpha,\beta}$ are given by the formula

$$x^{iR}_{\alpha,\beta} = x^{iR}_{\alpha\beta} - \Gamma^{\delta}_{\alpha\beta}x^{iR}_{\delta} . \tag{M-7}$$

Evaluating the term $\Gamma^{\delta}_{\alpha\beta}x^{iR}_{\delta}$ we have

$$\Gamma^{\delta}_{\alpha\beta}x^{iR}_{\delta} = \begin{bmatrix} 0 & 0 \\ 0 & w^1\Big/\left[(w^1)^2+k^2\right] \\ 0 & w^1\Big/\left[(w^1)^2+k^2\right] \\ -w^1 & 0 \end{bmatrix}_{\alpha\beta,\delta} \begin{bmatrix} \cos w^2 & \sin w^2 & 0 \\ -w^1\sin w^2 & w^1\cos w^2 & k \end{bmatrix}_{\delta,i}$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ \dfrac{-(w^1)^2\sin w^2}{(w^1)^2+k^2} & \dfrac{(w^1)^2\cos w^2}{(w^1)^2+k^2} & \dfrac{kw^1}{(w^1)^2+k^2} \\ \dfrac{-(w^1)^2\sin w^2}{(w^1)^2+k^2} & \dfrac{(w^1)^2\cos w^2}{(w^1)^2+k^2} & \dfrac{kw^1}{(w^1)^2+k^2} \\ -w^1\cos w^2 & -w^1\sin w^2 & 0 \end{bmatrix}_{\alpha\beta,i} . \qquad \text{(M-8)}$$

Substituting this result and Equation (M-4) into Equation (M-7) gives

$$x^{iR}_{\alpha,\beta} = \begin{bmatrix} 0 & 0 & 0 \\ \dfrac{-k^2\sin w^2}{(w^1)^2+k^2} & \dfrac{k^2\cos w^2}{(w^1)^2+k^2} & \dfrac{-kw^1}{(w^1)^2+k^2} \\ \dfrac{-k^2\sin w^2}{(w^1)^2+k^2} & \dfrac{k^2\cos w^2}{(w^1)^2+k^2} & \dfrac{-kw^1}{(w^1)^2+k^2} \\ 0 & 0 & 0 \end{bmatrix}_{\alpha\beta,i} . \qquad \text{(M-9)}$$

This equation can be expressed as

$$x^{iR}_{\alpha,\beta} = \frac{-k}{(w^1)^2+k^2}\begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix}_{\alpha\beta}\left[k\sin w^2,\ -k\cos w^2,\ w^1\right]_i \ . \qquad \text{(M-10)}$$

Comparing this result with Equation (F-1) we see that the unit normal vector is given by

$$\nu^i(w^1,w^2) = \left[(w^1)^2+k^2\right]^{-\frac{1}{2}}\left[k\sin w^2,\ -k\cos w^2,\ w^1\right]_i \ , \qquad \text{(M-11)}$$

and the second fundamental form is given by

$$b^{*}_{\alpha\beta} = -k\left[(w^1)^2+k^2\right]^{-\frac{1}{2}}\begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix}_{\alpha\beta} . \qquad \text{(M-12)}$$

Equations (V-3.22) - (V-3.25) may be written

$$\frac{\partial w^\delta}{\partial v^\alpha} = \begin{bmatrix}\dfrac{-kv^1\left[1+(v^1)^2+(v^2)^2\right]}{2\left[(v^1)^2+(v^2)^2\right]^{3/2}} & \dfrac{v^2}{(v^1)^2+(v^2)^2}\\[2ex] \dfrac{-kv^2\left[1+(v^1)^2+(v^2)^2\right]}{2\left[(v^1)^2+(v^2)^2\right]^{3/2}} & \dfrac{-v^1}{(v^1)^2+(v^2)^2}\end{bmatrix}_{\alpha\delta} \qquad \text{(M-13)}$$

The following expression which appears in Equation (M-12) is expressed in terms of the v^α-coordinates.

$$(w^1)^2+k^2 = \frac{k^2\left[1-(v^1)^2-(v^2)^2\right]^2}{4\left[(v^1)^2+(v^2)^2\right]} + \frac{4k^2\left[(v^1)^2+(v^2)^2\right]}{4\left[(v^1)^2+(v^2)^2\right]}$$

$$= \frac{k^2\left[1+(v^1)^2+(v^2)^2\right]^2}{4\left[(v^1)^2+(v^2)^2\right]} \ . \qquad \text{(M-14)}$$

When this expression is substituted into Equation (M-12) it becomes

$$b^*_{\delta\gamma} = \frac{-2\left[(v^1)^2+(v^2)^2\right]^{1/2}}{1+(v^1)^2+(v^2)^2}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}_{\delta\gamma} . \qquad \text{(M-15)}$$

We obtain the second fundamental form in the v^α-coordinate system by substituting Equations (M-13) and (M-15) into Equation (V-1.3).

$$b_{\alpha\beta} = k\left[(v^1)^2+(v^2)^2\right]^{-2}\begin{bmatrix} 2v^1v^2 & -\left[(v^1)^2-(v^2)^2\right] \\ -\left[(v^1)^2-(v^2)^2\right] & -2v^1v^2 \end{bmatrix}_{\alpha\beta} . \qquad \text{(M-16)}$$

From this result we have Equations (V-3.26) and (V-3.27).

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