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ABSTRACT

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ARRIVAL-TIME ESTIMATION BY ADAPTIVE THRESHOLDING

INTRODUCTION

One method of determining the arrival time of a signal is to measure the time at which the signal crosses a certain threshold level. Figure 1 shows a pulse with arrival time defined to be the time at which the pulse exceeds level A . Suppose a series of pulses are received. In a practical system the amplitude E will usually vary from pulse to pulse. If the rise time does not change, the arrival time relative to the leading edge will vary from pulse to pulse, even if no noise is present. To remedy the situation, adaptive thresholding can be used. Adaptive thresholding is defined to be arrival-time estimation by thresholding in which the level A is always a fixed fraction of E . Typically, A/E is set equal to $1/2$.

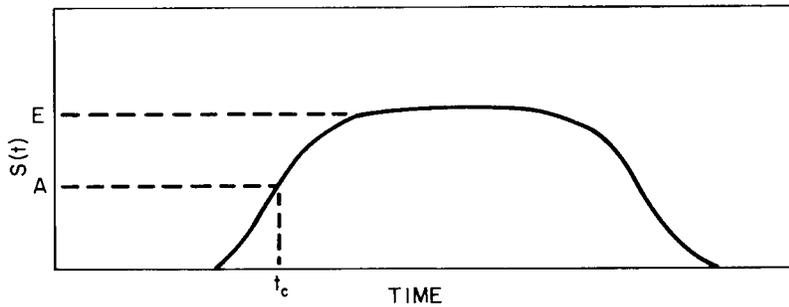


Fig. 1—Received pulse with a threshold crossing

A practical adaptive-thresholding system is shown in Fig. 2. The received signal is split into two branches. In the lower branch the signal is first delayed by an amount exceeding its rise time, which is assumed to be known. Then the delayed signal passes through an amplifier with a gain of G . The latter is a negative quantity such that $|G|^{-1}$ is equal to the desired value of A/E . The amplifier output is added to the original received signal. The adder output crosses the zero level at a time equal to the time at which the level A is crossed plus the constant delay. In the important special case of a half-amplitude arrival time, $A/E = 1/2$ and $G = -2$. A detailed analysis of this adaptive thresholder and an alternative one can be found in the literature (1).

Arrival-time estimation by adaptive thresholding is not the optimum estimation procedure (2-6). However, it will be shown that adaptive thresholding provides an asymptotically efficient estimate at least in some important cases. Furthermore, the simplicity of implementing adaptive-thresholding systems gives the method practical importance.

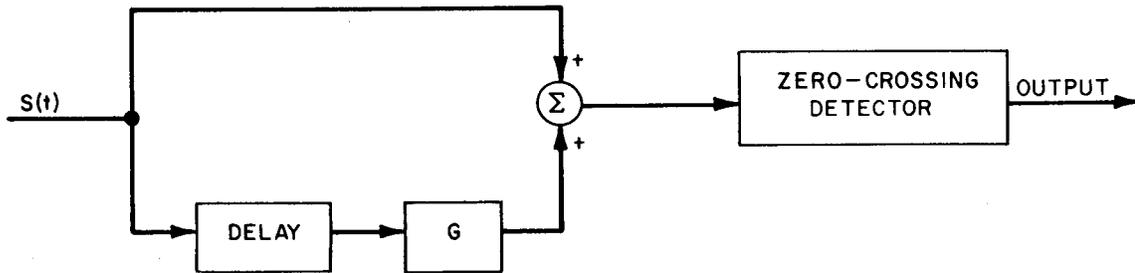


Fig. 2—Adaptive thresholder

An approximate treatment of adaptive thresholding in the presence of Gaussian noise has previously been done (1). In this report the general case of differentiable Markov processes is studied.

GENERAL FORMULATION

Figure 3 shows an exaggerated example of a multiple threshold crossing, which may occur when noise corrupts the received signal. The combined signal and noise is denoted by $r(t)$. When multiple crossings occur close together, there is some ambiguity as to which crossing represents the actual pulse position, i.e., which crossing is closest to that crossing which exists in the absence of noise. A somewhat arbitrary but simple strategy is for the adaptive thresholder to detect only the first level crossing for each pulse it receives. Thus we seek to derive the probability density of the first threshold crossing time of a received signal.

With respect a received pulse, we assume that the adaptive-thresholding system is capable of rejecting any level crossing which occurs too prematurely, that is, before a certain starting time. Under this assumption, we may, without loss of generality, focus attention on a single received pulse. We seek to determine the time of the first threshold crossing to occur after a starting time defined to be $t=0$. A practical system to accomplish this rejection is shown in Fig. 4. The combined signal and noise, $r(t)$, is applied to a fixed

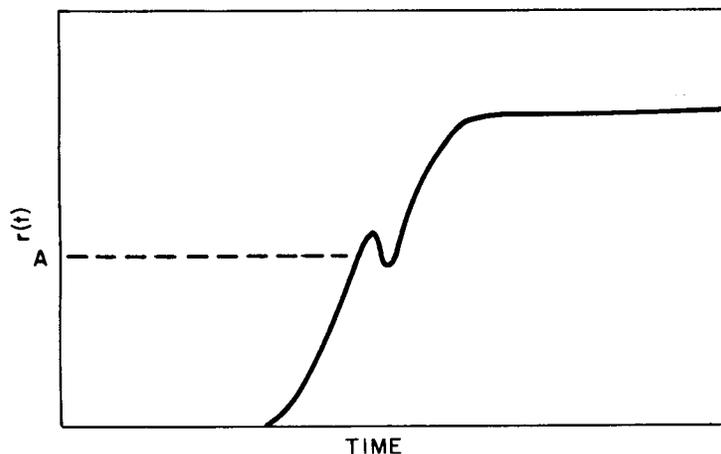


Fig. 3—Exaggerated example of a multiple threshold crossing

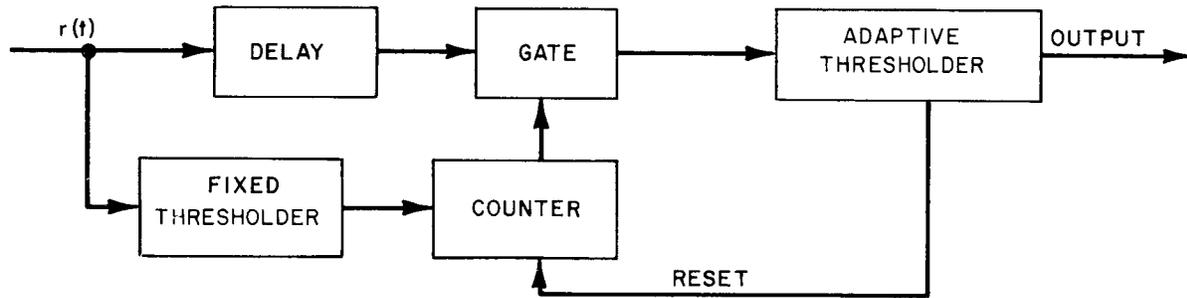


Fig. 4—System for the elimination of false alarms

threshold, the level of which is set to a fixed fraction of the minimum expected amplitude of the received pulses. The output of the fixed threshold initiates a counter. This counter continues to count as long as the fixed threshold level is exceeded by $r(t)$. If the count reaches a certain value, the gate is enabled, and $r(t)$ is allowed to pass to the adaptive threshold. A reasonable critical value of the count is one-half of the minimum signal duration. If the minimum signal duration is large enough such that there is negligible probability of a noise pulse exceeding this duration, the system of Fig. 4 will reject spurious pulses occurring outside the immediate vicinity of the signal edge. A reset pulse prepares the counter for the next received signal. In keeping with our strategy to detect the first level crossing, the delay in the upper branch should exceed somewhat the critical value of the count.

With t_c a random variable equal to the first threshold crossing time of a received pulse, $f_c(t)$ is defined to be its probability density function. To derive an expression for the probability density, we relate it to another function more readily obtained. Referring to Fig. 5, let $\beta(t)\epsilon$ represent the probability that a first threshold crossing of $r(t)$ occurs in the interval between t and $t + \epsilon$, assuming no crossing has occurred before time t . From the above definitions, it follows (7) that

$$\beta(t) = \frac{f_c(t)}{\int_t^{\infty} f_c(x) dx} \quad (1)$$

Since $f_c(t)$ is a probability density function and we have assumed $f_c(t) = 0$ for $t \leq 0$, it is required that

$$\int_0^{\infty} f_c(x) dx = 1. \quad (2)$$

Using Eqs. (2) as a boundary condition, Eqs. (1) may be inverted to yield

$$f_c(t) = \beta(t) \exp \left[- \int_0^t \beta(x) dx \right]. \quad (3)$$

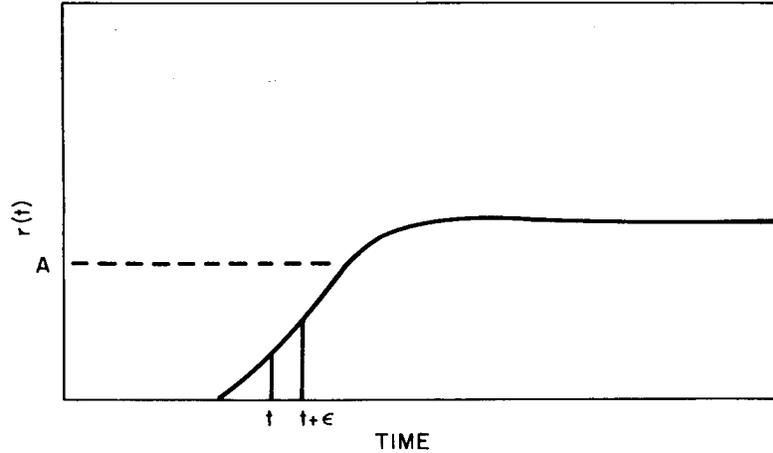


Fig. 5—Small time interval of a received pulse

Once an expression for $\beta(t)$ is derived, Eqs. (3) will provide all the information needed for our evaluation of the performance of adaptive thresholding.

If the time interval between t and $t + \epsilon$ is chosen small enough and $r(t)$ is assumed to be differentiable, we can write

$$r(t + \epsilon) = r(t) + r'(t)\epsilon, \quad (4)$$

where $r'(t)$ is the time derivative of $r(t)$. This equation implies that we may ignore the possibility of more than one threshold crossing in the small time interval. We now assume that $r(t)$ is a Markov process. In the definition of $\beta(t)\epsilon$, it follows from the Markov property that the condition that no crossing occurred before time t is equivalent to the condition that $r(t)$ is less than the threshold at time t . Thus if $P[\]$ signifies the probability of the event in the brackets,

$$\begin{aligned} \beta(t)\epsilon &= P[r(t + \epsilon) > A | r(t) < A] \\ &= \frac{P[r(t) + r'(t)\epsilon > A | r(t) < A]}{P[r(t) < A]}, \end{aligned} \quad (5)$$

where Eq. (4) has been used in the last step. We define $f_{rr'}(x, y)$ to be the joint probability density of $r(t)$ and $r'(t)$. The probability density of $r(t)$ is represented by $f_r(x)$. In general the densities will be functions of time. In terms of the density functions, Eqs. (5) can be rewritten as

$$\beta(t)\epsilon = \frac{\int_0^\infty \int_{A-y\epsilon}^A f_{rr'}(x, y) dx dy}{\int_{-\infty}^A f_r(x) dx} \quad (6)$$

where the region of integration in the numerator can be determined by inspection of the rr' plane. The time dependence of the density functions is implicit. In the limit of small ϵ , the inner integral of Eq. (6) can be evaluated. Taking the limit of Eq. (6), we obtain

$$\beta(t) = \frac{\int_0^{\infty} y f_{rr'}(A, y) dy}{\int_{-\infty}^A f_r(x) dx}. \quad (7)$$

The substitution of Eq. (7) into Eq. (3) gives a complete solution to our problem. However, the result will seldom be a simple closed-form expression, since the density functions of Eq. (7) are time dependent, making the evaluation of the integral in Eq. (3) difficult in most cases.

We could have attempted to derive $f_c(t)$ from the Fokker-Planck equation for continuous Markov processes and first passage time theory. However, the present approach is easier and yields greater physical insight.

STATIONARY NOISE

In most applications the noise can be considered stationary over the observation interval. In this case the general results can be simplified somewhat. Consider the received pulse shown in Fig. 6. Time is divided into four periods. Region I is defined by $0 \leq t < t_0$, the time before the occurrence of the leading edge. Region II is defined as the interval of the leading edge, $t_0 \leq t \leq t_1$. Region III is the interval between the leading and trailing edges, $t_1 < t < t_2$. Region IV is the remaining time period. These definitions are not precise but are often useful when the pulse has a reasonably fast rise.

In Region I, $r(t)$ has no signal component, by definition. Since the noise is stationary, neither $f_{rr'}(A, y)$ nor $f_r(x)$ has a time dependence. Thus

$$\beta(t) = C_0, \quad 0 \leq t < t_0, \quad (8)$$

and

$$f_c(t) = C_0 e^{-C_0 t} \quad 0 \leq t < t_0, \quad (9)$$

where C_0 is a constant. Using Eq. (8) in Eq. (3), it follows that for Region II

$$f_c(t) = e^{-C_0 t} \beta(t) \exp \left[- \int_{t_0}^t \beta(t) dt \right], \quad t_0 \leq t \leq t_1. \quad (10)$$

In Region III the signal component of $r(t)$ is approximately constant, by definition. Once again the stationarity of the noise leaves no time dependence in $\beta(t)$. It is then easy to see that $f_c(t)$ has the form given by

$$f_c(t) = C_1 e^{-C_2 t}, \quad t_1 < t < t_2, \quad (11)$$

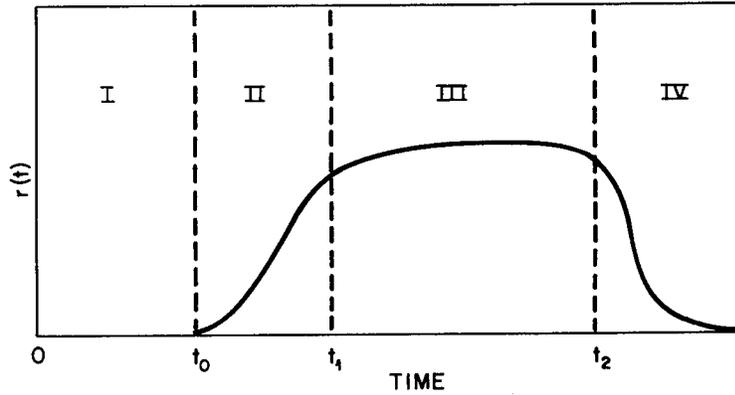


Fig. 6—Time regions of a pulse

where C_1 and C_2 are constants. In most applications the value of $f_c(t)$ is negligibly small in Region IV.

The general features of the variation of $f_c(t)$ are seen in the previous expressions. We will now consider some important special cases.

STATIONARY, ADDITIVE NOISE

If the noise is additive, we can write $r(t) = s(t) + n(t)$, where $s(t)$ is the signal and $n(t)$ is the noise. In the presence of additive noise, Eq. (7) becomes

$$\beta(t) = \frac{\int_{-s'(t)}^{\infty} [y + s'(t)] f_{nn'}(A - s(t), y) dy}{\int_{-\infty}^{A-s(t)} f_n(x) dx}, \quad (12)$$

where $s'(t)$ is the time derivative of $s(t)$, $f_{nn'}(x, y)$ is the joint probability density of the noise and its derivative, and $f_n(x)$ is the probability density of the noise. Note that, in contrast to Eq. (7), the time dependence is exhibited explicitly in Eq. (12). Because the noise is stationary, there is no implicit time variation in $f_{nn'}(x, y)$ or $f_n(x)$.

Equation (12) can easily be evaluated in two special cases. In the first one, it is assumed that $P[n'(t) < -s'(t)]$ is negligible for all $s(t)$ and $n(t)$ in Region II. Formally, we can write

$$f_{nn'}(x, y) \approx 0, \quad y < -s'(t), \quad t_0 \leq t \leq t_1. \quad (13)$$

It is further assumed that n' is symmetrically distributed about the origin, that is,

$$f_{nn'}(x, y) = f_{nn'}(x, -y). \quad (14)$$

Under these two assumptions, Eq. (12) is approximately given by

$$\beta(t) = \frac{s'(t)f_n(A - s(t))}{\int_{-\infty}^{A-s(t)} f_n(x) dx} \quad (15)$$

Looking at the right side of Eq. (15), we notice that we can write

$$\beta(t) = -\frac{d}{dt} \ln \left[\int_{-\infty}^{A-s(t)} f_n(x) dx \right]. \quad (16)$$

Using the Eq. (16), $\beta(t)$ can be readily integrated. Performing the integration, noting that $s(t_0) = 0$ by definition, and substituting into Eq. (10) yields

$$f_c(t) = s'(t)f_n(A - s(t)) \left[\frac{e^{-C_0 t_0}}{\int_{-\infty}^A f_n(x) dx} \right], \quad t_0 \leq t \leq t_1, \quad (17)$$

where the factor in brackets is a constant. Equations (9), (11), and (17) constitute a closed-form solution for the probability density in this special case.

For the second special case, we assume $s'(t)$ is a constant M (not necessarily large) in Region II. We further assume that $n(t)$ and $n'(t)$ are statistically independent. The latter assumption can be stated formally as

$$f_{nn'}(x, y) = f_n(x)f_{n'}(y) \quad (18)$$

Under these two assumptions, if $M \neq 0$, Eq. (12) reduces to

$$\beta(t) = K \frac{Mf_n(A - s(t))}{\int_{-\infty}^{A-s(t)} f_n(x) dx}, \quad (19)$$

where K is a constant defined by

$$K = \int_{-M}^{\infty} \left(1 + \frac{y}{M} \right) f_{n'}(y) dy, \quad M \neq 0. \quad (20)$$

Clearly, Eq. (19) can be put in the form of Eq. (16). It then follows from Eq. (10) that

$$f_c(t) = f_n(A - s(t)) \left[\int_{-\infty}^{A-s(t)} f_n(x) dx \right]^{K-1} \left[\frac{KMe^{-C_0 t_0}}{\left(\int_{-\infty}^A f_n(x) dx \right)^K} \right], \quad t_0 \leq t \leq t_1, \quad (21)$$

where the last factor in brackets is a constant. Equations (9), (11), and (21) constitute a closed-form solution for the probability density in this special case.

It is noticed that in the limit of large M , Eq. (20) indicates that $K \approx 1$. Then it is seen that Eq. (21) reduces to Eq. (17).

It can easily be shown (1) that additive Gaussian noise fulfills the conditions specified by Eq. (14) and (18). Thus if Eq. (13) is satisfied, we can use Eq. (17) for the Gaussian-noise problem. Alternatively, if M is constant, we can use Eq. (21) for the Gaussian-noise problem. A stationary Gaussian Markov process must have a power spectrum of the form (7)

$$S(\omega) = \frac{a}{\omega^2 + b^2}$$

where a and b are constants. A flat noise spectrum may be closely approximated over any finite frequency range by a suitable choice of values for the two constants. Of course, stationary white Gaussian noise, which is flat over the infinite frequency range, is neither differentiable nor Markov.

In many practical situations the received pulse has a very short rise time. Rather than use the formulas already established, we seek a direct derivation of the formulas for this special condition.

DIRECT DERIVATION FOR LARGE-SLOPE LEADING EDGES

If the leading edge of the received pulse has a large slope, it is intuitively reasonable to expect that there is only a small probability of more than one threshold crossing in Region II of Fig. 6. Formally, if $f_r(y)$ is the probability density of the slope of $r(t)$, we assume that

$$f_r(y) \approx 0, \quad y \leq 0, \quad t_0 \leq t \leq t_1. \quad (22)$$

In other words, the signal slope is sufficiently great for the combined signal and noise to have negligible probability of nonpositive slope. To simplify matters further, we set $t_0 = 0$, that is, we eliminate Region I.

Let $F_r(x, t)$ be the probability that $r(t) \leq x$. Thus $F_r(x, t)$ is a time-varying probability distribution function. Since there can be only one crossing between $0 \leq t \leq t_1$, the probability that no crossing occurs before time t is equal to $F_r(A, t)$, for if $r(t) < A$, then no crossing could have occurred before time t . Let $F_c(t)$ represent the probability distribution function of the first crossing, that is, $F_c(t)$ is equal to the probability that a crossing occurred before time t . From this definition and the preceding statements,

$$1 - F_c(t) = F_r(A, t), \quad 0 \leq t \leq t_1. \quad (23)$$

Differentiation of a distribution function gives a density function. Thus at all points where a derivative is defined, we have the following probability density function for a threshold crossing:

$$f_c(t) = -\frac{d}{dt} F_r(A, t) \quad 0 \leq t \leq t_1. \quad (24)$$

We may ignore the possibility of a first crossing for $t > t_1$ if

$$1 - \int_0^{t_1} f_c(t) dt \ll 1. \quad (25)$$

In deriving Eq. (24), we have not invoked the Markov property. Thus Eq. (24) is valid for processes with short rise times and differentiable distribution functions. We now turn to examples of the applications of this equation.

APPLICATIONS

As a first example of the application of Eq. (24), consider additive Gaussian noise. Then $F_r(A, t) = F_n(A - s(t))$, and it follows that

$$f_c(t) = s'(t) f_n(A - s(t)), \quad 0 \leq t \leq t_1. \quad (26)$$

Except for a multiplicative constant, Eq. (26) is identical to Eq. (17), which was derived in a more rigorous manner.

Zero-mean additive Gaussian noise has a probability density specified by

$$f_n(x) = \frac{1}{\sqrt{2\pi}\sigma} \exp\left(-\frac{x^2}{2\sigma^2}\right). \quad (27)$$

Suppose that the leading edge of a received pulse can be approximated by

$$s(t) = Mt, \quad 0 \leq t \leq t_1, \quad (28)$$

where M is a large constant. Then Eqs. (26) through (28) yield

$$f_c(t) = \frac{M}{\sqrt{2\pi}\sigma} \exp\left[-\frac{(A - Mt)^2}{2\sigma^2}\right], \quad 0 \leq t \leq t_1. \quad (29)$$

We now assume that the threshold is at the half-amplitude level, that is, $A = Mt_1/2$. Under this assumption, Eq. (25) gives the condition

$$2 \operatorname{erfc}\left(\frac{A}{\sigma}\right) \ll 1, \quad (30)$$

where we define

$$\operatorname{erfc}(x) = \frac{1}{\sqrt{2\pi}} \int_x^\infty \exp\left(-\frac{t^2}{2}\right) dt. \quad (31)$$

Equation (30) gives the condition under which we may ignore the region $t > t_1$.

We can easily calculate the expected value of t_c , the random variable specifying the pulse position. We have, if Eq. (25) is valid and $A = Mt_1/2$,

$$E[t_c] \approx \int_0^{2A/M} t f_c(t) dt. \quad (32)$$

Using Eq. (29) in Eq. (32), we obtain

$$E[t_c] \approx \frac{A}{M} \left[1 - 2 \operatorname{erfc} \left(\frac{A}{\sigma} \right) \right]. \quad (33)$$

Thus the arrival-time estimation by adaptive thresholding at the half-amplitude level is biased. However, Eq. (30) indicates that the amount of bias is small and that this bias decreases rapidly as signal amplitude increases.

A measure of the fluctuation in pulse position is provided by the standard deviation of the arrival time, which is defined to be

$$s = \sqrt{E[t_c^2] - \{E[t_c]\}^2}. \quad (34)$$

When Eq. (25) is satisfied, we can write

$$E[t_c^2] = \int_0^{2A/M} t^2 f_c(t) dt. \quad (35)$$

From Eqs. (29) and (33) through (35), it follows that for additive Gaussian noise,

$$\begin{aligned} s^2 = & \left(\frac{A}{M} \right)^2 2 \operatorname{erfc} \left(\frac{A}{\sigma} \right) \left[1 - 2 \operatorname{erfc} \left(\frac{A}{\sigma} \right) \right] \\ & + \left(\frac{\sigma}{M} \right)^2 \left[1 - 2 \operatorname{erfc} \left(\frac{A}{\sigma} \right) - \sqrt{\frac{2}{\pi}} \left(\frac{A}{\sigma} \right) \exp \left(\frac{-A^2}{2\sigma^2} \right) \right]. \end{aligned} \quad (36)$$

Additive Gaussian noise appears in the models for coherent amplitude-modulation (AM) systems. However, noncoherent AM systems usually contain envelope detectors. At the output of the detector, $r(t)$ is described by a Rice distribution. It can be shown that the probability distribution function is (8)

$$F_r(A, t) = \exp \left(-\frac{s^2(t)}{2\sigma^2} \right) \int_0^{A/\sigma} x I_0 \left(\frac{xs(t)}{\sigma} \right) \exp \left(-\frac{x^2}{2} \right) dx, \quad (37)$$

where $I_0(x)$ is the zero-order Bessel function of imaginary type. From Eqs. (24) and (28),

$$f_c(t) = \frac{M}{\sigma} \exp\left(-\frac{M^2 t^2}{2\sigma^2}\right) \int_0^{A/\sigma} \left\{ x \exp\left(-\frac{x^2}{2}\right) \times \left[\frac{Mt}{\sigma} I_0\left(\frac{xMt}{\sigma}\right) - x I_1\left(\frac{xMt}{\sigma}\right) \right] \right\} dx. \quad (38)$$

where $I_1(x)$ is the first-order Bessel function. The substitution of Eq. (38) into Eqs. (32) and (35) results in integrals which cannot be evaluated analytically. However, with the aid of a digital computer, it is possible to numerically compute the value of t_n , the normalized expected arrival time, defined by

$$t_n = \frac{E[t_c]}{t_1} = \frac{M}{2A} E[t_c]. \quad (39)$$

It is also possible to determine the function $g(A/\sigma)$, which is defined by

$$s = \frac{\sigma}{M} g\left(\frac{A}{\sigma}\right). \quad (40)$$

The results of the numerical computations are summarized in Table 1. The first column is the signal-to-noise ratio (S/N) as defined in the usual AM system. Thus

$$S/N = 10 \log_{10} \frac{E^2}{2\sigma^2} = 10 \log_{10} \frac{2A^2}{\sigma^2}. \quad (41)$$

The fifth column is calculated from Eqs. (36) and (40). It is noticed that the standard deviation for the Gaussian distribution (coherent AM system) is uniformly less than the standard deviation for the Rice distribution (noncoherent AM system). For a fixed value of S/N, the degradation in standard deviation for the Rice distribution is defined to be the additional power required to achieve the standard deviation that exists in the Gaussian case for the specified S/N. The sixth column lists the degradation in decibels as a function

Table 1
Signal-to-Noise Dependency of Various Quantities

S/N (dB)	A/σ	t_n (Rice)	$g(A/\sigma)$ (Rice)	$g(A/\sigma)$ (Gauss)	Approximate Degradation (db)
12	2.815	0.462	1.048	0.997	0.44
14	3.544	0.478	1.028	1.000	0.24
16	4.462	0.487	1.015	1.000	0.13
18	5.617	0.492	1.009	1.000	0.08
20	7.071	0.495	1.005	1.000	0.05

of S/N. For S/N < 12 dB, it is found that Eq. (25) is not well satisfied for either the Rice or the Gaussian distribution; hence, these values of S/N have not been listed.

Examination of the third column in Table 1 shows that the estimation procedure is biased. However, the bias steadily decreases as S/N increases.

ASYMPTOTIC RESULTS

We conclude with a heuristic discussion of the asymptotic properties of arrival-time estimation by adaptive thresholding. It has been seen in the last section that the estimation procedure is biased. Of course, some of the apparent bias is due to our having assumed a negligible probability of a first threshold crossing for $t > t_1$. Some bias is also due to setting the threshold at the half-amplitude level. For both the Rice and Gaussian distributions, the arrival-time estimate approaches the unbiased value of one-half the rise time, in the limit of large S/N. Thus we can say that the estimate is asymptotically unbiased.

For the unbiased estimate of arrival time of a low-pass pulse with large S/N in white Gaussian noise, the Cramer-Rao bound requires (6) that

$$s^2 \geq \left\{ \frac{1}{N} \int_0^{t_1} [s'(t)]^2 dt \right\}^{-1}, \quad (42)$$

where N is the noise power spectral density. If $s'(t) = M$ is a constant and $t_1 = 2A/M$, Eq. (42) reduces to

$$s^2 \geq \frac{N}{Mt_1} \approx \frac{\sigma^2}{M^2}, \quad (43)$$

where we have assumed that the rise time is approximately equal to the inverse of the system noise bandwidth B . Thus we have set

$$\frac{N}{t_1} \approx NB = \sigma^2, \quad (44)$$

where σ^2 is the mean noise power. A comparison of the fifth column of Table 1 and Eq. (40) with Eq. (43) indicates that the adaptive-thresholding estimate approaches the minimum variance expressed by the Cramer-Rao bound. Thus adaptive thresholding is asymptotically efficient for white Gaussian noise if Eq. (44) is valid. Clearly the performance of a noncoherent system is bounded by the performance of the corresponding coherent system. Thus from a comparison of the fourth and fifth columns of Table 1, it appears that adaptive thresholding is also asymptotically efficient for the Rician process.

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