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Optimal Piecewise-Constant Control of Dynamic Systems with Applications to Kalman Filtering

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Abstract: This report presents the theory and applications of a suboptimal control algorithm for a general nonlinear control system. This suboptimal algorithm requires the optimal control to be piecewise constant. Necessary conditions are derived for both discrete-time and continuous-time systems. The application to Kalman filtering problems is specifically carried out to illustrate the theoretical results.

INTRODUCTION

In the control of a dynamic system, the optimal controls, in general, are found to be time-varying functions. Difficulties are frequently encountered implementating these time-varying controls, especially in high-order systems. The reasons are twofold; first, the difficulty in storing the complete histories of the values of these time-varying controls; second, the difficulty of synchronizing the time-varying control with the real time (refer to [1] and [2]). Suboptimal algorithms were suggested to overcome these difficulties. Most of the known suboptimal algorithms are derived for a specific class of problems. Johansen [3] gives a lower dimensional controller for the class of linear stochastic systems. Athans et al. [2] restrict the feedback gain of linear systems to be a linear combination of known time functions, and in their other paper [4], they restrict the feedback gain of the linear system to be piecewise-constant. The concept of piecewise-constant control seems more appealing than other suboptimal algorithms.

This report intends to extend the idea of optimal piecewise-constant control to *nonlinear* control systems. Both discrete-time and continuous-time systems are considered. Necessary conditions are derived for each case. It is shown directly that the problems treated in [2] and [4] can be solved by applying the necessary conditions derived in this report. Numerous applications are expected. The application to the Kalman filtering problem is specifically treated in this report where the Kalman gain matrices are restricted to be piecewise-constant.

Notation

Small letters $x, y, \ldots$ represent vectors. Capital letters $A, B, \ldots$ represent matrices. The transpose of A is A^T. The trace of A is Trace (A). The expected value of x is $E[x]$.

2. OPTIMAL PIECEWISE-CONSTANT CONTROL OF DISCRETE-TIME NONLINEAR SYSTEMS

Consider the following problem:

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NRL Problem B01-10; Project RR 003-02-41-6152. This is a final report on one phase of the problem; work is continuing on other phases. Manuscript submitted April 19, 1972.

$$\text{minimize} \quad \mathcal{J} = \phi(x_N) + \sum_{i=0}^{N-1} L(x_i, u_i) , \tag{1}$$

subject to

$$x_{i+1} = f_i(x_i, u_i) ; \quad x_0 = c , \tag{2}$$

for $i = 0, 1, \ldots, (N-1)$, where x_i is an n vector and u_i is an m-dimensional control vector to be chosen to minimize $\mathcal{J}$. Instead of optimizing $\mathcal{J}$ with respect to u_i for $i = 0, 1, \ldots, (N-1)$, the u_i are restricted to be piecewise-constant, i.e.,

$$u_i = \alpha_j \quad \text{for} \quad k_j \leqslant i < k_{j+1} , \tag{3}$$

where k_j, for $j = 1, 2, \ldots, J$, divides the interval $[0, N]$ into $(J-1)$ subintervals, and $k_1 < k_2 < k_3 \ldots < k_J$, $k_1 = 0$ and $k_J = N$. In this section, the set $\{k_j\}$ is predetermined. The problem is to choose the set $\{\alpha_j / j = 1, 2, \ldots, (J-1)\}$ to minimize $\mathcal{J}$.

Necessary conditions and sufficient conditions are easily derived in the following by applying the general technique of parameter optimization [5].

Adjoining Eq. (2) to $\mathcal{J}$ with the Lagrange multiplier sequence λ_i, Eq. (1) becomes

$$\mathcal{J} = \phi(x_N) + \sum_{i=0}^{N-1} \left[L_i(x_i, u_i) + \lambda_{i+1}^T (f_i(x_i, u_i) - x_{i+1}) \right] . \tag{4}$$

For convenience, we use Eq. (3) and define the Hamiltonian sequence

$$H_i(x_i, \alpha_j, \lambda_{i+1}) = L_i(x_i, \alpha_j) + \lambda_{i+1}^T f_i(x_i, \alpha_j) , \quad k_j \leqslant i < k_{j+1} . \tag{5}$$

For a differential change in α_j, the differential change in $\mathcal{J}$ is

$$d\mathcal{J} = \left(\frac{\partial \phi(x_N)}{\partial x_N} - \lambda_N^T \right) dx_N + \sum_{j=1}^{J-1} \left(\sum_{i=k_j}^{k_{j+1}-1} \frac{\partial H_i}{\partial \alpha_j} \right) d\alpha_j$$

$$+ \sum_{i=1}^{N-1} \left(\frac{\partial H_i}{\partial x_i} - \lambda_i^T \right) dx_i . \tag{6}$$

Let

$$\lambda_i^T = \frac{\partial H_i(x_i, \alpha_j, \lambda_{i+1})}{\partial x_i} , \quad k_j \leqslant i < k_{j+1} \tag{7}$$

and

$$\lambda_N^T = \frac{\partial \phi(x_N)}{\partial x_N} . \tag{8}$$

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For $d\mathcal{J} = 0$, we must have

$$\sum_{i=k_j}^{k_{j+1}-1} \frac{\partial H_i}{\partial \alpha_j} = 0 \quad \text{for} \quad j = 1, \ldots, (J-1) , \tag{9}$$

where

$$\sum_{i=k_j}^{k_{j+1}-1} \frac{\partial H_i}{\partial \alpha_j}$$

gives the gradient of J with respect to α_j. Equations (2), (7), (8), and (9) are the necessary conditions. It should be noted that Eq. (9) is replaced by $\partial H_i/\partial u_i = 0$ when the piecewise-constant constraint is not imposed.

3. OPTIMAL PIECEWISE-CONSTANT CONTROL OF CONTINUOUS-TIME NONLINEAR SYSTEMS

Consider the following problem:

$$\text{minimize} \quad \mathcal{J} = \phi(x(t_f)) + \int_0^{t_f} L(x(t), u(t))\, dt , \tag{10}$$

subject to

$$\dot{x} = f(x, u), \quad x(0) = x_0 , \tag{11}$$

where x is an n-dimensional state vector and u is an m-dimensional control vector. The final time t_f is fixed. The problem is to find an optimal piecewise-constant $u(t)$. Let

$$u(t) = \alpha_j , \quad t_j \leqslant t < t_{j+1} , \quad \text{for} \quad j = 1, 2, \ldots, J , \tag{12}$$

where the set $\{t_j\}$ divides the interval $[0, t_f)$ into $(J-1)$ subintervals and $t_1 < t_2 < t_3 \ldots < t_J$, $t_1 = 0$ and $t_J = t_f$. The set $\{\alpha_j / j = 1, 2, \ldots, (J-1)\}$ and the set $\{t_j / i = 2, \ldots, (J-1)\}$ are to be chosen to minimize $\mathcal{J}$. The necessary conditions for this class of problems are obtained by the same technique of Section 2. We define the Hamiltonian

$$H(x(t), \alpha_j, \lambda(t)) = L(x(t), \alpha_j) + \lambda^T(t) f(x(t), \alpha_j), \tag{13}$$

where $t_j \leqslant t < t_{j+1}$. Then the following necessary conditions are obtained (see the Appendix and [6]).

$$\dot{\lambda}^T = -\frac{\partial H(x(t), \alpha_j, \lambda(t))}{\partial x(t)} , \tag{14}$$

$$\lambda^T(t_f) = \frac{\partial \phi}{\partial x(t_f)} , \tag{15}$$

$$\int_{t_j}^{t_{j+1}} \frac{\partial H(\tau)}{\partial \alpha_j} \, d\tau = 0 \tag{16}$$

for $j = 1, 2, \ldots, (J-1)$, and

$$H(x(t_j), \alpha_j, \lambda(t_j)) - H(x(t_j), \alpha_{j-1}, \lambda(t_j)) = 0 \tag{17}$$

for $j = 2, 3, \ldots, (J-1)$.

The left-hand sides of Eqs. (16) and (17) provide the gradient of J with respect to α_j and t_j respectively. The proof of these necessary conditions is given in the Appendix. An alternative proof has been given by Chang [7], using the generalized version of Pontryagin's maximum principle, for the class of curve-fitting problems. Equation (17) is also derived by Jacobson [8] as a necessary condition to obtain the optimal switch time in a bang-bang control problem.

Remarks:

1. The constraint of piecewise-constant control does not complicate the computing process for numerical solution. Rather, it simplifies the process, since the dimensions of the optimization variables are greatly reduced under the piecewise-constant assumption.

2. As has been mentioned, the necessary conditions for the suboptimal *linear* feedback system considered by Athans et al. [2,4] can be directly obtained by applying the necessary conditions (Eqs. (14)-(16)) without complicated derivations. Furthermore, by using Eq. (17), it is possible to optimize numerically the set $\{t_j / j = 2, \ldots, (J-1)\}$ by applying the gradient method.

4. APPLICATIONS TO LINEAR FILTERING PROBLEMS

Numerous applications of this suboptimal control algorithm can be found. As an example, however, we consider the application to linear filtering problems (Kalman filter). One of the difficulties in implementing Kalman filters is the computational complexity, specifically the process of updating the covariance matrix for the purpose of calculating Kalman gain matrices. Hence, in this section, an algorithm for computing a piecewise-constant gain matrix is developed, and a second-order numerical example is presented.

Consider the well-known linear filtering problem:

plant

$$x_{i+1} = \Phi_i x_i + w_i \,, \tag{18}$$

measurement

$$z_i = M x_i + v_i \,, \tag{19}$$

where the n-dimensional x_i is the state to be estimated from the measurements $(z_0, z_1, \ldots, z_i)$, w_i and v_i are white sequences with zero means and correlations $Q_i \delta_{ij}$ and $R_i \delta_{ij}$ respectively, and the initial state x_0 has mean $\bar{x}_0$ and covariance P_0. Let $\hat{x}_i$ be the estimate of x_i using z_j for $j = 1, \ldots, i$. Then it is reasonable to assume $\hat{x}_{i+1}$ to be a linear combination of $\hat{x}_i$ and z_{i+1}, or

$$\hat{x}_{i+1} = A_i \hat{x}_i + B_i z_{i+1} \,; \quad \hat{x}_0 = \bar{x}_0 \,. \tag{20}$$

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The A_i and B_i are to be optimally chosen to minimize the following performance index:

$$\mathcal{J} = E\left[(x_N - \hat{x}_N)^T D_N (x_N - \hat{x}_N)\right] + \sum_{i=0}^{N-1} E\left[(x_i - \hat{x}_i)^T D_i (x_i - \hat{x}_i)\right], \tag{21}$$

where D_i, for $i = 0, \ldots, N$, is a sequence of prescribed weighting matrices and is assumed to be positive semidefinite. The presence of D_i in Eq. (21) increases the generality of $\mathcal{J}$. For example, letting $D_i = 0$ for $i = 0, \ldots, (N-1)$ would result in a problem of minimizing terminal variances.

It has been proved [6] that the optimal time-varying (A_i, B_i) would give a filter which is identical to a Kalman filter [9] no matter what D_i is used. In this report, for the purpose of illustration, only the B_i will be restricted to be piecewise-constant, whereas the A_i are obtained by requiring that $\hat{x}_i$ be an unbiased estimate of x_i, which implies

$$A_i = \Phi_i - B_i M \Phi_i . \tag{22}$$

Substituting Eq. (22) into Eq. (20) gives

$$\hat{x}_{i+1} = \Phi_i \hat{x}_i + B_i (z_{i+1} - M\Phi\hat{x}_i), \tag{23}$$

where B_i corresponds to the Kalman gain matrix.

Defining

$$P_i = E\left[(x_i - \hat{x}_i)(x_i - \hat{x}_i)^T\right] \tag{24}$$

and using Eqs. (18), (19), and (23), we can easily obtain

$$\begin{aligned} P_{i+1} &= (I - B_i M)\Phi_i P_i \Phi_i^T (I - B_i M)^T \\ &\quad + (I - B_i M) Q_i (I - B_i M)^T + B_i R_i B_i^T , \end{aligned} \tag{25}$$

where P_0 is given. Also, $\mathcal{J}$ in Eq. (21) can be written as

$$\mathcal{J} = \text{Trace } \{D_N P_N\} + \sum_{i=0}^{N-1} \text{Trace } \{D_i P_i\} . \tag{26}$$

We have now formulated the linear filtering problem as a member of the class of problems considered in Section 2. Equations (25) and (26) are the concerned equations. We define the Hamiltonian

$$H_i = \text{Trace } \{D_i P_i + \Lambda_{i+1} (\text{right-hand side of Eq. (25)})\} , \tag{27}$$

where Λ_i, the Lagrange multiplier, is an $n \times n$ symmetric matrix. Using the matrix differentiation rule and applying the results of Section 2, the necessary conditions for optimal piecewise-constant B_i can be easily obtained as follows:

$$\Lambda_i = D_i + (I - \beta_j M)^T \Phi_i^T \Lambda_{i+1} \Phi_i (I - \beta_j M) ; \quad k_j \leqslant i < k_{j+1} \tag{28}$$

and

$$\Lambda_N = D_N ; \tag{29}$$

the gradient is

$$\sum_{i=k_j}^{k_{j+1}-1} \left\{-2\Lambda_i(I-\beta_j M)(\Phi_i P_i \Phi_i^T + Q_i)M^T + 2\Lambda_i \beta_j R_i\right\} = 0 , \tag{30}$$

where for piecewise-constant B_i we have used $B_i = \beta_j$ for $k_j \leqslant i < k_{j+1}$. In general, a solution in closed form is not available. Numerical solutions can be computed by applying the steepest-descent method [5] or, for faster convergence, applying the conjugate-gradient method [10]. To illustrate the method developed in this section, we consider the following example.

Example. A second-order linear filtering problem is described by

$$\begin{bmatrix} x_{i+1} \\ y_{i+1} \end{bmatrix} = \begin{bmatrix} 1 & 0.641 \\ -0.641 & 9.74 \end{bmatrix} \begin{bmatrix} x_i \\ y_i \end{bmatrix} + \begin{bmatrix} 0 \\ w_i \end{bmatrix} \tag{31}$$

and

$$z_i = y_i + v_i ,$$

where

$$Q = \begin{bmatrix} 0 & 0 \\ 0 & 0.004 \end{bmatrix} ,$$

$$R = 2.0 ,$$

and

$$P_0 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

The linear filter is of the form

$$\begin{bmatrix} \hat{x}_{i+1} \\ \hat{y}_{i+1} \end{bmatrix} = \begin{bmatrix} 1 & 0.641 \\ -0.641 & 9.74 \end{bmatrix} \begin{bmatrix} \hat{x}_i \\ \hat{y}_i \end{bmatrix} + \begin{bmatrix} a_i \\ b_i \end{bmatrix} (z_{i+1} + 0.641\hat{x}_i - 9.74\hat{y}_i) ;$$

a_i and b_i are the weighting coefficients. The performance index is taken to be the cumulative mean-square error or

$$\mathcal{J} = \sum_{i=1}^{N} E\left\{(x_i - \hat{x}_i)^2 + (y_i - \hat{y}_i)^2\right\} .$$

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The optimal piecewise-constant a_i and b_i were computed by the steepest-descent method for 1, 2, 4, and 8 subintervals. These solutions are shown in Figs. 1, 2, and 3 for $N = 100$. For comparison, the optimal a_i and b_i, computed by Kalman's algorithm, are also included in these figures. It is seen that the piecewise-constant a_i and b_i are fairly close to the "average" value of optimal (Kalman) a_i and b_i. The costs associated with these cases are listed in Table 1.

Table 1
Costs for Various Subintervals

Algorithm	One Subinterval	Two Subintervals	Four Subintervals	Eight Subintervals	Optimal (Kalman)
$\mathcal{J}$	8.6213	8.5363	8.5100	8.4814	8.256

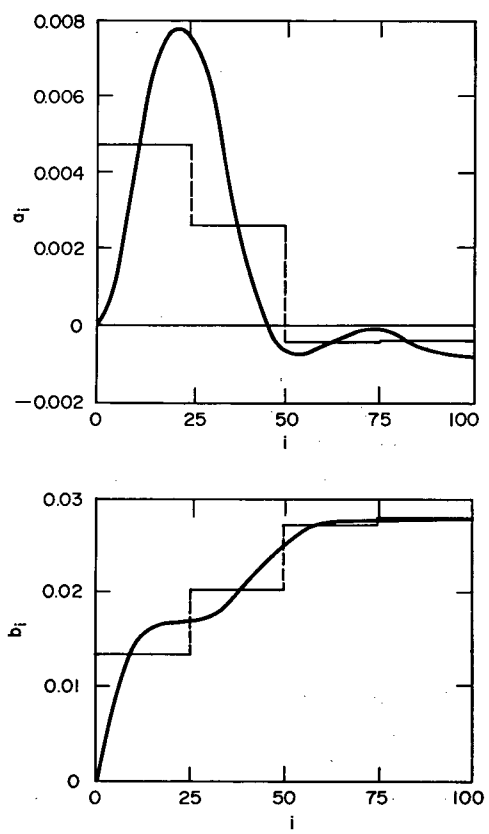

Fig. 1–Optimal piecewise-constant Kalman gain matrices calculated for one subinterval and two subintervals

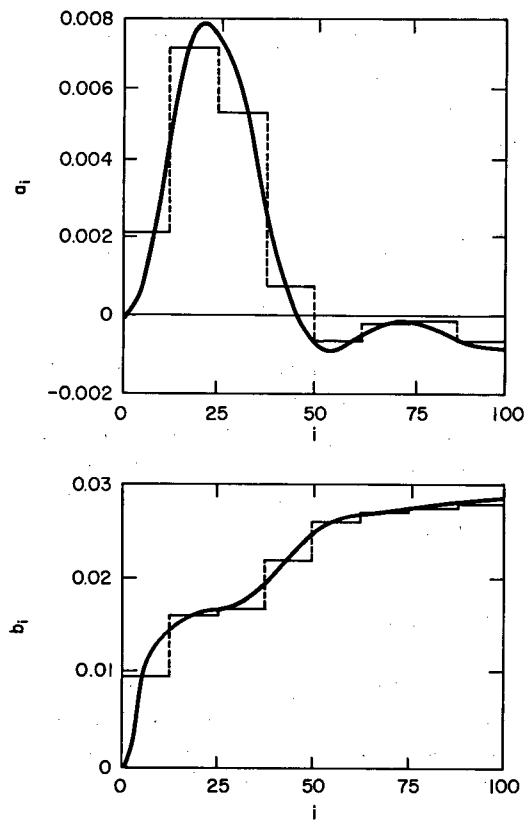

Fig. 2–Optimal piecewise-constant Kalman gain matrices calculated for four subintervals

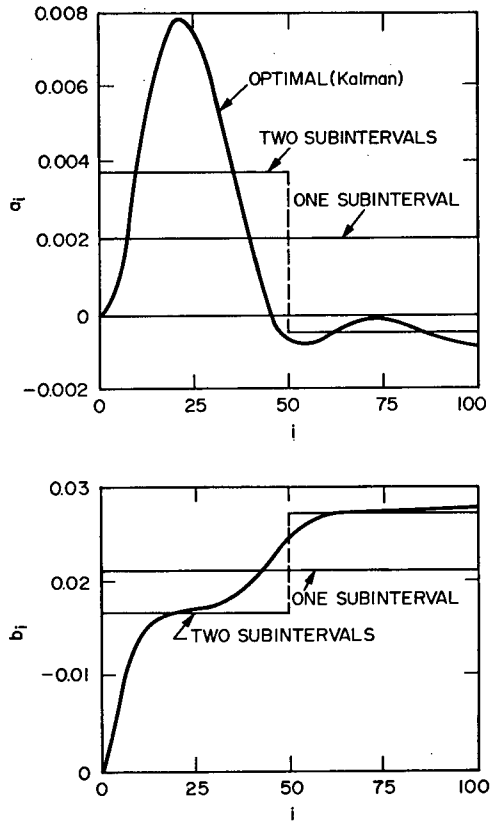

Fig. 3–Optimal piecewise-constant Kalman gain matrices calculated for eight subintervals

It is seen that if the gain is restricted to be constant over the whole interval, the cost is within 5% of the optimal (Kalman) cost. If eight piecewise-constant gains are used, the cost improved to within 3%. Further divisions would bring the cost even closer to the optimal cost but at the expense of implementation difficulties. Application of piecewise-constant control to the continuous-time filtering problem [11] can be similarly formulated and solved by using the necessary conditions derived in Section 3. For further information, refer to [12].

5. CONCLUSION

It has been shown that simple mathematical programming techniques can be used to derive the necessary conditions for nonlinear systems with piecewise-constant control. Applications to the Kalman filtering problem has been specifically considered where the Kalman gain matrices are taken to be piecewise-constant. It is believed that this suboptimal algorithm should find numerous applications in optimal control problems.

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Appendix

DERIVATION OF THE NECESSARY CONDITIONS OF THE SECTION ON OPTIMAL PIECEWISE-CONSTANT CONTROL OF CONTINUOUS-TIME SYSTEMS

This appendix is devoted to deriving the necessary conditions of Section 3. For clarity, we restate the problem:

$$\text{minimize} \quad \mathcal{J} = \phi(x(t_f)) + \int_0^{t_f} L(x(t)\,,\ u(t))\,dt\,, \tag{A1}$$

subject to

$$\dot{x} = f(x, u)\,, \quad x(0) = x_0\,, \tag{A2}$$

and

$$u(t) = \alpha_j \quad \text{for} \quad t_j \leqslant t < t_{j+1}\,, \tag{A3}$$

where $j = 1, 2, \ldots, J$, $t_1 = 0$ and $t_J = t_f$. We define the Hamiltonian

$$H(x(t), \alpha_j, \lambda(t)) = L(x(t), \alpha_j) + \lambda^T(t) f(x(t), \alpha_j) \tag{A4}$$

for $t_j \leqslant t < t_{j+1}$. Then Eq. (A1) becomes

$$\mathcal{J} = \phi(x(t_f)) + \sum_{J=1}^{J-1} \int_{t_j^+}^{t_{j+1}^-} \left[H - \lambda^T \dot{x}\right] d\tau\,. \tag{A5}$$

We consider the differential change in $\mathcal{J}$ for a differential change in $\{\alpha_j\}$ and $\{t_j\}$

$$d\mathcal{J} = \frac{\partial \phi(x(t_f))}{\partial x(t_f)}\, dx(t_f) + \sum_{j=1}^{J-2} L(t_{j+1}^-)\, dt_{j+1} - \sum_{j=2}^{J-1} L(t_j^+)\, dt_j$$

$$+ \sum_{j=1}^{J-1} \int_{t_j^+}^{t_{j+1}^-} \left[\frac{\partial H}{\partial x}\,\delta x + \frac{\partial H}{\partial \alpha_j}\, d\alpha_j - \lambda^T \delta \dot{x}\right] d\tau\,, \tag{A6}$$

where $L(t_j^-) = L(x(t_j), \alpha_{j-1})$ and $L(t_j^+) = L(x(t_j), \alpha_j)$.

Integrating $\int \lambda^T \delta \dot{x}\, d\tau$ by parts and collecting terms gives

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$$d\mathcal{J} = \frac{\partial\phi}{\partial x(t_f)}\, dx(t_f) + \sum_{j=1}^{J-2} \left[L(t_{j+1}^-)\, dt_{j+1} - \lambda^T(t_{j+1}^-)\delta x(t_{j+1})\right]$$

$$- \sum_{j=2}^{J-1} \left[L(t_j^+)\, dt_j - \lambda^T(t_j^+)\delta x(t_j)\right]$$

$$+ \sum_{j=1}^{J-1} \left\{ \int_{t_j^+}^{t_{j+1}^-} \left(\frac{\partial H}{\partial x} + \dot{\lambda}^T\right) \delta x d\tau \right\} + \sum_{j=1}^{J-1} \left\{ \int_{t_j^+}^{t_{j+1}^-} \frac{\partial H}{\partial \alpha_j}\, d\tau \right\} d\alpha_j$$

$$- \left[\lambda^T \delta x\right]_{t=t_f} + \left[\lambda^T \delta x\right]_{t=0} . \tag{A7}$$

Since $\delta x(t_j)$, the variation in $x(t_j)$, means "for t_j held fixed," $dx(t_j)$, the differential in $x(t_j)$, can be written (refer to Section 2.7 of [5]) as

$$dx(t_j) = \delta x(t_j) + \dot{x}(t_j) dt_j$$

for $j = 2, \ldots, (J-1)$ or

$$\delta x(t_j) = dx(t_j) - \dot{x}(t_j) dt_j . \tag{A8}$$

Furthermore, since $t_1(=0)$ and $t_J(=t_f)$ are fixed.

$$\delta x(t_f) = dx(t_f) ; \quad \delta x(0) = dx(0) .$$

Substituting Eq. (A8) into Eq. (A7) and using Eq. (A4), after manipulations Eq. (A7) becomes

$$d\mathcal{J} = \left(\frac{\partial\phi}{\partial x(t_f)} - \lambda^T(t_f)\right) dx(t_f) + \sum_{j=2}^{J-1} \left\{H(t_j^-) - H(t_j^+)\right\} dt_j$$

$$- \sum_{j=2}^{J-1} \left\{\lambda^T(t_j^-) - \lambda^T(t_j^+)\right\} dx(t_j) + \lambda^T(0)\delta x(0)$$

$$+ \sum_{j=1}^{J-1} \left\{ \int_{t_j^+}^{t_{j+1}^-} \left(\frac{\partial H}{\partial x} + \dot{\lambda}^T\right) \delta x d\tau \right\} + \sum_{j=1}^{J-1} \left\{ \int_{t_j^+}^{t_{j+1}^-} \frac{\partial H}{\partial \alpha_j}\, d\tau \right\} da_j . \tag{A9}$$

So the necessary conditions are

$$\dot{\lambda}^T = -\frac{\partial H}{\partial x} , \quad \lambda^T(t_f) = \frac{\partial\phi}{\partial x(t_f)} ,$$

$$\lambda^T(t_j^-) = \lambda^T(t_j^+) \quad \text{for} \quad j = 2, \ldots, (J-1) , \tag{A10}$$

$$\frac{\partial \mathcal{J}}{\partial t_j} = H(t_j^-) - H(t_j^+) = 0 \quad \text{for} \quad j = 2, \ldots, (J-1), \tag{A11}$$

and

$$\frac{\partial \mathcal{J}}{\partial \alpha_j} = \int_{t_j}^{t_{j+1}} \frac{\partial H}{\partial \alpha_j} \, d\tau = 0 \quad \text{for} \quad j = 2, \ldots, (J-1).$$

Equations (A10) and (A11) mean that $\lambda(t)$ and $H(t)$ are continuous at $\{t_j / j = 2, \ldots, (J-1)\}$.

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13. ABSTRACT

This report presents the theory and applications of a suboptimal control algorithm for a general nonlinear control system. This suboptimal algorithm requires the optimal control to be piecewise constant. Necessary conditions are derived for both discrete-time and continuous-time systems. The application to Kalman filtering problems is specifically carried out to illustrate the theoretical results.

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