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Tensor Manipulations in Complex Coordinates with Applications to the Mechanics of Materials

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CONTENTS

ABSTRACT

Research into the strength and failure of composite materials is benefited substantially by the development of tensor manipulation techniques in complex coordinates which insure a problem representation that is compact, simple, and invariant. Such techniques are developed through the introduction of several new notations and integer functions. The techniques are then applied to problems connected with analytic formulations in composite materials research, such as basic formulation of anisotropic plane-linear elasticity theory, the elastic constants of laminated composites, graphic representation of tensor transformations, and tensor-polynomial approximation functions to yield surfaces. The techniques are general and can be used whenever two-dimensional tensor formulations are desired.

PROBLEM STATUS

This is a final report on one phase of a continuing NRL problem.

AUTHORIZATION

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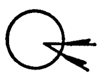
NOTATIONS
(in order of occurrence)

Section 2

x^u	contravariant Cartesian components of the position vector
z^v	contravariant complex components of the position vector
i	$\sqrt{-1}$, always
$\dfrac{\partial}{\partial x^v}$	partial derivative operator *wrt* (with respect to) the contravariant components of the position vector, which transforms as a set of covariant components
g^{uv}, g_{uv}	components of the metric tensor in complex coordinates
g	determinant of the contravariant metric tensor g^{uv}
$T^{u...v}{}_{w...z}(x^n)$	Cartesian components of the general tensor, contravariant *wrt* the indices $u...v$ and covariant *wrt* the indices $w...z$
$T^{u...v}{}_{w...z}(z^n)$	complex components of the general tensor, contravariant *wrt* the indices $u...v$ and covariant *wrt* the indices $w...z$
$T_{\overline{u}}$	use of the complement notation, where u is the set (1,2) and $\overline{u}$ is the set (2,1)
$\overline{T}_u$	complex conjugate of the complex components T_u
$\phi(z^n) = \phi(x^n)$	real-valued Airy's stress function
e^{rs}	Cartesian components of the alternating tensor ($e^{11} = e^{22} = 0$, $e^{12} = 1$, $e^{21} = -1$)
ϵ^{rs}	complex components of the alternating tensor
E_{st}	covariant components of the strain tensor
$p(u)$	p integer function, $p(1) = 1$, $p(2) = -1$
$\vert_u$	covariant components of the covariant differentiation operator which is equivalent to $\partial/\partial x^n$ in Cartesian coordinates, or $\partial/\partial z^n$ in complex coordinates
θ	angle of counterclockwise rotation of the Cartesian reference frame
$s(u...v)$	s integer function: $s(u...v) = p(u) + ... + p(v)$
$\perp$	normal notation, used to denote the transformation $ip(u)T^u(z^n) = T^u(w^m) = T^{u\perp}(z^n)$, where w^m are the complex components of the position vector when the reference frame is rotated clockwise 90°

${}^{k}w_{\ell}(z^n)$	set of components which transform as a vector *wrt* both k and ℓ

Section 3

$T^{up}(z^n)$	contravariant complex components of the stress tensor
S^{st}_{up}	complex components of the material compliance tensor
$D^s(z^n)$	complex components of the displacement vector
$T^{up}(z^n)$ 2	node of a directed graph, which stands for the components $T^{up}(z^n)$
$\xrightarrow[C^{up}_{st}]{S^{st}_{up}}$	directed edge of a graph, which connects two nodes and stands for the components of transformations S^{st}_{up} to the right and C^{up}_{st} to the left
	nodes with edges connected internally stands for a node whose value is the sum of the values obtained by the several paths
$V^u(z^n)$ 	double circle denotes that the value of the node is to be taken as the zero element of $V^u(z^n)$
$\psi(z^n)$	a zero-order tensor which provides the zero element for the compatibility condition
$T^u(z^n)$	complex components of the stress vector
γ_p	complex components of the tangent vector to the arc across which the stress vector acts
$\gamma_{p^\perp}$	normal vector to the arc across which the stress vector acts

Section 4

$\vert_{k^\perp}$	differential operator, $ip(k)\partial/\partial z^k$
C^{up}_{st}	complex components of the material stiffness tensor, the inverse of the material compliance tensor S^{up}_{st}.
λ^u	the characteristic vector of the polynomial differential field equations for the theory of plane linear anisotropic elasticity
${}^{k}\lambda^u$	a set of characteristic vectors which transform as a vector *wrt* k

Section 5

${}^{0}t$	thickness of a laminate
${}^{\alpha}t$	thickness of the α lamina which is a lamina of the laminate whose thickness is ${}^{0}t$
${}^{\alpha}\theta$	angular displacement between the reference frames ${}^{\alpha}y^{j}$ and x^{j}
θ	alternating angle of layup of a laminate
ϕ	2θ

Section 6

z^{u}, w^{u}	complex components of vectors not necessarily transformable into Cartesian coordinates
T	general tensor whose complex components are $T^{u\ldots v}{}_{w\ldots x}(z^{n})$
$\rho e^{is(u)\theta}$	polar representation of the complex numbers, z^{u}, with magnitude ρ and argument $s(u)\theta$
P, Q	complex values of two tensor polynomials in T

Section 7

$f(T^{up})$	a second-order polynomial in T^{up} used to approximate a yield surface
F, F_{up}, F_{upst}	coefficients of the polynomial
a_{f}	volume fraction of fibers in a fiber-reinforced-resin composite

TENSOR MANIPULATIONS IN COMPLEX COORDINATES
WITH APPLICATIONS TO THE MECHANICS OF MATERIALS

1. INTRODUCTION

In the formulation of solutions to mechanics problems there are two accepted steps: (1) attempt to formulate the problem using the tensor calculus to obtain the compactness of the representation and the invariance with respect to allowable coordinate transformations of any resulting equations, and (2) once having the tensor formulation of the problem, seek a coordinate transformation which will render the important transformations of the problem diagonal (a coordinate system such that the tensor components under the tensor transformation are carried into new components that are scalar multiples of the old components as opposed to sums of scalar multiples of all the old components). Hence, when successful, the problem representation is compact, simple, and invariant.

This report is restricted to two-dimensional problems, for which it is shown that tensor manipulations in complex coordinates when enriched by several new notations and integer functions affords a powerful means of accomplishing simple tensor formulations of mechanics problems. Section 2 discusses tensor manipulations in complex coordinates that is different from the usual approach. Particularly emphasized is that not all tensor quantities in complex coordinates are transformable to Cartesian coordinates. Sections 3 and 6 are applications of the formulations of Section 2 to important problems of mechanics. Section 3 discusses the differential constraints of plane linear anisotropic elasticity in which the structure of the relationships is enhanced through the graphic presentations of exact sequences. Section 4 discusses two-dimensional homogeneous partial-differential equations and introduces the use of the characteristic vectors in place of the usual characteristic roots to obtain a hitherto unobtainable tensor formulation of the problem. Section 4 further reveals that such formulations are not generally obtainable in Cartesian reference systems. Section 5 introduces an exceptionally compact and simple expression for the elastic constants of a laminate of anisotropic sheets in terms of the constants of the individual laminas using the diagonal rotation tensors of Section 2. Section 6 develops a graphic representation of the effect of tensor transformations on input vectors that allows for the first time complete control over visualization of such operators. In connection with the graphic representation, two types of characteristic vectors are considered: vectors which come out parallel and vectors which come out perpendicular with respect to the input vectors. The first are the characteristic vectors of homogeneous polynomials, and the second are the classic eigenvectors. Section 7 presents the use of truncated high-order multivariant complex tensor polynomials as desirable candidates for functions approximating yield surfaces in a popular approach to modeling the strength of composite materials. Section 8 summarizes Sections 2 through 7 and outlines some of the progress toward extending the tools developed in this report to finite-dimensional spaces in general.

This report assumes a fair amount of familiarity on the part of the reader with all of the subjects presented here insofar as the usual approach to these problems goes.

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As shown in Section 1.9 of Green and Zerna, the metric tensors and their determinants then follow as

$$\{g^{uv}\} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \tag{2.3a}$$

$$\{g_{uv}\} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \tag{2.3b}$$

$$\det \{g^{uv}\} = (-i)^2 = -1 = 1/g, \tag{2.4a}$$

$$\det \{g_{uv}\} = (i)^2 = -1 = g. \tag{2.4b}$$

The metric derived from the Cartesian system by Eq. 2.2a is extended to those tensors in the complex system not transformable by Eq. 2.2b into the Cartesian system. An apparent disadvantage of this is that vectors of nonreal length can occur, but such occurrence is actually an advantage in that tensors not transformable to Cartesian coordinates can be distinguished.

Associated Tensors and the Complement Notation

The associated tensors (Section 1.10 of Green and Zerna) defined by the relationship*

$$T^{u}{}_{w\ldots z} = g^{uv} T_{vw\ldots z} \tag{2.5a}$$

when using Eq. 2.3a become

$$\begin{Bmatrix} T^{1}{}_{w\ldots z} \\ T^{2}{}_{w\ldots z} \end{Bmatrix} = \begin{Bmatrix} T_{2w\ldots z} \\ T_{1\,w\ldots z} \end{Bmatrix}. \tag{2.5b}$$

By defining a complement notation such that if u is the set (1,2), then $\bar{u}$ is the set (2,1), Eq. 2.5b can be written as

$$T^{u}{}_{w\ldots z} = T_{\bar{u}w\ldots z}. \tag{2.6}$$

Similarly for lowering an index the relationship is

$$T_{u}{}^{w\ldots z} = T^{\bar{u}w\ldots z}. \tag{2.7}$$

Tensors Not Transformable to Cartesian Coordinates

The associated tensors and the complement notation provide a ready means for detecting which tensors are transformable to Cartesian coordinates (that is, for finding the conditions on the complex components of a tensor such that under the transformation 2.1b the components become real). When the x^u are real, the relations

*If an index appears one or more times on only one side of an equation, then a summation over the range of the index is implied.

$$\bar{z} = \frac{1}{\sqrt{2}}(x^1 - ix^2) = z^2 \tag{2.8a}$$

and

$$z = \frac{1}{\sqrt{2}}(x^1 + ix^2) = z^1, \tag{2.8b}$$

where $\bar{z}$ means the complex conjugate of z, are true. Using the complement notation, Eqs. 2.8a and 2.8b become

$$\bar{z}^u = z^{\bar{u}}. \tag{2.9}$$

For a general tensor $T^{s\ldots t}{}_{u\ldots v}(z^n)$ the corresponding statement to Eq. 2.9 is that

$$T^{s\ldots t}{}_{k\ldots \ell} = \overline{T}^{\bar{s}\ldots \bar{t}}{}_{\bar{k}\ldots \bar{\ell}} \tag{2.10}$$

is true when $T^{s\ldots t}{}_{u\ldots v}(z^n)$ is transformable to Cartesian coordinates. This is easily shown by the following argument. Under the transformation 2.1b we have that

$$T^{u\ldots v}{}_{w\ldots z}(x^n) = \frac{\partial x^u}{\partial z^s} \cdots \frac{\partial x^v}{\partial z^t} \frac{\partial z^k}{\partial x^w} \cdots \frac{\partial z^\ell}{\partial x^z} T^{s\ldots t}{}_{k\ldots \ell}(z^n). \tag{2.11}$$

When the x^n are real, it follows from Eq. 2.9 that

$$\overline{\frac{\partial x^u}{\partial z^v}} = \frac{\partial x^u}{\partial z^{\bar{v}}}, \tag{2.12}$$

so that the complex conjugate of Eq. 2.11 is

$$\overline{T}^{u\ldots v}{}_{w\ldots z}(x^n) = \frac{\partial x^u}{\partial z^{\bar{s}}} \cdots \frac{\partial x^v}{\partial z^{\bar{t}}} \frac{\partial z^{\bar{k}}}{\partial x^w} \cdots \frac{\partial z^{\bar{\ell}}}{\partial x^z} \overline{T}^{s\ldots t}{}_{k\ldots \ell}(z^n). \tag{2.13}$$

When the tensor components are Cartesian and hence real, the left sides of Eqs. 2.11 and 2.13 are equal. Substituting $\bar{s} \to s$, $\bar{t} \to t$, $\bar{k} \to k$, ..., $\bar{\ell} \to \ell$ into Eq. 2.13 and equating the right sides of Eqs. 2.11 and 2.13, Eq. 2.10 is obtained.

The Alternating Tensors and the p Function

As given in Section 7.5 of Green and Zerna, the Airy's stress function $\phi(z^n)$ for zero body forces is defined in terms of the stress tensor $T^{up}(z^n)$:

$$T^{up} = \epsilon^{us}\epsilon^{pt}\phi\,|_{st}. \tag{2.14}$$

The alternating tensor ϵ^{rs} is obtained from the tensor e^{rs} of the Cartesian system by the transformation (Eq. 1.7.14 of Green and Zerna)

$$\epsilon^{rs} = \frac{1}{\sqrt{g}} e^{rs} = ie^{rs}. \tag{2.15}$$

Since r and s must be either 1 or 2 and r cannot be equal to s, an obvious simplification of Eq. 2.15 is obtained using the complement notation, so that Eq. 2.15 becomes

$$\epsilon^{rs} = \epsilon^{r\bar{r}} \, \delta^{s}{}_{\bar{r}} = ie^{r\bar{r}} \, \delta^{s}{}_{\bar{r}}. \tag{2.16}$$

Since $e^{r\bar{r}}$ is +1 or —1 depending on whether $r = 1$ or $r = 2$, a further simplification can be arrived at by introducing the integer function $p(r)$ defined as

$$p(1) = +1 \tag{2.17a}$$

and

$$p(2) = -1. \tag{2.17b}$$

Using Eqs. 2.4b and 2.17, Eq. 2.16 then becomes

$$\epsilon^{rs} = - \, ip(r) \, \delta^{s}{}_{\bar{r}}. \tag{2.18}$$

The Airy's stress function now becomes

$$T^{up} = - \, p(u) \, p(p) \, \phi \, |_{\bar{u}\bar{p}}. \tag{2.19}$$

The p function has a further use in simplifying the compatibility equation (Eq. IV-33 of Pearson (1959))

$$\epsilon^{su} \, \epsilon^{tv} \, E_{st} \, |_{st} = 0. \tag{2.20}$$

In terms of the p function and complement notation Eq. 2.20 becomes

$$-p(s) \, p(t) \, E_{st} \, |_{\bar{s}\bar{t}} = 0. \tag{2.21}$$

Further simplifications are possible when the rotation transformations are introduced.

Rotations and the s Function

When the Cartesian coordinates are rotated through an angle θ, the derivative of the transformation is given by*

$$\left\{ \frac{\partial y^u}{\partial x^v} \right\} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}, \tag{2.22}$$

and the corresponding derivative in complex coordinates is

$$\frac{\partial w^u}{\partial z^v} = \frac{\partial z^u}{\partial x^s} \frac{\partial x^t}{\partial z^v} \frac{\partial y^s}{\partial x^t} \tag{2.23}$$

Using Eq. 2.2a, Eq. 2.2b, and the p function, this becomes

*Covariant differentiation *wrt* contravariant components of the position vector produces covariant components, so that the tensor $\partial y^u / \partial x^v$ is contravariant *wrt* u and covariant *wrt* v.

$$\left\{\frac{\partial w^u}{\partial z^v}\right\} = \begin{bmatrix} e^{-i\theta} & 0 \\ 0 & e^{i\theta} \end{bmatrix} = e^{-ip(v)\theta}\ \delta^u{}_v. \tag{2.24}$$

The general tensor $T^{u\ldots v}{}_{w\ldots z}(z^n)$ becomes under the transformation 2.24

$$\begin{aligned} T^{u\ldots v}{}_{w\ldots z}(w^n) &= \frac{\partial w^u}{z^s} \cdots \frac{\partial w^v}{z^t}\,\frac{\partial z^k}{w^w} \cdots \frac{\partial z^\ell}{w^z}\, T^{s\ldots t}{}_{k\ldots \ell}(z^n) \\ &= e^{i[p(\bar{u}) + \ldots + p(\bar{v}) + p(w) + \ldots + p(z)]\theta}\, T^{u\ldots v}{}_{w\ldots z}. \end{aligned} \tag{2.25}$$

If the integer function s is defined by

$$s(u\ldots v) = p(u) + \ldots + p(v), \tag{2.26}$$

then Eq. 2.25 becomes

$$T^{u\ldots v}{}_{w\ldots z}(w^n) = e^{is(\bar{u}\ldots\bar{v}\ w\ldots z)\theta}\, T^{u\ldots v}{}_{w\ldots z}(z^n). \tag{2.27}$$

Of particular interest is the case for $\theta = -\pi/2$, which corresponds to rotating the coordinate system clockwise through 90° or by rotating the tensors counterclockwise by 90°. For this case Eq. 2.27 becomes (using the fact that $e^{-ip(u)\pi/2} = ip(\bar{u})$)

$$\begin{aligned} T^{u\ldots v}{}_{w\ldots \ell}(w^n) &= i^n p(u) \ldots p(v) p(\bar{w}) \ldots p(\bar{\ell}) T^{u\ldots v}{}_{w\ldots \ell}(z^n) \\ &= T^{u^\perp \ldots v^\perp}{}_{w^\perp \ldots \ell^\perp}(z^n), \end{aligned} \tag{2.28}$$

where the symbol $\perp$ is used to indicate the transformation

$$T^{u^\perp \ldots v} = ip(u) T^{u\ldots v}. \tag{2.29}$$

By use of Eq. 2.29 and associated tensors, Airy's stress function as defined by Eq. 2.19 can be written as

$$T^{up} = \phi\ |^{u^\perp p^\perp}. \tag{2.30}$$

Similarly the compatibility equation 2.22 becomes

$$E_{st}\,|^{s^\perp t^\perp} = 0. \tag{2.31}$$

3. DIFFERENTIAL CONSTRAINTS OF PLANE LINEAR ANISOTROPIC ELASTICITY

The differential relationships of plane linear elasticity are represented by the graph of Fig. 1. The linear constitutive relationship of the state of stress, T^{up}, and the state of strain E^{st} is given in the figure by the sequence

$$\begin{array}{ccc} T^{up}(z^n) & & E^{st}(z^n) \\ 2 & \xrightarrow{\quad S^{st}_{up} \quad} & 7 \end{array} \tag{3.1}$$

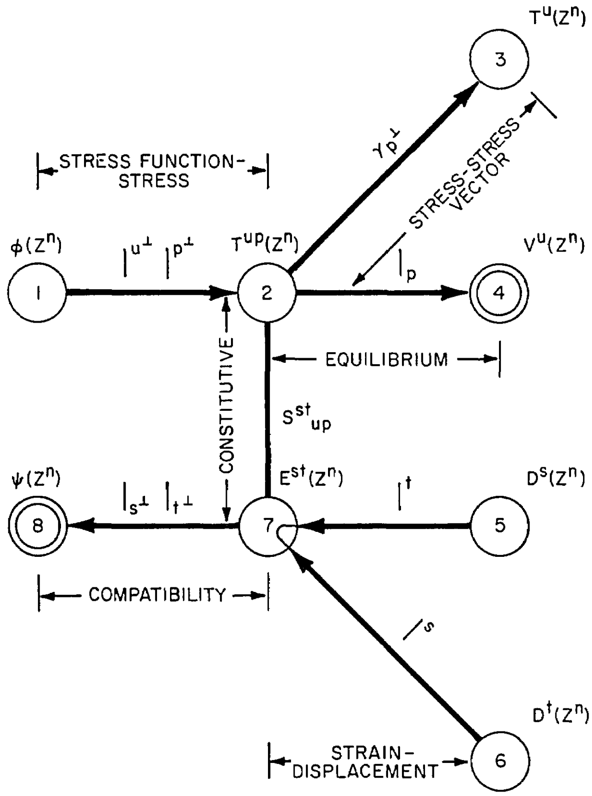
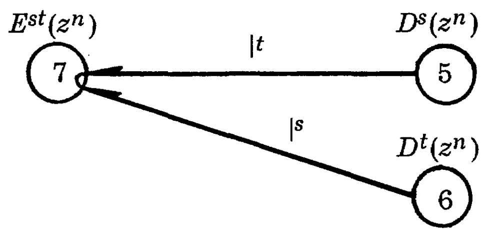


Fig. 1 — Graph of the relationship and differential constraints for plane linear elasticity of anisotropic bodies. Node numbering will be the same throughout the text. The half arrow indicates in which direction the transformation written on that side of the edge takes place. In addition the value of a node is to be taken as the sum of any edges which are connected within the node symbol.

and the strain displacement relations are given by



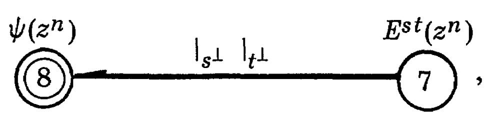
(3.2)

which is read as

$$E^{st} = |^t\, D^s + |^s\, D^t$$

$$= D^s\, |^t + D^t\, |^s\,. \tag{3.3}$$

The compatibility condition given by Eq. 2.31 is represented by the sequence



(3.4)

where $\psi(z^n)$ is a zero-order tensor which provides the zero for the compatibility condition, and where the double circle indicates that the value of the node is to be taken as zero.

The Airy's stress function, $\phi(z^n)$ is given by the sequence

$$\begin{tikzcd} \underset{1}{\phi(z^n)} \arrow[r, "|^{u^\perp}\,|^{p^\perp}"] & \underset{2}{T^{up}(z^n)} \end{tikzcd} \qquad (3.5)$$

using the previously derived form given by Eq. 2.30.

The equilibrium conditions are given by the sequence

$$\begin{tikzcd} \underset{2}{T^{up}(z^n)} \arrow[r, "|_p"] & \underset{4}{v^u(z^n)} \end{tikzcd} \,. \qquad (3.6)$$

Here $v^u(z^n)$ is the first-order tensor, which provides the zero vector for the relationship.

The last sequence of Fig. 1 is the stress-vector- and stress-relationship given by

$$\begin{tikzcd} \underset{2}{T^{up}(z^n)} \arrow[r, "\gamma_{p^\perp}"] & \underset{3}{T^u(z^n)} \end{tikzcd} \,. \qquad (3.7)$$

The vector γ_p is the tangent vector to the arc across which the stress vector $T^u(z^n)$ acts. And $\gamma_{p^\perp}$ is by Eq. 2.28 the normal vector to this arc.

The character of the sequences

$$\begin{tikzcd} \underset{1}{\phi} \arrow[r, "|^{u^\perp}\,|^{p^\perp}"] & \underset{2}{T^{up}} \arrow[r, "|_p"] & \underset{4}{V^u} \end{tikzcd} \qquad (3.8a)$$

and

$$\begin{tikzcd} \underset{8}{\psi} & \underset{7}{E^{st}} \arrow[l, "|_{s^\perp}\,|_{t^\perp}"'] & \underset{5}{D^s} \arrow[l, "|^t"'] \\ & & \underset{6}{D^t} \arrow[ul, "|^s"] \end{tikzcd} \qquad (3.8b)$$

should be noted. Such sequences are called exact. An exact sequence is one in which the image of a node under the edge transformation connecting that node to the next node is the kernel of the next node, where the kernel of a node is the domain of that node which transforms into the zero element of the next node (the identity element for the node). Such sequences have been observed to occur in many areas of mathematical analysis. The occurrence of exact sequences in the formulation of elasticity does not seem to have been pointed out before; use should be made of it in stating clearly the structures of continuum mechanics. For example, given the equilibrium condition, it is obvious from the notation what the stress function definition should be in order that the stress function definition and the equilibrium conditions form an exact sequence. The property of an exact sequence that is used here is that in the sequence 3.8a any path covering the edge (1,2) insures that the edge (2,4) maps node 2 onto the zero element of node 4, and similarly in the sequence 3.8b any path covering edges (5,7) and (6,7) insures that node 7 maps onto the zero element of node 8.

4. PARTIAL DIFFERENTIAL EQUATIONS

In continuation of the theory of linear anisotropic elasticity started in the last section, the governing fourth-order partial-differential equation of the theory (Eq. 4.3) will be considered here in detail. That the discussion applies directly to any-order homogeneous partial-differential equation is obvious if the polynomial of Eq. 4.3 is converted to an nth-order polynomial by introducing additional indices and vectors $|_{k^\perp}$.

When the sequences 3.8 are connected by the edge (2,7), two paths through the graph satisfy both the compatibility and equilibrium conditions. They are the paths (1,2,7,8) and (5,7,2,4). The first gives the stress-function formulation, and the second gives the displacement-vector formulation. The paths are represented by the differential equations

$$S^{st}_{up}\ \phi(z^n)\ |^{u^\perp p^\perp}{}_{s^\perp t^\perp} = 0 \tag{4.1}$$

and

$$C^{up}_{st}\ D^s(z^n)\ |^t_p = 0 \tag{4.2}$$

formed by taking the product of the edges connecting node 1 to node 8 and nodes 5 and 6 to node 4. C^{up}_{st} is the inverse of S^{st}_{up}, and use is made of the symmetry of C^{up}_{st}. Only the stress-function formulation (Eq. 4.1) will be investigated.

The differential equation 4.1 becomes, after lowering indices and noting the equivalence of the symbolization $\phi\ |_{u^\perp p^\perp} \equiv (|_{u^\perp} |_{p^\perp})\phi$,

$$(S^{stup}\,|_{u^\perp}\,|_{p^\perp}\,|_{s^\perp}\,|_{t^\perp})\phi(z^n) = 0. \tag{4.3}$$

Equation 4.3 is a fourth-order homogeneous polynomial in $|_{u^\perp}$ and can be factored into a product of linear forms $\lambda^u\,|_{u^\perp}$ (that is, $-i\lambda^1\,\partial/\partial z^1 + i\lambda^2\,\partial/\partial z^2$), provided λ^u is a vector in complex coordinates. When factored, Eq. 4.3 becomes

$$\left(\prod^{k\ell} {}^{k\ell}\lambda^u\,|_{u^\perp}\right)\phi(z^n) = 0, \tag{4.4}$$

which can be inverted by successive integration (quadratures) to yield

$$\phi(z^n) = \sum^{k\ell} {}^{k\ell}\phi(z^u\ {}^{k\ell}\lambda_u) \tag{4.5}$$

once it is noted that the integral of the differential equation

$$(\lambda^u\,|_{u^\perp})\,\phi(z^n) = 0 \tag{4.6}$$

is

$$\phi(z^n) = \phi(z^u\ \lambda_u). \tag{4.7}$$

After taking the complex conjugate of 4.6 and requiring ϕ to be real ($\bar{\phi} = \phi$), it is seen that if λ^u is a solution vector of 4.4, then $\bar{\lambda}^{\bar{u}}$ is also. A convenient way of denoting this is by the relation

$$^{k\ell}\lambda^{u} = {}^{\bar{k}\bar{\ell}}\bar{\lambda}^{\bar{u}}. \tag{4.8}$$

In addition the requirement that $^{k\ell}\phi$ be real is satisfied by imposing the relationship

$$^{k\ell}\bar{\phi} = {}^{\bar{k}\bar{\ell}}\phi. \tag{4.9}$$

5. THE ELASTIC CONSTANTS OF LAMINATED ANISOTROPIC SHEETS

The usual approach to computing the elastic constants (stiffnesses C_{st}^{up} or compliances S_{up}^{st}) of a laminate of anisotropic sheets is to assume either a constant state of stress T^{up} or a constant state of strain E^{st} through the thicknesses of the laminate.

The requirement that the stored strain energy of the laminate be the sum of the stored strain energies of the individual laminas leads easily to the equations

$$^{0}t\,^{0}C_{st}^{up}(x^{j}) = {}^{\alpha}t\,^{\alpha}C_{st}^{up}(x^{j}) \tag{5.1}$$

and

$$^{0}t\,^{0}S_{up}^{st}(x^{j}) = {}^{\alpha}t\,^{\alpha}S_{up}^{st}(x^{j}). \tag{5.2}$$

Equation 5.1 applies to the case of constant strain through the thickness and Eq. 5.2 applies to the case of constant stress through the thickness. In these equations t is the thickness of an individual lamina or of the laminate; the index α to the upper left of t and of the elastic constants indicates that those thicknesses and constants are for the α lamina, which designates one of all the laminas being summed, and the index 0 indicates the composite laminate.

A general statement of Eqs. 5.1 and 5.2 is that the elastic constants of a laminate are the weighted means of the elastic constants of the individual laminas.

The constants of Eqs. 5.1 and 5.2 are all with respect to the same reference frame x^{j}. Since the elastic constants in all likelihood will be available only with respect to some preferred reference frame $^{\alpha}y^{j}$ (the α indicates the preferred reference frame for each lamina), Eqs. 5.1 and 5.2 can be modified to

$$^{0}t\,^{0}C_{st}^{up}(x^{j}) = {}^{\alpha}t\,\frac{\partial x^{u}}{\partial\,^{\alpha}y^{k}}\,\frac{\partial x^{p}}{\partial\,^{\alpha}y^{\ell}}\,\frac{\partial\,^{\alpha}y^{m}}{\partial x^{s}}\,\frac{\partial\,^{\alpha}y^{n}}{\partial x^{t}}\,^{\alpha}C_{mn}^{k\ell}(^{\alpha}y^{j}) \tag{5.3}$$

and

$$^{0}t\,^{0}S_{up}^{st}(x^{j}) = {}^{\alpha}t\,\frac{\partial x^{s}}{\partial\,^{\alpha}y^{k}}\,\frac{\partial x^{t}}{\partial\,^{\alpha}y^{\ell}}\,\frac{\partial\,^{\alpha}y^{m}}{\partial x^{u}}\,\frac{\partial\,^{\alpha}y^{n}}{\partial x^{t}}\,^{\alpha}S_{mn}^{k\ell}(^{\alpha}y^{j}). \tag{5.4}$$

When the transformations indicated in Eqs. 5.1 and 5.2 are carried out in Cartesian coordinates, the computations can be quite lengthy. But if complex coordinates are used and the reference frame $^{\alpha}y^{j}$ is transformable into the frame x^{j} by a rotation through the angle $^{\alpha}\theta$, Eqs. 5.3 and 5.4 become

$$ {}^{0}t\,{}^{0}C^{up}_{st}(z^j) = {}^{\alpha}t\; e^{is(\bar{u}\bar{p}st)^{\alpha}\theta}\; {}^{\alpha}C^{up}_{st}({}^{\alpha}w^j) \tag{5.5} $$

and

$$ {}^{0}t\,{}^{0}S^{st}_{up}(z^j) = {}^{\alpha}t\; e^{is(\bar{s}\bar{t}up)^{\alpha}\theta}\; {}^{\alpha}S^{st}_{up}({}^{\alpha}w^j). \tag{5.6} $$

These expressions are sums of only α terms each.

When the laminas are all of the same material, then the elastic constants can be factored out of the expression, giving

$$ {}^{0}t\,{}^{0}C^{up}_{st}(z^j) = [{}^{\alpha}t\; e^{is(\bar{u}\bar{p}st)^{\alpha}\theta}]\; C^{up}_{st}({}^{\alpha}w^j) \tag{5.7} $$

and

$$ {}^{0}t\,{}^{0}S^{st}_{up}(z^j) = [{}^{\alpha}t\; e^{is(\bar{s}\bar{t}up)^{\alpha}\theta}]\; S^{st}_{up}({}^{\alpha}w^j). \tag{5.8} $$

A practical case is where not only are all the laminas of the same material but also of the same thickness ${}^{\alpha}t = {}^{0}t/n$ (n even) and constructed by alternately laying up the laminas at $\pm\theta$ angles from the x^j reference frame. For this case Eqs. 5.7 and 5.8 become still further simplified to the forms

$$ {}^{0}C^{up}_{st}(z^j) = \cos[s(\bar{u}\bar{p}st)\theta]\; C^{up}_{st}(w^j) \tag{5.9} $$

and

$$ {}^{0}S^{st}_{up}(z^j) = \cos[s(\bar{s}\bar{t}up)\theta]\; S^{st}_{up}(w^j). \tag{5.10} $$

For greater clarity as to the simplification effected, the individual stiffness constants ${}^{0}C^{up}_{st}(z^j)$ are tabulated as

$$ {}^{0}C^{11}_{11} = C^{11}_{11}, \tag{5.11a} $$

$$ {}^{0}C^{11}_{12} = (\cos 2\theta)\; C^{11}_{12}, \tag{5.11b} $$

$$ {}^{0}C^{11}_{22} = (\cos 4\theta)\; C^{11}_{22}, \tag{5.11c} $$

$$ {}^{0}C^{12}_{12} = C^{12}_{12}. \tag{5.11d} $$

The neglected constants are accounted for by the symmetry relations

$$ {}^{0}C^{up}_{st} = {}^{0}C^{pu}_{st} = {}^{0}C^{up}_{ts} = {}^{0}C^{st}_{up}; \tag{5.12} $$

and that the ${}^{0}C^{up}_{st}(z^j)$ must be transformable into the Cartesian components ${}^{0}C^{up}_{st}(x^j)$ is expressed by the relationship

$$ {}^{0}\overline{C}^{up}_{st}(z^j) = {}^{0}C^{\bar{u}\bar{p}}_{\bar{s}\bar{t}}(z^j). \tag{5.13} $$

A general statement about the elastic constants of a laminated composite constructed by alternating the layup angle $\phi = 2\theta$ of an even number of identical anisotropic sheets of equal thickness is that the elastic constants of the laminate are proportional to the elastic constants of the laminas and that the proportionality coefficients are one of the set ($\cos 0$, $\cos \phi$, $\cos 2\phi$) when the elastic constants are the stiffness ${}^0C^{up}_{st}$ for the assumption of constant strain through the thickness and are the compliance ${}^0S^{st}_{up}$ for the assumption of constant stress through the thickness, *provided a complex coordinate system is used.*

6. GRAPHIC REPRESENTATIONS OF TENSOR TRANSFORMATIONS AND TWO TYPES OF CHARACTERISTIC VECTORS

The classic example of a graphic representation of a tensor transformation is the Mohr-circle construction, familiar to engineering sophomores, which uses as an input vector to the stress tensor the normal to a surface and returns as an output the stress vector acting across that surface. In that construction the vector local to the origin with its end resting on the Mohr circle is the stress vector. This vector traces out a circle as the input vector rotates about some fixed point. Early successful efforts by the author (1962) and later by Mulville (1966) and Wu (1970) were directed at generalizing on the Mohr-circle construction so that the case of third-, fourth-, and in general nth-order tensors could be considered. But the graphic representations obtained were cumbersome.

The difficulties are attributable wholly to the fact that the tensors were being considered in Cartesian coordinates. When the tensors are considered in complex coordinates, the apparatus of Section 2 can be applied. This allows the use of a quite simple graphical technique for representing the transformation properties of general-order tensors, including the previously unconsidered case when the tensor is not transformable to Cartesian coordinates.

In the case of the general tensor

$$T^{u\ldots v}{}_{s\ldots t} \tag{6.1}$$

in complex coordinates, the polynomial

$$T^{uv\ldots w}{}_{s\ldots t}\, z_v \ldots z_w\; z^s \ldots z^t = w^u \tag{6.2}$$

is an operator equation, where T operates on the input z^u to produce the output w^u. By ranging the vector z^u over the set of complex numbers (z^1, z^2), the vector w^u is made to describe a locus of points which graphically represents the transformation T.

Two questions can be asked of z^n: for what values of z^n is w^u normal to z^n, and for what values of z^n is w^u parallel to z^n?

The condition for a normal output can be stated as

$$z_u w^u = 0. \tag{6.3}$$

When applied to Eq. 6.2, the polynomial

$$T^{uv\ldots w}{}_{s\ldots t}\, z_u z_v \ldots z_w z^s \ldots z^t = 0 \tag{6.4}$$

results, which provides as its characteristic vectors (those which satisfy Eq. 6.4) the required z^n satisfying Eq. 6.3.

The condition for a parallel output can be stated as

$$z_{u^\perp} w^u = 0, \tag{6.5}$$

where $z_{u^\perp} = -ip(u)z_u$ by Section 2 is the vector normal to z_u.

When applied to Eq. 6.2 the polynomial

$$ip(u)T^{uv\ldots w}{}_{s\ldots t}\, z_u z_v \ldots z_w z^s \ldots z^t = 0 \tag{6.6}$$

results, which provides as its characteristic vectors the required z^n satisfying condition 6.5.

Equation 6.2 can be mechanized via interactive computer graphics to render a graphic display such that the input and output vectors are displayed. Then a trial-and-error search can be undertaken to locate those input vectors satisfying either Eq. 6.3 or Eq. 6.5.

A more useful approach is to automatically scan over some input domain for the operator and display the locus of the output vectors. The Mohr circle for example considers as inputs a rotating unit vector (or points on the unit circle in the real plane). In addition, since it is the output relative to the input that is of primary interest, the output vector should use the input vector as a reference frame. This is the case for the Mohr-circle construction.

This report will consider the case of all input vectors

$$z^u = \rho e^{is(u)\theta} \tag{6.7}$$

for $0 \leqslant \rho < \infty$ and $0 \leqslant \theta < 2\pi$ which are transformable into Cartesian coordinates. Substituting Eq. 6.7 into Eq. 6.4, the polynomial

$$T^{uv\ldots w}{}_{s\ldots t}\rho^{n-1} e^{is(\bar{v}\ldots\bar{w},s\ldots t)\theta} = w^u \tag{6.8}$$

is obtained, where n is the order of the tensor T. To display w^u relative to z^u rotate the real space through the angle θ and scale it by $1/\rho$. w^u becomes then $(1/\rho)e^{-is(u)}w^u = v^u$ and Eq. 6.8 becomes

$$T^{uv\ldots w}{}_{s\ldots t}\, \rho^n e^{is(\bar{u}\bar{v}\ldots\bar{w},s\ldots t)\theta} = v^u. \tag{6.9}$$

Equation 6.9 has a simple graphic interpretation. It can be viewed as n unit vectors $e^{i[s(\bar{u}\ldots\bar{w},s\ldots t)\theta + \arg(T^{u\ldots w}{}_{s\ldots t})]}\,|T|\rho^n$ connected together, with each vector rotating at an angular frequency of $+2\theta$ greater than the preceding vector.

Similarly, Eqs. 6.4 and 6.6 become

$$T^{uv\ldots w}{}_{s\ldots t} e^{is(\bar{u}\ldots\bar{w},s\ldots t)\theta} = 0 \tag{6.10}$$

and

$$ip(u)T^{uv\ldots w}{}_{s\ldots t}\ e^{is(\bar{u}\ldots\bar{w},s\ldots t)\theta} = 0. \qquad (6.11)$$

Let T^k be the sum of all $T^{uv\ldots w}{}_{s\ldots t}$ such that $k = s(\bar{u}\ldots\bar{w},s\ldots t)$, let the polynomial P be (for $k \geqslant 0$)

$$P = T^n e^{in\theta} + T^{n-2}e^{i(n-2)\theta} + \ldots + T^k e^{ik\theta} + \ldots + \begin{Bmatrix} T^0 e^{i0\theta}, \ n \text{ even} \\ T^1 e^{i\theta}, \ n \text{ odd} \end{Bmatrix}, \qquad (6.12)$$

and let the polynomial Q be $Q = P(-k)$.

Then Eqs. 6.10 and 6.11 can be stated in terms of the polynomials P and Q as

$$\frac{P+Q}{2} = 0 \qquad (6.13)$$

and

$$\frac{P-Q}{2i} = 0. \qquad (6.14)$$

When the tensor T is transformable into Cartesian coordinates ($\overline{T}^{u\ldots w}{}_{s\ldots t} = T^{\bar{u}\ldots\bar{w}}{}_{\bar{s}\ldots\bar{t}}$), then $P = \overline{Q}$ and Eqs. 6.13 and 6.14 become

$$\frac{P+\overline{P}}{2} = 0 \qquad (6.15)$$

and

$$\frac{P-\overline{P}}{2i} = 0. \qquad (6.16)$$

That is, the required vectors are such that the real part of P is zero or the imaginary part of P is zero, depending on whether output vectors normal or parallel to the input vectors are desired.

If n is even, the polynomial P has the form

$$P(\theta) = T^0 + T^2 e^{2i\theta} + T^4 e^{4i\theta} + \ldots + T^n e^{ni\theta}, \qquad (6.17)$$

and if n is odd, P has the form

$$P(\theta) = T^1 e^{i\theta} + T^3 e^{3i\theta} + \ldots + T^n e^{ni\theta}. \qquad (6.18)$$

Equations 6.17 and 6.18 have been mechanized via a computer time-sharing terminal with a plotter output. Figures 2 are for n odd and Figs. 3 are for n even. The purpose of the plots is to show the effectiveness of the techniques of Section 2 (tensor manipulations in complex coordinates) in giving graphic interpretation to the nature of tensor operators. In the plots the complex value of the polynomial P is represented by a point. The initial value is for $\theta = 0$, and the arrows give the direction in which θ is increasing. By counting dots from the initial condition, the value of the vector $z^u = e^{is(u)\theta}$ can be determined which causes the polynomial to have the value of the dots (interpreted as a complex number). For example when $\theta = 45°$, $90°$, or $135°$ (12-1/2 × 3.6°, 25 × 3.6°, or 37-1/2 × 3.6°) the value of the polynomial for $T^1 = e^{i(0)}$ and $T^3 = e^{i(0)}$ is pure imaginary, so that the polynomial $(P + \bar{P})/2$ has the value zero. And for $\theta = 0°, 90°$, or $180°$ the polynomial P has a real value and the polynomial $(P - \bar{P})/2i$ has the value zero. The figures explore some simple exercise of varying the number and value of the coefficients T^k of the polynomial. It is left to the reader to explore the plots and determine the value of any insight into the nature of tensor operators imparted by the presented technique.

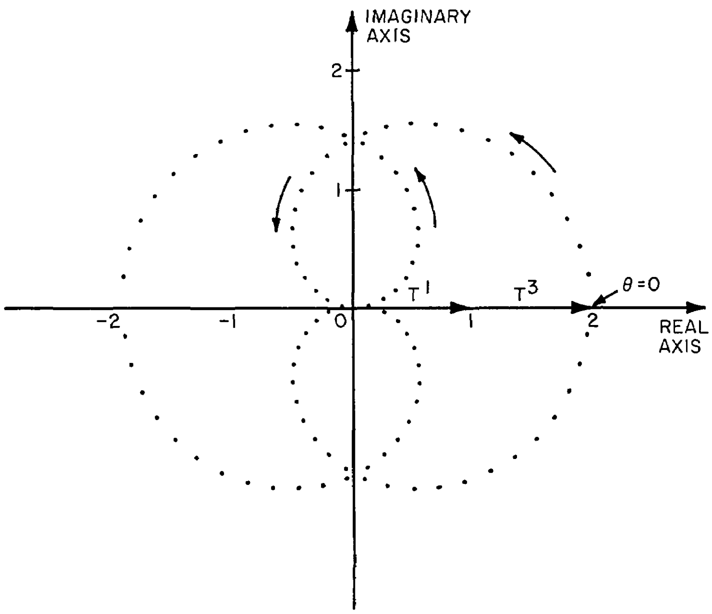


Fig. 2a — Outputs obtained from a computer mechanization of Eq. 6.18 for values of θ increasing in increments $\Delta\theta = 3.6°$ starting from $\theta = 0$ with $T_1 = (0°,1)$, $T_2 = (0°,1)$, and all other T's = 0, where $T = (\arg T, |T|)$

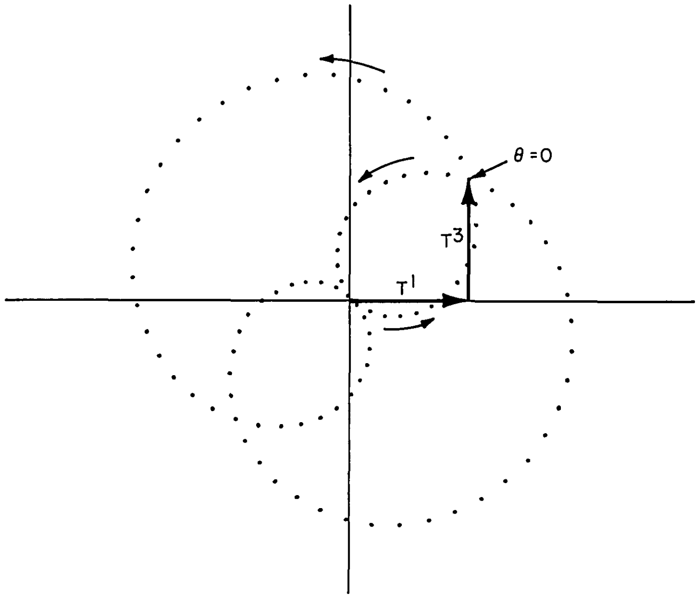


Fig. 2b — Outputs for Eq. 2.18 with $\Delta\theta = 3.6^\circ$, $T_1 = (0^\circ,1)$, $T_3 = (90^\circ,1)$, and all other T's = 0

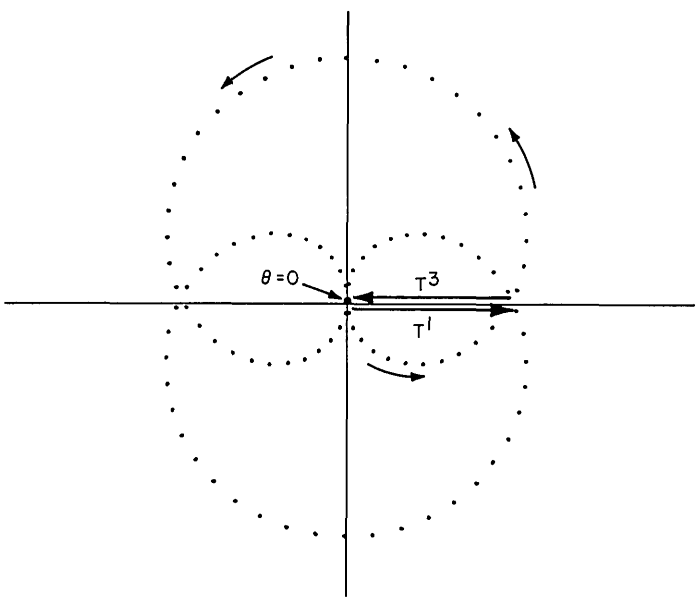


Fig. 2c — Outputs for Eq. 2.18 with $\Delta\theta = 3.6^\circ$, $T^1 = (0^\circ,1)$, $T^3 = (180^\circ,1)$, and all other T's = 0. A comparison of this plot with those of Figs. 2a and 2b is of interest.

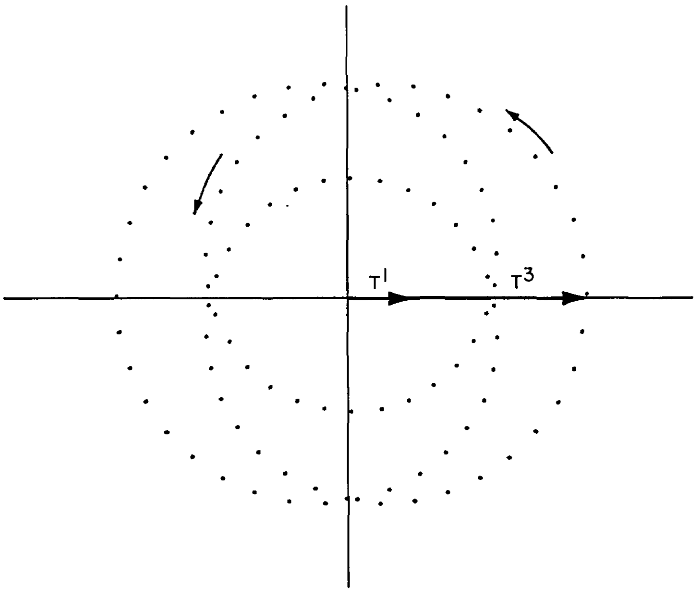


Fig. 2d — Outputs for Eq. 2.18 with $\Delta\theta$ = 3.6°, T^1 = (0°,0.5), T^3 = (0°,1.5), and all other T's = 0

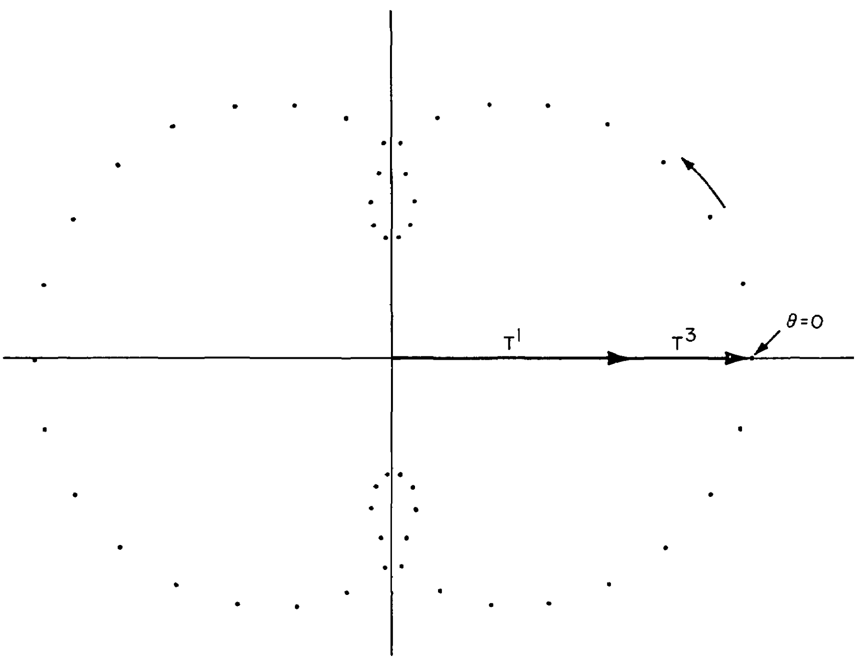


Fig. 2e — Outputs for Eq. 2.18 with $\Delta\theta$ = 7.2°, T^1 = (0°,2), T^3 = (0°,1), and all other T's = 0

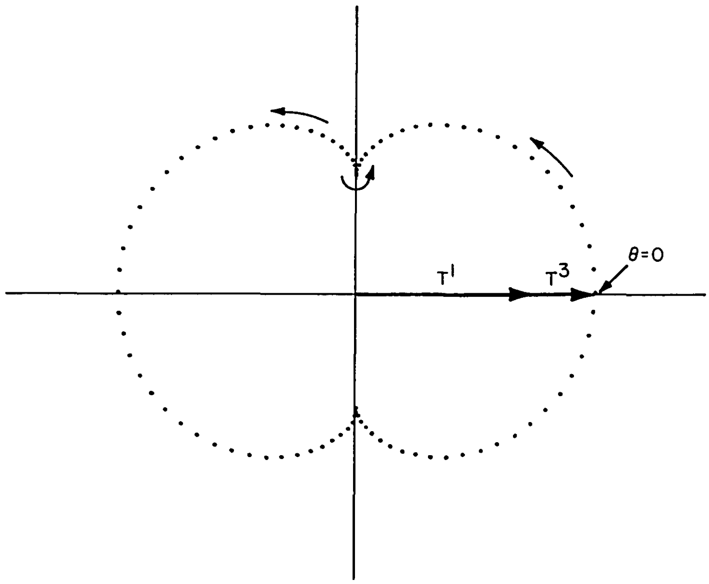


Fig. 2f — Outputs for Eq. 2.18 with $\Delta\theta$ = 3.6°, T^1 = (0°,1.5), T^3 = (0°,0.5), and all other T's = 0

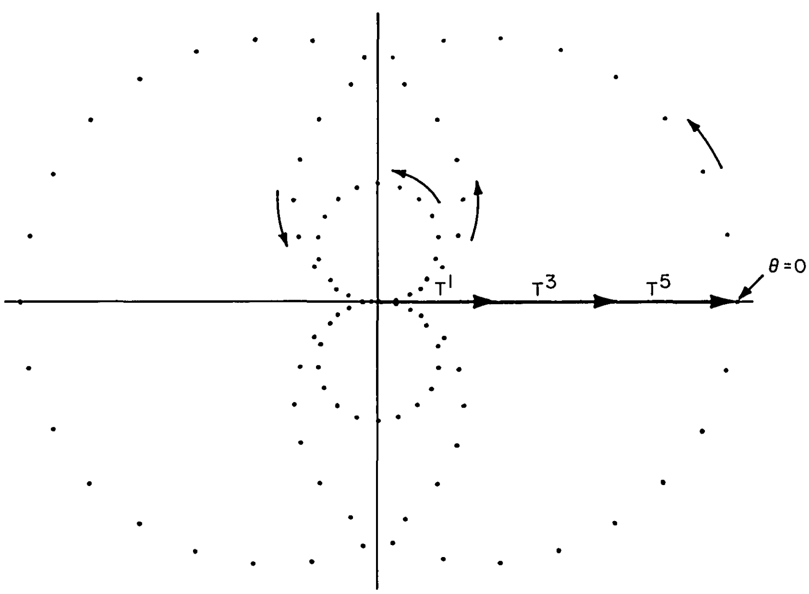


Fig. 2g — Outputs for Eq. 2.18 with $\Delta\theta$ = 3.6°, T^1 = (0°,1), T^3 = (0°1), T^5 = (0°,1), and all other T's = 0

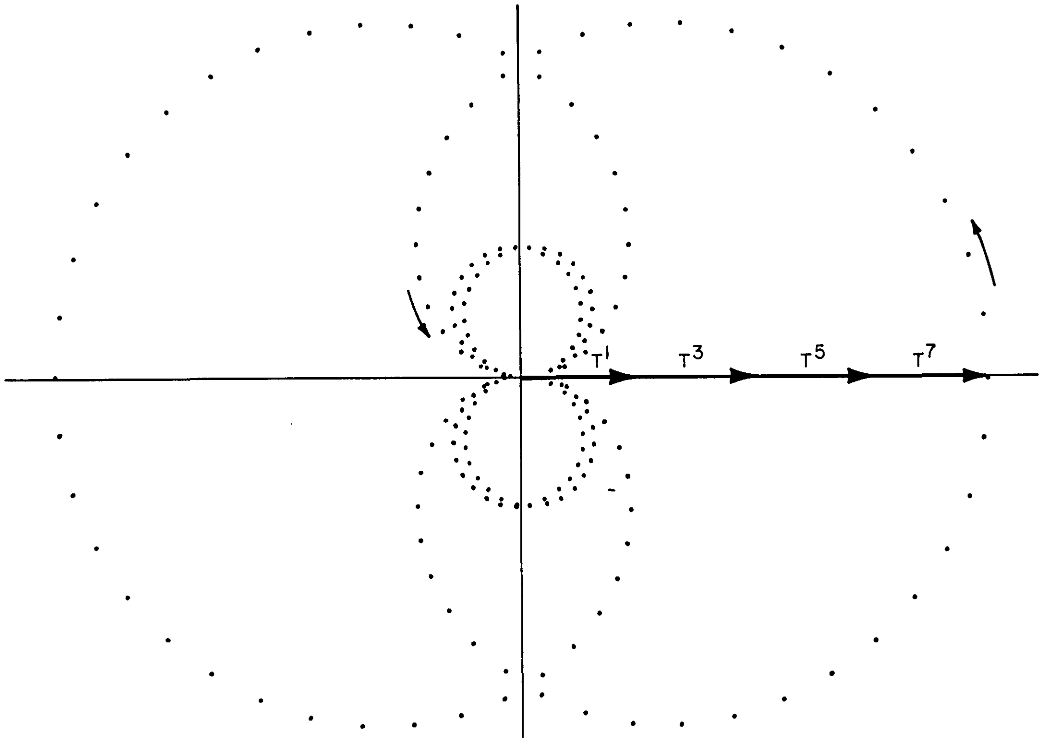


Fig. 2h — Outputs for Eq. 2.18 with $\Delta\theta$ = 3.6°, $T^1 = T^3 = T^5 = T^7$ = (0°,1), and all other T's = 0

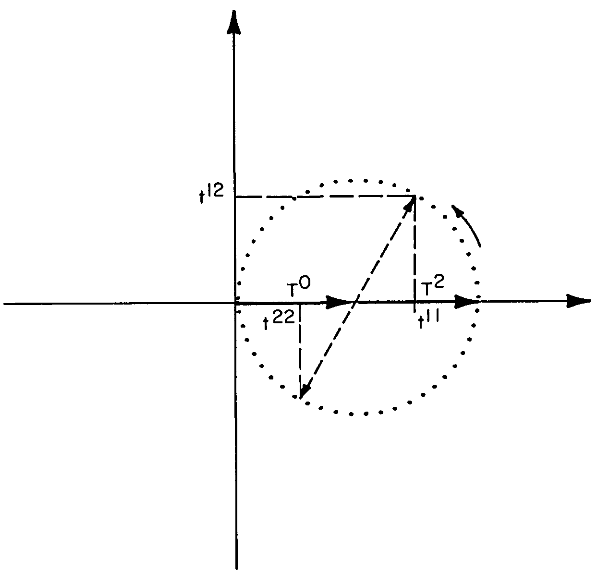


Fig. 3a — Outputs for Eq. 2.17 with $\Delta\theta$ = 3.6°, $T^0 = T^2$ = (0°,1), and all other T's = 0

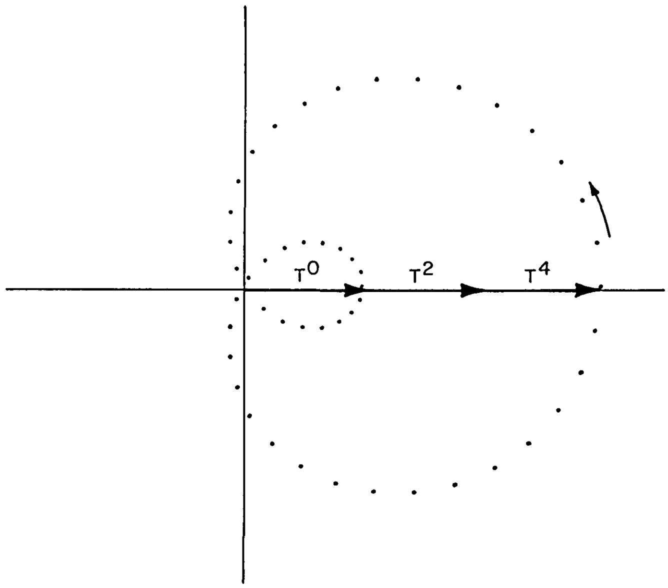


Fig. 3b — Outputs for Eq. 2.17 with $\Delta\theta$ = 3.6°, $T^0 = T^2 = T^4 = (0^\circ,1)$, and all other T's = 0

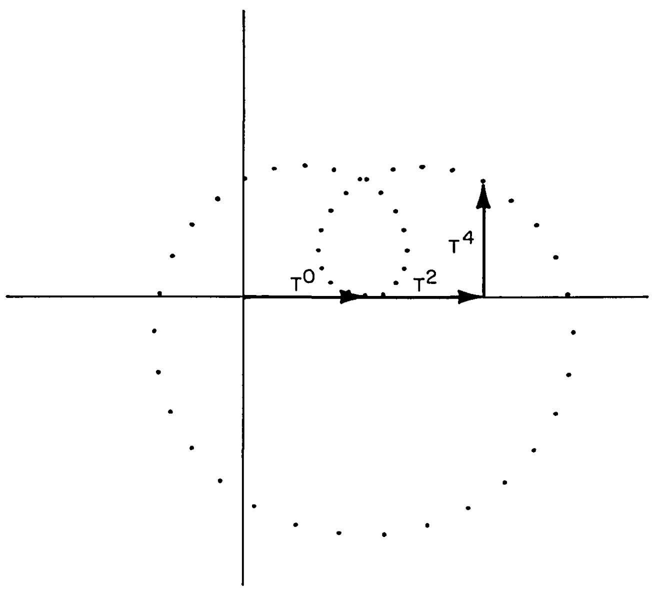


Fig. 3c — Outputs for Eq. 2.17 with $\Delta\theta$ = 3.6°, $T^0 = T^2 = (0^\circ,1)$, $T^4 = (90^\circ,1)$, and all other T's = 0. A comparison of this plot with that of Fig. 3b is of interest.

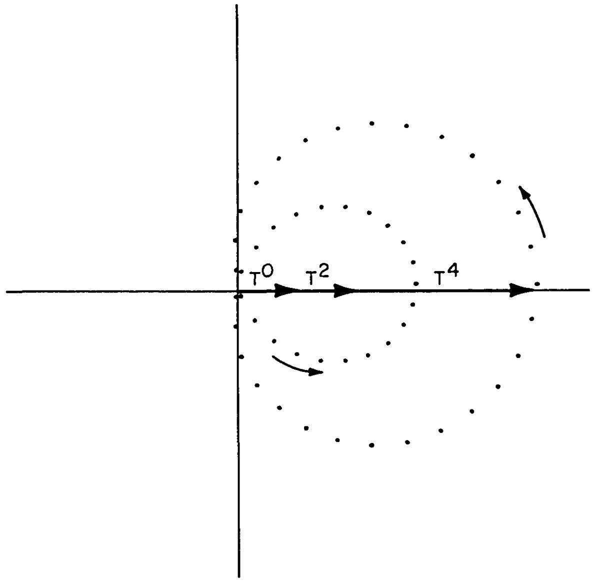


Fig. 3d — Outputs for Eq. 2.17 with $\Delta\theta = 3.6^\circ$, $T^0 = T^2 = (0^\circ, 0.5)$, $T^4 = (0^\circ, 1.5)$, and all other T's = 0. A comparison of this plot with that of Fig. 3b is of interest.

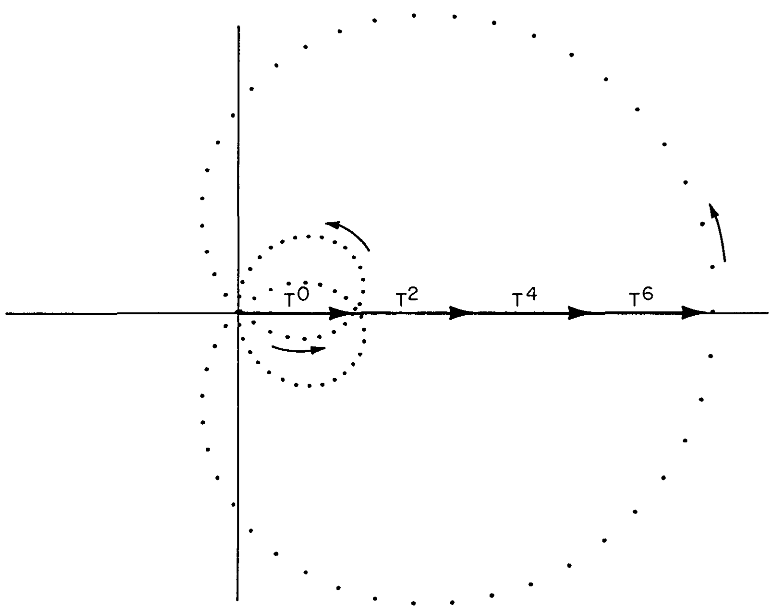


Fig. 3e — Outputs for Eq. 2.17 with $\Delta\theta = 1.8^\circ$, $T^0 = T^2 = T^4 = T^6 = (0^\circ, 1)$, and all other T's = 0

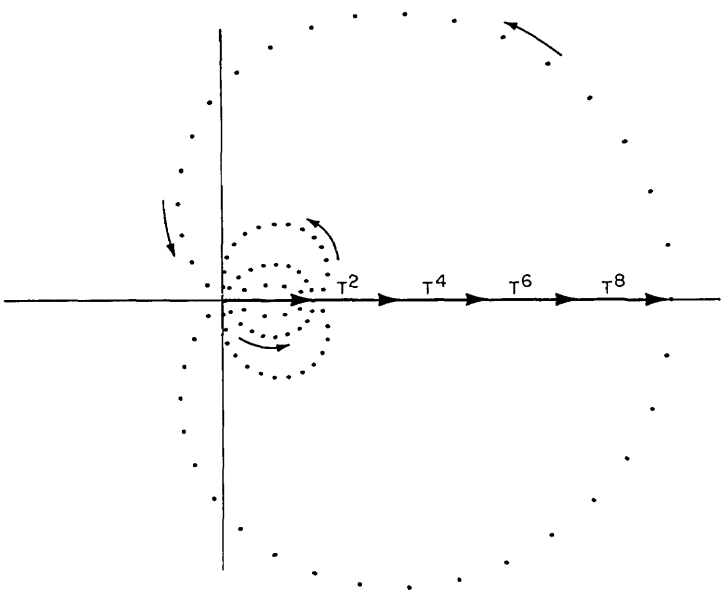


Fig. 3f — Outputs for Eq. 2.17 with $\Delta\theta = 1.8^\circ$, $T^0 = T^2 = T^4 = T^6 = T^8 = (0^\circ, 0.8)$, and all other T's = 0

7. MULTIVARIANT TENSOR POLYNOMIALS APPROXIMATING YIELD SURFACES FOR COMPOSITE MATERIALS

The final application in this report of the techniques of Section 2 is to the generation of candidate polynomials in approximating yield surfaces for composite materials. Tsai and Wu (1971) demonstrated the effectiveness of using second-order Cartesian-tensor polynomials in the state of stress to describe the failure (yield) surface of composite materials when the candidate polynomials are of the form

$$f(T^{up}) = F_{upst}T^{up}T^{st} + F_{up}T^{up} + F = 0. \tag{7.1}$$

Though more effective than previous equations for approximating the yield surface of composites, Eq. 7.1 still falls far short of what is needed for the problem. The basic problem is one of obtaining a means of representing an experimental data base and providing a means of interpolating between data points which requires, hopefully, far less computer storage space than it takes to represent the data alone. One approach is to build as much a priori information as possible into as small a candidate approximating function as possible.

Since the present problems of materials experimentation limits useful yield-surface information to that obtained on two-dimensional specimens, the two-dimensional techniques of the report are strongly applicable. Using the techniques of Section 2, much a priori information can be simply constructed into a more powerful approximating polynomial for two-dimensional composite yield-surface problems.

The first extension is to consider tensor polynomials in complex coordinates. Then by Section 2 the coefficients of the polynomial which are general tensors $F^{u\dots v}{}_{s\dots t}$ have the simplest transformation properties with respect to rotations given by

$$F^{u\ldots v}{}_{s\ldots t}(e^{ip(k)\theta} z^k) = e^{is(\bar{u}\ldots\bar{v},s\ldots t)\theta} F^{u\ldots v}{}_{s\ldots t}(z^k), \tag{7.2}$$

which is at this point quite familiar to the reader.

The second extension is to make the polymonials multivariant in that not only will they be functions of the state of stress but also of the elastic constants (the stiffnesses) C^{st}_{mn}. When the material is a laminated anisotropic body of the type considered in Section 5, the tensor C^{st}_{mn} takes the form

$$^{0}C^{up}_{mn}(z^k) = \cos[s(\bar{u}\bar{p}mn)\theta]\ C^{up}_{mn}(w^k), \tag{7.3}$$

where w^k is the preferred coordinate system for elastic constants.

The candidate approximation polynomials to the yield surface are then

$$P\left\{\cos[s(\bar{u}\bar{p}mn)\theta]\, C^{up}_{mn}(w^k),\ T^{up}(z^k)\right\} = 0 \tag{7.4}$$

When the lamina are fiber-reinforced plastics (such as glass reinforced plastics), the elastic stiffnesses C^{up}_{mn} are assumed to be homogeneous of rank one *wrt* the volume fraction a_f of fiber of the composite; that is,

$$C^{up}_{mn}(a_f) = a_f C^{up}_{mn}(a_f{=}1). \tag{7.5}$$

Then the candidate polynomial of degree n becomes

$$P\left\{a_f \cos[s(\bar{u}\bar{p}mn)\theta]\, C^{up}_{mn}(a_f{=}1, w^k),\ T^{up}(z^k)\right\} = 0. \tag{7.6}$$

One last extension can be made to Eq. 7.6, in allowing it to be truncated. To provide the high-order convolutions needed to fit the data and to avoid the problem of an exploding number of coefficients, all terms need not be other than zero. This follows after a discussion by Gilstrup (1972) on generally the same problem when providing predictions for developing machines with high-level artificial intelligence.

8. EXTENSIONS OF THE WORK IN PROGRESS

The explanation of the applications of the tensor manipulation techniques presented in Section 2 to engineering mechanics problems is rather terse and is meant to establish the power of the technique for two-dimensional problems rather than to be exhaustively informative about the actual applications.

Though the two-dimensional nature of the technique meets the present needs of the author and his associates in dealing with problems of research on composite materials, the technique has been extended to finite-dimensional tensor spaces, where the rotation operators are again diagonal and the tensor components come from the complex number field.

The finite-dimensional theory is not complete, but typically the technique involves mapping a given tensor onto a higher order one; so that a first-order tensor in Cartesian coordinates in three dimensions is mapped onto a fifth-order tensor in complex coordinates.

Since the need for the extension of the theory will probably coincide with a general use of computer symbolic manipulation, the apparent complication of raising the order of the tensor will present no real problems.

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13. ABSTRACT

Research into the strength and failure of composite materials is benefited substantially by the development of tensor manipulation techniques in complex coordinates which insure a problem representation that is compact, simple, and invariant. Such techniques are developed through the introduction of several new notations and integer functions. The techniques are then applied to problems connected with analytic formulations in composite materials research, such as basic formulation of anisotropic plane-linear elasticity theory, the elastic constants of laminated composites, graphic representation of tensor transformations, and tensor-polynomial approximation functions to yield surfaces. The techniques are general and can be used whenever two-dimensional tensor formulations are desired.

14. KEY WORDS	LINK A		LINK B		LINK C	
	ROLE	WT	ROLE	WT	ROLE	WT
Tensor manipulations						
Complex coordinates						
Differential constraints						
Linear anisotropic elasticity						
Exact sequences						
Partial differential equations						
Graphic representation						
Characteristic vectors						
Multivariant tensor polynomials						
Composite materials						
Laminates						

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