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A Complete Orthonormal Set of Two-Dimensional Haar-Like Functions

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ABSTRACT

A set of two-dimensional functions is defined. Expansions of two-dimensional functions $f(x, y)$ in terms of this set have convergence properties analogous to those of one-dimensional Haar series. The Nth partial sum $P_N(x, y)$ is a step function of 2^{2N} square steps each of area $1/2^{2N}$. The value of $P_N(x, y)$ on any step is the mean value of $f(x, y)$ over the area covered by the step. If $f(x, y)$ is continuous or has a finite number of discontinuities along binary-rational line segments, then $P_N(x, y)$ converges uniformly. If $f(x, y)$ has a finite number of discontinuities along binary-irrational line segments, then $P_N(x, y)$ converges pointwise, except along the discontinuities. Accuracy of the estimate P_N and the convergence rate are analogous to those for one-dimensional Haar series.

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A COMPLETE ORTHONORMAL SET OF TWO-DIMENSIONAL HAAR-LIKE FUNCTIONS

1. INTRODUCTION

Engineering interest in alternative representations of time-domain functions, which used to center on the Fourier and Laplace transforms, has extended in recent years to Walsh functions (1-3). In two-dimensional image processing, use of the two-dimensional Fourier transform is being supplemented by the Walsh-Hadamard transform (4). Haar functions (5-7) are closely related to Walsh functions and have properties that suggest they may be useful in applications such as multiplexing, data transmission, and pattern recognition (8). In this report we introduce a set of two-dimension functions that has properties analogous to the set of Haar functions. These two-dimensional functions may be particularly well suited for image coding and pattern recognition.

Haar functions are briefly reviewed in Section 2. In Section 3 we introduce the two-dimensional functions. Convergence properties of the two-dimensional partial sums are discussed in Section 4. Before continuing, we introduce some definitions that will be used throughout the report.

DEFINITION. *Let the interval* $\left((m-1)/2^{n-1}, m/2^{n-1}\right)$ *be noted by* i_n^m, *where* $m = 1, 2, \ldots, 2^{n-1}$. *Thus the interval* $[0, 1]$ *is subdivided into* 2^{n-1} *equal parts labeled* $i_n^1, i_n^2, \ldots, i_n^{2^{n-1}}$.

DEFINITION. *Let the square defined by* $(\ell - 1)/2^{n-1} < x < \ell/2^{n-1}$ *and* $(k-1)/2^{n-1} < y < k/2^{n-1}$ *be denoted by* $\mathcal{S}_n^{k\ell}$, *where* $k, \ell = 1, 2, \ldots, 2^{n-1}$. *The square* $\mathcal{S}_n^{k\ell}$ *is the intersection of* i_n^ℓ *along the* x *axis and* i_n^k *along the* y *axis. The unit square will be denoted by* $\mathcal{S}_1$. *When the boundaries of the square are to be included, one writes* $[\mathcal{S}_n^{k\ell}]$ *instead of* $\mathcal{S}_n^{k\ell}$.

DEFINITION. *If* $x \in [0, 1]$ *for some* k *and if* M *satisfies* $x = k/2^M$, *where* $k = 0, 1, 2, \ldots, 2^M$, *then is called a binary rational, in particular an* Mth*-order dyadic. If* k *is odd,* x *may be called an odd-*Mth*-order dyadic and similarly where* k *is even. Any* Mth*-order dyadic is also an* (M + P)*th-order dyadic for any positive integer* P. *If* x *does not satisfy* $x = k/2^M$ *for any* M, *then it is called a binary irrational. A point* $(x, y) \in \mathcal{S}_1$ *is called a binary rational if either* x *or* y *is a binary rational. It is called a binary irrational if both* x *and* y *are binary irrationals. A binary-rational line segment is a set of colinear points* (x, y) *such that either* x *or* y *is a binary-rational constant and the segment end-points are binary rationals in both* x *and* y.

DEFINITION. *Let* $f(x, y) \in L^2[\diamond_1]$, *then*

$$\bar{f}(\diamond_n^{k\ell}) \equiv 2^{2n-2} \int_{\diamond_n^{k\ell}} f(x, y)\, dxdy \tag{1}$$

is the mean value of f *over the square* $\diamond_n^{k\ell}$.

DEFINITION. *If* $f, g \in L^2[0, 1]$, *then let*

$$\langle f|g \rangle \equiv \int_0^1 f(x)g(x)\, dx. \tag{2}$$

If $f, g \in L^2[\diamond_1]$, *then let*

$$\langle f|g \rangle \equiv \int_{\diamond_1} f(x, y)g(x, y)\, dxdy. \tag{3}$$

2. BRIEF REVIEW OF HAAR FUNCTIONS

2.1 Haar Functions

The Haar orthonormal sequence is defined on the closed interval $[0, 1]$ and is composed of functions labeled by two indices:

$$\{\varphi_n^m\} = \varphi_0; \varphi_1^1; \varphi_2^1, \varphi_2^2; \ldots; \varphi_k^1, \ldots, \varphi_k^{2^{k-1}}; \ldots . \tag{4}$$

The functions are defined as follows.

$$\varphi_0(x) = 1, \qquad \text{for } 0 \leqslant x \leqslant 1$$

$$\varphi_1^1(x) = \begin{cases} 1, & \text{for } 0 \leqslant x < 1/2 \\ -1, & \text{for } 1/2 < x \leqslant 1 \end{cases}$$

$$\varphi_2^1(x) = \begin{cases} \sqrt{2}, & \text{for } 0 \leqslant x < 1/4 \\ -\sqrt{2}, & \text{for } 1/4 < x < 1/2 \\ 0, & \text{for } 1/2 < x \leqslant 1 \end{cases} \tag{5}$$

(cont'd.)

$$\varphi_2^2(x) = \begin{cases} 0, & \text{for } 0 \leqslant x < 1/2 \\ \sqrt{2}, & \text{for } 1/2 < x < 3/4 \\ -\sqrt{2}, & \text{for } 3/4 < x \leqslant 1 \end{cases}$$

$$\vdots \tag{5}$$

$$\varphi_n^m(x) = \begin{cases} 2^{(n-1)/2}, & \text{for } \dfrac{m-1}{2^{n-1}} < x < \dfrac{m-(1/2)}{2^{n-1}} \\ -2^{(n-1)/2}, & \text{for } \dfrac{m-(1/2)}{2^{n-1}} < x < \dfrac{m}{2^{n-1}} \\ 0, & \text{for } 0 < x < \dfrac{m-1}{2^{n-1}} \text{ and } \dfrac{m}{2^{n-1}} < x < 1. \end{cases}$$

At points of discontinuity, the Haar functions are defined to be the average of the limits approached on the two sides of the discontinuity. The first few Haar functions are shown in Fig. 1. We note that $\varphi_n^m(x)$ is nonzero over the interval i_n^m. The Haar functions are a complete orthonormal basis of $L^2[0, 1]$.

2.2 Haar Series Convergence

Any function can be expressed as an infinite series in terms of Haar functions:

$$f(x) = c_0 + \sum_{n=1}^{\infty} \sum_{m=1}^{2^{n-1}} c_n^m \varphi_n^m(x), \tag{6}$$

where

$$c_n^m = \langle f | \varphi_n^m \rangle. \tag{7}$$

For the purposes of this report, convergence is best discussed by means of the partial sums

$$S_N(x) = c_0 + \sum_{n=1}^{N} \sum_{m=1}^{2^{n-1}} c_n^m \varphi_n^m(x), \tag{8}$$

which contain 2^N terms; S_{N+1} contains 2^N more terms than S_N, namely all Haar functions with the subscript $N + 1$. More general partial sums are discussed in Refs. 5 and 7.

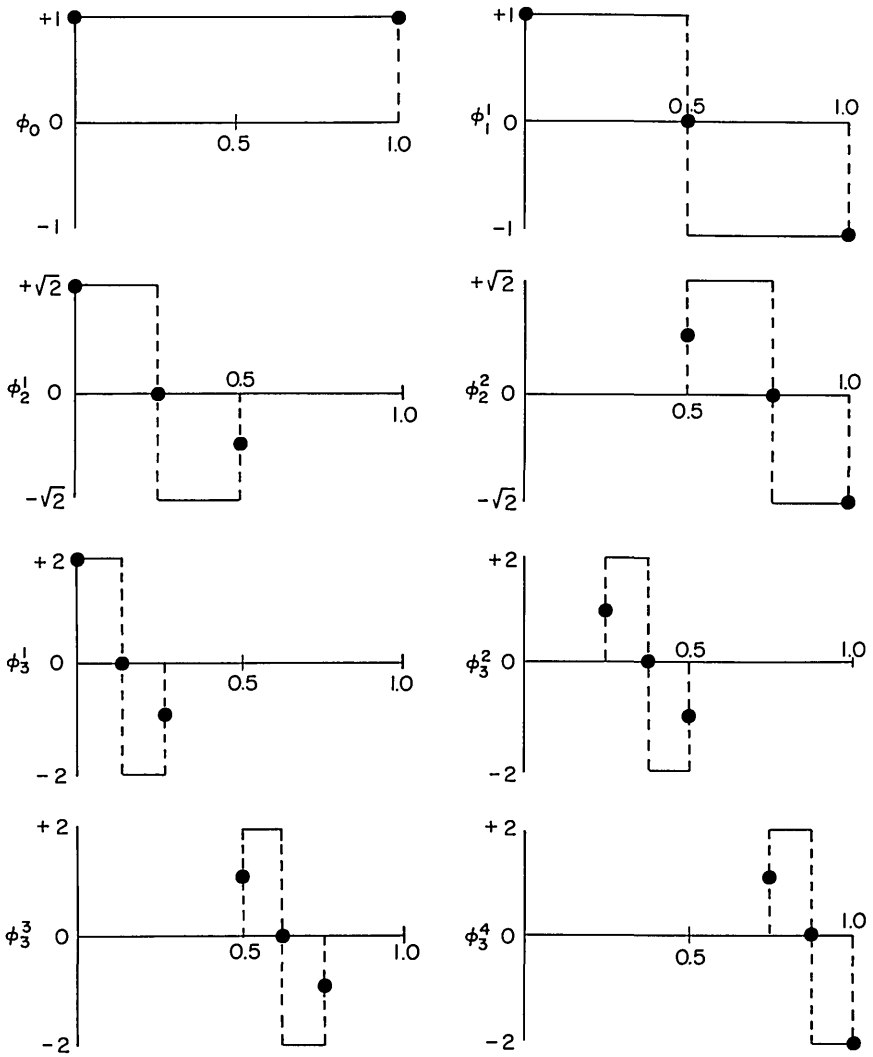

Fig. 1—The first few Haar functions

For continuous functions, the sequence of partial sums S_N is uniformly convergent to the given function. This means that given a required accuracy of approximation ϵ there is a value M such that for all $N \geqslant M$ we have $|S_N(x) - f(x)| < \epsilon$ for all x in $[0, 1]$. For discontinuous functions, $\{S_N\}$ will still converge uniformly to $(1/2)[f(x+) + f(x-)]$, provided all discontinuities are at binary-rational points. This convergence property for discontinuous functions derives from the fact that all Haar-function discontinuities are at binary-rational points.

For functions with discontinuities at binary-irrational points, $S_N(x)$, though no longer uniformly convergent, is still pointwise convergent everywhere except at the binary-irrational discontinuities. This means that given an approximation accuracy ϵ, that must be satisfied at a particular point x_1, there is a value M such that for all $N \geqslant M$ we have $|S_N(x_1) - f(x_1)| < \epsilon$. We cannot, however, guarantee that the required accuracy is obtained simultaneously at all points in $[0, 1]$.

2.3 Mean-Value Properties of Partial Sums and Coefficients

Several aspects of the potential utility of Haar functions derive from an important property of the partial sum. In the expansion of $f(x)$, the Nth Haar partial sum $S_N(x)$ is a step function with 2^N equal-length steps. The value of $S_N(x)$ on each step is simply the mean value of $f(x)$ in the interval covered by the step. The value of $S_N(x)$ at a discontinuity between adjacent steps is halfway between the adjacent steps. Stated differently, $S_N(x)$ is the step function of 2^N steps which is the best approximation to $f(x)$ in the mean-square-error sense. The effect of additional terms is simple and intuitive, unlike the effect of additional terms when the function is expanded as a trigonometric Fourier series (or as a series in terms of other continuous bases of $L^2[0, 1]$). We note that it follows from this mean-value property that if $f(x)$ is constant in the interval covered by any step, then $S_N(x) = f(x)$ exactly on this step.

The coefficients in the Haar series also have a simple relationship with the mean value of $f(x)$ over subintervals of $[0, 1]$. This is easily seen as follows:

$$c_\ell^m = \int_0^1 f(x)\varphi_\ell^m(x)\,dx$$

$$= 2^{(\ell-1)/2}\left[\int_{(2m-2)/2^\ell}^{(2m-1)/2^\ell} f(x)\,dx - \int_{(2m-1)/2^\ell}^{2m/2^\ell} f(x)\,dx\right] \tag{9a}$$

$$= 2^{-(\ell+1)/2}\left[\bar{f}\left(\frac{2m-2}{2^\ell}, \frac{2m-1}{2^\ell}\right) - \bar{f}\left(\frac{2m-1}{2^\ell}, \frac{2m}{2^\ell}\right)\right], \tag{9b}$$

where $\bar{f}(a, b)$ is the mean value of $f(x)$ in the interval (a, b). Thus c_ℓ^m is proportional to the difference in the mean value of $f(x)$ over adjacent subintervals of length $1/2^\ell$. Stated differently, c_ℓ^m is proportional to the difference of two adjacent steps of $S_\ell(x)$, namely the steps on either side of $x = (2m - 1)/2^\ell$.

3. A SET OF TWO DIMENSIONAL, HAAR-LIKE FUNCTIONS

3.1 Definitions

The attractive qualities of Haar functions derive from two related properties. First, in the expansion of $f(x)$, $S_N(x)$ is a step function of 2^N equal-length steps. The value of S_N on each step is the mean value of f in the interval covered by the step. Second, the expansion coefficients are proportional to the difference in the mean value of $f(x)$ over adjacent subintervals of $[0, 1]$. As discussed in Ref 8, these properties are useful in applications such as data coding and pattern recognition. Whatever the application, Haar functions require a one-dimensional data stream. If the data is intrinsically two dimensional, it seems more appropriate to use a two-dimensional set of functions with analogous properties.

We therefore look for a set of orthonormal functions that have the following properties. When a function $f(x, y)$ on the unit square is expanded in terms of these functions, the Nth partial sum $P_N(x, y)$ is a step function of 2^{2N} square steps each of area $1/2^{2N}$. The value of $P_N(x, y)$ on any step is the mean value of $f(x, y)$ over the area covered by the step. The expansion coefficients are proportional to the difference in the mean value of $f(x, y)$ over adjacent subareas of subsquares of the unit square.

In analogy with the double trigonometric Fourier series, we might expect that such a set is provided by all products of the form

$$\Phi_{nk}^{m\ell}(x, y) = \varphi_n^m(y)\varphi_k^\ell(x). \tag{10}$$

The set $\Phi_{nk}^{m\ell}$ is the basis of the two-dimensional transform used in Ref. 9. The set $\Phi_{nk}^{m\ell}$ has all of the desirable properties just enumerated except for that concerning the expansion coefficients. To obtain all of the desired properties, we define three sets of functions: $S_n^{ij}(x, y), H_n^{ij}(x, y)$, and $V_n^{ij}(x, y)$. The first is a subset of the products defined in Eq. (10):

$$S_0(x, y) = \varphi_0(y)\varphi_0(x) \tag{11}$$

$$S_n^{ij}(x, y) = \varphi_n^i(y)\varphi_n^j(x) \tag{12}$$

where $(x, y) \in \mathcal{S}_n^{ij}$ and $n > 0$. The first few $\mathcal{S}_n^{ij}$ are shown in Fig. 2; the symbol S was chosen because the functions have a saddle shape.

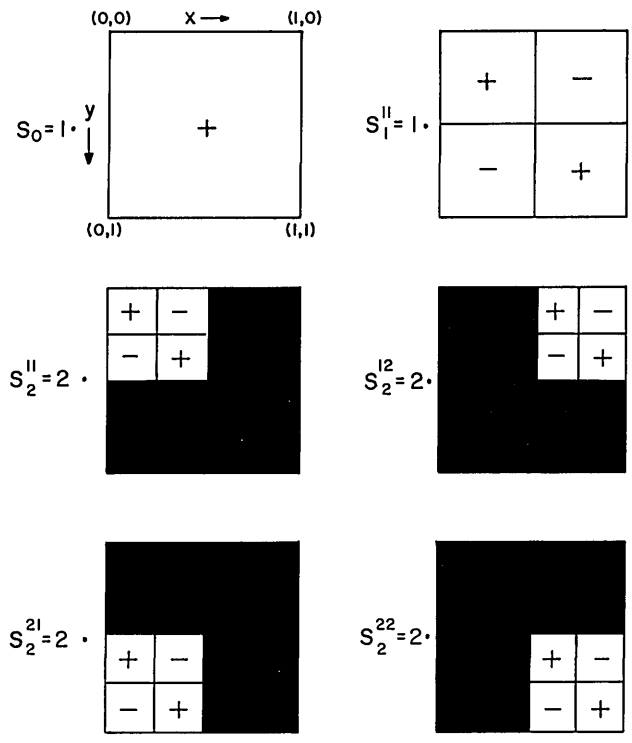

Fig. 2—The first few saddle functions $S_n^{ij}(x, y)$. The shaded areas have the value zero. Other areas have values of plus or minus the constant written outside the square.

Introducing the block functions

$$\alpha_n^m(x) = \begin{cases} 2^{(n-1)/2}, & \text{for } x \in i_n^m \\ 0, & \text{elsewhere} \end{cases} \tag{13}$$

we define

$$H_n^{ij}(x, y) = \varphi_n^i(y)\alpha_n^j(x) \tag{14}$$

$$V_n^{ij}(x, y) = \alpha_n^i(y)\varphi_n^j(x) \tag{15}$$

where $(x, y) \in \diamond_n^{ij}$ and $n > 0$. The first few H_n^{ij} are shown in Fig. 3; the symbol H was chosen because the rectangular nonzero areas are horizontally oriented. The first few V_n^{ij} are shown in Fig. 4; the symbol V was chosen because the rectangular areas are vertically oriented. The form of the definitions was chosen so that the superscripts locate the nonzero square $\diamond_n^{ij}$ according to the familiar row-column matrix notation.

On the four borders of $\diamond_n^{ij}$, all S_n^{ij}, H_n^{ij}, and V_n^{ij} are defined as the average of the limits approached on the two sides of the border. For example, if $\diamond_n^{ij}$ has no borders in common with the unit square $\diamond_1$, then the functions take one of the values $0, \pm 2^{n-2}$. On a common border of $\diamond_n^{ij}$ and $\diamond_1$ they take one of the values $0, \pm 2^{n-1}$.

The functions S_n^{ij}, H_n^{ij}, and V_n^{ij} may be expressed as simple linear combinations of the functions $\Phi_{nk}^{m\ell}$:

$$S_n^{ij} = \Phi_{nn}^{ij}$$

$$H_1^{11} = \Phi_{10}^{10}$$

$$V_1^{11} = \Phi_{01}^{01}$$

$$H_2^{11} = \frac{\sqrt{2}}{2}\left[\Phi_{20}^{10} + \Phi_{21}^{11}\right]$$

$$V_2^{11} = \frac{\sqrt{2}}{2}\left[\Phi_{02}^{01} + \Phi_{12}^{11}\right]$$

etc.

In general, H_n^{uv} is a linear combination of all functions $\Phi_{nk}^{u\ell}$ with $k = 0, 1, 2, \ldots, n - 1$ and $i_n^v \subset i_k$. Similarly, V_n^{uv} is a linear combination of all functions $\Phi_{kn}^{\ell v}$ with $k = 0, 1, 2, \ldots, n = 1$ and $i_n^u \subset i_k^\ell$.

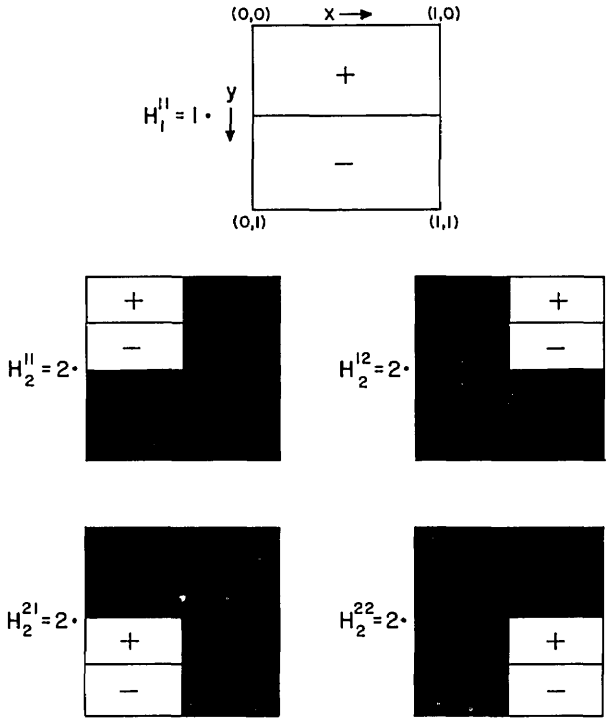

Fig. 3—The first few horizontal functions $H_n^{ij}(x, y)$

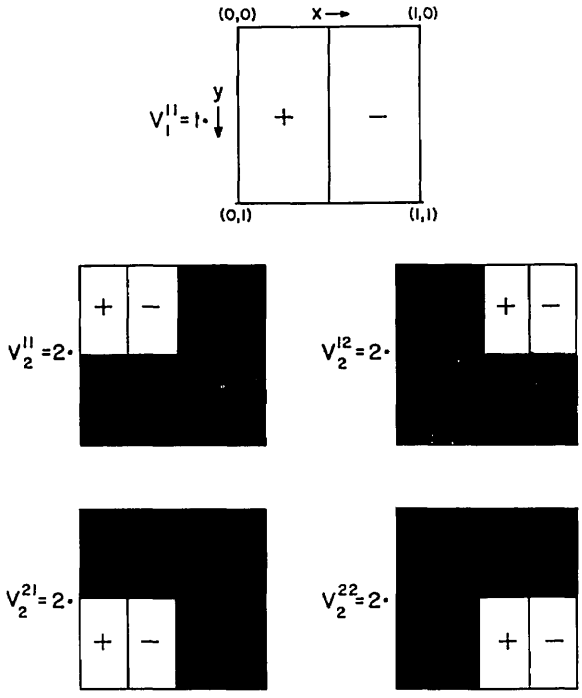

Fig. 4—The first few vertical functions $V_n^{ij}(x, y)$

3.2 Properties

It is convenient to introduce a set of two-dimensional block functions that are zero everywhere except on $\diamond_n^{ij}$:

$$B_n^{ij}(x, y) \equiv 2^{n-1}\alpha_n^i(y)\alpha_n^j(x). \tag{16}$$

We note that

$$\int_{\diamond_1} B_n^{ij}(x, y)\, dxdy = 1 \tag{17}$$

and that the mean value of any function $f(x, y)$ over the square $\diamond_n^{ij}$ may be written

$$\overline{f}(\diamond_n^{ij}) = \int_{\diamond_1} f(x, y) B_n^{ij}(x, y)\, dxdy. \tag{18}$$

We now consider the multiplicative relations that exist between the functions S_n^{ij}, H_n^{ij}, and V_n^{ij}. We ignore function values on the border of $\diamond_n^{ij}$. For like functions with the same subscript, we have

$$S_n^{ij}S_n^{k\ell} = \delta_{ik}\delta_{j\ell}B_n^{ij} \tag{19a}$$

$$H_n^{ij}H_n^{k\ell} = \delta_{ik}\delta_{j\ell}B_n^{ij} \tag{19b}$$

$$V_n^{ij}V_n^{k\ell} = \delta_{ik}\delta_{j\ell}B_n^{ij} \tag{19c}$$

For different functions with the same subscript, we have

$$S_n^{ij}H_n^{k\ell} = 2^{n-1}\delta_{ik}\delta_{j\ell}V_n^{ij} \tag{20a}$$

$$H_n^{ij}V_n^{k\ell} = 2^{n-1}\delta_{ik}\delta_{j\ell}S_n^{ij} \tag{20b}$$

$$V_n^{ij}S_n^{k\ell} = 2^{n-1}\delta_{ik}\delta_{j\ell}H_n^{ij} \tag{20c}$$

To obtain the products for different subscripts, we note that for $n > m$, either $\diamond_n^{ij}$ and $\diamond_m^{k\ell}$ have no area in common, or $\diamond_n^{ij} \subset \diamond_m^{k\ell}$ and is contained entirely within one quadrant of $\diamond_m^{k\ell}$.

For like functions with different subscripts we then have

$$S_n^{ij}S_m^{k\ell} = \begin{cases} \pm 2^{m-1}S_n^{ij}, & \text{if } \diamond_n^{ij} \subset \diamond_m^{k\ell} \\ 0, \text{ otherwise} \end{cases} \tag{21a}$$

$$H_n^{ij} H_m^{k\ell} = \begin{cases} \pm 2^{m-1} H_n^{ij}, \text{ if } \mathcal{S}_n^{ij} \subset \mathcal{S}_m^{k\ell} \\ 0, \text{ otherwise} \end{cases} \tag{21b}$$

$$V_n^{ij} V_m^{k\ell} = \begin{cases} \pm 2^{m-1} V_n^{ij}, \text{ if } \mathcal{S}_n^{ij} \subset \mathcal{S}_m^{k\ell} \\ 0, \text{ otherwise} \end{cases} \tag{21c}$$

where $n > m$ in all cases. The signs on the right sides of Eqs. (21a)-(21c) depend on which quadrant of $\mathcal{S}_m^{k\ell}$ contains $\mathcal{S}_n^{ij}$. For different functions with different subscripts, we have the relations

$$S_n^{ij} H_m^{k\ell} = \begin{cases} \pm 2^{m-1} S_n^{ij}, \text{ if } n > m \text{ and } \mathcal{S}_n^{ij} \subset \mathcal{S}_m^{k\ell} \\ \pm 2^{n-1} H_n^{ij}, \text{ if } n < m \text{ and } \mathcal{S}_m^{k\ell} \subset \mathcal{S}_n^{ij} \\ 0, \text{ otherwise} \end{cases} \tag{22a}$$

$$H_n^{ij} V_m^{k\ell} = \begin{cases} \pm 2^{m-1} H_n^{ij}, \text{ if } n > m \text{ and } \mathcal{S}_n^{ij} \subset \mathcal{S}_m^{k\ell} \\ \pm 2^{n-1} V_m^{k\ell}, \text{ if } n < m \text{ and } \mathcal{S}_m^{k\ell} \subset \mathcal{S}_n^{ij} \\ 0, \text{ otherwise} \end{cases} \tag{22b}$$

$$V_n^{ij} S_m^{k\ell} = \begin{cases} \pm 2^{m-1} V_n^{ij}, \text{ if } n > m \text{ and } \mathcal{S}_n^{ij} \subset \mathcal{S}_m^{k\ell} \\ \pm 2^{n-1} S_m^{k\ell}, \text{ if } n < m \text{ and } \mathcal{S}_m^{k\ell} \subset \mathcal{S}_n^{ij} \\ 0, \text{ otherwise} \end{cases} \tag{22c}$$

It is now easy to see that the set $\{S_n^{ij}, H_n^{ij}, V_n^{ij}\}$ is orthonormal. Using Eq. (17), integration of Eqs. (19a)-(19c) demonstrates normalization for all functions and orthogonality for the subsequences of like functions. Since

$$\int_{\mathcal{S}_1} S_n^{ij}\, dxdy = \int_{\mathcal{S}_1} H_n^{ij}\, dxdy = \int_{\mathcal{S}_1} V_n^{ij}\, dxdy = 0, \tag{23}$$

integration of Eqs. (20a)-(20c), (21a)-(21c), and (22a)-(22c) shows orthogonality for different-type functions.

The orthonormal set $\{S_n^{ij}, H_n^{ij}, V_n^{ij}\}$ happens to be complete in $L^2[\mathcal{S}_1]$. This means that there can exist no element $f(x, y) \in L^2[\mathcal{S}_1]$ which is orthogonal to all $\{S_n^{ij}, H_n^{ij}, V_n^{ij}\}$ and at the same time satisfies $f \neq 0$. We shall prove completeness by assuming

$$\langle f | S_n^{ij} \rangle = \langle f | H_n^{ij} \rangle = \langle f | V_n^{ij} \rangle = 0 \tag{24}$$

and by showing that $f = 0$ necessarily.

We define

$$F(x, y) = \int_{r(x,y)} f(x', y')\, dx'\, dy',$$

where $r(x, y)$ is the area of $\triangle_1$ in which $x' \leqslant x$ and $y' \leqslant y$ are satisfied. Now $F(0, 0) = 0$, and, applying Eq. (24), we have

$$\langle f | S_0 \rangle = F(1,1) = 0 \tag{25}$$

$$\begin{aligned}\langle f | H_1^{11} \rangle &= F\left(1, \frac{1}{2}\right) - \left[F(1,1) - F\left(\frac{1}{2}\right)\right] \\ &= 2F\left(1, \frac{1}{2}\right) = 0\end{aligned} \tag{26}$$

$$\begin{aligned}\langle f | V_1^{11} \rangle &= F\left(\frac{1}{2}, 1\right) - \left[F(1,1) - F\left(\frac{1}{2}, 1\right)\right] \\ &= 2F\left(\frac{1}{2}, 1\right) = 0\end{aligned} \tag{27}$$

$$\begin{aligned}\langle f | S_1^{11} \rangle &= F\left(\frac{1}{2}, \frac{1}{2}\right) - \left[F\left(1, \frac{1}{2}\right) - F\left(\frac{1}{2}, \frac{1}{2}\right)\right] - \left[F\left(\frac{1}{2}, 1\right) - F\left(\frac{1}{2}, \frac{1}{2}\right)\right] \\ &\quad + \left\{F(1,1) - F\left(\frac{1}{2}, 1\right) - \left[F\left(1, \frac{1}{2}\right) - F\left(\frac{1}{2}, \frac{1}{2}\right)\right]\right\} \\ &= 4F\left(\frac{1}{2}, \frac{1}{2}\right) = 0\end{aligned} \tag{28}$$

Thus $F(0, 0) = F(1/2, 1/2) = F(1/2, 1) = F(1, 1/2) = F1, 1) = 0$. By induction

$$F\left(\frac{k}{2^n}, \frac{\ell}{2^m}\right) = 0,$$

where $k = 1, 2, \ldots, 2^n$; $\ell = 1, 2, 3, \ldots, 2^m$; and $n, m = 0, 1, 2, \ldots$. Since $F(x, y)$ is continuous and since the binary rationals are dense in $\triangle_1$, $F = 0$ everywhere. By a theorem of analysis, we conclude that $f(x, y) = 0$ almost everywhere (10). The orthonormal sequence $\{S_n^{ij}, H_n^{ij}, V_n^{ij}\}$ is therefore complete in $L^2[\triangle_1]$.

4. SERIES CONVERGENCE

In expanding any function $f(x, y) \in L^2[\triangle_1]$ in terms of $\{S_n^{ij}, H_n^{ij}, V_n^{ij}\}$, we shall take the following partial sums:

$$P_N(x, y) = a_0 S_0 + \sum_{n=1}^{N} \sum_{i=1}^{2^{n-1}} \sum_{j=1}^{2^{n-1}} \left[a_n^{ij} S_n^{ij} + b_n^{ij} H_n^{ij} + c_n^{ij} V_n^{ij} \right], \tag{29}$$

where $a_0 = \langle f | S_0 \rangle$, $a_n^{ij} = \langle f | S_n^{ij} \rangle$, $b_n^{ij} = \langle f | H_n^{ij} \rangle$, and $c_n^{ij} = \langle f | V_n^{ij} \rangle$. Since $\{ S_n^{ij}, H_n^{ij}, V_n^{ij} \}$ is complete in $L^2[\diamond_1]$, these partial sums approximate f in the mean:

$$\| f - P_N \| \to 0 \quad \text{as} \quad N \to \infty, \tag{30}$$

where

$$\| g \| \equiv \langle g | g \rangle^{1/2}. \tag{31}$$

Furthermore

$$\| f \|^2 = a_0^2 + \sum_{n=1}^{\infty} \sum_{i,j=1}^{2^{n-1}} \left[(a_n^{ij})^2 + (b_n^{ij})^2 + (c_n^{ij})^2 \right] \tag{32}$$

(Parseval's formula).

We note that P_N is exactly equal to the partial sum obtained when expanding $f(x, y)$ in terms of the first 2^{2n} elements of the set $\{ \Phi_{nk}^{m\ell} \}$. The convergence properties that we derive for P_N will therefore also hold for the set $\{ \Phi_{nk}^{m\ell} \}$.

For a certain class of functions, we can prove a much stronger convergence than mean convergence, namely uniform convergence. As in Haar functions, this results from a mean-value property of the partial sums. Here $P_N(x, y)$ has a constant value over each of 2^{2N} square steps. The value of P_N over a particular square $\diamond_{N+1}^{nm}$ is the mean value of $f(x, y)$ over $\diamond_{N+1}^{nm}$. This property is fully stated in the following theorem.

THEOREM 1. *Let* $f(x, y) \in L^2[\diamond_1]$ *be expanded in terms of the set* $\{ S_n^{ij}, H_n^{ij}, V_n^{ij} \}$, *so that the* N*th partial sum is given by*

$$P_N(x, y) = a_0 S_0 + \sum_{n=1}^{N} \sum_{i=1}^{2^{n-1}} \sum_{j=1}^{2^{n-1}} \left[a_n^{ij} S_n^{ij} + b_n^{ij} H_n^{ij} + c_n^{ij} V_n^{ij} \right], \tag{33}$$

where $a_0 = \langle f | S_0 \rangle$, $a_n^{ij} = \langle f | S_n^{ij} \rangle$, $b_n^{ij} = \langle f | H_n^{ij} \rangle$, *and* $c_n^{ij} = \langle f | V_n^{ij} \rangle$. *Then for any point* $(x, y) \in \diamond_1$ *such that neither* x *nor* y *are* N*th-order dyadics, we have*

$$P_N(x, y) = \bar{f}(\diamond_{N+1}^{nm}), \tag{34}$$

where n, m *satisfy* $(x, y) \in \diamond_{N+1}^{nm}$. *For interior* N*th-order dyadics, the value of* $P_N(x, y)$ *is the average of the values of* P_N *on the two or four squares adjacent to the point.*

Theorem 1 is proved in the appendix. Two corollaries follow directly.

COROLLARY 1. *Let* $f(x, y) \in L^2[\diamond_1]$ *be expanded as a series in the form of Eq. (33). Then in a subarea of* $\diamond_1$ *defined by any set of contiguous squares* $\diamond_{N+1}^{ij}$, *the convergence behavior of* $P_N(x, y)$ *and of all higher partial sums depends only on the behavior of* $f(x, y)$ *within this subarea. For large* N *the convergence behavior at any point depends only on* $f(x, y)$ *in the neighborhood of that point.*

COROLLARY 2. *Let* $f(x, y) \in L^2[\diamond_1]$ *be expanded as a series in the form of Eq. (33). If* $f(x, y) = B$, *a constant, within a subarea of* $\diamond_1$ *defined by any set of contiguous squares* $\diamond_{N+1}^{ij}$, *then within this subarea,* P_N *satisfies* $P_N(x, y) = f(x, y) = B$ *exactly, as do all higher partial sums.*

Just as in the theory of Haar series, a variety of convergence properties follow directly from the mean-value property of partial sums; the proofs given are analogous to those given in Ref. 7 for Haar series. We begin by showing that P_N converges uniformly in the expansion of continuous functions.

THEOREM 2. *Let* $f(x, y) \in L^2[\diamond_1]$ *be continuous over* $[\diamond_1]$. *Then the sequence of partial sums* P_N *is uniformly convergent to* $f(x, y)$.

Proof. For points that are not interior Nth-order dyadics, we have for $(x, y) \in \diamond_{N+1}^{ij}$,

$$P_N(x, y) = \bar{f}(\diamond_{N+1}^{ij}) = f(\mathbf{u}_{N+1}^{ij}), \tag{35}$$

where $\mathbf{u}_{N+1}^{ij}$ is some point in $\diamond_{N+1}^{ij}$. The last step in Eq. (35) is by the mean-value theorem for integrals.

Since f is continuous over $[\diamond_1]$, it is uniformly continuous there. Thus for every $\epsilon > 0$ there exists a $\delta > 0$ such that

$$|f(\mathbf{r}_1) - f(\mathbf{r}_2)| < \epsilon \tag{36}$$

whenever

$$|\mathbf{r}_1 - \mathbf{r}_2| < \delta. \tag{37}$$

If $\mathbf{r}$ and $\mathbf{u}_{N+1}^{ij}$ are both in $\diamond_{N+1}^{ij}$,

$$|\mathbf{u}_{N+1}^{ij} - \mathbf{r}| \leqslant \frac{\sqrt{2}}{2^N}. \tag{38}$$

Therefore, if we choose a fixed integer M so that $\sqrt{2}/2^M < \delta$, we will have

$$|P_M(\mathbf{r}) - f(\mathbf{r})| = |f(\mathbf{u}_{M+1}^{ij}) - f(\mathbf{r}) < \epsilon \tag{39}$$

for all $\mathbf{r} \in \diamond_{M+1}^{ij}$. This inequality also holds on the boundary of the unit square.

As for interior Mth-order dyadics, let $x_k = k/2^M$, and let y be a non-Mth-order dyadic. Then the point (x_k, y) must be on the common boundary of two squares $\diamond_{M+1}^{ik}$ and $\diamond_{M+1}^{i\,k+1}$. Defining, for positive x,

$$f(x+, y) = \lim_{\alpha \to 0} f(x + \alpha, y) \tag{40}$$

and

$$f(x-, y) = \lim_{\alpha \to 0} f(x - \alpha, y), \tag{41}$$

we have

$$|P_M(\diamond_{M+1}^{ik}) - f(x_k-, y)| < \epsilon \tag{42}$$

and

$$|P_M(\diamond_{M+1}^{i\,k+1}) - f(x_k+, y)| < \epsilon. \tag{43}$$

Here we have used Eq. (39) and have written $P_N(\diamond_n^{ij})$ for $P_N(x, y)$ when $(x, y) \in \diamond_n^{ij}$. Combining Eqs. (42) and (43),

$$|P_M(\diamond_{M+1}^{ik}) + P_M(\diamond_{M+1}^{i\,k+1}) - f(x_k-, y) - f(x_k+, y)| < 2\epsilon, \tag{44}$$

and since f is continuous,

$$\left| \frac{1}{2}\left[P_M(\diamond_{M+1}^{ik}) + P_M(\diamond_{M+1}^{i\,k+1})\right] - f(x_k, y) \right| = |P_M(x_k, y) - f(x_k, y)| < \epsilon, \tag{45}$$

where we have used Theorem 1. Similar conditions hold for points that are Mth-order dyadics in y only and for points that are Mth-order dyadics in both x and y. Details are omitted.

We have therefore shown that for every $\epsilon > 0$, there exists an integer M such that

$$|P_M(x, y) - f(x, y)| < \epsilon \tag{46}$$

for all $(x, y) \in [\diamond_1]$. That this hold for $N > M$ can be seen by noting that for $\mathbf{r} \in \diamond_{N+1}^{ij}$, we also have $|\mathbf{u}_{N+1}^{ij} - \mathbf{r}| < \sqrt{2}/2^M < \delta$. The steps leading to Eq. (46) therefore follow for $N > M$. This completes the proof of Theorem 2.

If $f(x, y)$ is differentiable everywhere, we have the following convergence estimate.

THEOREM 3. *Let* $f(x, y) \in L^2[\diamond_1]$ *be differentiable everywhere in* $[\diamond_1]$. *and let* $\vec{\nabla} f$ *be bounded on* $[s_1]$. *Then*

$$|P_N(x, y) - f(x, y)| \leqslant \frac{\Xi(\diamond_{M+1}^{ij})}{2^N} \tag{47}$$

for all (x, y) $\in \mathcal{A}_{N+1}^{ij}$. *Here*

$$\Xi(\mathcal{A}) \equiv \sqrt{2} \sup_{(x,y)\in\mathcal{A}} |\nabla f(x, y)|, \tag{48}$$

where $\mathcal{A}$ *is a subarea of* $\mathcal{A}_1$.

Proof. By the mean-value theorem for integrals,

$$|P_N(\mathbf{r}) - f(\mathbf{r})| = |f(\mathbf{u}_{N+1}^{ij}) - f(\mathbf{r})| \tag{49}$$

for $\mathbf{r} \in \mathcal{A}_{N+1}^{ij}$, where $\mathbf{u}_{N+1}^{ij}$ is a particular point in $\mathcal{A}_{N+1}^{ij}$. The mean-value theorem states that

$$|f(\mathbf{r}_1) - f(\mathbf{r}_2)| = |\nabla f(\mathbf{r}_0) \cdot (\mathbf{r}_1 - \mathbf{r}_2)|, \tag{50}$$

where $\mathbf{r}_0$ is on the line segment joining $\mathbf{r}_1$ and $\mathbf{r}_2$. Applying this to Eq. (49) and noting that when $\mathbf{r}_1$ and $\mathbf{r}_2$ are both in $\mathcal{A}_{N+1}^{ij}$

$$|\nabla f(\mathbf{r}_0) \cdot (\mathbf{r}_1 - \mathbf{r}_2)| \leqslant \frac{\sqrt{2}|\nabla f(\mathbf{r}_0)|}{2^N} \leqslant \frac{\Xi(\mathcal{A}_{N+1}^{ij})}{2^N}, \tag{51}$$

we have

$$|P_N(x, y) - f(x, y)| \leqslant \frac{\Xi(\mathcal{A}_{N+1}^{ij})}{2^N}, \tag{52}$$

which proves Theorem 3.

COROLLARY 3. *If* f(x, y) $\in L^2[\mathcal{A}_1]$ *is continuous and differentiable everywhere in* $[\mathcal{A}_1]$,

$$|P_N(x, y) - f(x, y)| \leqslant \frac{\Xi(\mathcal{A}_1)}{2^N} \tag{53}$$

for all (x, y) $\in [\mathcal{A}_1]$.

We state without proof the following theorem.

THEOREM 4. *Let* f(x, y) $\in L^2[\mathcal{A}_1]$ *be a plane over the square* $\mathcal{A}_{N+1}^{ij}$. *Then for* (x, y) $\in \mathcal{A}_{N+1}^{ij}$,

$$|P_N(x, y) - f(x, y)| \leqslant \frac{\sqrt{2}|\nabla f(x, y)|}{2^{N+1}}. \tag{54}$$

This theorem is useful when expanding any well-behaved function, since Eq. (54) then will be approximately true for large N, except at points where $\nabla f = 0$.

We now consider the convergence of P_N in the case of discontinuous $f(x, y)$. Suppose there is a single discontinuity which occurs along the line $x = k/2^n$, beginning at $y_1 = \ell/2^m$ and ending at $y_2 = u/2^v$. Let N be the largest of n, m, and v. Then P_N and all higher partial sums will be discontinuous exactly on this binary-rational line segment. For $(x, y) \in [\diamond_1]$, including the discontinuity,

$$P_N(x, y) \to \frac{1}{2}[f(x-, y) + f(x+, y)] \tag{55}$$

uniformly on $[\diamond_1]$, where we have used Theorems 1 and 2. A similar condition holds for discontinuities on binary-rational line segments parallel to the x axis. We therefore have the following theorem.

THEOREM 5. *Let* $f(x, y) \in L^2[\diamond_1]$ *have a finite number of discontinuities of the first kind along binary-rational line segments in an area* $a \in \diamond_1$. *Then the sequence of partial sums* $\{P_N\}$ *is uniformly convergent*,

$$P_N(x, y) \to \frac{1}{4}[f(x-, y-) + f(x-, y+) + f(x+, y-) + f(x+, y+)], \tag{56}$$

on a.

The four terms on the right of Eq. (56) are required if the theorem is to hold at points that are binary rationals in both x and y.

It is easy to see that Theorem 5 does not hold if $f(x, y)$ is discontinuous along binary-irrational line segments. For example, suppose f is discontinuous along $x = x_1$, a binary irrational. Then for no value of N will $P_N(x, y)$ be discontinuous everywhere on this line. The line of discontinuity cuts across all squares $\diamond_{N+1}^{nm}$, where m is given by $x_1 \in i_{N+1}^m$. As N increases, x_1 falls at relatively different positions within i_{N+1}^m, resulting in different values for $P_N(x_1, y) = \bar{f}(\diamond_{N+1}^{nm})$. The sequence does not approach a limit, and P_N does not converge at (x_1, y). It is, however, pointwise convergent elsewhere. For any point $(x_2, y) \neq (x_1, y)$, we can increase N until $(x_2, y) \in \diamond_{N+1}^{ij}$ while $(x_1, y) \notin \diamond_{N+1}^{ij}$. Then, by Theorem 2, $P_N(x_2, y) \to f(x_2, y)$.

THEOREM 6. *Let* $f(x_1, y) \in L^2[\diamond_1]$ *have a finite number of discontinuities of the first kind along binary-irrational line segments in the subarea* $a \in \diamond_1$. *Then the sequence of partial sums* $\{P_N(x, y)\}$ *is pointwise convergent to* $f(x, y)$ *everywhere in* a *except along the discontinuities, where there is no convergence.*

5. CONCLUSION

The orthonormal set $\{S_n^{ij}(x, y), H_n^{ij}(x, y), V_n^{ij}(x, y)\}$ is complete in $L^2[\diamond_1]$. Series expansions in terms of this set have convergence properties analogous to those of one-dimensional Haar series. The Nth partial sum $P_N(x, y)$ is a step function of 2^{2N} square steps each of area $1/2^{2N}$. The value of $P_N(x, y)$ on any step is the mean value of $f(x, y)$ over the area covered by the step. If $f(x, y)$ is continuous or has a finite number of discontinuities along binary-rational line segments, then $P_N(x, y)$ converges uniformly. If $f(x, y)$ has a finite number of discontinuities along binary-irrational line segments, then

$P_N(x, y)$ converges pointwise except along the discontinuities. The accuracy of the estimate $P_N(x, y)$ in the neighborhood of some point is proportional to the gradient at that point. Also, the accuracy of the estimate P_{N+1} is bounded by a value which is half that which bounds the estimate P_N.

6. ACKNOWLEDGMENT

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APPENDIX
PROOF OF THEOREM 1

The proof is by induction. We first consider points that are binary irrational in both x and y. For $N = 0$

$$P_0(x, y) = a_0 = \langle f | S_0 \rangle = \int_{\mathcal{S}_1} f(x, y)\, dx dy = \bar{f}(\mathcal{S}_1). \tag{A1}$$

We now assume

$$P_R(x, y) = \bar{f}(\mathcal{S}_{R+1}^{nm}), \tag{A2}$$

where $(x, y) \in \mathcal{S}_{R+1}^{nm}$. For $N = R + 1$

$$P_N(x, y) = P_R(x, y) + \sum_{i=1}^{2^R} \sum_{j=1}^{2^R} \left[a_N^{ij} S_N^{ij} + b_N^{ij} H_N^{ij} + c_N^{ij} V_N^{ij} \right]. \tag{A3}$$

For a particular point $(x, y) \in \mathcal{S}_N^{nm}$, only one term survives each of the three summations in Eq. (A3). We then have

$$P_N(x, y) = \bar{f}(\mathcal{S}_N^{nm}) + a_N^{nm} S_N^{nm} + b_N^{nm} H_N^{nm} + c_N^{nm} V_N^{nm}, \tag{A4}$$

where we have also used Eq. (A2). We now introduce the following representation:

$$\begin{pmatrix} S_N^{nm}(x, y) \\ H_N^{nm}(x, y) \\ V_N^{nm}(x, y) \end{pmatrix} = 2^{N-1} \operatorname{sgn} \begin{pmatrix} S_N^{nm}(x, y) \\ H_N^{nm}(x, y) \\ V_N^{nm}(x, y) \end{pmatrix} \tag{A5}$$

for $(x, y) \in \mathcal{S}_N^{nm}$. The function sgn operates on each element of the column vector, which therefore takes on one of the four values

$$\begin{pmatrix} - \\ + \\ - \end{pmatrix}, \begin{pmatrix} + \\ + \\ + \end{pmatrix}, \begin{pmatrix} - \\ - \\ + \end{pmatrix}, \begin{pmatrix} + \\ - \\ - \end{pmatrix},$$

depending on the quadrant of $\triangle_N^{nm}$ that $(x, y) \in \triangle_N^{nm}$ is in (Fig. A1). Thus in the first quadrant,

$$P_N(x, y) = \bar{f}(\triangle_N^{nm}) + 2^{N-1}\left[-a_N^{nm} + b_N^{nm} - c_N^{nm}\right]. \tag{A6}$$

Fig. A1—Values of sgn $\begin{pmatrix} S_N^{nm} \\ H_N^{nm} \\ V_N^{nm} \end{pmatrix}$ in the four quadrants of $\triangle_N^{nm}$.

$\begin{pmatrix}+\\+\\+\end{pmatrix}$	$\begin{pmatrix}-\\+\\-\end{pmatrix}$
$\begin{pmatrix}-\\-\\+\end{pmatrix}$	$\begin{pmatrix}+\\-\\-\end{pmatrix}$

The coefficients a_N^{nm}, b_N^{nm}, and c_N^{nm} can be written in terms of the average value of $f(x, y)$ over the quadrants of $\triangle_N^{nm}$. Using Eqs. (12), (14), and (15) (see also Figs. 2 - 4), we obtain

$$a_N^{nm} = \frac{1}{2^{N+1}}\left[\bar{f}(\triangle_{N+1}^{2n-1\ 2m-1}) - \bar{f}(\triangle_{N+1}^{2n-1\ 2m}) - \bar{f}(\triangle_{N+1}^{2n\ 2m-1}) + \bar{f}(\triangle_{N+1}^{2n\ 2m})\right] \tag{A7a}$$

$$b_N^{nm} = \frac{1}{2^{N+1}}\left[\bar{f}(\triangle_{N+1}^{2n-1\ 2m-1}) + \bar{f}(\triangle_{N+1}^{2n-1\ 2m}) - \bar{f}(\triangle_{N+1}^{2n\ 2m-1}) - \bar{f}(\triangle_{N+1}^{2n\ 2m})\right] \tag{A7b}$$

$$c_N^{nm} = \frac{1}{2^{N+1}}\left[\bar{f}(\triangle_{N+1}^{2n-1\ 2m-1}) - \bar{f}(\triangle_{N+1}^{2n-1\ 2m}) + \bar{f}(\triangle_{N+1}^{2n\ 2m-1}) - \bar{f}(\triangle_{N+1}^{2n\ 2m})\right] \tag{A7c}$$

Substituting Eqs. (A7a) - (A7c) into Eq. (A6) and expressing $\bar{f}(\triangle_N^{nm})$ as

$$\bar{f}(\triangle_N^{nm}) = \frac{1}{4}\left[\bar{f}(\triangle_{N+1}^{2n-1\ 2m-1}) + \bar{f}(\triangle_{N+1}^{2n-1\ 2m}) + \bar{f}(\triangle_{N+1}^{2n\ 2m-1}) + \bar{f}(\triangle_{N+1}^{2n\ 2m})\right], \tag{A8}$$

we obtain, for $(x, y) \in \triangle_{N+1}^{2n-1\ 2m}$,

$$P_N(x, y) = \bar{f}(\triangle_{N+1}^{2n-1\ 2m}). \tag{A9}$$

Similar results hold for (x, y) in each of the other three quadrants. Theorem 1 is therefore proved for points in $\triangle_1$ that are not Nth-order dyadics.

All Nth-order dyadics are points of discontinuity in $P_N(x, y)$. Of the 2^{2N} lines of discontinuity in P_N, the odd lines defined by $x = k/2^N$ and $y = \ell/2^N$, for $k, \ell = 1, 3, \ldots, 2^N - 1$, are appearing for the first time. By this we mean that $P_R(x, y) = P_{N-1}(x, y)$ is not discontinuous along these lines.

We consider the square $\triangle_{R+1}^{nm} = \triangle_N^{nm}$. The lines $y = (2n - 1)/2^N$ and $x = (2m - 1)/2^N$ are not lines of discontinuity in $P_R(x, y)$, whereas they are so in $P_N(x, y)$ (see Fig. A2). Values of P_N on these lines may be calculated using Eqs. (A4) and (A5). Values of the function sgn for points on these lines are given in Fig. A3. For a point located on the common boundary of quadrants one and four, where $y = (2n - 1)/2^N$ and $x \in ((2m - 1)/2^N, 2m/2^N)$

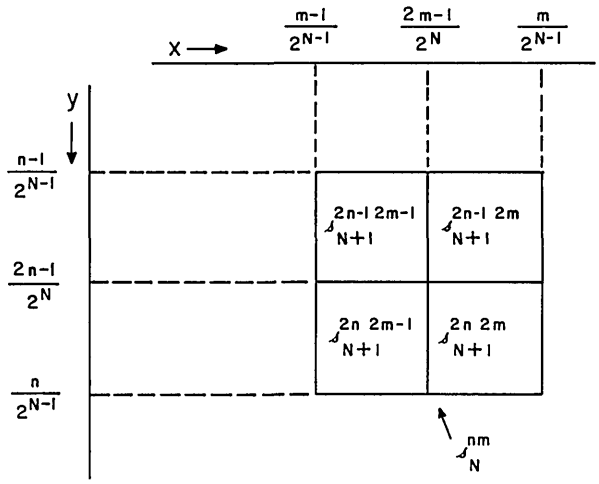

Fig. A2—The square $\triangle_N^{nm}$ and its four quadrants

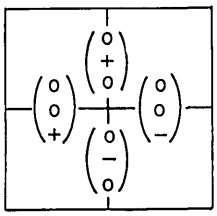

Fig. A3—Values of sgn $\begin{pmatrix} S_N^{nm} \\ H_N^{nm} \\ V_N^{nm} \end{pmatrix}$ on the interquadrant boundaries of $\triangle_N^{nm}$

$$\begin{aligned} P_N(x, y) &= \bar{f}(\triangle_N^{nm}) - 2^{N-1}c_N^{nm} \\ &= \frac{1}{2}\left[\bar{f}(\triangle_{N+1}^{2n-1\ 2m}) + \bar{f}(\triangle_{N+1}^{2n\ 2m})\right]. \\ &= \frac{1}{2}\left[P_N(\triangle_{N+1}^{2n-1\ 2m}) + P_N(\triangle_{N+1}^{2n\ 2m})\right]. \end{aligned} \tag{A10}$$

In the second step we have used Eqs. (A7c) and (A8). Equation (A10), which holds on the boundary between quadrants one and four of $\triangle_N^{nm}$, states that $P_N(x, y)$ is the average of the values of P_N on the two squares $\triangle_{N+1}^{ij}$ adjacent to (x, y). Similar results hold for points on the other three interquadrant boundaries. At the center of $\triangle_N^{nm}$, we have

$$P_N(x, y) = P_R(x, y) = \bar{f}(\triangle_N^{nm}), \tag{A11}$$

since S_N^{nm}, H_N^{nm}, and V_N^{nm} are all zero there. From Eq. (A8) we see that $\bar{f}(\triangle_N^{nm})$ is equivalent to the average of the values of P_N on each of the four squares $\triangle_{N+1}^{ij}$ adjacent to the center point of $\triangle_N^{nm}$.

So far we have shown that Theorem 1 holds for binary-rational points at which $P_N(x, y)$ is discontinuous but at which $P_{N-1}(x, y)$ is not. That is, it holds at every binary-rational point whenever that point first becomes a discontinuity of $P_K(x, y)$. To continue

the proof by induction, we now assume that Theorem 1 holds at binary-rational points that are discontinuities of an arbitrary partial sum $P_L(x, y)$. We then show that it also holds for $P_{L+1}(x, y)$.

Consider the two adjacent squares $\diamond_{L+1}^{nm}$ and $\diamond_{L+1}^{n\ m+1}$ (Fig. A4). Along their common boundary $x = m/2^L$, we have by assumption

$$P_L(x, y) = \frac{1}{2}\left[\bar{f}(\diamond_{L+1}^{nm}) + \bar{f}(\diamond_{L+1}^{n\ m+1})\right], \tag{A12}$$

except at the two end points. For $M = L + 1$,

$$P_M(x, y) = P_L(x, y) + \sum_{i=1}^{2^L}\sum_{i=1}^{2^L}\left[a_M^{ij}S_M^{ij} + b_M^{ij}H_M^{ij} + c_M^{ij}V_M^{ij}\right]. \tag{A13}$$

When $x = m/2^L$, two terms survive each of the three summations in Eq. (A13), specifically

$$\begin{aligned} P_M\left(\frac{m}{2^L}, y\right) &= P_L\left(\frac{m}{2^L}, y\right) + a_M^{nm}S_M^{nm} + b_M^{nm}H_M^{nm} + c_M^{nm}V_M^{nm} \\ &\quad + a_M^{n\ m+1}S_M^{n\ m+1} + b_M^{n\ m+1}H_M^{n\ m+1} + c_M^{n\ m+1}V_M^{n\ m+1}, \end{aligned} \tag{A14}$$

where $y \in ((n - 1)/2^L, n/2^L)$.

At a point on the boundary of a square $\diamond_N^{ij}$,

$$\begin{pmatrix} S_N^{ij} \\ H_N^{ij} \\ V_N^{ij} \end{pmatrix} = 2^{N-2}\,\mathrm{sgn}\begin{pmatrix} S_N^{ij} \\ H_N^{ij} \\ V_N^{ij} \end{pmatrix}. \tag{A15}$$

Except at the midpoint of the boundary, the function sgn has the same values on the boundary as those on the quadrant of $\diamond_N^{ij}$ whose external boundary contains the point in question (Fig. A1). At the midpoint the value of sgn is the same as that given in Fig. A3.

Restricting our attention to the region $y \in \left((2n - 2)/2^M, (2n - 1)/2^M\right)$, Eq. (A14) now becomes

$$\begin{aligned} P_M\left(\frac{m}{2^L}, y\right) &= P_L\left(\frac{m}{2^L}, y\right) + 2^{N-2}\left[-a_M^{nm} + b_M^{nm} - c_M^{nm}\right] \\ &\quad + 2^{N-2}\left[a_M^{n\ m+1} + b_M^{n\ m+1} + c_M^{n\ m+1}\right]. \end{aligned} \tag{A16}$$

Using Eqs. (A7), (A8), and (A12), this becomes

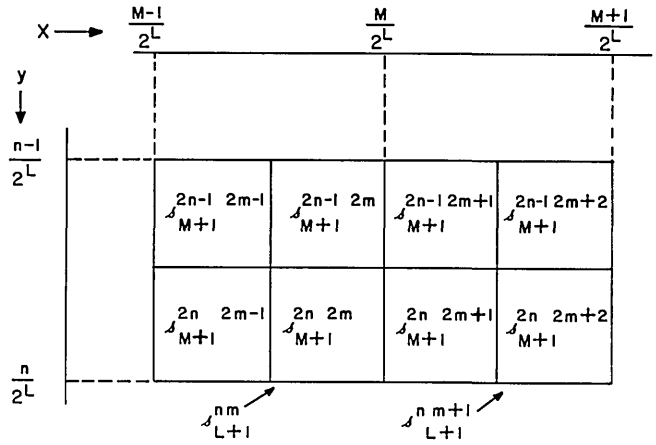

Fig. A4—The two adjacent squares $\triangle^{nm}_{L+1}$ and $\triangle^{nm}_{L+1}$

$$P_M\left(\frac{m}{2^L}, y\right) = \frac{1}{2}\left[\bar{f}(\triangle^{2n-1\ 2m}_{M+1}) + \bar{f}(\triangle^{2n-1\ 2m+1}_{M+1})\right]. \tag{A17}$$

Similarly, for $y \in ((2n-1)/2^M, 2n/2^M)$,

$$\begin{aligned} P_M\left(\frac{m}{2^L}, y\right) &= P_L\left(\frac{m}{2^L}, y\right) + 2^{N-2}\left[a_M^{nm} - b_M^{nm} - c_M^{nm}\right] \\ &\quad + 2^{N-2}\left[-a_M^{n\ m+1} - b_M^{n\ m+1} + c_M^{n\ m+1}\right] \\ &= \frac{1}{2}\left[\bar{f}(\triangle^{2n\ 2m}_{M+1}) + \bar{f}(\triangle^{2n\ 2m+1}_{M+1})\right]. \end{aligned} \tag{A18}$$

At the midpoint $\left((2n-1)/2^M, (2m-1)/2^M\right)$,

$$\begin{aligned} P_M &= P_L + 2^{n-2}\left[-c_M^{nm} + c_M^{n\ m+1}\right] \\ &= \frac{1}{4}\left[\bar{f}(\triangle^{2n-1\ 2m}_{M+1}) + \bar{f}(\triangle^{2n-1\ 2m+1}_{M+1}) + \bar{f}(\triangle^{2n\ 2m}_{M+1} + \bar{f}(\triangle^{2n\ 2m+1}_{M+1})\right]. \end{aligned} \tag{A19}$$

Eqs. (A16), (A18), and (A19) show that Theorem 1 holds for P_{L+1} on the boundary of $\triangle^{nm}_{L+1}$ that we consider. Similar results are easily obtained for points on the other boundaries, as well as for the four corners. This completes the proof of Theorem 1.

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13. ABSTRACT

A set of two-dimensional functions is defined. Expansions of two-dimensional functions $f(x, y)$ in terms of this set have convergence properties analogous to those of one-dimensional Haar series. The Nth partial sum $P_N(x, y)$ is a step function of 2^{2N} square steps each of area $1/2^{2N}$. The value of $P_N(x, y)$ on any step is the mean value of $f(x, y)$ over the area covered by the step. If $f(x, y)$ is continuous or has a finite number of discontinuities along binary-rational line segments, then $P_N(x, y)$ converges uniformly. If $f(x, y)$ has a finite number of discontinuities along binary-irrational line segments, then $P_N(x, y)$ converges pointwise, except along the discontinuities. Accuracy of the estimate P_N and the convergence rate are analogous to those for one-dimensional Haar series.

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