Electromagnetic Beam Fields

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ELECTROMAGNETIC BEAM FIELDS

INTRODUCTION

Although electromagnetic beams play an important role in laser technology, our present theoretical understanding of beams is not satisfactory. For example the electromagnetic fields associated with even the most common laser beams are not well known. And the term "beam" describes an empirical concept which is not precisely defined mathematically.

The electromagnetic fields which are most commonly thought of as beams are probably the Hermite Gaussian and Laguerre Gaussian beams studied in laser theory [1,2]. Even so, the mathematical models used to describe them are only scalar theories*, employ a paraxial approximation, and can be justified only if the beam is rather well collimated. It was found recently [4,5] that these beams cease to behave according to these theories as the divergence from focus is increased. In fact it was found that the beams described in these theories cannot exist at all in free space except as asymptotic limits. If an actual Gaussian laser beam in free space diverges from focus, forming a cone of light that makes a half angle of as much as 10° with the beam axis, then the electromagnetic field associated with this beam becomes clearly anomalous in that it does not agree with the paraxial theories. As the divergence is increased still further, the electromagnetic field becomes increasingly anomalous and approaches a modified dipole field. Thus the usual concepts regarding beams require modification.

In the present report, two, mutually exclusive, classes of beam fields are defined through mathematical restrictions on transverse components of the associated electromagnetic fields. A certain connection is found between these classes of beam fields and multipole fields. The paraxial theories describing Gaussian beams are replaced by more general models which involve prolate spheroidal wave functions. It is found that beam fields exist which obey Maxwell's equations in free space, maintain a consistent amplitude cross section, and also behave asymptotically like Gaussian beams in the limit as the degree of collimation is increased. Expressions for the electromagnetic components associated with these beam fields are given which are valid for arbitrary divergence from focus.

ANGULAR SPECTRUM REPRESENTATION FOR ELECTROMAGNETIC FIELDS

An electromagnetic field in free space can be represented as a superposition of monochromatic plane waves in several equivalent ways by expanding different scalar functions associated with the field. This study of beam fields will employ two such representations differing by expansion of either the electric or the magnetic component which is transverse to the direction of propagation. Although either representation may be used to describe a particular electromagnetic field, each offers its own formal advantages.

*A vector treatment of the Laguerre Gaussian beam was done using a paraxial approximation by Gobau and Schwering [3].

Both representations employ expansions of components of an electromagnetic field into an angular spectrum of plane waves. Such expansions were used as early as 1909, in a paper by Debye [6], and more recently by Stratton [7], Borgiotti [8], Rhodes [9], and Baños [10]. However in the recent optics literature the angular spectrum expansion has been used primarily to represent only scalar fields [11] (electromagnetic fields in a scalar approximation).

A general electromagnetic field in free space can be decomposed into a linear superposition of monochromatic fields. Thus without loss of generalitv the following considerations will be limited to a monochromatic field. According to Maxwell's equations the electric and magnetic field vectors associated with a monochromatic field must satisfy the equations

$$\nabla \cdot \mathbf{E}\ (x,y,z) = 0 \tag{1a}$$

$$\nabla \times \mathbf{E}\ (x,y,z) = ik\ \mathbf{B}\ (x,y,z) \tag{1b}$$

$$\nabla \cdot \mathbf{B}\ (x,y,z) = 0 \tag{1c}$$

$$\nabla \times \mathbf{B}\ (x,y,z) = -ik\ \mathbf{E}\ (x,y,z) \tag{1d}$$

where $k = \omega/c = 2\pi/\lambda$ and the $e^{-i\omega t}$ time dependence has been suppressed. In Eqs. (1) only two rectangular components of either the electric or the magnetic field vector can be independently specified over an infinite plane as will soon be shown.

Consider an electromagnetic field propagating away from sources limited to a region where $z = -\infty$ and traveling through free space. If we wish to specify the transverse components of the electric field in the $z = z_0$ plane, then it is convenient to expand the x and y rectangular components of the electric field vector in the manner

$$E_x(x,y,z) = \iint_{-\infty}^{\infty} \mathscr{E}_x(p,q) e^{ik(px+qy+mz)}\, dp\, dq, \tag{2a}$$

$$E_y(x,y,z) = \iint_{-\infty}^{\infty} \mathscr{E}_y(p,q) e^{ik(px+qy+mz)}\, dp\, dq, \tag{2b}$$

where

$$m = \sqrt{1-p^2-q^2},\ \mathit{if}\ p^2+q^2 \leqslant 1, \tag{3a}$$

$$= i\sqrt{p^2+q^2-1},\ \mathit{if}\ p^2+q^2 > 1. \tag{3b}$$

It then follows from Eqs. (2a) and (2b) that each plane wave amplitude is given by the Fourier transform relations

$$\mathscr{E}_x(p,q) = (e^{-ikmz_0}/\lambda^2) \iint\limits_{-\infty}^{\infty} E_x(x,y,z_0) e^{-ik(px+qy)}\, dx\, dy, \tag{4a}$$

$$\mathscr{E}_y(p,q) = (e^{-ikmz_0}/\lambda^2) \iint\limits_{-\infty}^{\infty} E_y(x,y,z_0) e^{-ik(px+qy)}\, dx\, dy. \tag{4b}$$

The other rectangular components of the electric and magnetic fields cannot now be chosen arbitrarily but are determined by substituting Eqs. (2a) and (2b) into (1). The longitudinal electric field, which is the component in the z direction, is found by substituting Eqs. (2a) and (2b) into (1a) to be

$$E_z(x,y,z) = -\iint\limits_{-\infty}^{\infty} \left[\frac{p}{m}\mathscr{E}_x(p,q) + \frac{q}{m}\mathscr{E}_y(p,q)\right] e^{ik(px+qy+mz)}\, dp\, dq, \tag{2c}$$

whereas the components of the magnetic field are found similarly by substituting Eqs. (2a) and (2b) into (1b) to be

$$B_x(x,y,z) = -\iint\limits_{-\infty}^{\infty} \left[\frac{pq}{m}\mathscr{E}_x(p,q) + \frac{1-p^2}{m}\mathscr{E}_y(p,q)\right] e^{ik(px+qy+mz)}\, dp\, dq, \tag{2d}$$

$$B_y(x,y,z) = \iint\limits_{-\infty}^{\infty} \left[\frac{1-q^2}{m}\mathscr{E}_x(p,q) + \frac{pq}{m}\mathscr{E}_y(p,q)\right] e^{ik(px+qy+mz)}\, dp\, dq, \tag{2e}$$

$$B_z(x,y,z) = -\iint\limits_{-\infty}^{\infty} [q\,\mathscr{E}_x(p,q) - p\,\mathscr{E}_y(p,q)]\, e^{ik(px+qy+mz)}\, dp\, dq. \tag{2f}$$

Thus the electromagnetic field is completely specified by $E_x(x,y,z_0)$ and $E_y(x,y,z_0)$ or equivalently by $\mathscr{E}_x(p,q)$ and $\mathscr{E}_y(p,q)$.

For many fields the surface integrals appearing in Eqs. (2) can be evaluated asymptotically in the limit as kr becomes large, where $r = (x^2 + y^2 + z^2)^{1/2}$ is the magnitude of the radius vector $\mathbf{r}$ to the point (x,y,z). This procedure, which employs the method of stationary phase [12,13], is valid for fields in which $\mathscr{E}_x(\bar{p},\bar{q})$ and $\mathscr{E}_y(\bar{p},\bar{q})$ can be extended as analytic functions of complex $\bar{p},\bar{q}$ which are real on the real axis (or vary sufficiently slowly in phase along the real axis) and regular in some neighborhood about the path of integration. For such fields we have

$$E_x(x,y,z) \sim -i\lambda\frac{z}{r}\mathscr{E}_x\left(\frac{\pm x}{r},\frac{\pm y}{r}\right)\frac{e^{\pm ikr}}{r}, \tag{5a}$$

$$E_y(x,y,z) \sim -i\lambda \frac{z}{r} \mathscr{E}_y\left(\frac{\pm x}{r}, \frac{\pm y}{r}\right) \frac{e^{\pm ikr}}{r}, \tag{5b}$$

$$E_z(x,y,z) \sim i\lambda\left[\frac{x}{r} \mathscr{E}_x\left(\frac{\pm x}{r}, \frac{\pm y}{r}\right) + \frac{y}{r} \mathscr{E}_y\left(\frac{\pm x}{r}, \frac{\pm y}{r}\right)\right] \frac{e^{\pm ikr}}{r}, \tag{5c}$$

$$B_x(x,y,z) \sim \pm i\lambda\left[\frac{xy}{r^2} \mathscr{E}_x\left(\frac{\pm x}{r}, \frac{\pm y}{r}\right) + \left(1 - \frac{x^2}{r^2}\right) \mathscr{E}_y\left(\frac{\pm x}{r}, \frac{\pm y}{r}\right)\right] \frac{e^{\pm ikr}}{r}, \tag{5d}$$

$$B_y(x,y,z) \sim \mp i\lambda\left[\left(1 - \frac{y^2}{r^2}\right) \mathscr{E}_x\left(\frac{\pm x}{r}, \frac{\pm y}{r}\right) + \frac{xy}{r^2} \mathscr{E}_y\left(\frac{\pm x}{r}, \frac{\pm y}{r}\right)\right] \frac{e^{\pm ikr}}{r}, \tag{5e}$$

$$B_z(x,y,z) \sim \pm i\lambda \frac{z}{r}\left[\frac{y}{r} \mathscr{E}_x\left(\frac{\pm x}{r}, \frac{\pm y}{r}\right) - \frac{x}{r} \mathscr{E}_y\left(\frac{\pm x}{r}, \frac{\pm y}{r}\right)\right] \frac{e^{\pm ikr}}{r} \tag{5f}$$

as kr approaches infinity (for a similar derivation see Ref. 3, Appendix A). In these equations the upper sign is chosen if $z > 0$, the lower sign is chosen if $z < 0$, and we require that $z \neq 0$. For light waves, in which the wavelength is much smaller than the observable spatial intervals, Eqs. (5) are useful everywhere except in the $z = 0$ plane and over a neighborhood about the origin.

From Eqs. (5a) and (5b) it is clear that the plane wave amplitudes directly determine how the transverse electric components vary as a function of the angle that r makes with the z axis. Hence this representation is particularly useful if we wish to specify a beam for which the transverse electric field is localized about the z axis. Once this is done however, no additional similar restrictions may be placed on the other field components. Within the paraxial region, where $x \ll r$ and $y \ll r$, Eqs. (5) are approximately given by

$$E_x(x,y,z) \sim \mp i\lambda\, \mathscr{E}_x\left(\frac{\pm x}{r}, \frac{\pm y}{r}\right) \frac{e^{\pm ikr}}{r}, \tag{6a}$$

$$E_y(x,y,z) \sim \mp i\lambda\, \mathscr{E}_y\left(\frac{\pm x}{r}, \frac{\pm y}{r}\right) \frac{e^{\pm ikr}}{r}, \tag{6b}$$

$$E_z(x,y,z) \sim 0, \tag{6c}$$

$$B_x(x,y,z) \sim \pm i\lambda\, \mathscr{E}_y\left(\frac{\pm x}{r}, \frac{\pm y}{r}\right) \frac{e^{\pm ikr}}{r}, \tag{6d}$$

$$B_y(x,y,z) \sim \mp i\lambda\, \mathscr{E}_x\left(\frac{\pm x}{r}, \frac{\pm y}{r}\right) \frac{e^{\pm ikr}}{r}, \tag{6e}$$

$$B_z(x,y,z) \sim 0. \tag{6f}$$

These equations indicate that the transverse magnetic components in this region must obey the same constraints as the transverse electric field and that the longitudinal components can be neglected. Outside this region, according to Eqs. (5), the transverse magnetic components are no longer equal to the transverse electric components; however the entire field can always be localized to some region about the z axis by requiring that $\mathscr{E}_x$ and $\mathscr{E}_y$ are sufficiently small outside of some region about the origin in p,q space where $p^2 + q^2 << 1$.

If we wish to specify the transverse components of the magnetic field instead of the electric, it is more convenient to expand the x and y components of the magnetic field vector, namely

$$B_x(x,y,z) = \iint_{-\infty}^{\infty} \mathscr{B}_x(p,q)\, e^{ik(px+qy+mz)}\, dp\, dq, \tag{7a}$$

$$B_y(x,y,z) = \iint_{-\infty}^{\infty} \mathscr{B}_y(p,q)\, e^{ik(px+qy+mz)}\, dp\, dq, \tag{7b}$$

instead of the transverse electric components as in Eqs. (2), where m is given by Eqs. (3) as before. In this new representation the plane wave amplitudes are given by

$$\mathscr{B}_x(p,q) = (e^{-ikmz_0}/\lambda^2) \iint_{-\infty}^{\infty} B_x(x,y,z_0)\, e^{-ik\,(px+qy)}\, dx\, dy, \tag{8a}$$

$$\mathscr{B}_y(p,q) = (e^{-ikmz_0}/\lambda^2) \iint_{-\infty}^{\infty} B_y(x,y,z_0)\, e^{-ik(px+qy)}\, dx\, dy, \tag{8b}$$

which follows directly from Eqs. (7). The longitudinal component of the magnetic field vector is determined by substituting Eqs. (7a) and (7b) into (1c) to be

$$B_z(x,y,z) = -\iint_{-\infty}^{\infty} \left[\frac{p}{m}\mathscr{B}_x(p,q) + \frac{q}{m}\mathscr{B}_y(p,q)\right] e^{ik(px+qy+mz)}\, dp\, dq, \tag{7c}$$

and the components of the electric field vector are found by substituting Eqs. (7a) and (7b) into (1d) to be

$$E_x(x,y,z) = \iint_{-\infty}^{\infty} \left[\frac{pq}{m}\mathscr{B}_x(p,q) + \frac{1-p^2}{m}\mathscr{B}_y(p,q)\right] e^{ik(px+qy+mz)}\, dp\, dq, \tag{7d}$$

$$E_y(x,y,z) = -\iint_{-\infty}^{\infty} \left[\frac{1-q^2}{m}\mathscr{B}_x(p,q) + \frac{pq}{m}\mathscr{B}_y(p,q)\right] e^{ik(px+qy+mz)}\, dp\, dq, \tag{7e}$$

$$E_z(x,y,z) = \iint_{-\infty}^{\infty} [q\mathscr{B}_x(p,q) - p\mathscr{B}_y(p,q)]\, e^{ik(px+qy+mz)}\, dp\, dq. \tag{7f}$$

Thus in this representation the electromagnetic field is completely specified by $B_x(x,y,z_0)$ and $B_y(x,y,z_0)$, or equivalently by $\mathscr{B}_x(p,q)$ and $\mathscr{B}_y(p,q)$.

The surface integrals in Eqs. (8) can be evaluated using the method of stationary phase as before, provided $\mathscr{B}_x(\bar{p},\bar{q})$ and $\mathscr{B}_y(\bar{p},\bar{q})$ can be extended as analytic functions of complex $\bar{p},\bar{q}$ which are real on the real axis and regular in some neighborhood about the path of integration. In this manner we find that

$$B_x(x,y,z) \sim -i\lambda \frac{z}{r} \mathscr{B}_x\left(\frac{\pm x}{r}, \frac{\pm y}{r}\right) \frac{e^{\pm ikr}}{r}, \tag{9a}$$

$$B_y(x,y,z) \sim -i\lambda \frac{z}{r} \mathscr{B}_y\left(\frac{\pm x}{r}, \frac{\pm y}{r}\right) \frac{e^{\pm ikr}}{r}, \tag{9b}$$

$$B_z(x,y,z) \sim i\lambda \left[\frac{x}{r} \mathscr{B}_x\left(\frac{\pm x}{r}, \frac{\pm y}{r}\right) + \frac{y}{r} \mathscr{B}_y\left(\frac{\pm x}{r}, \frac{\pm y}{r}\right)\right] \frac{e^{\pm ikr}}{r}, \tag{9c}$$

$$E_x(x,y,z) \sim \mp\, i\lambda \left[\frac{xy}{r} \mathscr{B}_x\left(\frac{\pm x}{r}, \frac{\pm y}{r}\right) + \left(1 - \frac{x^2}{r^2}\right) \mathscr{B}_y\left(\frac{\pm x}{r}, \frac{\pm y}{r}\right)\right] \frac{e^{\pm ikr}}{r}, \tag{9d}$$

$$E_y(x,y,z) \sim \pm\, i\lambda \left[\left(1 - \frac{y^2}{r^2}\right) \mathscr{B}_x\left(\frac{\pm x}{r}, \frac{\pm y}{r}\right) + \frac{xy}{r^2} \mathscr{B}_y\left(\frac{\pm x}{r}, \frac{\pm y}{r}\right)\right] \frac{e^{\pm ikr}}{r}, \tag{9e}$$

$$E_z(x,y,z) \sim \mp\, i\lambda \frac{z}{r}\left[\frac{y}{r} \mathscr{B}_x\left(\frac{\pm x}{r}, \frac{\pm y}{r}\right) - \frac{x}{r} \mathscr{B}_y\left(\frac{\pm x}{r}, \frac{\pm y}{r}\right)\right] \frac{e^{\pm ikr}}{r} \tag{9f}$$

as kr approaches infinity. Again the $\pm$ sign is taken according to whether $z \gtrless 0$, and $z \neq 0$.

From Eqs. (9) it is clear that this representation is more useful if we wish to localize the transverse magnetic field to some region about the z axis. Within the paraxial region Eqs. (9) may be approximated by

$$B_x(x,y,z) \sim \mp\, i\lambda\, \mathscr{B}_x\left(\frac{\pm x}{r}, \frac{\pm y}{r}\right) \frac{e^{\pm ikr}}{r}, \tag{10a}$$

$$B_y(x,y,z) \sim \mp\, i\lambda\, \mathscr{B}_y\left(\frac{\pm x}{r}, \frac{\pm y}{r}\right) \frac{e^{\pm ikr}}{r}, \tag{10b}$$

$$B_z(x,y,z) \sim 0, \tag{10c}$$

$$E_x(x,y,z) \sim \mp\, i\lambda\, \mathscr{B}_y\left(\frac{\pm x}{r}, \frac{\pm y}{r}\right) \frac{e^{\pm ikr}}{r}, \tag{10d}$$

$$E_y(x,y,z) \sim \pm\, i\lambda\, \mathscr{B}_x\left(\frac{\pm x}{r}, \frac{\pm y}{r}\right)\frac{e^{\pm ikr}}{r}, \tag{10e}$$

$$E_z(x,y,z) \sim 0, \tag{10f}$$

which represent a field similar to that given by Eqs. (6). Thus the transverse electric field is everywhere proportional to the transverse magnetic field inside of the paraxial region. Outside this region, as shown by Eqs. (9), this is no longer true; however the field can always be localized by suitably localizing $\mathscr{B}_x$ and $\mathscr{B}_y$ to some region about the origin in p,q space where $p^2 + q^2 << 1$.

The two representations given here are equivalent in that any field expressed in terms of one representation can be easily transformed to the other. By comparison of Eqs. (2) with (7), it is seen that the representation can be transformed by using

$$\mathscr{E}_x(p,q) = \frac{pq}{m}\mathscr{B}_x(p,q) + \frac{1-p^2}{m}\mathscr{B}_y(p,q), \tag{11a}$$

$$\mathscr{E}_y(p,q) = -\frac{1-q^2}{m}\mathscr{B}_x(p,q) - \frac{pq}{m}\mathscr{B}_y(p,q) \tag{11b}$$

or equivalently

$$\mathscr{B}_x(p,q) = -\frac{pq}{m}\,\mathscr{E}_x(p,q) - \frac{1-p^2}{m}\,\mathscr{E}_y(p,q), \tag{11c}$$

$$\mathscr{B}_y(p,q) = \frac{1-q^2}{m}\,\mathscr{E}_x(p,q) + \frac{pq}{m}\,\mathscr{E}_y(p,q). \tag{11d}$$

Each plane wave in either representation individually satisfies the wave equation; thus Eqs. (2) and (7) are modal expansions. This can be varified by substituting Eqs. (2) and (7) into the homogeneous wave equation while using Eqs. (3).

The flow of energy carried by the field can be described by use of the Poynting vector

$$\mathbf{S}(x,y,z) = \frac{c}{4\pi}\,\mathbf{E}(x,y,z) \times \mathbf{B}(x,y,z). \tag{12}$$

By substituting Eqs. (5) into (12), we have

$$\mathbf{S}(x,y,z) \sim \frac{\pm\lambda^2 c}{4\pi}\left[\left|\,\mathscr{E}_x\left(\frac{\pm x}{r}, \frac{\pm y}{r}\right)\right|^2 + \left|\mathscr{E}_y\left(\frac{\pm x}{r}, \frac{\pm y}{r}\right)\right|^2\right](\sin^2\phi_B)\,\frac{\mathbf{r}}{r^3}, \tag{13}$$

as kr approaches infinity in one representation, and by substituting Eqs. (9) into (12), we have

$$\mathbf{S}(x,y,z) \sim \frac{\pm\lambda^2 c}{4\pi}\left[\left|\mathscr{B}_x\left(\frac{\pm x}{r}, \frac{\pm y}{r}\right)\right|^2 + \left|\mathscr{B}_y\left(\frac{\pm x}{r}, \frac{\pm y}{r}\right)\right|^2\right](\sin^2\phi_E)\,\frac{\mathbf{r}}{r^3} \tag{14}$$

as kr approaches infinity in the other. In these relations ϕ_B and ϕ_E are the angles that $\mathbf{r}$ makes respectively with the vectors $\mathbf{k} \times \underline{\mathscr{E}}\ (\pm x/r, \pm y/r)$ and $\mathbf{k} \times \underline{\mathscr{B}}\ (\pm x/r, \pm y/r)$ where $\underline{\mathscr{E}} = \mathbf{i}\ \mathscr{E}_x + \mathbf{j}\ \mathscr{E}_y$, $\underline{\mathscr{B}} = \mathbf{i}\ \mathscr{B}_x + \mathbf{j}\ \mathscr{B}_y$, and $(\mathbf{i}, \mathbf{j}, \mathbf{k})$ are the usual rectangular unit vectors. For the important special case where the two angular spectra ($\mathscr{E}_x$ and $\mathscr{E}_y$, or $\mathscr{B}_x$ and $\mathscr{B}_y$) are proportional, symmetric about the origin, and real over all p, q, the angles ϕ_B and ϕ_E become the angles $\mathbf{r}$ makes respectively with the projection of $\mathbf{B}(0,0,0)$ and $\mathbf{E}(0,0,0)$ into the $z = 0$ plane.

THE ANGULAR SPECTRA FOR A BEAM FIELD

The representations developed in the last section are valid for any electromagnetic field propagating in free space away from sources limited to a region where $z = -\infty$. A particular field is specified uniquely by the angular spectra $\mathscr{E}_x(p,q)$ and $\mathscr{E}_y(p,q)$, or by the angular spectra $\mathscr{B}_x(p,q)$ and $\mathscr{B}_y(p,q)$. In this section we will define two, mutually exclusive, classes of beam fields, which we will name consistent electric beams and consistent magnetic beams, by making certain mathematical restrictions on their associated angular spectra.

First we will consider the class of consistent magnetic beams which are defined such that their angular spectra, $\mathscr{E}_x(p,q)$ and $\mathscr{E}_y(p,q)$, satisfy four conditions.

We require that the field be represented by the angular spectra containing only homogeneous plane waves:

$$\mathscr{E}_x(p,q) = \mathscr{E}_y(p,q) = 0, \text{ if } p^2 + q^2 > 1. \tag{I}$$

This is consistent with the usual assumption that an electromagnetic field in free space contains no evanescent plane waves due to the rapid exponential decay of these plane waves as they propagate away from the sources [14].

It is convenient to also require that the field produce a plane wavefront in the $z = 0$ plane. This is always the case if the plane wave amplitudes are real, that is,

$$\begin{aligned} \mathscr{E}_x(p,q) &= \mathscr{E}_x^*(p,q), \\ \mathscr{E}_y(p,q) &= \mathscr{E}_y^*(p,q), \end{aligned} \tag{II}$$

which can be observed by substituting Condition (II) into Eqs. (2). In addition we assume that $\mathscr{E}_x(\bar{p},\bar{q})$ and $\mathscr{E}_y(\bar{p},\bar{q})$ can be extended as analytic functions of complex $\bar{p},\bar{q}$ which are regular in some neighborhood about the segment of the real axis within the domain of support*.

These restrictions, together with Condition (II), are made so that the asymptotic behavior of the field is correctly given by Eqs. (5). In addition to permitting the use of Eqs. (5), Condition (II) places a useful restriction on the field. By substituting Condition (II) into Eqs. (2), it is found that

*The domain of support for some function $f(x)$ is all x for which $f(x) \neq 0$.

$$\mathbf{E}(x,y,z) = \mathbf{E}^*(-x,-y,-z), \tag{15a}$$

$$\mathbf{B}(x,y,z) = \mathbf{B}^*(-x,-y,-z), \tag{15b}$$

a property closely related to a reciprocity theorem due to Shewell and Wolf [14]. Since the origin is the center of symmetry for this field, it is defined here to be the focus. It follows that because of Condition (II) the field is focused in the $z = 0$ plane and that Eqs. (5) give the field components asymptotically away from focus where kr is large.

Next we will formulate a condition under which a beam field is localized to a region about the z axis. It was noted that Eqs. (5) show a simple relationship between the angular spectra and the angular dependence of the transverse electric field away from focus. After observing that $\theta = \sin^{-1}((x/r)^2 + (y/r)^2)^{1/2}$ is the angle that the radius vector $\mathbf{r}$ makes with the z axis, it is evident that on any spherical surface of constant r away from focus, E_x and E_y vary with θ in exactly the same manner as $\mathscr{E}_x(p,q)$ and $\mathscr{E}_y(p,q)$ vary with the parameter $\sin^{-1}(p^2 + q^2)^{1/2}$. Therefore the transverse electric component of the field can be localized to the interior of a cone, as shown in Fig. 1, which makes a half angle of θ_0 with the z axis, by the condition*

$$\begin{aligned} \mathscr{E}_x(p,q) &<< \mathrm{Max}[\mathscr{E}_x(p,q)]/\sqrt{e}, \text{ if } p^2 + q^2 > \sin^2\theta_0, \\ \mathscr{E}_y(p,q) &<< \mathrm{Max}[\mathscr{E}_y(p,q)]/\sqrt{e}, \text{ if } p^2 + q^2 > \sin^2\theta_0. \end{aligned} \tag{III}$$

If the angular spectra associated with a field obey Conditions (I) through (III), we will refer to the field as a beam, but not necessarily a consistent beam.

Finally we will formulate a condition under which a beam will maintain a consistent amplitude cross section both at focus and away from focus. More specifically we require that a consistent magnetic beam must produce a transverse electric field which is distributed about the z axis in the same manner on both the focal ($z = 0$) plane and on all spherical surfaces of constant r away from focus (S in Fig. 1). From Eqs. (5) it is clear that this can be accomplished by setting

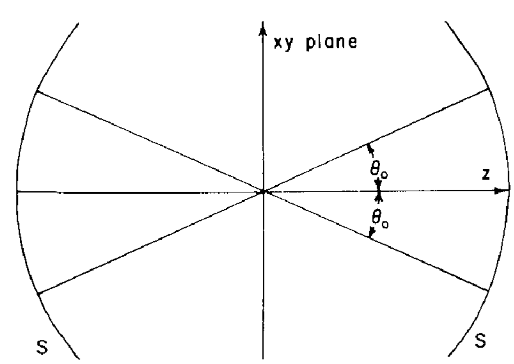

Fig. 1 Cone subtending an angle θ_0 with the z axis. Condition (III) (or (VIII)) localizes a field such that it is small in amplitude outside of this cone.

*The symbol $\mathrm{Max}[f(x)]$ indicates the maximum value of the function $f(x)$.

$$E_x(x,y,0) = \gamma \mathscr{E}_x(x/d_x, y/d_y), \tag{16a}$$

$$E_y(x,y,0) = \gamma \mathscr{E}_y(x/d_x, y/d_y) \tag{16b}$$

inside some simply connected domain D of x,y which includes the origin, such as that defined by

$$(x/d_x)^2 + (y/d_y)^2 \leqslant \sin^2\theta_0', \tag{17}$$

where γ is a complex constant. Thus, by substituting Eqs. (16) into (2), the condition

$$\left.\begin{aligned}\gamma\mathscr{E}_x(x/d_x, y/d_y) &= \iint_{-\infty}^{\infty} \mathscr{E}_x(p,q)\, e^{ik(px+qy)}\, dp\, dq, \text{ if } x,y \in D,\\ \gamma\mathscr{E}_y(x/d_x, y/d_y) &= \iint_{-\infty}^{\infty} \mathscr{E}_y(p,q)\, e^{ik(px+qy)}\, dp\, dq, \text{ if } x,y \in D\end{aligned}\right\} \tag{IV}$$

is obtained for the angular spectra.

Since, according to Conditions (II) and (IV), $E_x(\bar{x},\bar{y},0)$ and $E_y(\bar{x},\bar{y},0)$ are Fourier transforms of functions which vanish outside of a finite domain (functions with bounded support), then it follows that they can be continued as entire functions of complex $\bar{x},\bar{y}$. Although Eqs. (16) give $E_x(x,y,0)$ and $E_y(x,y,0)$ only over D, they are also defined outside D. Thus the distribution of the transverse electric component over the focal plane is proportional over D to the distribution over all spherical surfaces of constant r away from focus and is the analytic continuation of this function outside D. The concept of consistency would be more meaningful if this function were small enough outside D so that the field there could be neglected.

Therefore to insure that a beam maintains a truly consistent amplitude cross section, it is necessary to replace Condition (III) with the stronger condition

$$\left.\begin{aligned}E_x(x,y,0) &<< \mathrm{Max}[E_x(x,y,0)]/\sqrt{e}, \text{ if } x,y \in D,\\ E_y(x,y,0) &<< \mathrm{Max}[E_y(x,y,0)]/\sqrt{e}, \text{ if } x,y \in D,\end{aligned}\right\} \tag{V}$$

so that the transverse electric component of the field over the focal plane will be localized within the domain D. However this condition is so restrictive that we will consider it desirable but not necessary.

Any beam field which obeys Condition (IV), in addition to (I), (II), and (III), is defined to be a consistent magnetic beam. If a consistent magnetic beam also obeys Condition (V), we will call it strongly consistent.

A second general class of beam field can be defined in a similar manner by the use of the angular spectra $\mathscr{B}_x(p,q)$ and $\mathscr{B}_y(p,q)$. Any electromagnetic field which can be represented by the angular spectra which obey the conditions

$$\mathscr{B}_x(p,q) = \mathscr{B}_y(p,q) = 0, \text{ if } p^2 + q^2 > 1, \tag{VI}$$

$$\begin{aligned} \mathscr{B}_x(p,q) &= \mathscr{B}_x^*(p,q), \\ \mathscr{B}_y(p,q) &= \mathscr{B}_y^*(p,q), \end{aligned} \tag{VII}$$

and

$$\begin{aligned} \mathscr{B}_x(p,q) &<< \mathrm{Max}[\mathscr{B}_x(p,q)]/\sqrt{e}, \text{ if } p^2 + q^2 > \sin^2\theta_0, \\ \mathscr{B}_y(p,q) &<< \mathrm{Max}[\mathscr{B}_y(p,q)]/\sqrt{e}, \text{ if } p^2 + q^2 > \sin^2\theta_0 \end{aligned} \tag{VIII}$$

is defined to be an electric beam field. We assume that $\mathscr{B}_x(\bar{p},\bar{q})$ and $\mathscr{B}_y(\bar{p},\bar{q})$ can be extended as analytic functions of complex $\bar{p},\bar{q}$ which are regular in some neighborhood about the segment of the real axis within the domain of support.

Consistent electric beam fields are defined as a special case of electric beam fields which obey the additional condition

$$\begin{aligned} \gamma\mathscr{B}_x(x/d_x, y/d_y) &= \iint_{-\infty}^{\infty} \mathscr{B}_x(p,q)\, e^{ik(px+qy)}\, dp\, dq, \text{ if } x,y \in D, \\ \gamma\mathscr{B}_y(x/d_x, y/d_y) &= \iint_{-\infty}^{\infty} \mathscr{B}_y(p,q)\, e^{ik(px+qy)}\, dp\, dq, \text{ if } x,y \in D. \end{aligned} \tag{IX}$$

Such beams will be called strongly consistent if, in addition to Condition (VIII), they obey the stronger condition

$$\begin{aligned} B_x(x,y,0) &<< \mathrm{Max}[B_x(x,y,0)]/\sqrt{e}, \text{ if } x,y \in D, \\ B_y(x,y,0) &<< \mathrm{Max}[B_y(x,y,0)]/\sqrt{e}, \text{ if } x,y \in D. \end{aligned} \tag{X}$$

Thus consistent electric beam fields must satisfy Conditions (VI), (VII), (VIII), and (IX).

Thus for each magnetic beam field there is an electric beam field with closely related properties. We will call two beams complementary if the angular spectra $\mathscr{E}_x(p,q)$ and $\mathscr{E}_y(p,q)$ for one are equal at each point to the angular spectra $\mathscr{B}_x(p,q)$ and $\mathscr{B}_y(p,q)$ respectively for the other. Complementary beams have very different field components. The field components in an electric beam field are related to those in the complementary magnetic beam field in the same manner as the field components of an electric multipole field are related to those of the complementary magnetic multipole:

$$\mathbf{E}_e \to -\mathbf{B}_m, \tag{18a}$$

$$\mathbf{B}_e \to \mathbf{E}_m \tag{18b}$$

[15, p. 547]. This is readily observed by comparing Eqs. (2) with (7). It is this property which suggested the names that we have chosen for the two classes of beams.

Complementary beams have the same field components over the paraxial region about the z axis away from focus. This is evident from Eqs. (6) or (10). Thus two complementary beam fields are asymptotically the same in the limit as the degree of collimation is increased.

RECTANGULAR AND CYLINDRICAL CONSISTENT BEAMS

Some consistent beams which are closely related to the fields observed from some lasers may now be introduced using the formulation described in the earlier sections. These Hermite Gaussian and Laguerre Gaussian laser beams have been studied previously by theories which employ a paraxial approximation [for example, Ref. 2, p. 1316]. However the theory developed here is much more general. It is valid when the paraxial approximation fails, and it gives results in agreement with the paraxial theories for very well collimated beams.

The electromagnetic field given by the angular spectra

$$\mathscr{E}_x(p,q) = \frac{e_x}{\lambda^2}\, \alpha_{N,n}\, \Phi_{N,n}\, (k^2\sigma^2\rho_0^2,\rho)\, \begin{matrix} \sin N\phi \\ \cos N\phi \end{matrix}\, \mathrm{Rect}(\rho/2\rho_0), \tag{19a}$$

$$\mathscr{E}_y(p,q) = \frac{e_y}{\lambda^2}\, \alpha_{N,n}\, \Phi_{N,n}\, (k^2\sigma^2\rho_0^2,\rho)\, \begin{matrix} \sin N\phi \\ \cos N\phi \end{matrix}\, \mathrm{Rect}(\rho/2\rho_0) \tag{19b}$$

is an important magnetic beam which is closely related to the Laguerre Gaussian beam of the paraxial theory. In Eq. (19), p,q are given in polar coordinates by the transformation

$$p = \rho\cos\phi, \tag{20a}$$

$$q = \rho\sin\phi, \tag{20b}$$

$\Phi_{N,n}(c,\rho)$ are the circular prolate spheroidal functions as given by Frieden [16, p. 313], and

$$\mathrm{Rect}(\rho/2\rho_0) = 1, \text{ if } -\rho_0 \leqslant \rho \leqslant \rho_0, \tag{21a}$$

$$= 0, \text{ otherwise.} \tag{21b}$$

The integers N,n, the real constants ρ_0,σ, and the complex constants e_x,e_y which appear in Eqs. (19) can be chosen to fit a particular field. The constant $\alpha_{N,n}$ will be specified in the following.

The circular prolate functions have the following mathematical properties which recommend them for this application. They are eigenfunctions for the finite Hankel transform:

$$\int_0^{\rho_0} \Phi_{N,n}(c,\rho)\, J_N(kr\rho)\rho\, d\rho = (-1)^n\, (\rho_0^2/c)\sqrt{\lambda_{N,n}}\ \Phi_{N,n}(c,\, kr\rho_0^2/c). \tag{22}$$

For very large values of c they have the asymptotic form

$$\Phi_{N,n}(c,\rho) \sim c^{1/4}\sqrt{\frac{2n!\lambda_{N,n}}{(N+n)!\rho_0}}\left(\frac{c}{\rho_0^2}\right)^{(N/2)+(1/4)} \rho^N L_n^{(N)}\left(\frac{c\rho^2}{\rho_0^2}\right) e^{-c\rho^2/2\rho_0^2}, \tag{23}$$

within the domain $0 \leqslant \rho \leqslant \rho_0 c^{-1/4}$, and are relatively small in value outside of this domain. And the expression

$$\Phi_{Nn}(c,\rho) \begin{array}{c}\sin N\phi \\ \cos N\phi\end{array}$$

can be extended as an entire function of complex $\bar{p},\bar{q}$. In Eq. (23), $L_n^{(N)}(x)$ are Laguerre polynomials of degree n (Szegö's notation). For large c the eigenvalues in Eq. (22) have the asymptotic form

$$\lambda_{N,n} \sim 1 - \frac{\pi 2^{2N+4n+3}\, c^{N+2n+1}\, e^{-2c}}{n!(N+n)!}. \tag{24}$$

The angular spectra given by Eqs. (19) obey Conditions (I) and (III), because of Eqs. (21), if we set $\rho_0 = \sin\theta_0$. These spectra also obey Condition (II), since $\Phi_{N,n}(c,\rho)$ and the other functions appearing in Eqs. (19) are real. Thus the field satisfies the definition for a magnetic beam field.

The transverse electric component of the field in the focal plane is found by substituting Eqs. (19) into (2) as

$$E_x(x,y,0) = \frac{e_x}{\lambda^2}\alpha_{N,n}\ \frac{(-1)^n i^N\ 2\pi\sqrt{\lambda_{N,n}}}{k^2\sigma^2}\,\Phi_{N,n}(k^2\sigma^2\rho_0^2, r'/k\sigma^2) \begin{array}{c}\sin N\phi' \\ \cos N\phi'\end{array}, \tag{25a}$$

$$E_y(x,y,0) = \frac{e_y}{\lambda^2}\alpha_{N,n}\ \frac{(-1)^n i^N\ 2\pi\sqrt{\lambda_{N,n}}}{k^2\sigma^2}\,\Phi_{N,n}(k^2\sigma^2\rho_0^2, r'/k\sigma^2) \begin{array}{c}\sin N\phi' \\ \cos N\phi'\end{array}, \tag{25b}$$

where x,y are given in polar coordinates by the transformation

$$x = r'\cos\phi', \tag{26a}$$

$$y = r'\sin\phi'. \tag{26b}$$

The transverse electric component away from focus (as kr approaches infinity) is found by substituting Eqs. (19) into (5) as

$$E_x(x,y,z) \sim -i\,\frac{e_x\alpha_{N,n}}{\lambda}\,\Phi_{N,n}\!\left(k^2\sigma^2\rho_0^2,\frac{\pm r'}{r}\right)\begin{matrix}\sin N\phi'\\ \cos N\phi'\end{matrix},\ \mathrm{Rect}(r'/2\rho_0 r)\,\frac{e^{\pm ikr}}{r}, \tag{27a}$$

$$E_y(x,y,z) \sim -i\,\frac{e_y\alpha_{N,n}}{\lambda}\,\Phi_{N,n}\left(k^2\sigma^2\rho_0^2,\frac{\pm r'}{r}\right)\begin{matrix}\sin N\phi'\\ \cos N\phi\end{matrix},\ \mathrm{Rect}(r'/2\rho_0 r)\,\frac{e^{\pm ikr}}{r}. \tag{27b}$$

In deriving Eq. (25), we have made use of the expression

$$\iint_{p^2+q^2\leqslant\rho_0^2}\left[\Phi_{N,n}(c,\rho)\begin{matrix}\sin N\phi\\ \cos N\phi\end{matrix}\right]e^{ik(px+qy)}\,dp\,dq$$

$$= (-1)^n i^N 2\pi\left(\frac{\rho_0^2}{c}\right)\sqrt{\lambda_{N,n}}\left[\Phi_{N,n}(c,kr\rho_0^2/c)\begin{matrix}\sin N\phi'\\ \cos N\phi'\end{matrix}\right], \tag{28}$$

which follows directly from Eq. (22) [17, p. 3015].

By substituting Eqs. (19) into Condition (IV), it is clear from Eq. (28) that the field obeys the condition within the domain D, given through Eq. (17) by the parameters

$$d_x = d_y = k\,\sigma^2,\ \text{with}\ \theta_0' = \theta_0. \tag{29}$$

However, the field does not obey Condition (V) except in the limit as $k^2\sigma^2\rho_0^2$ becomes large, because the prolate function which appears in Eqs. (25) is large outside D except in this limit [16, p. 338]. Thus this beam is consistent but not strongly consistent except in this limit.

The field becomes a strongly consistent magnetic beam field in the limit as $k\sigma$ becomes large, if ρ_0 is held constant. In this limit the asymptotic representation for the angular spectra, by substituting Eq. (23) into Eqs. (19), is

$$\mathscr{E}_x(p,q) \sim \frac{e_x}{\lambda^2}\begin{matrix}\sin N\phi\\ \cos N\phi\end{matrix}\quad(k\sigma\rho)^N\left[\frac{n!N!}{(n+N)!}\,L_n^{(N)}(k^2\sigma^2\rho^2)\right]e^{-k^2\sigma^2\rho^2/2}, \tag{30a}$$

$$\mathscr{E}_y(p,q) \sim \frac{e_y}{\lambda^2}\begin{matrix}\sin N\phi\\ \cos N\phi\end{matrix}\quad(k\sigma\rho)^N\left[\frac{n!N!}{(n+N)!}\,L_n^{(N)}(k^2\sigma^2\rho^2)\right]e^{-k^2\sigma^2\rho^2/2} \tag{30b}$$

within the domain $0 \leqslant \rho/\rho_0 \leqslant 1/\sqrt{k\sigma\rho_0}$, where we have defined

$$\alpha_{N,n} = \frac{N!}{k\sigma}\ \sqrt{\frac{n!}{2(n+N)!\ \lambda_{N,n}}}. \tag{31}$$

The field in the focal plane in this limit becomes, by substituting from Eq. (23) into Eqs. (25),

$$E_x(x,y,0) \sim e_x \alpha'_{N,n} \begin{matrix}\sin N\phi' \\ \cos N\phi'\end{matrix} \left[\frac{n!N!}{(n+N)!} L_n^{(N)}\left(\frac{r^2}{\sigma^2}\right)\right] \left(\frac{r}{\sigma}\right)^N \frac{e^{-r^2/2\sigma^2}}{2\pi\sigma^2}, \tag{32a}$$

$$E_y(x,y,0) \sim e_y \alpha'_{N,n} \begin{matrix}\sin N\phi' \\ \cos N\phi'\end{matrix} \left[\frac{n!N!}{(n+N)!} L_n^{(N)}\left(\frac{r^2}{\sigma^2}\right)\right] \left(\frac{r}{\sigma}\right)^N \frac{e^{-r^2/2\sigma^2}}{2\pi\sigma^2}, \tag{32b}$$

within the domain $0 \leqslant r'/\sigma \leqslant \sqrt{k\sigma\rho_0}$, and the field away from focus becomes, by substituting Eqs. (23) and (24) into (27),

$$E_x(x,y,z) \sim -i\,\frac{e_x}{\lambda}\frac{z}{r} \begin{matrix}\sin N\phi' \\ \cos N\phi'\end{matrix} \left(\frac{k\sigma r}{r'}\right)^N$$

$$\times \left[\frac{n!N!}{(n+N)!} L_n^{(N)}\left(\frac{k^2\sigma^2 r'^2}{r^2}\right)\right] e^{-\frac{k^2\sigma^2 r'^2}{2r^2}} \frac{e^{\pm ikr}}{r}, \tag{33a}$$

$$E_y(x,y,z) \sim -i\,\frac{e_y}{\lambda}\frac{z}{r} \begin{matrix}\sin N\phi' \\ \cos N\phi'\end{matrix} \left(\frac{k\sigma r}{r'}\right)^N$$

$$\times \left[\frac{n!N!}{(n+N)!} L_n^{(N)}\left(\frac{k^2\sigma^2 r'^2}{r^2}\right)\right] e^{-\frac{k^2\sigma^2 r'^2}{2r^2}} \frac{e^{\pm ikr}}{r} \tag{33b}$$

within the domain $0 \leqslant r'/r \leqslant \sqrt{\rho_0/k\sigma}$, where

$$\begin{aligned}\alpha'_{N,n} &= (-1)^n\, i^N \sqrt{\lambda_{N,n}} \\ &\sim (-1)^n\, i^N, \text{ as } k\sigma \to \infty.\end{aligned} \tag{34}$$

Finally we determine the most readily observable feature for a light beam, the intensity, defined as the magnitude of the Poynting vector. The intensity of this beam away from focus, by substituting Eqs. (30) into (13), is

$$I(x,y,z) = \frac{\pm ck^2}{16\pi^3 r^2} \left(|e_x|^2 + |e_y|^2\right) \begin{matrix}\sin^2 N\phi' \\ \cos^2 N\phi'\end{matrix} \left(\frac{k\sigma r'}{r}\right)^{2N}$$

$$\times \left[\frac{n!N!}{(n+N)!} L_n^{(N)}\left(\frac{k^2\sigma^2 r'^2}{r^2}\right)\right]^2 e^{-k^2\sigma^2 r'^2/r^2} \sin\phi_B. \tag{35}$$

From Eq. (35) we observe that as the parameter $k\sigma$ is increased, the field becomes more localized about the z axis; that is, it becomes more highly collimated. Thus Eqs. (30) through (35) give the asymptotic approximation for the magnetic beam field defined by Eqs. (19) in the limit as it becomes more highly collimated. In this limit we have observed that the field becomes a strongly consistent magnetic beam.

The strongly consistent magnetic beam field described by Eqs. (30) through (35) is the same Laguerre Gaussian beam which appears in the paraxial theory [2, p. 1317]. The formulation we have derived is a generalization of the paraxial theory to beams of arbitrary divergence from focus. The physical significance of some of the parameters appearing in these equations can be seen by comparing them with the similar equations appearing in the paraxial theory. It is clear that e_x and e_y specify the maximum amplitude of the beam, σ specifies the divergence from focus,* and ρ_0 does not appear in the paraxial expressions and therefore can affect only the poorly collimated beam.

A second, very different, magnetic beam field is defined by the angular spectra

$$\mathscr{E}_x(p,q) = \frac{e_x}{\lambda^2}\,\beta_{nm}\;\psi_n(k^2\sigma_x^2 p_0^2,p)\;\psi_m\;(k^2\sigma_y^2 q_0^2,q)\;\mathrm{Rect}\left(\frac{p}{2p_0}\right)\mathrm{Rect}\left(\frac{q}{2q_0}\right), \quad (36a)$$

$$\mathscr{E}_y(p,q) = \frac{e_y}{\lambda^2}\,\beta_{nm}\;\psi_n\;(k^2\sigma_x^2 p_0^2,p)\;\psi_m\;(k^2\sigma_y^2 q_0^2,q)\;\mathrm{Rect}\left(\frac{p}{2p_0}\right)\mathrm{Rect}\left(\frac{q}{2q_0}\right). \quad (36b)$$

This type of beam exhibits rectangular rather than cylindrical symmetry about the z axis and is closely related to the Hermite Gaussian beam of paraxial theory. In Eqs. (36), $\psi_n(c,p)$ are linear prolate functions as given by Frieden [16], and β_{nm} is a constant which will be defined in the following. The integers n,m, the complex constants e_x,e_y, and the real constants $p_0,q_0,\sigma_x,\sigma_y$ may be chosen at will to fit a particular physical beam field.

The linear prolate functions are recommended for this application by the following mathematical properties. They are eigenfunctions for the finite Fourier transform:

$$\int_{-p_0}^{p_0} \psi_n\;(c,p)\,e^{ikxp}\;dp = i^n\sqrt{\frac{2\pi\,\lambda_n}{c}}\,p_0\;\psi_n\left(c,\frac{kxp_0^2}{c}\right). \quad (37)$$

They are entire functions of complex $\bar{p},\bar{q}$ which are real on the real axis. And for large c they have the asymptotic form

$$\psi_n(c,p) \sim \sqrt{\frac{\lambda_n}{p_0\;\bar{N}_n\;2^n}}\;H_n\,(\sqrt{c}p/p_0)\,e^{-\frac{cp^2}{2p_0^2}} \quad (38)$$

within the domain $0 \leqslant p \leqslant p_0 c^{-1/3}$ and are small at points outside of this domain. In Eq. (38), $H_n(x)$ are Hermite polynomials and

$$\bar{N}_n = \sqrt{\pi/c}\,n!\;[1+2^{-7}(4c^2)^{-1}\;(n^4+2n^3+23n^2+22n+12)]. \quad (39)$$

For large values of c the eigenvalues in Eqs. (30) have the asymptotic form

$$\lambda_n(c) \sim 1 - 4\sqrt{\pi}\;(n!)^{-1}2^{3n}\;c^{n+(1/2)}e^{-2c}\;[1-(32c)^{-1}\;(6n^2-2n+3)]. \quad (40)$$

*In the usual paraxial theory, parameters like σ, σ_x, and σ_y are given as the standard deviations of Gaussian functions which describe the amplitude distribution of some field component over the focal plane. However, as shown in Ref. 4, this is meaningful only for very well collimated beams. The divergence of a beam from focus is given by these parameters much more generally.

It can be readily proven that the angular spectra given by Eqs. (36) obey Conditions (I) through (III), so that the associated field is a magnetic beam, provided that we set $p_0^2 + q_0^2 = \sin^2 \theta_0$.

Proceeding as before, the transverse electric component of the field is found, by substituting Eqs. (36) into (2), to be

$$E_x(x,y,0) = e_x \beta_{nm} \frac{i^n i^m \sqrt{\lambda_n \lambda_m}}{2\pi \sigma_x \sigma_y} \psi_n(k^2\sigma_x^2 p_0^2, x/k\sigma_x^2)\ \psi_m(k^2\sigma_y^2 q_0^2, y/k\sigma_y^2), \quad \text{(41a)}$$

$$E_y(x,y,0) = e_y \beta_{nm} \frac{i^n i^m \sqrt{\lambda_n \lambda_m}}{2\pi \sigma_x \sigma_y} \psi_n(k^2\sigma_x^2 p_0^2, x/k\sigma_x^2)\ \psi_m(k^2\sigma_y^2 q_0^2, y/k\sigma_y^2), \quad \text{(41b)}$$

where we have used the expression

$$\int_{-p_0}^{p_0}\int_{-q_0}^{q_0} [\psi_n(c_x,p)\ \psi_m\ (c_y,q)]\, e^{ik(px+qy)}\, dp\, dq$$

$$= i^{n+m}\, 2\pi p_0 q_0 \sqrt{\frac{\lambda_n \lambda_m}{c_x c_y}} \left[\psi_n\left(c_x, \frac{kx p_0^2}{c_x}\right) \psi_m\left(c_y, \frac{ky q_0^2}{c_y}\right)\right], \quad \text{(42)}$$

which follows directly from Eq. (37). The transverse electric field away from focus (as kr approaches infinity) is, by substituting Eqs. (36) into (5),

$$E_x(x,y,z) \sim \frac{-ie_x}{\lambda} \frac{z}{r} \beta_{nm}\ \psi_n\left(k^2\sigma_x^2 p_0^2, \frac{\pm x}{r}\right) \psi_m\left(k^2\sigma_y^2 q_0^2, \frac{\pm y}{r}\right) \frac{e^{\pm ikr}}{r}, \quad \text{(43a)}$$

$$E_y(x,y,z) \sim \frac{-ie_y}{\lambda} \frac{z}{r} \beta_{nm}\ \psi_m\left(k^2\sigma_x^2 p_0^2, \frac{\pm x}{r}\right) \psi_m\left(k^2\sigma_y^2 q_0^2, \frac{\pm y}{r}\right) \frac{e^{\pm ikr}}{r}. \quad \text{(43b)}$$

By substituting Eqs. (36) into Condition (IV), it is clear from Eq. (37) that the field obeys this condition everywhere within the domain D given through Eq. (17) by the parameters

$$\begin{aligned} \frac{x}{d_x} &\equiv \frac{x}{k\sigma_x^2} \leqslant p_0, \\ \frac{y}{d_y} &\equiv \frac{y}{k\sigma_y^2} \leqslant q_0. \end{aligned} \quad \text{(44)}$$

However this field does not obey Condition (V) except in the limit as $k^2\sigma_x^2 p_0^2$ and $k^2\sigma_y^2 p_0^2$ becomes large because the prolate functions which appear in Eqs. (41) are large outside D except in this limit [16, p. 322]. Thus this beam is consistent but is not strongly consistent except in this limit.

The field becomes a strongly consistent magnetic beam field in the limit as $k\sigma_x$ and $k\sigma_y$ becomes large if p_0 and q_0 are held constant. In this limit the angular spectra, by substituting Eq. (38) into Eqs. (36), are given asymptotically by

$$\mathscr{E}_x(p,q) \sim \frac{e_x}{\lambda^2}\left[\frac{(-1)^{(n+m)/2}\,(n/2)!\,(m/2)!}{n!\,m!} H_n(k\sigma_x p)\, H_m(k\sigma_y q)\right] e^{-(k^2/2)(\sigma_x^2 p^2+\sigma_y^2 q^2)}, \tag{45a}$$

$$\mathscr{E}_y(p,q) \sim \frac{e_y}{\lambda^2}\left[\frac{(-1)^{(n+m)/2}\,(n/2)!\,(m/2)!}{n!\,m!} H_n(k\sigma_x p)\, H_m(k\sigma_y q)\right] e^{-(k^2/2)(\sigma_x^2 p^2+\sigma_y^2 q^2)} \tag{45b}$$

within the domain where $|k\sigma_x p| \leqslant \sqrt[3]{k\sigma_x p_0}$ and $|k\sigma_y q| \leqslant \sqrt[3]{k\sigma_y q_0}$, where β_{nm} has been defined as

$$\beta_{nm} = \frac{(-1)^{(n+m)/2}\,(n/2)!\,(m/2)!}{n!\,m!}\sqrt{\frac{2^{n+m}p_0 q_0 \overline{N}_n \overline{N}_m}{\lambda_n\,\lambda_m}}. \tag{46}$$

Similarly, by substituting Eq. (38) into Eqs. (41), we have

$$E_x(x,y,0) \sim e_x\beta_{nm}'\left[\frac{(-1)^{(n+m)/2}\,(n/2)!\,(m/2)!}{n!\,m!} H_n\left(\frac{x}{\sigma_x}\right) H_m\left(\frac{y}{\sigma_y}\right)\right] \times \frac{e^{-(1/2)[(x^2/\sigma_x^2)+(y^2/\sigma_y^2)]}}{2\pi\sigma_x\sigma_y}, \tag{47a}$$

$$E_y(x,y,0) \sim e_y\beta_{nm}'\left[\frac{(-1)^{(n+m)/2}\,(n/2)!\,(m/2)!}{n!\,m!} H_n\left(\frac{x}{\sigma_x}\right) H_m\left(\frac{y}{\sigma_y}\right)\right] \times \frac{e^{-(1/2)[(x^2/\sigma_x^2)+(y^2/\sigma_y^2)]}}{2\pi\sigma_x\sigma_y} \tag{47b}$$

within the domain where $|x/\sigma_x| \leqslant \sqrt[3]{k\sigma_x p_0}$ and $|y/\sigma_y| \leqslant \sqrt[3]{k\sigma_y p_0}$, in the focal plane, and with

$$\begin{aligned}\beta_{nm}' &= i^{n+m}\sqrt{\lambda_n\,\lambda_m}\\ &\sim i^{n+m}, \text{ as } k^2\sigma_x^2 p_0^2 \to \infty \text{ and } k^2\sigma_y^2 q_0^2 \to \infty.\end{aligned} \tag{48}$$

And, by substituting Eq. (38) into Eqs. (43), we have

$$E_x(x,y,z) \sim \frac{-ie_x}{\lambda}\frac{z}{r}\left[\frac{(-1)^{(n+m)/2}\,(n/2)!\,(m/2)!}{n!\,m!}\,H_n\left(\frac{\pm k\sigma_x x}{r}\right)H_m\left(\frac{\pm k\sigma_y y}{r}\right)\right]$$

$$\times\, e^{-(k^2/2r^2)(\sigma_x^2x^2+\sigma_y^2y^2)}\,\frac{e^{\pm ikr}}{r}, \qquad (49a)$$

$$E_y(x,y,z) \sim \frac{-ie_y}{\lambda}\frac{z}{r}\left[\frac{(-1)^{(n+m)/2}\,(n/2)!\,(m/2)!}{n!m!}\,H_n\left(\frac{\pm k\sigma_x x}{r}\right)H_m\left(\frac{\pm k\sigma_y y}{r}\right)\right]$$

$$\times\, e^{-(k^2/2r^2)(\sigma_x^2x^2+\sigma_y^2y^2)}\,\frac{e^{\pm ikr}}{r} \qquad (49b)$$

within the domain where $|x/r| \leqslant \sqrt[3]{p_0/k^2\sigma_x^2}$ and $|y/r| \leqslant \sqrt[3]{q_0/k^2\sigma_y^2}$, for the region away from focus where kr is large. The intensity in this region may now be found, by substituting Eqs. (49) into (13), to be

$$I(x,y,z) = \frac{\pm ck^2}{16\pi^3 r^2}\left(|e_x|^2+|e_y|^2\right)\left[\frac{(-1)^{(n+m)/2}\,(n/2)!\,(m/2)!}{n!m!}\,H_n\left(\frac{\pm k\sigma_x x}{r}\right)H_m\left(\frac{\pm k\sigma_y y}{r}\right)\right]^2$$

$$\times\, e^{-(k^2/r^2)(\sigma_x^2x^2+\sigma_y^2y^2)}\,\sin^2\phi_B. \qquad (50)$$

This equation indicates that as the parameters $k\sigma_x$ and $k\sigma_y$ increase, the beam becomes more highly collimated. Thus Eqs. (45) through (50) give the asymptotic approximation for the beam defined by Eqs. (36) in the limit as the beam becomes well collimated. Although the beam is not always strongly consistent, it is in this limit.

The strongly consistent magnetic beam field described by Eqs. (45) through (50) is the same as the Hermite Gaussian beam which appears in the paraxial theory [2, p. 1316]. Thus the field defined by Eqs. (36) is a generalization of the Hermite Gaussian beam to fields of arbitrary divergence from focus. By comparison of the equations appearing here with those appearing in the paraxial theory, we see that σ_x and σ_y specify the divergence from focus,* while p_0,q_0 do not appear in the paraxial expressions; therefore they can affect only the poorly collimated beam. For the special case where $n = m = 0$ this field becomes an elliptical Gaussian beam. This special case has also been studied through the use of a different magnetic beam which is not consistent [4]. As this beam becomes very well collimated, it is asymptotically the same as that given here.

In this section the treatment has been limited to only magnetic beam fields. However for each of these beams there is a complementary electric beam with similar properties. Lasers produce beams of light which frequently have intensity distributions similar to

*In the usual paraxial theory, parameters like σ, σ_x, and σ_y are given as the standard deviations of Gaussian functions which describe the amplitude distribution of some field component over the focal plane. However, as shown in Ref. 4, this is meaningful only for very well collimated beams. The divergence of a beam from focus is given by these parameters much more generally.

those in Eqs. (35) and (50). But, upon comparing Eq. (13) with (14), we see that for well collimated beams, where ϕ_B and ϕ_E are approximately 90° over the range of x,y for which the intensity is not negligible, complementary beams produce indistinguishable intensity distributions. Thus the difference between complementary beams becomes physically significant for light only as the divergence from focus is greatly increased.

The anomalies which appear in the intensity patterns due to failure of the paraxial approximation are not so difficult to observe. In an earlier work [5] it was found that as the half angle which the beam boundary* makes with the z axis is increased to only 10°, clearly observable failure of the paraxial approximation occurs within the focal region.

CONNECTION WITH MULTIPOLE FIELDS

It was noted in two earlier papers [4,5] that the field distributions for some beams approach a modified dipole field as the beam divergence is increased. It was shown in the present report that complementary beams have electromagnetic compounds which are related like those of similar electric and magnetic multipole fields. The connection between beam and multipole fields is discussed here in more detail.

The electromagnetic field due to an electric dipole at the origin with a dipole moment $\mathbf{u} = (u_x, y_y, u_z)$ is given by the vector potential [15, p. 271]

$$\mathbf{A}(x,y,z) = -ik\,\mathbf{u}\,\frac{e^{ikr}}{r}. \tag{51}$$

By using an identity due to Weyl [18], namely

$$\frac{e^{ikr}}{r} = \frac{-ik}{2\pi}\int\!\!\int_{-\infty}^{\infty} \frac{1}{m}\, e^{ik(px+qy+mz)}\, dp\, dq, \tag{52}$$

where m is given by Eqs. (3) and $z \geqslant 0$, and using the relations

$$\mathbf{B}(x,y,z) = \nabla \times \mathbf{A}(x,y,z), \tag{53}$$

$$\mathbf{E}(x,y,z) = (i/k)\,\nabla \times \mathbf{B}(x,y,z), \tag{54}$$

we find that the dipole field can be given in the $z \geqslant 0$ half space by

$$E_x(x,y,z) = \frac{-ik^3}{2\pi}\int\!\!\int_{-\infty}^{\infty} \left[\left(\frac{u_y p - u_x q}{m}\right) q + (u_z p - u_x m)\right] e^{ik(px+qy+mz)}\, dp\, dq, \tag{55a}$$

$$E_y(x,y,z) = \frac{-ik^3}{2\pi}\int\!\!\int_{-\infty}^{\infty} \left[(u_z q - u_y m) + \left(\frac{u_x q - u_y p}{m}\right) p\right] e^{ik(px+qy+mz)}\, dp\, dq, \tag{55b}$$

*The beam boundary, as defined in Ref. 4, is the surface about the beam axis which is generated by the locus of all beam radii. A beam radius is the maximum distance out perpendicular from the beam axis to a point at which the intensity has fallen to 1/e of the maximum intensity within the same plane of

$$E_z(x,y,z) = \frac{-ik^3}{2\pi} \iint_{-\infty}^{\infty} \left[\left(u_x - \frac{u_z p}{m}\right) p + \left(u_y - \frac{u_z q}{m}\right) q\right] e^{ik(px+qy+mz)}\, dp\, dq, \quad (55c)$$

$$B_x(x,y,z) = \frac{ik^3}{2\pi} \iint_{-\infty}^{\infty} \left(\frac{u_z q}{m} - u_y\right) e^{ik(px+qy+mz)}\, dp\, dq, \quad (55d)$$

$$B_y(x,y,z) = \frac{ik^3}{2\pi} \iint_{-\infty}^{\infty} \left(u_x - \frac{u_z p}{m}\right) e^{ik(px+qy+mz)}\, dp\, dq, \quad (55e)$$

$$B_z(x,y,z) = \frac{ik^3}{2\pi} \iint_{-\infty}^{\infty} \left(\frac{u_y p - u_x q}{m}\right) e^{ik(px+qy+mz)}\, dp\, dq. \quad (55f)$$

Thus, by comparison of Eqs. (55) with (8), we see that the field of an electric dipole polarized in the xy plane ($u_z = 0$) can be described within the half space by the angular spectra

$$\mathscr{B}_x(p,q) = \frac{-ik^3}{2\pi} u_y, \quad (56a)$$

$$\mathscr{B}_y(p,q) = \frac{ik^3}{2\pi} u_x, \quad (56b)$$

which are constant over all p,q. It is evident that the dipole has produced a field containing evanescent plane waves.

In exactly the same manner, the field associated with a magnetic dipole at the origin with a dipole moment $\mathbf{v} = (v_x, v_y, v_z)$ can be expanded throughout the $z \geqslant 0$ half space as

$$E_x(x,y,z) = \frac{ik^3}{2\pi} \iint_{-\infty}^{\infty} \left(\frac{v_z q}{m} - v_y\right) e^{ik(px+qy+mz)}\, dp\, dq, \quad (57a)$$

$$E_y(x,y,z) = \frac{ik^3}{2\pi} \iint_{-\infty}^{\infty} \left(v_x - \frac{v_z p}{m}\right) e^{ik(px+qy+mz)}\, dp\, dq, \quad (57b)$$

$$E_z(x,y,z) = \frac{ik^3}{2\pi} \iint_{-\infty}^{\infty} \left(\frac{v_y p - v_x q}{m}\right) e^{ik(px+qy+mz)}\, dp\, dq, \quad (57c)$$

$$B_x(x,y,z) = \frac{ik^3}{2\pi} \iint_{-\infty}^{\infty} \left[\left(\frac{v_y p - v_x q}{m}\right) q + (v_z p - v_x m)\right] e^{ik(px+qy+mz)}\, dp\, dq, \quad (57d)$$

$$B_y(x,y,z) \frac{ik^3}{2\pi} \iint_{-\infty}^{\infty} \left[(v_z q - v_y m) + \left(\frac{v_x q - v_y p}{m}\right) p\right] e^{ik(px+qy+mz)}\, dp\, dq, \quad (57e)$$

$$B_z(x,y,z) = \frac{ik^3}{2\pi} \iint_{-\infty}^{\infty} \left[\left(v_x - \frac{v_z p}{m}\right) p + \left(v_y - \frac{v_z q}{m}\right) q\right] e^{ik(px+qy+mz)}\, dp\, dq. \quad (57f)$$

So, by comparison of Eqs. (57) with (2), we observe that the field of a magnetic dipole polarized in the xy plane ($v_z = 0$) can be described by the angular spectra

$$\mathscr{E}_x(p,q) = \frac{-ik^3}{2\pi} v_y, \quad (58a)$$

$$\mathscr{E}_y(p,q) = \frac{ik^3}{2\pi} v_x, \quad (58b)$$

which are constant for all p,q. By comparison of Eqs. (55) and (57), we see that these two fields are complementary in the sense which was defined for beams. They have a close connection with beam fields, which we will now explore.

Consider an electric beam field with the angular spectra

$$\mathscr{B}_x(p,q) = e_x\, f(x/a,\, y/b) \text{ Rect } (p/2) \text{ Rect } (q/2), \quad (59a)$$

$$\mathscr{B}_y(p,q) = e_y\, f(x/a,\, y/b) \text{ Rect } (p/2) \text{ Rect } (q/2), \quad (59b)$$

where $f(p/a, q/b)$ is some function chosen such that Conditions (VI) through (VIII) are obeyed. Then as the beam divergence approaches infinity, such that a and b approach infinity, these spectra become

$$\mathscr{B}_x(p,q) \sim e_x f(0,0) \text{ Rect } (p/2) \text{ Rect } (q/2), \quad (60a)$$

$$\mathscr{B}_y(p,q) \sim e_y f(0,0) \text{ Rect } (p/2) \text{ Rect } (q/2). \quad (60b)$$

By comparing these equations with Eqs. (56), we see that any electric beam which does not vanish in this limit ($f(0,0) \neq 0$) approaches an electric dipole field from which all evanescent plane waves have been removed.

The anomalies which occur as the divergence of a beam is increased are related to the field distribution of this dipole field. Such a field does not resemble the empirical concept of a beam. Scalar fields which correspond to B_x and B_y in this field have been studied previously [19].

The magnetic beam complementary to that given by Eqs. (59) bears the same relationship to a magnetic dipole field; that is, as the divergence is increased, the beam approaches a magnetic dipole field from which all evanescent plane waves have been removed.

Since the dipole expansions are valid only in the half space $z \geqslant 0$ and the beam field expansions are valid everywhere but near the plane $z = -\infty$, a beam field does not approach a dipole field in the $z < 0$ half space. In fact the waves travel in opposite directions. However, from the symmetry property given by Eqs. (15), it is clear that such a beam field is otherwise very similar to a dipole field even where $z < 0$.

By substituting Eqs. (60) into (5) and comparing the results with Eqs. (55), we find that this electric beam is exactly the same field away from focus as that produced by this electric dipole. This is because the evanescent plane waves do not contribute significantly to the dipole field away from the source. The same observation can be made for the complementary magnetic beam field.

CONCLUSIONS

In this report we have pointed out that there is more than one representation for an electromagnetic field in free space based on angular spectrum expansions and that each of these representations has particular formal advantages when used to study certain fields, such as beams. The two representations used here are physically equivalent and differ only in the choice of independent variables; however each has a unique value, so that the distinction is not trivial. The equations for the field components and Poynting vector in the first section are sufficiently general to be useful in other theories which do not involve beams.

The empirical concept of a beam is given a precise mathematical definition in the second section. The two classes of beams, electric and magnetic, are mutually exclusive, since Maxwell's equations do not allow both the transverse electric and transverse magnetic components of the beam to have exactly the same distribution about the beam axis. Although complementary beams are asymptotically the same as they become very well collimated, they are never truly identical. The difference between complementary beams may not be measurable near this limit, so that the distinction may appear somewhat academic. However, as the divergence is allowed to increase, complementary beams will become significantly different. Thus the classifications are important.

The theory of Laguerre Gaussian and Hermite Gaussian beams in the third section is much more general than the usual scalar, paraxial treatment. It is valid for beams of arbitrary divergence from focus and provides the following conclusions. Although the beams described by the usual paraxial theories cannot exist in free space [4], consistent beam fields do exist which, as they become very well collimated, asymptotically approach these paraxial beams. As such a beam is made to diverge more rapidly from focus, it remains consistent but it changes in most other respects. It is no longer strongly consistent. The Laguerre Gaussian or Hermite Gaussian amplitude cross section is lost and the beam takes on, more and more, the characteristics of a modified dipole field. Expressions are given for the field components and intensity distribution away from focus which can be used to plot the spatial distribution of these beams, quite accurately and under a wide range of conditions, using the tabulated data for the prolate functions [16].

It is evident from earlier work [4] that other beam fields exist which also asymptotically approach a Gaussian beam as the beam becomes very well collimated. Apparently several different beams can approach the same field in this limit. However the beams given here remain consistent as the divergence is increased. They are the only beams known to have this property. As the divergence from focus is increased, the beam described in this earlier work maintains a Gaussian cross section away from focus but not over the focal plane.

Because of Conditions (VI) and (IX) any consistent electric beam produces a transverse magnetic component which is distributed over the focal plane as an entire function of x,y. Thus this component cannot vanish over the region away from the focal point but must be nonzero almost everywhere in the focal plane. A similar conclusion results for the transverse electric component in a consistent magnetic beam. It follows that the intensity of the beam cannot be completely concentrated in some region about the focus but can only be localized in the sense demanded by Conditions (V) or (X).

The connection between beam fields and multipole fields is interesting and rather curious. Any beam (which does not vanish in this limit) takes on the characteristics of a modified dipole field as it is allowed to diverge more rapidly from focus. In addition the electric and magnetic components of an electric beam are related to those of the complementary magnetic beam just like the components of an electric multipole are related to those of the complementary magnetic multipole. This suggests that a more basic relationship may exist between these two types of fields.

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