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Some Labeling Theorems for Planar Linear Graphs: A Contribution to the Four-Color Problem

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FOREWORD

Horace M. Trent, who died in December 1964, made a number of important contributions to graph theory and its applications. Much of this work was related to or motivated by his long-standing interest in the four-color problem. The present paper, the original version of which was completed on January 27, 1963, represents his approach to, and attempt to prove, the four-color conjecture. The original version was later revised to take into account some deficiencies in the proof, and it is this version which is given here. There still remained some doubtful points in the proof, however (see the historical outline for details), so neither version was published at that time. Instead, the author made numerous attempts, which continued until shortly before his death, to correct and complete the proof. However, it was not possible for him to complete this task in a systematic and satisfactory manner, but merely in the form of isolated notes and comments. These are collected in the appendix, and the reader may judge for himself to what extent they fulfilled their purpose.

This manuscript is being published in somewhat the same spirit as the recent excellent book by the late Oystein Ore on the same topic ["The Four-Color Problem," Academic Press, 1967]. We feel that the approach and methods used by Dr. Trent contain sufficient novelty and ingenuity to be of interest and of possible benefit to those working in this and related areas. The manuscript is also being offered to the scientific community as the final work, although unfortunately incomplete, of an eminent mathematician and physicist.

It is a pleasure to acknowledge the advice and assistance given us by the author's widow, Eva Mae Trent, and by several former members of the Applied Mathematics Staff of this Laboratory, especially Alvin Owens, Betty Anderson, and Violet Hicks.

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PREFACE

According to Courant and Robbins*, the four-color problem was probably proposed by Möbius in 1840. It was proposed again by De Morgan in 1850 and by Cayley in 1878. The problem may be stated as follows: Given any linear graph which will map on a plane (or sphere), it is possible to assign one of four colors to each face of the map so that no two faces colored alike have a common edge on their boundaries.

In 1879 Kempe† published a proof of the conjecture. However, in 1890, Heawood‡ showed that Kempe's proof was defective, by use of a counterexample that disproved one of Kempe's assertions. The portion of this manuscript from Comment 18 through Lemma 24 is designed to take care of the analog to Heawood's counterexample. In the same paper, Heawood showed that five colors were sufficient to color a planar map.

The problem has received much attention in the intervening years. Although a number of interesting results have been obtained since 1890, no one has been able to exhibit a planar graph that required more than four colors, nor has anyone been able to prove that four colors were sufficient in all cases. The situation up to now is summed up rather well by Courant and Robbins*, "Despite the efforts of many famous mathematicians, the matter essentially rests with this more modest result: It has been proved that five colors suffice for all maps and it is *conjectured* that four will likewise suffice." This manuscript presents a new approach to and a partial proof of the conjecture.

Heawood's paper of 1890 included some other remarkable results. He was able to define upper and lower limits for the number of distinct colors needed to solve the coloring problem on a surface of any genus. Furthermore, for all surfaces of greater complexity than the sphere, he was able to show that the two limits were the same, provided it was possible to construct a map in which there were this many faces each of which touched all others. In the case of the sphere, however, Heawood's upper limit was six and his lower limit was four. He used a separate argument to reduce the upper limit to five. In a real sense then, a solution to the four-color problem amounts to showing that Heawood's lower limit is also the upper limit. In this work, the concept of the stably triangulated graph (the ST-graph) is introduced, and its properties are derived. It is shown that the arbitrary planar graph can be converted to an ST-graph by the insertion of appropriate edges.

Next the notion of the valid edge labeling for an ST-graph is defined, and algebraic conditions sufficient to insure the existence of a valid edge labeling are obtained. It is shown that its existence implies the affirmative answer to the four-color conjecture. Finally, an inductive, but incomplete, proof is given that there exists a valid edge labeling for every finite graph.

Horace M. Trent

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*Courant and Robbins, "What is Mathematics," Oxford: Oxford University Press, pp. 246-248, 1941.

† A. B. Kempe, "On the Graphical Problem of the Four Colours," Am. J. Math. 2:193-200 (1879).

‡ P. J. Heawood, "Map-Colour Theorem," Quart. J. Pure Appl. Math. 24:332-338 (1890).

SOME LABELING THEOREMS FOR PLANAR LINEAR GRAPHS.
A CONTRIBUTION TO THE FOUR-COLOR PROBLEM.

APPROACH TO, AND PROJECTED PROOF OF, THE FOUR-COLOR CONJECTURE

Introduction

The concept of a stably triangulated graph (called an ST-graph) is developed and then defined formally. Next it is shown that every planar graph can be converted to an ST-graph by the addition of extra edges. Then it is shown that any ST-graph can be created by an ordered insertion of vertices having incidences of 3, 4, and 5. These developments set the stage for an inductive argument later in the proof.

The concept of a valid edge labeling is introduced and defined. The conditions required to assure the existence of a valid edge labeling are established as a set of algebraic conditions. These conditions are reduced to the existence of a multilinear, characteristic polynomial which is subject to some side conditions. An inductive proof follows in which it is shown that if every ST-graph with $K + 2$ vertices ($K \geq 2$) has a valid edge labeling, then so does a graph with $K + 3$ vertices which is obtained from the graph having $K + 2$ vertices by the insertion of a vertex with incidence 3, 4, or 5.* Thus there would exist a valid edge labeling for every finite graph.

Given the foregoing result, it is shown that each vertex of an ST-graph can be given one of four labels such that no pair of edge-connected vertices are labeled alike. The removal of the edges added in the first step (in order to create an ST-graph) recovers the original planar graph for which the vertex version of the four-color conjecture would be true.

The validity of the four-color conjecture for the faces of a planar graph would then follow easily by the principle of duality.

It may be remarked that by far the greater portion of the proof is devoted to defining the problem algebraically and to proving the edge-labeling theorem by induction.

Throughout this paper when the word *graph* is used it shall refer to a finite linear graph, containing at least four vertices, which can be mapped on a plane or a sphere.

Sections I and II are concerned with the labeling of edges, Section III concerns the labeling of vertices, and the final section concerns the labeling of faces.

I. Stable Triangulations of Graphs and Some of Their Properties

Definition 1. The subgraph g of a graph G is a graph obtained by selecting a subset of the edges in G together with their bounding vertices.

Definition 2. The subgraph G—g consists of those edges from G, not contained in g, together with their bounding vertices.

Comment 1. It is clear from Definitions 1 and 2 that g and G-g can have vertices in common.

Definition 3. A subgraph g of a graph G is said to be K-vertex connected to (G—g) if it satisfies the following conditions:

*The proof of the case for vertices having incidence 5 (Lemmas 17 through 25) appears to be incomplete. Most of the supplementary notes given in the appendix appear to be a partially successful attempt to fill this gap. B.L. & H.H.

1. The subgraph g is connected and contains at least two edges,

2. Any simple path beginning in g and ending in G–g must pass through one of K vertices common to g and G–g.

Definition 4. A graph G having the three properties

1. G is connected,
2. Every face has three bounding edges,
3. Every connected subgraph g having two or more edges is at least 3-vertex connected to G–g,

is said to be a *triangulated graph* with *stable faces.* Hereafter, such a graph will be called an ST-graph; i.e., a stably triangulated graph.

Comment 2. An ST-graph cannot contain a simple series connection; i.e., it cannot contain a vertex with only two edges incident on it. This result follows from the fact that the two edges, together with their common vertex, constitute a two-vertex connected subgraph. Thus every vertex in an ST-graph must have at least three edges incident on it.

Comment 3. An ST-graph cannot contain a parallel connection; i.e., it cannot have two or more edges incident on the same pair of vertices. This follows from the fact that if g is a subgraph of G consisting of the two parallel edges, then g is two-vertex connected to G.

Comment 4. The faces of an ST-graph are stable in the sense that no topological rearrangement of the graph on the plane can change any of the three edges that bound the face. If this were not the case, then if an edge e_1 bounded face f_1 in some mapping on the plane, another mapping would replace e_1 by an edge e_2. This however, would imply a parallel connection between e_1 and e_2.

Definition 5. If two vertices in a graph are end points of the same edge, the vertices are said to be *edge connected.*

Definition 6. If there are ι edges which have a vertex ν as one of their endpoints, then the vertex is said to have *incidence* ι and such a vertex is called an *ι-vertex.*

Lemma 1. *Let V be the number of vertices, F the number of faces, E the number of edges in an ST-graph and let $K = V - 2$. Then $F = 2K$ and $E = 3K$.*
Proof. Since $V \geqslant 4$, $K \geqslant 2$, every edge is a boundary to two faces. Hence $3F = 2E$, since every face is triangular. Now let the values of E and V be substituted into the Euler relation

$$E + 2 = F + V,$$

giving

$$\frac{3}{2}F + 2 = F + K + 2,$$

from which it follows that

$$F = 2K$$

and

$$E = 3K.$$

Definition 7. The integer K is called the *characteristic number* for the ST-graph.

Lemma 2. *There is at least one vertex in an ST-graph with an incidence of 5 or less.*
Proof. This lemma follows readily from the relation for the average incidence on a vertex, $\bar{\iota}$. The total incidence provided by the edges is 6K, since each edge has two endpoints. Hence

$$\bar{\iota} = \frac{6K}{K+2} = \frac{6}{1 + \frac{2}{K}} < 6, \text{ K finite.}$$

Since the average incidence is less than 6, and since the incidence on any one vertex is an integer, the lemma follows immediately.

Definition 8. Let e be an edge in an ST-graph which connects vertices v_1 and v_2 and then let e be shrunk to zero, thus making v_1 and v_2 coincident. This operation creates two parallel connections, which are then removed by deleting two edges. This sequence of operations is called *removing a vertex.* Clearly, it leaves the graph planar.

Definition 9. Let a sequence of vertices be removed, one after the other, from an ST-graph, subject to the restriction that if a three-vertex exists, it is removed before a four-vertex and all four-vertices in turn are removed before a five-vertex is removed. Such a sequence is called an *ordered removal of vertices.*

Lemma 3. *If the characteristic number for an ST-graph is greater than 2, then two two-vertices cannot be edge connected.*

Proof. Assume the converse. Since every face is triangular, there must be a subgraph as shown in Fig. 1, where v_1 and v_2 are the two 3-vertices under discussion. Since $K > 2$, there is at least one vertex external to the subgraph. (Then in order for every face to be triangular, there must be two edges connecting v_3 and v_4.) But this is not an ST-graph, and the contradiction proves the lemma.

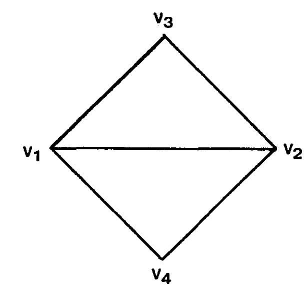


Fig. 1—Two edge-connected three-vertices

Comment 5. If K = 2, then the unique ST-graph with this characteristic number has four 3-vertices and each is edge connected to the other three.

Lemma 4. *An ordered removal of K − 2 vertices from an ST-graph whose characteristic number is K produces an ST-graph at every step in the sequence of removals.*

Proof. The limitation to (K − 2) removals assures that the last step in the sequence leaves K = 2. Since a removal leaves a planar graph, it remains only to be shown that the ordered removal of a vertex from an ST-graph leaves an ST-graph There are three cases to be considered.

1. Let the vertex to be removed, say v, be of incidence 3, and let the edges incident on v be e_1, e_2, and e_3. Let the vertices at the other ends of e_1, e_2, and e_3 be v_1, v_2, and v_3, respectively. By Lemma 3, the incidence on v_1, v_2, and v_3 must be 4 or greater. Now let e_1 be shrunk to zero, thus making v and v_1 coincident, after which e_2 and e_3 are deleted in order to remove the two parallel connections created by shrinking e_1. Now each incidence on v_1, v_2, and v_3 has been reduced by 1 and hence is 3 or greater in each case. Finally, it must be shown that no two of these incidences is 3 unless the characteristic number of the resulting graph is 2. But this requirement is equivalent to asserting that no pair of vertices in the set v_1, v_2, and v_3 had incidences of 4 in the original ST-graph unless its characteristic number was 3. Assume, therefore, that v_1 and v_2, say, are four-vertices in the initial graph. Then a direct construction shows that this condition can be satisfied only if the characteristic number of the given graph is 3, in which case v_3 has an incidence of 4 as well. Hence, if the characteristic number for the given graph is 4, or more, no pair of vertices from the set v_1, v_2, and v_3 can have incidences of 4 or less. It follows that the removal of a three-vertex leaves an ST-graph in every instance.

2. Let the vertex to be removed, say v, be of incidence 4, let the edges incident on v be labeled e_1, e_2, e_3, and e_4 in a clockwise direction, and let their second endpoints be v_1, v_2, v_3, and v_4, respectively. Since the removal is ordered, the incidence on each of v_1, v_2, v_3, and v_4 must be 4 or greater. Now let e_1 be shrunk to zero, thus making e_2 and e_4 parts of parallel connections. Now remove e_2 and e_4. The incidence on v_1 and v_3 has not been changed by this operation, and hence is 4 or greater for both vertices. The incidence on v_2 and v_3 has each been reduced by 1 and hence may be 3 or more. If, however, one or both of these is now of incidence 3, this does not violate the conditions for an ST-graph, since v_2 and v_4 are not adjacent. Hence, if the removal is ordered, it can always be done and the result is an ST-graph.

3. Let the vertex to be removed, say v, be of incidence 5. Let the edges incident on v be labeled e_1, $e_2, \ldots, e_5$ in a clockwise direction, and let their second endpoints be $v_1, v_2, \ldots, v_5$, respectively. Since the removal is ordered, the incidence on each of $v_1, v_2, \ldots, v_5$ must be 5 or greater. Now shrink e_1 to zero, making v and v_1 coincident. Follow this by a removal of edges e_2 and e_5 in order to eliminate parallel connections. At the end of this process, the incidence on v_1 has been increased by 1, the incidence on each of v_2 and v_5 has been reduced by 1 and the incidences on v_3 and v_4 remain the same. The process, therefore, can create no vertices with incidence less than 4; consequently, the result is an ST-graph.

Finally, it is to be noted that if a removal is ordered and if it is made on an ST-graph, the result is an ST-graph in every possible case.

Definition 10. Let the operations defined in Definition 9 be applied to an ST-graph in the reverse order. The result is an ST-graph containing one more vertex. This operation is called an *ordered addition* of a vertex.

Lemma 5. *Any ST-graph with characteristic number $K > 2$ can be created by the ordered addition of $K - 2$ vertices in turn, each of which has an incidence of 3, 4, or 5, starting with the ST-graph with characteristic number 2.*
Proof. The proof is constructive. All that is needed is to reverse the ordered removals implied by Lemma 4.

Definition 11. If all of the edges which connect vertices v_1 and v_2 are deleted from a graph, the two vertices are said to be *disconnected.*

Theorem 1. *Let $\overline{G}$ be a planar linear graph containing at least four vertices. Then an ST-graph G can be created from $\overline{G}$ such that (a) the number of vertices in G is the same as in $\overline{G}$ and (b) no pair of vertices that is edge connected in $\overline{G}$ is disconnected in G.*
Proof. The proof is based upon a constructive process which at each step neither adds nor removes a vertex, nor does it disconnect an already connected pair of vertices.

Step 1. Remove all loops from $\overline{G}$; i.e., all edges whose endpoints are coincident. The result leaves the graph planar. Call it G''.

Step 2. In every parallel connection in G'', remove all but one edge. The result is a planar graph. Call it G_0', where the subscript suggests that the result may contain subgraphs g and G_0-g, which are not connected.

Step 3. If two disjoint subgraphs exist, select a vertex in each subgraph in such a way that an edge connecting them can be mapped on the plane. Clearly, this can always be done. Repeat the process until the graph becomes connected and call the result G_1', where the subscript suggests that subgraphs g and $G_1'-g$ may exist, which have only one vertex in common.

Step 4. Identify any subgraph g which has only one vertex, say v, in common with $G_1'-g$. This subgraph contains at least one edge and hence at least two vertices. Thus it contains at least one vertex other than v. The same condition holds in $G_1'-g$. Next note that it is meaningful to speak of the outside boundaries of g or $G_1'-g$ as they are mapped in any specific fashion on the plane. The vertex v will be on these boundaries. But each boundary will contain at least one other vertex. Join two such vertices with an edge. Clearly, this edge can be mapped on the plane. Continue this process until every subgraph is at least two-vertex connected. Call the result G_2'.

Step 5. Identify any two-vertex connected subgraph g and let v_1 and v_2 be the vertices which it shares in common with $G_2'-g$. It is assumed that if v_1 and v_2 are edge connected, that edge is a part of $G_2'-g$. Note that g is connected and contains at least two edges; hence, it contains at least one vertex other than v_1 and v_2. Of course the same condition holds in $G_2'-g$. Next note that $G_2'-g$ has a face with v_1 and v_2 on its boundary. This face contains at least one other vertex, for otherwise there would be a parallel connection between v_1 and v_2. Call this vertex v_3. Now consider G_2' and its face, which contains v_1, v_2, v_3, and edges from g on its boundary. There is at least one vertex in this boundary which is in g; this by virtue of the fact that g contains no edge which connects v_1 and v_2. Call this vertex in g, v_4. Finally, join v_3 and v_4. Clearly, this edge maps into the plane and does not create a parallel connection. The subgraph g is now three-vertex connected. This process is repeated until no two-vertex subgraph exists. Call the result G_3'.

Step 6. The faces in G_3' are now identified. Let f_1 be such a face with $\ell\,(\ell > 3)$ edges on its boundary. Let the vertices on the boundary of f_1 be labeled $v_1, v_2, \ldots, v_\ell$ in a clockwise fashion. Now select $\ell - 3$ distinct pairs of vertices, v_i and v_j $(j > i)$, subject to the conditions

1. $j \neq i + 1, (i = 1, \ldots, (\ell - 1))$,
2. $j \neq \ell$ if $i = 1$,
3. If the pair v_α and v_β $(\beta > \alpha)$ has already been selected subject to 1 and 2, then
   if $i < \alpha$, either $j < \alpha$ or $j > \beta$,
   if $\alpha < i < \beta, j < \beta$
   if $i > \beta, j \leqslant \ell$

Now connect each pair of vertices by an edge. The conditions on the vertices assure that the edges will map into the plane. That this is always possible is clear from the one set of choices $i = 1$ and $j = 3, 4, \ldots (\ell - 1)$ in turn.

This operation triangulates f_1. The process is repeated until all faces are triangular. Call the result G. It is an ST-graph, since the conditions of Definition 4 are satisfied. Furthermore, the conditions required by the theorem are satisfied at each step.

II. The Edge Labeling of ST Planar Linear Graphs.

Definition 12. Let there by three different, but not necessarily unrelated, kinds of labels, say $\underline{a}$, $\underline{b}$ and $\underline{c}$; let f be one face of an ST-graph with bounding edges e_1, e_2, and e_3; and let the label $\underline{a}$ be assigned to one of these edges, the label $\underline{b}$ to a second, and the label $\underline{c}$ to the third. This operation is called an *edge labeling of a face.*

Definition 13. Let one face of an ST-graph be given an edge labeling. Select a second face with an edge already labeled i ($i = \underline{a}$, $\underline{b}$, or $\underline{c}$), and label the remaining edges so that each of the labels $\underline{a}$, $\underline{b}$, and $\underline{c}$ has been used in the region. In this fashion, continue labeling edges which bound new regions, each edge being assigned one and only one label. If it is possible to assign labels to edges so that each of the labels $\underline{a}$, $\underline{b}$, and $\underline{c}$ appears somewhere on the boundary of each and every face, the ST-graph is said to have a *valid edge labeling.*

Comment 6. It is to be observed that the first step in the labeling process possesses two ambiguities. First of all, there is no requirement as to which edge is labeled $\underline{a}$; in fact, this choice is arbitrary. Assume that the label $\underline{a}$ is assigned to some edge. The second ambiguity now appears, for the label $\underline{b}$ can be assigned to either of two edges. Note that one choice for $\underline{b}$ makes the ordered sequence of labels ($\underline{a}$, $\underline{b}$, $\underline{c}$) proceed, say, counterclockwise around the face, while the second makes the sequence proceed clockwise. Thus the assignment of $\underline{b}$ establishes an *orientation* for the face. Actually, it is the orientation assigned to each face which is important, as we shall see later; the assignment of the label $\underline{a}$ to the first face is completely arbitrary and of no basic significance.

The foregoing ideas will now be reduced to an algebraic form.

Definition 14. Let λ be a multiplicative *linear* operator such that, in any one face, if ℓ ($\ell = \underline{a}$, $\underline{b}$, or $\underline{c}$) is the label assigned to an edge, $\lambda \cdot \ell$ is the label assigned to the adjacent edge on the boundary of the face in a counterclockwise direction.

Definition 15. Let ℓ ($\ell = \underline{a}$, $\underline{b}$, or $\underline{c}$) be the label assigned to an edge on the boundary, then $\lambda \cdot \ell$ is the label assigned to the next edge in a counterclockwise direction, $\lambda \cdot (\lambda \cdot \ell) = \lambda^2 \cdot \ell$ is the third edge, and $\lambda \cdot (\lambda^2 \cdot \ell) = \lambda^3 \ell$ must be the initial edge. If, then, λ is a linear algebraic operator such that $\lambda^3 \ell = \ell$, then λ is said to be *consistent* over the face.

Lemma 6. *The operator λ is a cube root of unity.*
Proof. From Definition 15,

$$\lambda^3 \ell = \ell.$$

By Definition 14, λ is a linear operator, hence

$$\lambda^3 \ell - \ell = 0$$

and

$$(\lambda^3 - 1)\ell = 0.$$

Since ℓ is an arbitrary label, the relation can be true always only if

$$\lambda^3 - 1 = 0.$$

It follows then that

$$\lambda = 1, -\frac{1}{2} + \frac{\sqrt{-3}}{2} = p, \text{ or } -\frac{1}{2} - \frac{\sqrt{-3}}{2} = q.$$

Comment 7. The root $\lambda = 1$ implies that all edges bounding a given face are labeled alike. This is not a valid labeling, however, and must be excluded. Hence, hereafter it will be required that only the roots p and q are used. Let it be recalled that $p^2 = q$ and $q^2 = p$; i.e., two applications of one operator is equivalent to the others. Therefore, it is permissible to interpret p as an operator that advances the labeling one edge in the counterclockwise direction while q advances it one step in the clockwise direction. Thus if ℓ is in fact the label $\underline{a}$, pa is identified as the label $\underline{b}$, and $p^2 a = qa$ is identified as $\underline{c}$ if the set $(\underline{a}, \underline{b}, \underline{c})$ is ordered in a counterclockwise direction.

Definition 16. Let v be a vertex in an ST-graph with incidence i. It is said that the edges are labeled consistently relative to the vertex v if orientations p^{x_i} can be assigned to the i faces incident on v such that the labeling is consistent on each of the i faces.

Comment 8. Let the i edges of Definition 16 be designated by the ordered set $(e_1, e_2, \ldots e_i)$, the ordering being in a clockwise direction around the vertex. If ℓ is the label of e_1, it follows that

$$\begin{aligned}&\text{the label of } e_2 \text{ is } p^{x_1}\ell,\\ &\text{the label of } e_3 \text{ is } p^{x_2}(p^{x_1}\ell) = p^{x_1+x_2}\ell, \text{ etc.,}\\ &\text{the label of } e_1 \text{ is } p^{x_1+x_2+\ldots+x_i}\ell = \ell.\end{aligned}$$

This gives

$$(p^{x_1+x_2+\ldots x_i} - 1)\ell = 0.$$

Since ℓ is any of the labels $\underline{a}$, $\underline{b}$, or $\underline{c}$, it follows that

$$p^{x_1+x_2+\ldots+x_i} = 1.$$

In some sense, then, $(x_1 + x_2 + \ldots + x_i)$ must be congruent to zero. Returning to the question of consistency over a face, we have

$$p^{3x}\ell = \ell$$

or

$$(p^{3x} - 1)\ell = 0$$

and

$$p^{3x} = 1,$$

and in this case 3x must be congruent to 0. This suggests that the exponents of p should be treated on a modulo 3 basis. Hence it is taken for granted hereafter that all algebraic processes on the exponents of p are carried out modulo 3. Then at the vertex v,

$$x_1 + x_2 + \ldots + x_i \equiv 0 \pmod 3.$$

In order that the labeling shall be valid in each of the i faces, it is further necessary to require that each of the x's be nonzero. This requirement can be stated by the i relations

$$(x_j)^2 \equiv 1 \pmod 3) \ (j = 1, 2, \ldots, i).$$

The same sort of analysis can be carried out at each vertex in the given ST-graph.

Definition 17. Let there be an x_i $(i = 1, 2, \ldots, 2K)$ associated with each face of an ST-graph. Consistency is maintained over each face and at each vertex, if at each vertex α $(\alpha = 1, 2, \ldots, K + 2)$ an equation of the form $d_{\alpha i}x_i \equiv 0 \pmod 3$ $(i = 1, 2, \ldots, 2K)$ is satisfied (it being assumed that the usual convention, that a repeated index is summed over its range, holds). Any coefficient $d_{\alpha i}$ may be a 1 or a 0, depending on whether or not the face i is incident on the vertex α. Hereafter the relations

$$d_{\alpha i}x_i \equiv 0 \pmod 3$$

will be called *consistency relations,* while the set

$$x_i^2 \equiv 1 \pmod 3$$

will be called *heterogeneous relations* because they prevent any two labels' being alike on the boundary of a face. Clearly, a valid labeling will satisfy the consistency and the heterogeneous relations.

Definition 18. By a simple, closed path P of length L in an ST-graph is meant an ordered sequence of L edges $(e_1, e_2, \ldots, e_L)$ such that the pairs of vertices which they connect can be ordered thus,

$$[(v_1,v_2), (v_2,v_3), (v_3,v_4), \ldots, (v_{L-1},v_L), (v_L,v_1)],$$

and in this ordering every vertex symbol appears exactly twice.

Comment 9. A simple closed path P has the obvious meaning implied by its name; it is a closed curve in the plane. The path divides the plane into an inside and an outside. However, topological warpings will allow the outside to become the inside and vice versa, and hence the distinction between them is purely artificial. Hereafter, the word inside will be used predominantly, and if any confusion is apt to arise, those faces of the ST-graph which constitute the inside will be identified.

Lemma 7. *If a set of x_i's exists which satisfies the heterogeneous and the consistency relations for an ST-graph, then that graph has a valid labeling.*
Proof. The proof is constructive; i.e., a procedure is given for assigning labels to each edge. The procedure involves k − 1 steps, and at each step the boundaries of all faces inside a simple closed path have been labeled.

Step 1. Select any edge and label it a. Let v_1 be one of the endpoints of a. Starting with a and proceeding in a counterclockwise direction, assign labels to the boundary of each face incident on v_2 using the x_i as given each face. This yields a valid labeling for the boundary of each face and a consistent labeling at v_1 by virtue of the corresponding consistency relations. Clearly, all of the labeled faces are contained within a simple closed path each of whose edges is labeled.

Step k (k = 2, 3, . . ., k − 1). In each of the k − 2 steps, a vertex v_k is selected according to the following scheme: (a) v_k is on the labeled, simple, closed path. (b) The path which includes all of the unlabeled faces incident on v_k contains only three vertices from the labeled simple path. It will now be shown that such a selection is possible.

Two cases are to be considered. First, if there is a vertex with only one unlabeled face incident upon it, this vertex is taken as v_k, for it satisfies the selection rule trivially. Second, if no vertex exists with only one unlabeled face incident upon it, there must be at least one interior vertex not on the simple labeled boundary. For if E_1 is the number of labeled edges in the boundary, if E_2 is the number of edges in the interior, if F_2 is the number of interior faces, and if there are no interior vertices so that $v = E_1$, it follows from Euler's formula that

$$E_2 = F_2 - 1.$$

Also

$$3F_2 = 2E_2 + E_1.$$

The elimination of E_2 from these relations yields

$$F_2 = E_1 - 2.$$

Since in such a subgraph each interior face contains at least one edge from the labeled boundary and at least one interior edge, it follows that two of the faces contain two edges each from the labeled boundary. Thus there are two vertices with only single faces incident upon each of them. It can be concluded therefore that if no such vertex exists, there must be at least one interior vertex. Vertex v_k is then chosen so that the closed path, including the faces incident on v_k, contains three consecutive vertices from the labeled boundary and at least one interior vertex. In either of the two cases, the boundaries of the unlabeled faces incident on v_k are labeled in a counterclockwise order, starting with the existing labeled simple closed path. It follows trivially that the new labeled boundary is also simple. Furthermore, the labeling is consistent about the vertex v_k.

After the step k = k − 1, the labeled boundary contains exactly three vertices, for at each step the labeling was completed about only one vertex. Thus the outside of the labeled boundary is a triangular region or face. At this step the boundaries of all faces have been labeled, and a valid labeling for the graph has been generated.

Comment 10. The rest of this section is devoted to showing that the equations of Lemma 7 always have a solution for an ST-graph. However, a number of new definitions and lemmas are needed in order to arrive at the desired end result.

Lemma 8. *Let G be an ST-graph with a valid edge labeling, let a simple closed path P in G contain n_i edges labeled i, i = a, b, or c, and let there be N_I faces inside P. Then, if n_i is even (odd) so also is N_I.*
Proof. Let $\overline{n}_i$ be the number of edges inside P which are labeled i, i = a, b, or c. Then

$$N_I = 2\overline{n}_i + n_i,$$

since each interior edge labeled i is a boundary of two faces, while an edge in P bounds only one face which is inside P. It follows that

$$N_I - n_i = 2\overline{n}_i.$$

Since the difference of N_I and n_i is an even integer, they are either both even or both odd. This fact proves the lemma.

Comment 11. The proof of Lemma 8 yields two bonuses. In the first place, since $\overline{n}_i$ cannot be negative, we have the inequality

$$N_I \geqslant n_i.$$

Second, it is to be noted that the analysis is valid no matter which of the three labels is considered. Hence if the number of edges in P labeled $\underline{a}$ is even (odd) so also is the number labeled $\underline{b}$ and the number labeled $\underline{c}$.

Lemma 9. *Let G be an ST-graph, f_1 one of its faces, and let v_1, v_2, and v_3 be the vertices on the boundary of f_1. Then, if a valid edge labeling can be found for each face other than f_1 and for each vertex other than v_1, v_2 and v_3, a valid edge labeling can be found for the entire graph.*
Proof. Let the boundary of f_1 be a simple closed path P in G and let the inside of P be all faces other than f_1 and all vertices other than v_1, v_2, and v_3. By assumption, a valid labeling exists for P and for the inside of P. Since P contains three edges, and this is an odd number, by Comment 11 it must contain one edge labeled $\underline{a}$, one labeled $\underline{b}$, and one labeled $\underline{c}$. But this assigns a valid labeling to f_1 and valid labelings about v_1, v_2, and v_3. Since valid labelings exist for every region and about every vertex, a valid labeling exists for G.

Comment 12. Lemma 9 shows that a valid labeling exists if a set of nonzero x_i's can be found which satisfies all but three of the relations $d_{ai}x_i \equiv 0 \pmod 3$, provided the three relations omitted are associated with the three vertices on the boundary of a single face. Hereafter, then, only K − 1 linear relations of the form $d_{ai}x_i \equiv 0 \pmod 3$ will be considered, the three relations not used being all of those which contain the x_q associated with the qth face. The comment assumes, of course, that the heterogeneous relations are satisfied for all of the x's which appear in the (K − 1) consistency relations.

Lemma 10. *Let G be an ST-graph. Then it is possible to select K − 1 independent consistency relations, $d_{ai}x_i \equiv 0$ (mod 3).*
Proof. The lemma will be proved by displaying a method that will always select a dependent set of x_i's, (K − 1) in number. Let v_1, v_2, and v_3 be the vertices on the boundary of the face whose x does not appear in the set of equations.

Step 1. Select a face which contains two vertices from the set (v_1, v_2, v_3) plus a third not in the set, and let this vertex be designated v_4. Identify the x associated with this region; it is the first number in the sought-after dependent set.

Step 2. Select a face which contains two vertices from the set (v_1, v_2, v_3, v_4) plus one not in the set, v_5, and identify the corresponding x. Add this x to the set.

. . . .

Step j. Select a face which contains two vertices from the set (v_1, v_2, . . ., v_{j+2}) and one not in the set, v_{j+3}, and identify the corresponding x. Add this x to the set.

. . . .

The foregoing steps are continued up to K – 1 in all. At each step a new vertex is introduced and in the way that it was selected, a new face will have been used. Thus at each step at least one new x is added to the set; hence the consistency relation in which it appears will be independent of all those used before. Thus the K – 1 equations are independent.

The foregoing procedure is a special case of a more general procedure as follows: Let the vertex v_1 be edge connected to v_2 and let v_2 be edge connected to v_3. The selection process is carried out as already outlined. This more general selection process will be used later. Note that, in the procedure outlined initially, v_1 is also connected to v_3.

Comment 13. Since the K – 1 consistency relations are independent, it is possible, by additions, to convert them to the form

$$x_a \equiv c_{ai}x_i \pmod 3, (a = 1, \ldots, K-1), (i = 1, 2, \ldots, K).$$

In other words the given relations can be solved for K – 1 of the x's. The K – 1 members of the set x_a will be called hereafter the *dependent variables* while the members of the set x_i will be called the *independent variables.* As a matter of practical fact, given an ST-graph the process described in the proof of Lemma 10 generates the set x_a and then the c_{ai}'s are easily written down by inspection.

Lemma 11. *Let G be an ST-graph and let the relations $x_a \equiv c_{ai}x_i$ (mod 3) have a solution which in addition satisfies the heterogeneous relation. Then if all x_a's and all x_i's are multiplied by 2, the new values also satisfy the consistency equations.*

Proof. Multiply the given relation by 2, giving

$$2x_a \equiv 2c_{ai}x_i \equiv c_{ai}(2x_i);$$

Now let $2x_a = \overline{x}_a$ and $2x_i = \overline{x}_i$ and substitute these new variables giving

$$\overline{x}_a \equiv c_{ai}\overline{x}_i.$$

Clearly, then, the two sets $\overline{x}_a$ and $\overline{x}_i$ satisfy the consistency requirements.

Lemma 12. *Let G be an ST-graph with a valid edge labeling, let P be a simple closed path in G with all of its edges labeled alike, and let x_γ be the set of x's associated with faces inside P. Then a valid labeling is obtained if all the x_γ are multiplied by 2 and the x's associated with the faces outside of P are left unchanged.*

Proof. By Lemma 8 the number of edges in P is even, since an even number (namely zero) has two of the three labels. Hence the number of inside faces is even and similarly for the number of outside faces. Let the number of inside faces be N_I. At each vertex on P the x's belonging to the faces on the inside of P and incident on the vertex add to zero, mod 3, since the edges of P are all labeled alike. Hence the relations $d_{ai}x_i = 0$ can be broken into two sets.

For the x's associated with the inside faces,

$$d_{ai}x_i \equiv 0 \pmod 3 (a = 1, 2, \ldots, K-1), (i = 1, 2, \ldots, N_I),$$

and for the x's associated with the outside faces,

$$d_{ai}x_i \equiv 0 \pmod 3 (a = 1, 2, \ldots, K-1), (i = N_{I+1}, \ldots, 2K).$$

As each summation is made, some values of a will give trivial null identities; i.e., for those values of a associated with vertices not on the boundary of an x_i in the set then being considered. Clearly for the inside faces,

$$2d_{ai}x_i \equiv d_{ai}(2x_i) \equiv d_{ai}\overline{x}_i \equiv 0.$$

Thus the new set of variables $\overline{x}_i$ satisfies the consistency relations and this without changing the relations associated with outside faces. They also satisfy the heterogeneous relations.

Definition 19. The operation defined in Lemma 12 is called a *reversal of orientations inside a closed path.*

Definition 20. Let G be an ST-graph and let P be a simple closed path in G which contains four edges and two interior faces. Let the vertices on P be labeled v_1, v_2, v_3, and v_4 in order and let the one inside edge connect v_1 and v_3. If the edge is removed from vertices v_1 and v_3 and then connected to vertices v_2 and v_4, the edge is said to be *rotated.*

Lemma 13. *Let G be an ST-graph with a valid edge labeling, let P be a simple closed path in G having two interior faces, and let the interior edge connect vertices v_1 and v_3, each of which has an incidence of at least 4. Then a new ST-graph with a valid labeling can be created by rotating the interior edge if the x's associated with the interior faces, no x's associated with exterior faces being changed in the process.*

Proof. The limitation on the incidences of vertices v_1 and v_3 assures that, after the edge rotation, the result is an ST-graph. Let $\overline{x}_1$ and $\overline{x}_2$ be associated with the interior faces after the rotation, with $\overline{x}_1$ being associated with the faces having v_1 on its boundary. If a valid labeling exists, then at v_1

$$\overline{x}_1 = x_1 + x_2 = 2x_1.$$

Likewise at v_3,

$$\overline{x}_2 = x_1 + x_2 = 2x_1.$$

At v_2 (or v_4),

$$\overline{x}_1 + \overline{x}_2 = x_1 = 2x_1 + 2x_1.$$

With these values of $\overline{x}_1$ and $\overline{x}_2$ and with no changes in the x's associated with any outside face, the consistency and the heterogeneous relations are satisfied everywhere. Hence by Lemma 7 a valid labeling exists for the entire new graph.

Definition 21. The creation of a new ST-graph with a valid labeling by changes made only inside a defined closed path is spoken of as a *local relabeling.* If more extensive changes have to be made in order to arrive at a valid labeling, the process is called a *global relabeling.*

Lemma 14. *Let G be an ST-graph with a valid edge labeling and let P be a closed path containing three edges and one interior face. Then a valid labeling of a new graph, obtained by adding a three-vertex inside P, can be obtained by a local relabeling.*

Proof. The addition of a three-vertex to an ST-graph yields an ST-graph. Let x be the variable associated with the inside face in G, let v_1, v_2, and v_3 be the vertices on P and let $\overline{x}_1$, $\overline{x}_2$, and $\overline{x}_3$ be the variables associated with the three inside faces of the new graph, with $\overline{x}_1$ and $\overline{x}_2$ associated with faces incident on v_1. For consistency at v_1 it is necessary that

$$\overline{x}_1 + \overline{x}_2 \equiv x.$$

The heterogeneous relations require that $\overline{x}_1^2 \equiv 1$ and $\overline{x}_2^2 \equiv 1$. Squaring the consistency relation and using the heterogeneous relations, it follows that

$$\begin{aligned}\overline{x}_1^2 + 2\overline{x}_1\overline{x}_2 + \overline{x}_2^2 &\equiv x^2,\\ 1 + 2\overline{x}_1\overline{x}_2 + 1 &\equiv 1,\\ 2\overline{x}_1\overline{x}_2 &\equiv 2,\end{aligned}$$

and

$$\overline{x}_1 \equiv \overline{x}_2 \equiv 2x,$$

by virtue of the consistency relation. At the vertex v_2, consistency requires that either

$$\overline{x}_2 + x_3 \equiv x$$

or

$$2x + \overline{x}_3 \equiv x,$$
$$\overline{x}_3 \equiv 2x.$$

It is trivial to show that consistency holds at v_3 and at the added vertex. Hence by Lemma 7, a valid labeling exists. Finally, it is observed that the foregoing operations were all local.

It is trivial to show that the argument can be reversed. Thus, if an ST-graph with a valid edge labeling contains a simple closed path containing a three-vertex and three interior faces, the interior three-vertex can be removed by a local relabeling, leaving an ST-graph with a valid labeling.

Definition 22. Let G be an ST-graph with a valid edge labeling. Then G_i (i = a, b, or c) is a subgraph of G obtained by deleting all edges of G labeled i. In a similar fashion G_{jk} is a subgraph of G obtained by deleting all edges of G labeled j or k.

Comment 14. Clearly every face in G_i has four bounding edges and there are K of these. The subgraph G_{jk} may contain simple closed paths; it will always have disjoint subgraphs, as the following argument shows.

Let μ be the number of independent simple closed paths in G_{jk} and let σ be the number of disjoint subgraphs. G_{jk} has K + 2 vertices and K edges. If these numbers are substituted into the well-known formula,

number of edges + number of subgraphs = number of vertices + number of independent closed paths,

we have

$$K + \sigma = K + 2 + \mu.$$

Hence

$$\sigma - \mu = 2.$$

Since μ cannot be negative, $\sigma \geqslant 2$; hence G_{ij} always has disjoint subgraphs. Finally it is noted that every time the number of closed paths is increased by 1, the number of disjoint subgraphs is increased by 1 also. This is just another way of saying that every closed path in G_{jk} has a non-null inside. This interior subgraph may be as simple as a single vertex; it also may be quite complicated.

Lemma 15. *Let G be an ST-graph with a valid edge labeling and let e be an edge in G; then if e is contained in a closed path P in G having all of its edges labeled alike, the rotation of e removes it from such a path and vice versa.*
Proof. The conditions of the lemma assert that e is in a closed path in G_{jk}. As pointed out in Comment 14, this closed path has a disjoint subgraph on its inside and on its outside. When e is rotated, one simple closed path is opened. Hence by Comment 14, the number of disjoint subgraphs in G_{jk} is reduced by 1; that is to say that the disjoint subgraphs which were on the inside and outside of P are connected after the rotation of e.

Conversely, assume that e is not an edge in a simple closed path. It is, however, a part of a disjoint subgraph in G_{jk}. Let v_1 and v_3 be the endpoints of e. After the rotation of e, the subgraph in which e was originally located is divided into two parts, since v_1 and v_3 are no longer connected, while the rest of the subgraph is unaffected. Thus the numbers of disjoint subgraphs has increased by 1; hence by Comment 14, the number of closed paths has been increased by 1. Clearly, the edge e has connected two vertices, say v_2 and v_4, which formerly were unconnected, and thereby has created the new closed path.

Lemma 16. *Let G be an ST-graph with a valid edge labeling, let P be a simple closed path in G having four edges and two interior faces, and let a new ST-graph be created by an ordered addition of a four-vertex inside P. Then the new graph has a valid edge labeling.*

Proof. Let the vertices on P be v_1, v_2, v_3, and v_4 in order, let the interior edge, say e, connect v_1 and v_3, and let x_1 and x_2 be the variables associated with the two faces in G with the face related to x_1 having v_2 on its boundary. Also, let the four interior faces in the new graph have the variables $\bar{x}_1$, $\bar{x}_2$, $\bar{x}_3$, and $\bar{x}_4$ associated with them, with $\bar{x}_1$ and $\bar{x}_2$ being associated with v_2, and $\bar{x}_2$ and $\bar{x}_3$ with v_3.

Three cases are to be considered.

Case 1. Let $x_1 \equiv 2x_2$. Then at vertex v_2, consistency requires that

$$\bar{x}_1 + \bar{x}_2 \equiv x_1,$$

and the heterogeneous relations require that

$$\bar{x}_1^2 \equiv \bar{x}_2^2 \equiv x_1^2 \equiv 1.$$

Then it is easy to show that $\bar{x}_1 \equiv \bar{x}_2 \equiv 2x_1$. A similar argument at v_4 gives

$$\bar{x}_3 + \bar{x}_4 \equiv x_2 \equiv 2x_1,$$

with

$$\bar{x}_3^2 \equiv \bar{x}_4^2 \equiv x_1^2 \equiv 1.$$

This yields $\bar{x}_3 \equiv \bar{x}_4 \equiv x_1$. Finally, these values of $\bar{x}_1$, $\bar{x}_2$, $\bar{x}_3$, and $\bar{x}_4$ are easily seen to be consistent at v_1, v_3 and the added vertex. In this case a valid labeling is obtained locally.

Case 2. Let $x_1 \equiv x_2$, and let e be in a closed path P with all of its edges labeled alike. Then by Lemma 12, the x's associated with faces on the inside of P can be multiplied by 2 and still have a valid labeling. Let x_2 be one of the x's inside of P. Then a valid labeling exists for G in which x_2 becomes $2x_2 \equiv x_2' \equiv x_2'$, say. Now $x_2' \equiv 2x_1$, and so the conditions of Case 1 apply. The valid addition of a four-vertex follows.

Case 3. Let $x_1 \equiv x_2$ and let e not be in a closed path p with all of its edges labeled alike. By Lemma 13, the edge e can be rotated and a valid labeling results. Let x_1' and x_2' be the variables associated with the new faces created by the rotation. By Lemma 13, $x_1' \equiv x_2' \equiv 2x_1$. By Lemma 15, the rotation of e put it in a closed path with all of its edges labeled alike. But now the situation is the same as in Case 2; the addition of a four-vertex with a valid labeling follows.

Comment 15. Let G be an ST-graph with a valid edge labeling, and let P be a simple closed path having five edges and three interior faces. Suppose a five-vertex is added inside of P as described in Lemma 5. But this lemma assumes that after the addition of the five-vertex, the result must be a graph with no four-vertices. Hence the assumption that the five-vertex is added according to Lemma 5 is equivalent to asserting that the number of edges outside of P and incident on each vertex in P is two or greater. This condition is assumed in all subsequent discussions of the addition of a five-vertex. To standardize these discussions, it will be assumed that the vertices on P, the interior edges, and the interior faces have associated x's as shown in Fig. 2a. After the additions of a five-vertex, the situation will be as shown in Fig. 2b for lemmas 17 through 19.

Lemma 17. *Let G be an ST-graph with a valid edge labeling, let P be a simple closed path in G containing five edges and 3 interior faces as shown in Fig. 2a, and let $x_1 \equiv 2x_2 \equiv x_3$. Then a five-vertex can be added as shown in Fig. 2b, and a valid edge labeling can be found using only local operations.*

Proof. To have consistency and to satisfy the heterogeneous relations at v_2, it is required that

$$\bar{x}_1 + \bar{x}_2 \equiv x_1$$

and

$$\bar{x}_1^2 \equiv \bar{x}_2^2 \equiv x_1^2 \equiv 1.$$

This yields the conditions

$$\bar{x}_1 \equiv \bar{x}_2 \equiv 2x_1.$$

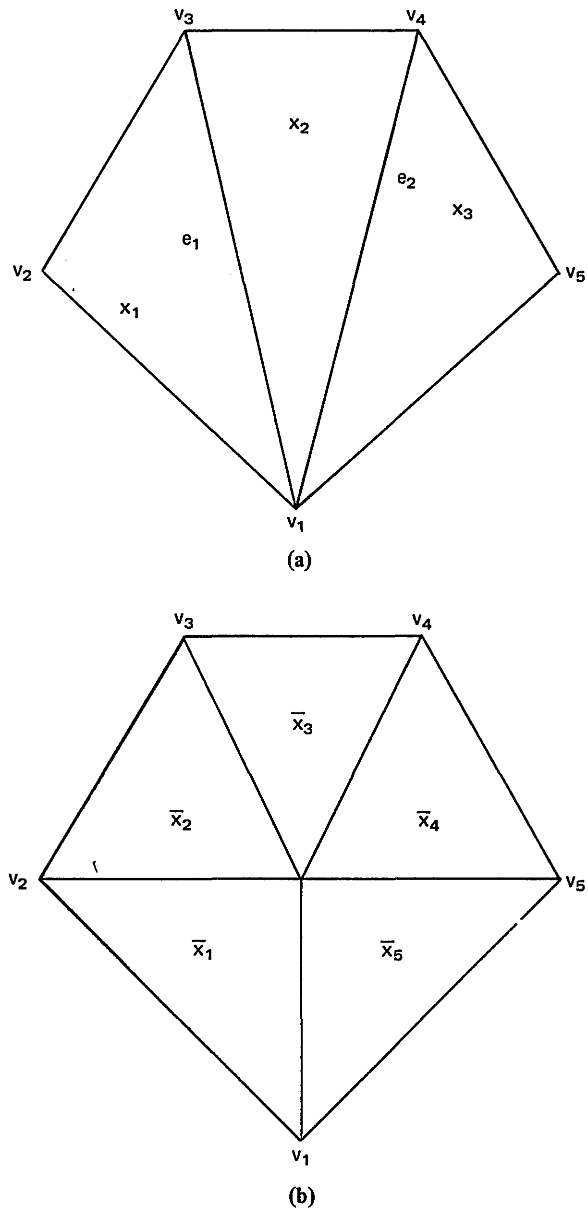


Fig. 2–Standard designations for five-sided figures.

At v_3 the requirements are

$$\bar{x}_2 + \bar{x}_3 \equiv x_1 + x_2 \equiv 0$$

and

$$\bar{x}_2^{\,2} \equiv \bar{x}_3^{\,2} \equiv x_1^{\,2} \equiv x_2^{\,2} \equiv 1.$$

This yields

$$\bar{x}_3 \equiv 2\bar{x}_2 \equiv x_1.$$

At v_4 the requirements are

$$\overline{x}_3 + \overline{x}_4 \equiv x_2 + x_3 \equiv 0$$

and

$$\overline{x}_3^{\,2} \equiv \overline{x}_4^{\,2} \equiv x_2^{\,2} \equiv x_3^{\,2} \equiv 1.$$

This yields

$$\overline{x}_4 \equiv 2\overline{x}_3 \equiv 2x_1.$$

At v_5 the requirements are

$$\overline{x}_4 + \overline{x}_5 \equiv x_3 \equiv x_1$$

and

$$\overline{x}_4^{\,2} \equiv \overline{x}_5^{\,2} \equiv x_3^{\,2} \equiv 1.$$

This gives

$$\overline{x}_5' \equiv 2x_3 \equiv 2x_1.$$

By using the values of $\overline{x}_1, \overline{x}_2, \overline{x}_3, \overline{x}_4$, and $\overline{x}_5$ it is easy to show that consistency holds at v_1 and at the added five-vertex.

Lemma 18. *Let G be an ST-graph with a valid edge labeling, let P be a simple closed path containing five edges and three interior faces as shown in Fig. 2a, and let $x_1 \equiv x_2 \equiv 2x_3$. Then a valid relabeling of G can be made so that the x's associated with the interior faces are all equal.*

Proof. By Lemma 13, edge e_1 can be rotated with a local relabeling. This yields a new graph with a five-sided figure like Fig. 2a but with the vertex and edge labels permuted. The x's associated with the three faces in the new graph, say $\overline{x}_1, \overline{x}_2$, and $\overline{x}_3$, are given by

$$\overline{x}_1 \equiv x_3, \overline{x}_2 \equiv 2x_1, \overline{x}_3 \equiv 2x_1.$$

Hence $\overline{x}_1 \equiv \overline{x}_2 \equiv \overline{x}_3$ by virtue of the given conditions.

Comment 16. It is clear from symmetry that if $2x_1 \equiv x_2 \equiv x_3$, then the conclusion of Lemma 18 is still valid.

Lemma 19. *Let G be an ST-graph with a valid edge labeling, let P be a simple closed path in G with five edges and three interior faces as shown in Fig. 2a, let $x_1 \equiv x_2 \equiv x_3$, and let e_1 not be in a simple closed path $\overline{P}$ with all of its edges labeled alike; then a five-vertex can be added inside of P with a valid edge labeling for the new graph.*

Proof. By Lemma 13, edge e_1 can be rotated. Let the x's for the new faces be $\overline{x}_1$ and $\overline{x}_2$. Then by Lemma 13, $\overline{x}_1 \equiv 2x_1 \equiv \overline{x}_2$. By Lemma 15, e_1 is now in a closed path, say $\overline{P}$, in which all edges are labeled alike. Let the face associated with $\overline{x}_1$ be on the inside of $\overline{P}$. Then by Lemma 12, the x's associated with the faces inside of $\overline{P}$ can be multiplied by 2 and yield a valid labeling. This operation makes $2\overline{x}_1 \equiv x_1 \equiv 2\overline{x}_2 \equiv x_3$. Now the condition of Lemma 17 holds and a five-vertex can be added with a valid labeling.

Comment 17. The conclusions of Lemma 19 are still true if the roles played by e_1 and e_2 are reversed. The truth of this statement is rather obvious from the symmetry of the given initial conditions.

Comment 18. At this point, every possible situation, except one, for the addition of a five-vertex has been examined. The exception is the combination $x_1 \equiv x_2 \equiv x_3$, with both e_1 and e_2 contained in closed paths and with all edges in each path labeled alike. Some additional lemmas are needed before this case can be treated.

Definition 23. Let $c(x_\alpha)$, $\alpha \equiv 1, 2, \ldots, K-1$, be a polynomial which is generated as follows:

$$c(x_\alpha) \equiv \prod_{\alpha \equiv 1}^{K-1} x_\alpha$$

Also, let the values of x_a, namely $x_a = c_{ai}x_i$, be substituted into the product, and then let the heterogeneous relations $x_i^2 = 1$ be substituted. The result is a multilinear polynomial in which the number of variables appearing in each term is either even or odd. Call the result the *characteristic polynomial* of G relative to the dependent x's. It is a function of the x_i's, say $[c(x_i)]$. Hereafter, brackets will be used to show that the substitutions $x_i^2 = 1$ have been made.

Comment 19. If $x_a^2 \equiv 1$ for every value of a, it follows that $[c(x_i)]$ must be nonzero. That this condition is also sufficient to assure a valid labeling of a graph for which $[c(x_i)] \not\equiv 0$ (mod 3) is given in Lemma 24.

Comment 20. If x_j is a variable in $[c(x_i)]$, then the characteristic polynomial can be written as

$$[c(x_i)] = [x_j\phi(x_m) + \theta(x_m)],$$

where $\phi(x_m)$ and $\theta(x_m)$ are multilinear polynomials neither of which contains the variable x_j.

Comment 21. From the definition of a characteristic polynomial, $[c(x_i)]$ is a product of linear factors followed by a substitution of the heterogeneous relations $x_i^2 = 1$. It is possible for the same characteristic polynomial to be obtained from a product of different linear factors. For example, a certain ST-graph with a particular choice of dependent variables yields the characteristic polynomial

$$\begin{aligned}[c_1(x_i)] &= [(x_1 + x_2)(x_1 + 2x_3 + 2x_4)(x_1 + 2x_4)] \\ &= 2x_1 + 2x_2 + 2x_3 + x_4 + 2x_1x_2x_3 + x_1x_2x_4 + x_1x_3x_4 + x_2x_3x_4.\end{aligned}$$

On the other hand the same result is obtained from

$$[c_2(x_i)] = [(x_1 + x_2)(x_1 + x_3)(2x_1 + x_4)].$$

Such possibilities prompt the following definition:

Definition 24. If the characteristic polynomial obtained from the product of k linear factors is the same as the polynomial obtained from the product of an equal number of other factors in which not all of the factors in the two sets of factors are identical, the two sets of linear factors are said to be *equivalent factorizations* of the polynomial.

Comment 22. If a characteristic polynomial is not identically zero, it is trivial that if the polynomial is multiplied by a constant (mod 3) the result is also nonzero. In addition, the multiplication by a variable x_j does not yield a zero result, for by Comment 20,

$$\{x_j[c(x_i)]\} = \{x_j[x_j\phi(x_m) + \theta(x_m)]\} = [\phi(x_m) + x_j\theta(x_m)] \neq 0.$$

Thus if the concern is to keep a polynomial nonzero, no basic change is made, whether a polynomial is multiplied by either a constant or a variable.

Definition 25. A characteristic polynomial is said to be reduced if a linear factor, not containing x_j is substituted for x_j in the polynomial and if the result obtained after the imposition of the heterogeneous relations is not identically zero.

Comment 23. A reduction is one step in the process of finding a set of ratios between variables, which, if substituted in a characteristic polynomial, leaves it nonzero. Clearly, the finding of a sufficient number of such ratios is equivalent to finding a set of x_i's that satisfies the heterogeneous and the consistency relations for (a) if the polynomial is nonzero every x_i is nonzero, (b) every independent x_i is kept nonzero in the reduction process, and (c) the factors in the polynomial are such as to satisfy the consistency relations. It will be shown in Lemma 23 that a sufficient set of ratios can be found from a sequence of reductions so that a ratio is established between every pair of variables. These ratios may or may not be unique.

Lemma 20. *If in $[c(x_i)] = [\phi x_j + \theta]$ there exists the condition $[\ell x_k \phi] = [\theta]$, $\ell^2 = 1$, then the substitution $x_j = \ell x_k$ will yield a reduced polynomial which is nonzero.*
Proof. $[c(x_i)] = [\phi x_j + \theta] = [\phi x_j + \ell x_k \phi] = [x_j + \ell(x_k)\phi]$.

The substitution $x_j = k x_k$ yields $[c(x_i)]_j = [2k x_k \phi]$, where the subscript j indicates the variable eliminated, and ϕ does not contain x_j. By Comment 22, the result is nonzero.

Comment 24. Since $x_j = k x_k$ it follows that $x_k = k x_j$. A similar argument will show that x_k can be eliminated instead of x_j.

Lemma 21. *If in $[c(x_j)] = [\phi x_j + \theta]$ there exists the condition $[\theta\lambda] = \theta$, where λ is a linear factor, and in addition, if $[\phi] = [\lambda\psi]$, where ψ is a polynomial, then the substitution $x_j = \lambda$ will yield a nonzero polynomial.*
Proof. $[c(x_i)] = [\phi x_j + \lambda\phi] = [(x_j + \lambda)\phi] = [(x_j + \lambda)(\lambda)\psi]$. It follows that $[c(x_i)]_j = [(2\lambda)(\lambda)\psi]$. From the given conditions $[\lambda\psi] \neq 0$, it follows directly that $[2(\lambda^2)\psi] \neq 0$.

Comment 25. If $[c(x_i)]$ has the factorization $[(x_j + \lambda)(\lambda)\psi]$ and if $\lambda = x_k + \mu$, where μ is a linear factor, then

$$[(x_j + \lambda)(\lambda)] = [(x_j + x_k + \mu)(x_k + \mu)] = [1 + x_j x_k + 2\mu x_k + \mu x_j + \mu^2].$$

But this has the equivalent factorization

$$[(x_k + x_j + 2\mu)(x_j + 2\mu)] = [1 + x_j x_k + 2\mu x_k + \mu x_j + \mu^2]$$

and hence

$$[c(x_i)] = [(x_k + x_j + 2\mu)(x_j + 2\mu)\psi].$$

This suggests the substitution $x_k = x_j + 2\mu$. If this is done, it follows that $[c(x_i)]_k = [2(x_j + 2\mu)^2 \psi] \neq 0$, since it is given that $[(x_j + 2\mu)\psi] \neq 0$.

Comment 26. Lemmas 20 and 21 are cases of what is called a *forced reduction.* The essential feature in both of these cases is the existence of a nonzero sum of two quantities, both of which are known to be nonzero. Thus if the two quantities are, say, ϕ and θ, and if in addition it is known that $\theta \neq 0$, then the only condition that will allow $\phi + \theta$ to be nonzero is that $\phi = \theta$. These conditions are met by the polynomial $[\phi\theta(\phi + \theta)]$. Now suppose that x_j is a variable from $\phi = \theta$ and that x_j occurs in ϕ but not θ so that $\phi = x_j + \mu$. Then

$$[(x_j + \mu)\theta(x_j + \mu + \theta)] = [\theta + 2x_j\mu\theta + x_j\theta^2 + \theta\mu^2].$$

This product has the equivalent factorization

$$[(x_j + \theta + 2\mu)(\theta + 2\mu)(x_j + \mu)] = [\theta + 2x_j\mu\theta + x_j\theta^2 + \mu\theta^2 + \theta\mu^2],$$

and hence is of the same form as in Lemma 21. No generality has been lost by assuming that θ did not contain x_j, for if one assumes to the contrary and if x_j is to remain in $\phi = \theta$, then θ must contain $2x_j$ and $(\phi + \theta)$ will not contain x_j. A redefinition of linear factors will recreate the form originally assumed. It is concluded therefore that Lemma 21 represents the most general case possible for (a) it includes Lemma 20 as a special case and (b) a sum of three or more linear factors, each of which is not zero and for which a linear combination of the three factors is known to be nonzero, yields no unique information about relations between the factors. The simplest case is $(x_i + k_1 x_j + k_2 x_k) \neq 0$, where the three nonzero factors are x_i, $k_1 x_j$, and $k_2 x_k$. No unique ratios between the x's exist.

Comment 27. It is clear at this point that there are always at least two ways to make a forced reduction.

Lemma 22. *If $[c(x_i)] = [\phi x_j + \theta]$ has no forced reduction, then the substitution $x_j = k x_k$, $k^2 = 1$, will yield a nonzero polynomial.*

Since $(k x_k)^2 = 1, k^2 = 1$, it follows that

$$\begin{aligned} [c(x_i)] &= [\phi x_j + (k x_k)^2 \theta] \\ &= [\phi x_j + k x_k \theta)] \\ &= [\theta x_j + k x_k \psi], \end{aligned}$$

where

$$\psi = x_k\theta.$$

Now if the substitution $x_j = \ell\, x_k$ is made, the result becomes

$$[c(x_i)]_j = [\ell x_k \phi + \ell x_k \psi] = [\ell x_k(\phi + \psi)].$$

This result can be zero only if $\phi = 2\psi = 2\ell x_k \theta$. But if this were the case, $[c(x_i)]$ would have a forced reduction which is contrary to the stated conditions of the lemma.

Comment 28. The reduction in Lemma 22 can be made in either of two ways, $x_j = x_k$ or $x_j = 2x_k$, and either substitution yields a nonzero result. Such a substitution is called an *arbitrary reduction.*

Comment 29. If $[c(x_i)]$ is of the product form $[c(x_i)] = \ell\, x_1 x_2 \ldots x_p$, then the substitution $x_i = \ell\, x_j$, $\ell_1^2 = 1$, will yield a new polynomial which is nonzero and in which the number of variables has been reduced by two. Such a substitution is called a *product reduction.* There are two ways to make such a reduction. Finally if p is odd, a succession of product reductions yields a polynomial containing a single variable; if p is even, the end result is a constant.

Comment 30. There are at least two possible ways to make any reduction. Thus it is possible in any forced or arbitrary reduction to make a substitution for some variable other than a distinguished one.

Comment 31. If, as a result of a forced or arbitrary reduction, some variable other than the one eliminated by the substitution does not appear in the reduced polynomial, then that variable can be given an arbitrary nonzero value without making any polynomial in the sequence of reductions zero. Thus, if x_m is such a variable, then one can make $x_m = \ell\, x_q$, $\ell^2 = 1$, where x_q is any other appropriate variable.

Comment 32. The set of substitutions associated with a sequence of reductions can be solved simultaneously to yield a set of ratios of the form $x_i = \ell x_B$, where the x_B's are a set of distinguished variables such that there is no ratio established between any two of them. If desired, ratios can be made arbitrarily among the x_B's so that there remains in fact only one distinguished variable. It may be noted that if the initial polynomial contains R variables, and if all reductions are forced or arbitrary, there are at most R – 1 substitutions. Each product reduction will reduce this number by one.

Lemma 23. *There exists a sequence of forced, arbitrary, and product reductions such that each reduction yields a nonzero polynomial and the end result is either a constant or a constant times a single variable.*
Proof. The proof is constructive. At each step in the sequence let the polynomial be tested for the existence of a forced reduction. If such exists let it be made. If none exists let an arbitrary reduction be made. These steps are continued until a product form is obtained. By Comment 29, this form may be reduced to either a constant or the product of a constant and a variable.

Comment 33. The tests for the existence of a forced reduction are made readily. Given $[c(x_i)] = [\phi x_j + \theta]$, one seeks a linear factor $\lambda = \Sigma f_i x_i$, $i \neq j$, such that $[\phi\lambda] = \theta$. Substituting the linear form of λ and equating coefficients of like terms on the two sides of the equation yield a set of linear equations with the f_i's as unknowns. If a solution exists for the f_i's, then the factor λ exists and is determinate. There may be multiple solutions. Next if λ consists of more than one term, then it must be ascertained if λ is also a factor of ϕ. In other words, one asks if there is a polynomial ψ such that $[\lambda\psi] = \phi$. This calculation is a bit more complicated than the determination of λ. Note that the number of variables in ϕ is known and that the number of variables appearing in each term of ϕ is either even or odd. It is clear that the parity of each term in ψ must be opposite to that of the terms in ϕ. Hence ψ must be a polynomial containing all of the variables that appear in ϕ, and each term must be of parity opposite to that for the terms in ϕ. Let ψ be a polynomial containing the sum of all possible terms of the permitted form with each term having a coefficient g_i. This polynomial is substituted for ψ and the coefficients of like terms on the two sides of the equations are set equal to each other. This yields a set of simultaneous equations involving the g_i's as unknowns. If a solution exists, then ψ exists and a forced reduction must be made. In theory a test for each x_j in $[c(x_i)]$ must be made before it can be asserted that no forced reduction exists.

Comment 34. The foregoing operations and ideas will now be illustrated using polynomials and factors derived from two different ST-graphs, each having a characteristic number 4.

Illustration 1

$$(x_1 + x_2 + x_3)(2x_1 + 2x_2 + 2x_3 + 2x_4)(2x_1 + 2x_2 + x_3 + 2x_4) = c(x_i).$$

Then

$$[c(x_i)] = x_1 + x_2 + 2x_3 + x_4 + 2x_1x_2x_3 + x_1x_2x_4 + 2x_1x_3x_4 + 2x_2x_3x_4.$$

Using Comment 20.

$$[c(x_i)] = [x_1(1 + 2x_2x_3 + x_2x_4 + 2x_3x_4) + (x_2 + 2x_3 + x_4 + 2x_2x_3x_4)].$$

If a forced reduction exists, there must exist a linear factor $\lambda = f_2x_2 + f_3x_3 + f_4x_4$ such that

$$[(f_2x_2 + f_3x_3 + f_4x_4)(1 + 2x_2x_3 + x_2x_4 + 2x_3x_4)] = x_2 + 2x_3 + x_4 + 2x_2x_3x_4.$$

By expanding the left-hand side and equating the coefficients of like terms, we find only one independent relation, namely, $f_2 + 2f_3 + f_4 = 1$, or $f_2 = 1 + f_3 + 2f_4$. Thus there are nine factors, each of which will satisfy the conditions since each of the variables f_3 and f_4 can take on the values 0, 1, or 2 arbitrarily. Let attention be focused on one possibility, namely, $f_3 = 2$, $f_4 = 1$, which gives $f_2 = 2$ and $\lambda = 2x_2 + 2x_3 + x_4$. Next, one seeks a polynomial $\psi = g_2x_2 + g_3x_3 + g_4x_4$ such that

$$[\lambda\psi] = 1 + 2x_2x_3 + x_2x_4 + 2x_3x_4 = [(2x_2 + 2x_3 + x_4)(g_2x_2 + g_3x_3 + g_4x_4)].$$

Again by equating coefficients of like terms one obtains four relations,

$$\begin{aligned} 2g_2 + 2g_3 + g_4 &\equiv 1, \\ 2g_2 + 2g_3 &\equiv 2, \\ g_2 + 2g_4 &\equiv 1, \end{aligned}$$

and

$$g_3 + 2g_4 \equiv 2.$$

The unique solution of this set is $g_2 = 0$, $g_3 = 1$, $g_4 = 2$. Thus $\psi = x_3 + 2x_4$, and there is a forced reduction. Now make the substitution $x_1 = 2x_2 + 2x_3 + x_4$, giving

$$[c(x_i)]_1 = 2x_2 + x_3 + 2x_4 + x_2x_3x_4 = [x_2(2 + x_3x_4) + (x_3 + 2x_4)].$$

Again it can be shown that a forced reduction exists. One such possibility is $\lambda = x_3 + 2x_4$, which gives the substitution $x_2 = x_3 + 2x_4$. The result of this substitution is $[c(x_i)]_{1,2} = 2x_3 + x_4$. Again there is a forced reduction $x_3 = 2x_4$. This substitution yields $[c(x_i)]_{1,2,3} = 2x_4$. Thus the end product is the product of a constant and a variable.

Now, looking at the three substitutions,

$$x_1 = 2x_2 + 2x_3 + x_4,$$
$$x_2 = x_3 + 2x_4,$$

and

$$x_3 = 2x_4,$$

makes it obvious that all x_i's can be expressed in terms of any one of the four variables, say x_4. Thus

$$x_1 = x_4,$$
$$x_2 = x_4,$$

and

$$x_3 = 2x_4.$$

Finally, it is noted that the three factors taken from the graph yield

$$x_1 + x_2 + x_3 \equiv x_4 + x_4 + 2x_4 \equiv x_4 \equiv y_1 \text{ (say)},$$
$$2x_1 + 2x_2 + 2x_3 + 2x_4 \equiv 2x_4 + 2x_4 + x_4 + 2x_4 \equiv x_4 \equiv y_2,$$

and

$$2x_1 + 2x_2 + x_3 + 2x_4 \equiv 2x_4 + 2x_4 + 2x_4 + 2x_4 \equiv 2x_4 \equiv y_3.$$

Thus it is verified that the dependent variables y_i are each nonzero. Using the values of $x_1, x_2, x_3, y_1, y_2,$ and y_3 in terms of x_4 and interpreting x_4 as some orientation assigned to the labeling of the boundary edges for the region corresponding to x_4, we determine a labeling for the whole graph.

Illustration 2

A certain ST-graph gives

$$c(x_i) \equiv (x_1 + x_2 + 2x_4)(2x_1 + x_3 + x_4)(2x_2 + 2x_3 + 2x_4).$$

It follows from performing the indicated multiplication that

$$[c(x_i)] \equiv x_1 + 2x_3 + 2x_4 + 2x_1x_2x_4 + 2x_2x_3x_4.$$

From this it follows that

$$[c(x_i)] \equiv [x_1(1 + 2x_2x_4) + (2x_3 + 2x_4 + 2x_2x_3x_4)].$$

If a linear factor $\lambda \equiv f_2x_2 + f_3x_3 + f_4x_4$ is sought so that

$$(1 + 2x_2x_4)(f_2x_2 + f_3x_3 + f_4x_4) \equiv 2x_3 + 2x_4 + 2x_2x_3x_4,$$

it is found that the f_i's must satisfy

$$2f_2 + f_4 \equiv 0,$$
$$2f_3 \equiv 1,$$
$$f_1 + 2f_4 \equiv 1,$$

and

$$f_3 \equiv 1.$$

Since these relations are inconsistent, no λ can be found that satisfies the required condition. It can be shown, by using each variable in turn, that no forced reduction exists. Thus an arbitrary reduction must be used. Let the substitution $x_1 = x_4$ be made, giving

$$[c(x_i)]_4 \equiv 2x_2 + 2x_3 + 2x_1x_2x_3 = [x_1(2x_2x_3) + 2(x_2 + x_3)].$$

If a factor $\lambda \equiv f_2x_2 + f_3x_3$ is sought so that

$$[(x_2x_3)(f_2x_2 + f_3x_3)] \equiv x_2 + x_3,$$

it is found that $\lambda \equiv x_2 + x_3$. If now a polynomial $\psi = g_2x_2 + g_3x_3$ is sought so that

$$[(x_2 + x_3)(g_2x_2 + g_3x_3)] = 2x_2x_3,$$

it is found that $g_2 + g_3 \equiv 0$ and $g_2 + g_3 \equiv 1$. It is concluded that even though a factor λ exists, no forced reduction exists. So a second arbitrary reduction must be made. Let $x_2 = x_3$ be substituted, giving $[c(x_i)]_{4,3} = 2x_1 + x_2$. This leads to the final substitution $x_2 = 2x_1$. Combining the results to this point, we find that one reduction is $x_1 = x_4, x_2 = 2x_4, x_3 = 2x_4, y_1 = 2x_4, y_2 = 2x_4, y_3 = x_4$.

Returning to the second arbitrary choice, we let $x_2 = 2x_3$ be substituted, giving $[c(x_i)]_{4,3} = x_1$. Here arises an example of a situation where a substitution removes some additional variable, x_2 in this illustration. Hence it is permissible to set x_2 equal to x_4 or to $2x_4$, for example. By using the first of these substitutions, it follows that $x_1 = x_4$, $x_2 = x_4$, $x_3 = 2x_4$, $y_1 = x_4$, $y_2 = 2x_4$, and $y_3 = 2x_4$. On the other hand, if $x_2 = 2x_4$, the results become $x_1 = x_4$, $x_2 = 2x_4$, $x_3 = x_4$, $y_1 = 2x_4$, $y_2 = x_4$, and $y_3 = x_4$.

Three independent possibilities have been found to this point, but this is not all. The first choice may be $x_1 = 2x_4$, giving

$$[c(x_i)]_4 = 2x_1 + x_2 + 2x_3 + x_1x_2x_3 = [x_1(2 + x_2x_3) + (x_2 + 2x_3)].$$

Without giving the details, it can be shown that

$$[c(x_i)]_4 = [x_1(2 + x_2x_3) + 2x_2(2 + x_2x_3)] = [(x_1 + 2x_2)(2 + x_2x_3)].$$

From this it follows that $x_2 = 2x_1$, and $x_3 = 2x_2$. In terms of x_4, $x_1 = 2x_4$, $x_2 = x_4$, $x_3 = 2x_4$, $y_1 = 2x_4$, $y_2 = x_4$, and $y_3 = 2x_4$. Thus, in all, there are four independent ways of assigning labels to the edges of the given graph which incidentally is a regular graph of six vertices with four edges incident on each vertex. Of course a special choice of dependent regions has been used here.

Lemma 24. *If the characteristic polynomial for an ST-graph is nonzero, then that graph has a valid edge labeling.*
Proof. By Lemma 22, if the characteristic polynomial is nonzero, there exists a sequence of forced, arbitrary, and product reductions that will reduce the polynomial to a constant or the product of a constant and a variable. From these reductions, at least one set of ratios can be found which when substituted into the polynomial leaves it nonzero. Such a solution exists because it is always possible to find R − 1 independent relations among the R variables of the polynomial which when solved determine R − 1 ratios. These ratios make every variable, both dependent and independent, nonzero, thus satisfying the heterogeneous relations. Since the factors in the given polynomial are derived from the consistency relations, it follows that these relations are satisfied also. Thus the reduction process yields at least one valid labeling for the graph.

Lemma 25. *Let G be an ST-graph with a valid edge labeling, let P be a simple closed path in G containing five edges and three interior faces as shown in Fig. 2a, let $x_1 = x_2 = x_3$, and let both e_1 and e_2 be in simple closed paths $\overline{P}_1$ and $\overline{P}_2$ with all of the edges in each path labeled alike. Then an ordered addition of a five-vertex can be made inside of P with a valid edge labeling for the new graph.*
Proof. By Lemma 12, the x's associated with the faces inside of $\overline{P}_1$ can be multiplied by 2 and still leave a valid labeling. Let the inside of $\overline{P}_1$ contain x_1 and let a reversal of orientation be made inside of $\overline{P}_1$. This situation is as shown in Fig. 3a. Either (a) edge e_2 is contained in a simple closed path P'_2, not necessarily $\overline{P}_2$, with all edges labeled alike, or (b) it is not. If (a) holds, then a reversal of orientation can be made on the inside of P'_2, which is taken to be the side containing x_3. This yields the situation shown in Fig. 3b. By Lemma 17, a five-vertex can be inserted, using only local relabelings, and a valid labeling is the result.

If (b) holds, then by Lemma 16 (Case 1) a four-vertex can be added inside the closed path v_1, v_2, v_3, v_4, v_1, giving the result shown in Fig. 4a. Again by Lemma 16 (Case 1) a four-vertex can be added inside the closed path $\overline{v}, v_4, v_5, v_1, \overline{v}$ as shown in Fig. 4b. Call the resulting graph $\overline{G}$. Let dependent variables be chosen in the order $\overline{x}_4$, $\overline{x}_5$, $\overline{x}_6$, $\overline{x}_7$ and then x's associated with faces outside P, but not including a face incident on v_5. $\overline{x}_1$, $\overline{x}_2$, and $\overline{x}_3$ belong to the set of independent variables. From Fig. 4b it is clear that

$$\overline{x}_6 \equiv 2\overline{x}_1 + 2\overline{x}_2 + \lambda_6,$$
$$\overline{x}_5 \equiv \overline{x}_1 + \overline{x}_2 + \lambda_5,$$
$$\overline{x}_4 \equiv 2\overline{x}_1 + 2\overline{x}_2 + \lambda_4,$$

and

$$\overline{x}_7 \equiv 2\overline{x}_1 + 2\overline{x}_2 + 2\overline{x}_3,$$

where λ_4, λ_5, and λ_6 are linear factors none of which contain $\overline{x}_1$, $\overline{x}_2$, or $\overline{x}_3$.

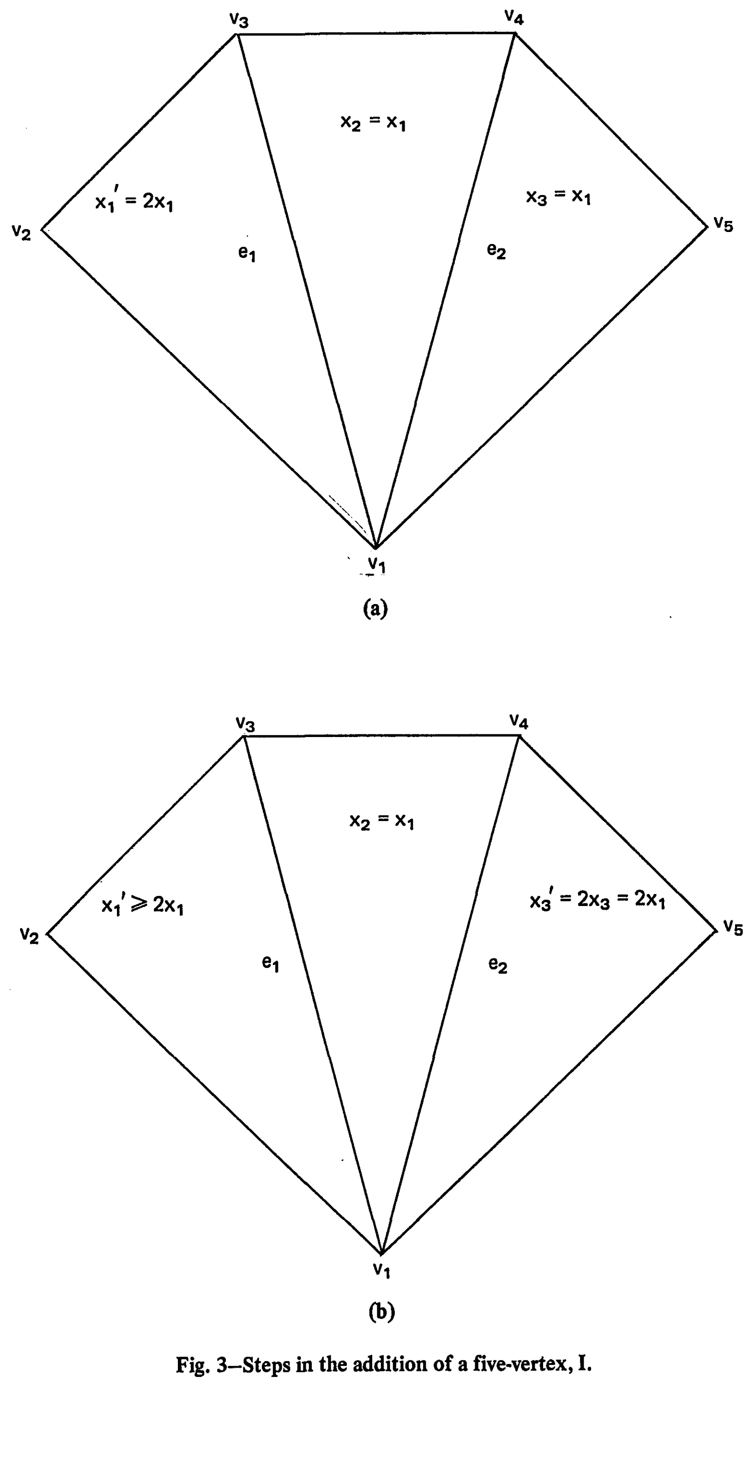


Fig. 3—Steps in the addition of a five-vertex, I.

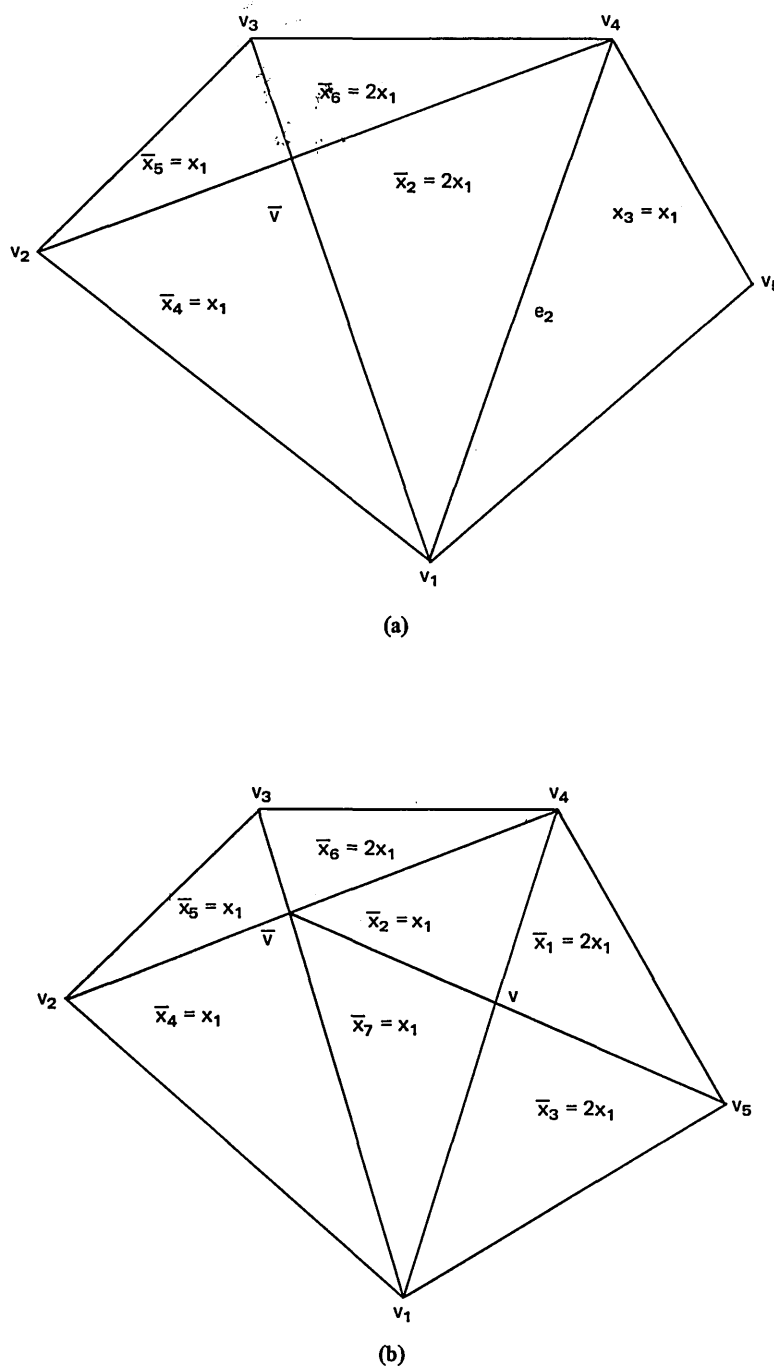


Fig. 4—Steps in the addition of a five-vertex, II.

The characteristic polynomial of $\overline{G}$ is of the form

$$[c(x_i)] = [(2\overline{x}_1 + 2\overline{x}_2 + 2\overline{x}_3)(2\overline{x}_1 + 2\overline{x}_2 + \lambda_4)(\overline{x}_1 + \overline{x}_2 + \lambda_5)(2\overline{x}_1 + 2\overline{x}_2 + \lambda_6)\psi] \neq 0,$$

where ψ does not contain $\overline{x}_1, \overline{x}_2$, or $\overline{x}_3$. It is important to note that the variables $\overline{x}_1$ and $\overline{x}_2$ always appear in the combination $\overline{x}_1 + \overline{x}_2$.

It is observed first that by Lemma 15 the edges in the path $v_3, \overline{v}, v', v_5$ lie in a simple closed path with all edges labeled alike. From this fact it can be concluded that $[c(x_i)]$ cannot involve a forced reduction at each step in a complete reduction. Hence the reduction process must include some arbitrary steps. Next it is observed that if any forced reductions exist, they cannot involve the factor $(\overline{x}_1 + \overline{x}_2)$ in a necessarily nonzero form, for if this were the case then of necessity $\overline{x}_1 = \overline{x}_2$, whereas it is known that at least two sequences of reductions exist in which $\overline{x}_1 = 2\overline{x}_2$. Furthermore by Comment 30, it is not required at any step in a complete reduction that $\overline{x}_1$ be eliminated by a substitution. Since $\overline{x}_1$ and $\overline{x}_2$ always appear in the combination $\overline{x}_1 + \overline{x}_2$, if follows that it is not necessary to eliminate either $\overline{x}_1$ or $\overline{x}_2$ by a substitution at any step in a complete reduction. It is theoretically possible that some reduction may remove $\overline{x}_1$ from all succeeding polynomials. If so, however, $\overline{x}_2$ will be removed also, and it will follow in that case, that, either $\overline{x}_1 = \overline{x}_2$ or $\overline{x}_1 = 2\overline{x}_2$ will be valid ratios.

Now consider a polynomial derived from $[c(x_i)]$ by deleting the first factor from its definition. Thus

$$[c_1(x_i)] = [(2\overline{x}_1 + 2\overline{x}_2 + \lambda_4)(\overline{x}_1 + \overline{x}_2 + \lambda_5)\ (2\overline{x}_1 + 2\overline{x}_2 + \lambda_6)\psi] \neq 0.$$

The inequality is true trivially since $[c(x_i)] \neq 0$. $[c_1(x_i)]$ can be interpreted as the characteristic polynomial for everything in Fig. 5a outside of the four-sided region $v_1, \overline{v}, v', v_5, v_1$. It is clear that every sequence of reductions that is valid for $[c(x_i)]$ will be valid also for $[c_1(x_i)]$. Since it must satisfy one less heterogeneous relation (it does not contain $\overline{x}_3$) and one less consistency relation, it is generally true that more different sequences of reductions exist for $[c_1(x_i)]$ than for $(c(x_i)]$, but this fact is not of importance here. It can be assumed that both $\overline{x}_1$ and $\overline{x}_2$ are contained in $[c_1(x_i)]$, for if this were not the case $\overline{x}_1$ and $\overline{x}_2$ would be arbitrary and the ratio $\overline{x}_1 = \overline{x}_2$ would be valid by an arbitrary assignment. Now let the substitution $\overline{z} = \overline{x}_1 + \overline{x}_2$ be made in $[c_1(x_i)]$, giving $[c_1(x_i)] = [(2\overline{z} + \lambda_4)(\overline{z} + \lambda_5 - (2\overline{z} + \lambda_4)\psi] \neq 0$. Now consider the polynomial defined by

$$[c'(x_i)] = [2\overline{z}c_1(x_i)] = [2\overline{z}(2\overline{z} + \lambda_4)(\overline{z} + \lambda_5)(2\overline{z} + \lambda_6)\psi] \neq 0.$$

The inequality follows from Comment 22. Now let the substitution $\overline{z} = \overline{x}_1 + \overline{x}_2$ be made in $[c'(x_i)]$, thus restoring these variables. The result is

$$[c'(x_i)] = [(2\overline{x}_1 + 2\overline{x}_2)(2\overline{x}_1 + 2\overline{x}_2 + \lambda_4)(\overline{x}_1 + \overline{x}_2 + \lambda_5)(2\overline{x}_1 + \overline{x}_2 + \lambda_6)\psi] \neq 0.$$

This quantity can be interpreted to be the characteristic polynomial for the graph shown in Fig. 5b in which the variable $\overline{z}$ is associated with a dependent face. Since $[c'(x_i)]$ is nonzero by Lemma 23, there exists a sequence of reductions which will completely reduce the polynomial. Such a sequence must involve the substitution $\overline{x}_1 = \overline{x}_2$, since the factor $(2\overline{x}_1 + 2\overline{x}_2)$ by Lemma 20 gives rise to a forced reduction. It follows that $\overline{z} = \overline{x}_1$. By Lemma 24, there exists a valid labeling for that part of the graph of Fig. 5b outside of the three-sided face $v_1, \overline{v}, v_5, v_1$. But by Lemma 9, a valid labeling exists for the entire graph.

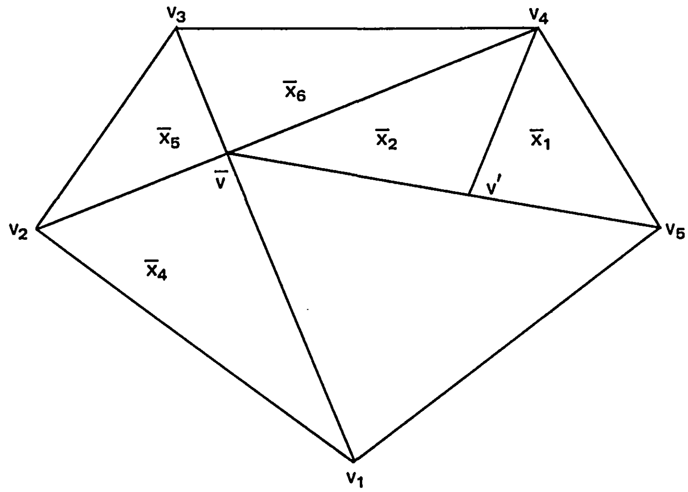
Finally, let the vertex v' of Fig. 5b be removed. This leaves a valid labeling for a graph in which a five-vertex has been inserted inside the closed path P.

Theorem 2. *Every ST-graph has a valid edge labeling.*
Proof. The proof is by induction on the characteristic number K.

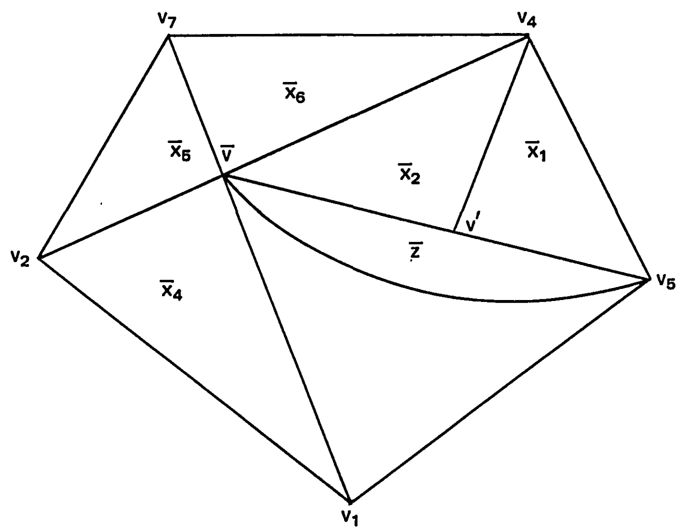
1. If K = 2, a valid labeling exists.

2. Assume that a valid edge labeling exists for every ST-graph up to and including K.

3. By Lemma 5, any ST-graph can be created by the ordered addition of vertices having incidences of 3, 4, or 5. By Lemma 14, the addition of a three-vertex yields a valid edge labeling; by Lemma 16, the introduction of a four-vertex yields a valid edge labeling; and by Lemmas 17, 19, and 25, we show that a five-vertex can be added so as to yield a valid edge labeling. Hence a valid edge labeling exists for an ST-graph with a characteristic number (K + 1).



(a)



(b)

Fig. 5—Steps in the addition of a five-vertex, III.

Comment 35. The proof of Theorem 2 only assures that there shall be one valid labeling for a given ST-graph. It is the exception rather than the rule if only one labeling is possible. Consider the technique of Lemma 20 which can generate all possible labelings. A unique labeling exists only if $\theta(x_m) = Kx_k\phi(x_m)$, where x_k can be any x other than x_j. It is obvious that this is a very special set of conditions and is not to be expected often in practice. It can happen, however, and does, for example, in the unique ST-graph for K = 3.

III. The Vertex Labeling of Planar Linear Graphs

Definition 26. Let $\overline{G}$ be a planar linear graph, let there be four distinct kinds of labels available, and let one label be assigned to each vertex in $\overline{G}$. Then if an assignment of labels to vertices can be made so that no pair of vertices which are edge connected are labeled alike, a *valid vertex labeling* for $\overline{G}$ is said to exist.

Comment 36. This section is concerned with proofs that valid vertex labelings exist for ST-graphs and for planar graphs in general.

Lemma 26. *Let G be an ST-graph with an assigned valid edge labeling. Then one of two labels can be assigned to each vertex of $G_i (i = a, b, \text{ or } c)$ so that no pair of edge-connected vertices are labeled alike.*
Proof. Every face of G_i has four edges on its boundary (Comment 14); hence, every closed path in G_i has an even number of edges. It is known that in any graph which is at least two-vertex connected, at least two simple paths can be found that join any two vertices. These two paths form one or more closed paths; the more-than-one case holds if the two paths share one or more vertices other than the two endpoints. In the case of two vertices in G_i, the number of edges in the two paths must be either both even or both odd, since the two paths taken together must give a set of simple closed paths, each of which has an even number of edges. These facts provide the basis for a constructive process of assigning labels.

Select a set of independent simple closed paths in G_i. They will be $(K - 1)$ in number and may, for example, be the boundaries of the faces of G_i, or, they may be based upon some topological tree in G_i. It really does not matter how they are selected. Now select an edge in one of these closed paths and label the endpoints of the edge, say, δ and ϵ. Calling the rest of the closed path a simple path from one already labeled vertex to the other, label the vertices along this path alternately ϵ, δ, ϵ, δ, etc., starting with the vertex already labeled δ. Since the number of edges along the path is odd, ϵ will be the next label in the sequence when the end of the simple path is reached, and this assignment will be consistent with that already made.

Now select a second independent closed path which has at least one edge in common with the closed path already labeled. The endpoints of this common edge will have been labeled already. Using it as a starting point, label all vertices in the closed path that have not already been labeled.

Repeat the process just described until all $(K - 1)$ independent closed paths have been labeled. A valid vertex labeling results, since every closed path contains an even number of edges.

Lemma 27. *Let G be an ST-graph with an assigned valid edge labeling. Then one of two distinct labels can be assigned to the vertices in G_{jk} $(j = a, b, \text{ or } c, k \neq j)$ so that no pair of vertices that are edge connected in G_{jk} are labeled alike.*
Proof. Since G_{jk} has at least two disjoint subgraphs, the assignment of labels can be made to each subgraph in turn. There are two cases to be considered.

Case 1. A subgraph in G_{jk} contains no simple closed paths. In this case there is one and only one simple path joining any pair of vertices. Let v_0 be an end vertex in G_{jk}, let the other vertices be called v_1, v_2, etc., and let the unique paths from v_0 to v_1, v_2, etc., be called P_1, P_2, etc. Label v_0, say, μ. Starting from v_0 and using path P_1, label the vertices alternately μ, υ, μ, υ, etc. Repeat the process with P_2, P_3, etc. If any two of the paths have edges in common, the labeling will be consistent, since the same starting point is used in both cases and the common vertices are labeled in the same order. Finally it is observed that if the subgraph consists of a single vertex, only one label is required.

Case 2. A connected subgraph of G_{jk} may contain one or more simple closed paths. Since every edge in one of these closed paths is labeled alike (by Lemma 8), the number of edges in each such closed path is even. This case differs from Case 1 in that there may not be a unique path from one vertex to another. However, if two or more such paths exist, the number of edges in the paths are either all even or all odd. Select some vertex, call if v_0 and label it, say, μ. Call the remaining vertices v_1, v_2, v_3, etc. Call the simple paths which join v_0 to v_i $(i = 1, 2, 3, \text{etc.})$ $P_{i\alpha}$, where α is an index that designates each of the separate simple paths connecting v_0 to v_i. It is assumed that the maximum number that α can have is the number of independent paths connecting v_0 to v_i.

Select the path P_{11}, and starting from v_0, label the vertices along the path alternately μ, υ, μ, υ, etc. Now select path P_{12} and repeat the process. Continue until all paths having 1 for a first subscript are labeled.

Now repeat the process with all paths $P_{2\alpha}$, then $P_{3\alpha}$, etc., until all paths have been labeled. Consistency will hold at every step, for if two paths have edges in common their endpoints which have been labeled in the same order, and if two paths take alternate paths through a simple closed path, the number of edges traversed by each path through the closed path will be both even or both odd.

The foregoing process is repeated for each disjoint subgraph of G_{jk}. The final result is a valid vertex labeling using only two distinct kinds of labels.

Theorem 3. *Let G be an ST-graph with an assigned valid edge labeling; then one of four distinct labels can be assigned to each vertex of G so as to give a valid vertex labeling to G; furthermore, three distinct such labelings can be found, no two of which are permutations of each other.*
Proof. Let the subscript i in Lemma 26 be $\underline{a}$ and the subscript j and k in Lemma 27 be $\underline{b}$ and $\underline{c}$. Let the labels δ and ϵ be assigned to the vertices in G_a according to Lemma 26 and the labels μ and υ be assigned to the vertices in G_{bc} according to Lemma 27. Every vertex in G is also in G_a and in G_{bc}. Hence every vertex in G has been assigned two labels. These assignments will appear in the four distinct pairs (δ, μ), (δ, υ), (ϵ, μ), and (ϵ, υ). Let these pairs be called A, B, C, and D in order. If a pair of vertices are edge connected in G_a then they will differ in the first of the pair of Greek labels; if the pair are edge connected in G_{bc}, they will differ in the second of the pair of Greek labels. Therefore any pair of edge-connected vertices will have different labels from the set of labels A, B, C, and D. Clearly then, the labeling so determined is a valid vertex labeling.

Now let the indices be changed in G_i and G_{jk} to G_b and G_{ac}. Half of the edges which were in G_a and had labels δ and ϵ at their endpoints are now in G_b and have labels μ and υ at their endpoints. It is clear, therefore, that the labeling in this case is different from that given before. The change is not a permutation, for the labelings assigned to the endpoints of one-third of the edges are not changed.

A third arrangement is obtained if G_i is made G_c and G_{jk} is made G_{ab}. The same arguments hold regarding its difference from the other arrangements and for the absence of a permutation. Thus there are three distinct ways in which an ST-graph can be given a valid vertex labeling.

Comment 37. Theorem 3 can be proved in several ways. The argument given above was called to the author's attention by Robert Busacker.

Comment 38. Steps will now be taken to remove the restriction to ST-graphs which is contained in the statement of Theorem 3.

Comment 39. It is almost trivial to remark that if a valid vertex labeling is determined for an ST-graph, the same labeling will be valid if any number of edges are removed from the graph.

Theorem 4. *Let $\overline{G}$ be a finite planar linear graph. Then one of four distinct labels can be assigned to each vertex in $\overline{G}$ so that no pair of vertices which are edge connected are labeled alike. Furthermore, the assignments can be made in three distinct ways, no two of which are permutations of each other.*
Proof. A constructive proof will be given.

Remove edges from $\overline{G}$ so as to remove all loop and all parallel connections. Call the result G'. Every pair of edge-connected vertices in $\overline{G}$ is still connected in G'. Using Theorem 1, let G' be converted to an ST-graph G. Now using Theorem 3, three distinct labelings are found for the vertices in G. Next remove edges from G until G' is recovered. By Comment 39, the three labelings are still valid. Finally, the parallel connections and the loops are added to G' to recreate $\overline{G}$. In this last step no new edge connections are made; hence, each of the three labelings is valid in $\overline{G}$.

Comment 40. Theorem 4 is a positive statement that the dual of the four-color conjecture is true.†

IV. The Labeling of Faces in a Planar Linear Graph.

Definition 27. Let G* be a finite planar linear graph, and let one of four labels be assigned to each face of G* such that no pair of faces with an edge in common are labeled alike. Then a *valid face labeling* of G* is said to exist.

Comment 41. This section is given over to a proof of the existence of a valid face labeling for any planar graph; i.e., to a proof of the four-color conjecture.†

Theorem 5. *Let G* be a finite planar linear graph. Then one of four distinct labels can be assigned to each face of G* in such a way that no two faces with an edge in common are labeled alike; furthermore, the labelings can be assigned in three distinct ways, no two of which are permutations of each other.*
Proof. The proof of the theorem is straightforward through the use of duality principles.

† See, however, the footnote on p. 1. BL. & H.H.

Since G^* is a planar graph, it has a physically realizable dual $\overline{G}$ in which every face of G^* becomes a vertex in $\overline{G}$. The number of edges remains invariant in the dual transformation; an edge which is common to two faces in G^* edge connects the corresponding vertices in $\overline{G}$. Now by Theorem 4, $\overline{G}$ has three distinct valid vertex labelings. Since the dual of $\overline{G}$ is G^*, if any of the three valid vertex labelings is assigned to $\overline{G}$, and it is then dualized, with the labels of vertices being carried over to the corresponding faces in G^*, it follows that a valid face labeling is obtained.

The same result is obtained if each of the other two possible valid vertex labelings of $\overline{G}$ is used in turn. Hence a valid face labeling for G^* can be found in three distinct ways, no two of which are permutations of each other, since the valid vertex labelings in $\overline{G}$ are also not permutations of each other.

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MATERIAL PERTAINING TO LEMMAS 17 THROUGH 25

Notation for Multilinear Polynomials

$\hat{\lambda}_k(x_i)$, $(i = 1, \ldots n)$ is a product of k linear factors modulo 3.

Hence $\hat{\lambda}_k(x_i)$ is a quadratic, at most, in any x_i. $\overline{\lambda}_k(x_i)$ is $\hat{\lambda}_k(x_i)$ with the substitution $x_i^2 = 1$. Any term is a linear product of variables k or less in number. $\overline{\lambda}_k(x_i)$ is reduced in the following fashion, assuming $\overline{\lambda}_k(x_i) \neq 0$ identically.

1. If $\overline{\lambda}_k(x_i)$ can be factored into the form (using $x_i^2 = 1$) $tx_k\,\psi(x_i)$, where $t^2 \equiv 1$, then multiply the polynomial by tx_k and set $x_k^2 = 1$, giving $\psi(x_i)$ as a result.

2. If $\overline{\lambda}_k(x_i)$ can be factored into the form (using $x_i^2 = 1$) $[\phi(x_i) + \theta(x_i)]\,\lambda(x_i)$ where $\phi(x_i) \neq 0$ and $\theta(x_i) \neq 0$, then set $\phi(x_i) = \theta(x_i)$ and solve for some variable x_j. This value of x_j is then substituted into $\overline{\lambda}_k(x_i)$, giving $\overline{\lambda}_{k-1}(x_i)$.

Special Case a. If $\phi(x_i) = x_k$ and $\theta(x_i) = tx_i$, with $t^2 \equiv 1$, then set $x_k = tx_i$ or $x_i = tx_k$.

Special Case b. If $\phi(x_i) = x_k$, then set $x_k = \theta(x_i)$ and solve for some variable.

$Y(x_i)$ is $\overline{\lambda}_k(x_i)$ reduced as far as it can be.

* * * *

The x_i's in $\overline{\lambda}_k(x_i)$ are said to be *unique to within a ratio* if the range of i is 1 to k + 1 and if the reduction process gives k linear relations. ($\lambda(x_i)$ is said to be exhausted if it is a linear polynomial.) Note that if the x_i's are unique to within a ratio, then $\lambda(x_i)$ is a constant.

Lemma. *If i in $\hat{\overline{\lambda}}_k(x_i)$ is equal to or greater than k + 2, then the x_i's are* **not** *unique to within a ratio.*
Proof. $\overline{\lambda}_k(x_i)$ is obtained from k linear factors. Hence at most k linear relations can be found in the reduction process. But it takes k + 1 relations to uniquely fix ratios. The result follows.

Definition. Any x_k in $\overline{\lambda}_k(x_i)$ which disappears in the reduction process, without having its ratio to some other variable determined, is said to be *arbitrary.*

Lemma. *If $\overline{\lambda}_k(x_i) = x_i\overline{\phi}(x_j)$, where $\overline{\phi}(x_j)$ does not contain x_i, then x_i is arbitrary and $\overline{\phi}(x_j)$ is of the form $\overline{\lambda}_{k-1}(x_j)$.*
Proof. That x_i is arbitrary follows from the definition of such a variable.

Every term in $\overline{\lambda}_k(x_i)$ is a linear product of not over k variables. Consequently, every term in $\overline{\phi}(x_j)$ is a linear product of at most (k – 1) variables. The result follows.

Lemma. *If $\overline{\lambda}_k(x_i) \neq 0$, $n \geqslant (k + 2)$; and if none of the variables are arbitrary, then $\lambda(x_i)$ contains at least three variables.*
Proof. At most, k linear relations can be found in the reduction process. Since no variable is arbitrary, at each step in the reduction process, the number of variables is reduced by exactly one. Hence after k – 1 steps, a linear factor results having at least three variables.

Let $\hat{\lambda}_k(x_i)$ be the product of k linear factors involving n unknowns. Let F(n, k) be the largest number of distinct terms in $\hat{\lambda}_k(x_i)$. Then

$$F(n, k+1) = F(n, k) + F(n-1, k+1),$$

with

$$F(p, 1) = p \text{ and } F(1, q) = 1.$$

Also, for example,

$$\begin{aligned} &[(x + y)(\lambda xy + \mu(x + y) + \omega)] \\ &= [\lambda y + \mu + \mu xy + \omega x + \lambda x + \mu xy + \mu + \omega y] \\ &= [2\mu xy + (\lambda + \omega)x + (\lambda + \omega)y + 2\mu] \\ &= [2\mu(xy + 1) + (\lambda + \omega)(x + y)] \\ &= [(2\mu y + \lambda + \omega)(x + y)] \equiv 0, \end{aligned}$$

if

$$[\mu] \equiv 0$$

and

$$[\lambda + \omega] \equiv 0.$$

Again,

$$[\lambda(\lambda + \omega)(\lambda + 2\omega)] \equiv [\lambda(\lambda^2 + 2)].$$

Two further lemmas are needed.

Lemma 1. *If $x_j = hx_n$ is one of a set of forced reductions prior to any arbitrary reductions, it can be found in the first reduction.*

Lemma 2. *If x_p and x_q appear symmetrically in a polynomial, then $x_q = 2x_p$ cannot be a forced reduction.*

(The material from 4 April to 8 July 1964 is believed to be supplementary to Comment 34.)

April 4, 1964

Lemma. *Let λ be a linear polynomial containing more than two terms; then, there exists no linear polynomial μ such that $[\lambda\mu] \equiv 0$.*

Proof. 1. Assume that λ contains a term z which is not contained in μ. Let $\lambda = \lambda_1 + z$, where λ_1 does not contain z. Then $[\lambda\mu] = [(\lambda_1 + z)\mu] = [\lambda_1\mu + z\mu] = [\lambda_1\mu] + [z\mu]$. Since the second term is not null and has no product in common with $[\lambda_1\mu]$, the result cannot be null.

2. Assume that μ contains a term ω which is not contained in λ. By a similar argument $[\lambda\mu] \neq 0$.

3. From (1) and (2), it follows that λ and μ have the same number of terms.

4. If $[\lambda\mu] = 0$, then so will $[\lambda(x_1\mu)]$, since this amounts to no more than $x_1[\lambda\mu]$. Note that $[x_1\mu]$ is of second degree and is of the form $\Sigma k_i x_1 x_i$.

5. Let λ be represented as $\Sigma\beta_i x_i$, where $\beta_1 = 1$ and $\beta_i \neq 0$. Then for the product to be null, the coefficient of each term in the product must be null. The coefficient of $x_i (i > 1)$ is $(k_i\beta_1 + \beta_i) = (\beta_i + k_i)$; hence for this coefficient to be zero we must have $k_i = 2\beta_i$. The coefficient of $x_1 x_j x_k$, $(1 < j < k)$ is $(k_j\beta_k + k_k\beta_j)$. But $k_j = 2\beta_j$ and $k_k = 2\beta_k$. Hence, the coefficient is $2\beta_j\beta_k + 2\beta_j\beta_k = \beta_j\beta_k \neq 0$. Thus it is impossible to make all coefficients zero simultaneously.

Comment. If

$$\lambda = x_1 + \beta_2 x_2$$

and

$$\mu = 1 + k_2 x_1 x_2$$

then

$$[\lambda\mu] \equiv [x_1 + \beta_2 x_2 + k_2 x_2 + k_2\beta_2 x_1] \equiv [(1 + k_2\beta_2)x_1 + (k_2 + \beta_2)x_2].$$

For this to be null, $k_2 = 2\beta_2$, and this value will make the first coefficient zero also. This explains the limitation in the theorem. On the other hand, if

$$\lambda = x_1 + \beta_2 x_2 + \beta_3 x_3$$

and

$$\mu = 1 + k_2 x_1 x_2 + k_3 x_1 x_3,$$

then

$$[\lambda\mu] \equiv [(1 + k_2\beta_2 + k_3\beta_3)x_1 + (k_2 + \beta_2)x_2 + (k_3 + \beta_3)x_3 + (k_2\beta_3 + k_3\beta_2)x_1 x_2 x_3].$$

The second and third coefficients can be made zero if $k_2 = 2\beta_2$ and $k_3 = 2\beta_3$. The last coefficient then becomes $\beta_2\beta$, which is not null. The first coefficient becomes $(1 + 2 + 2) \equiv 2$, which is also not null. However, if the number of terms had been $1 + 3N$ (N an integer), then this first coefficient would have been zero. It is clear that the crucial part of the argument requires that there be a triple term in the product. Note also that the coefficient of the term $x_i x_j x_k (1 < i < j < k)$ is always zero; an interesting fact of no great consequence.

April 21, 1964

Consider the following multilinear polynomial of second degree and fourth order:

$$ab + ac + ad + 2bc + 2bd + 2cd.$$

This polynomial can be obtained from any of the following products:

$$[(a + b + c + d)(b + c + d)],\ [(a + b + 2c + 2d)(a + 2c + 2d)],$$
$$[(a + 2b + c + 2d)(a + 2b + 2d)],\ \text{or}\ [(a + 2b + 2c + d)(a + 2b + 2c)].$$

This shows that the decomposition of a multilinear polynomial into a product of linear factors is *not* unique. On the other hand, it is legitimate to write, for example,

$$[(a + b + c + d)(b + c + d)] \equiv [(a + 2b + 2c + d)(a + 2b + 2c)].$$

It is not legitimate to remove the brackets, for this implies an equality of factors before the imposition of the constraint $(x_i)^2 = 1$.

April 22, 1964

Lemma. *Let $\psi = a\phi + \theta$ be a multilinear form and let $a + \lambda$ be a linear form. Then $[(a\phi + \theta)(a + \lambda)]$ can be identically null only if λ is a factor of ψ (i.e., $[\psi] \equiv [\lambda\psi_1]$), if $[\phi] \equiv [2\lambda\theta]$, $[\theta] \equiv [2\lambda\phi]$, and if $(a + 2\lambda)$ is a factor.*
Proof. $[(a\phi + \theta)(a + \lambda)] \equiv [\phi + a\lambda\phi + a\theta + \lambda\theta] \equiv [(\phi + \lambda\theta) + a(\lambda\phi + \theta)]$. The first factor does not contain a. Therefore, each term in parentheses must be zero independently. Thus $[\phi + \lambda\theta] \equiv 0$ and $[\lambda\phi + \theta] \equiv 0$. It follows that $[\phi] \equiv 2[\lambda\theta]$ and $[\theta] \equiv 2[\lambda\phi]$. Now return to the first term $[a\phi + \theta]$ and substitute the foregoing values, giving $[2a(\lambda\theta) + 2(\lambda\phi)] \equiv 2[\lambda(a\theta + \phi)] \equiv 2[[\lambda]\ [a\theta + \phi]]$. Q.E.D. Also, $[a\phi + \theta] \equiv [a\phi + 2\lambda\phi] \equiv [\phi(a + 2\lambda)] \equiv [2\lambda\theta(a + 2\lambda)] \equiv 2[\theta\lambda(a + 2\lambda)]$.

Comment. The foregoing lemma shows that if $[\psi(a + \lambda)]$ is to be zero identically, then the product can be written in the form $[(a\phi + \theta)(a + \lambda)] \equiv 2[\lambda(a\theta + \phi)(a + \lambda)] \equiv 2[\lambda(a + \lambda)(a + 2[\lambda\phi)] \equiv 2[\lambda(a + \lambda)(a + 2\lambda)\phi]$. This is identically zero because $[\lambda(a + \lambda)(a + 2\lambda)] \equiv [\lambda(1 + 2\lambda^2)] \equiv [\lambda + 2\lambda] \equiv 0$.

To give a feeling of confidence in the foregoing lemma, let us look at a product which at first glance does not appear to fall into the triple product category, yet we can show that in fact it does.

Let ϕ and θ be two linear factors, each of which has at least two terms. Hence, we cannot assert that the square of either term must of necessity be nonnull.

Now consider the product $[\phi\theta(\phi + 2\theta)(\phi + \theta)] \equiv [\phi\theta(\phi^2 + 2\theta^2)] \equiv [\phi\theta + 2\phi\theta] \equiv 0$.

Now let us play the game that ψ of the previous lemma is the product of the first three factors; namely $[\phi\theta(\phi + 2\theta)]$. We seek to show that if the last factor $[\phi + \theta] \equiv [a + \lambda]$, where a is a single term, then λ is a factor of ψ. We have $[\theta] \equiv [a + \lambda + 2\phi]$ and $[\phi + 2\theta] \equiv [2(a + \lambda + \phi)]$. Then $[\phi\theta(\phi + 2\theta)] \equiv 2[\phi(a + \lambda + 2\phi)(a + \lambda + \phi)] \equiv 2[\phi(1 + 2a\lambda + \lambda^2) + 2\phi^2)] \equiv 2[\phi + (2a\lambda + \lambda^2)\phi + 2\phi] \equiv 2[(2a\lambda + \lambda^2)\phi] \equiv 2[\lambda(2a + \lambda)\phi] \equiv [\lambda(a + 2\lambda)\phi]$. Hence λ is a factor, as theory says it should be.

Note if λ = a single term, say b, then only two factors are required, for if b is a factor of ψ, no generality is lost if the b is dropped and a new polynomial ψ_b is considered, which is obtained by the rule $\psi_b = [b\psi]$. Then we have $[(a\phi + \theta)(a + b)] \equiv [\phi + ab\phi + a\theta + b\theta] \equiv [(b\phi + \theta)a + (\phi + b\theta)] \equiv 0$. It follows that $[\theta] \equiv [2b\phi]$ and $[\phi] \equiv [2b\theta]$. Then $[(a\phi + \theta)(a + b)] \equiv [(a\phi + 2b\phi)(a + b)] \equiv [\phi(a + 2b)(a + b)] \equiv 0$, since $[(a + 2b)(a + b)] \equiv [a^2 + 2b^2] \equiv 0$.

Finally, note that in all but the case $\lambda = x_i$, the factors needed to make a null product are *dependent.*

May 28, 1964

Theorem. *Let $p(x_i)$ be a multilinear polynomial, involving three or more variables, which contains no linear factor, and let a be any of the variables in $p(x_i)$. Then there exists no linear polynomial $a + \lambda$ such that $[p(x_i)(a + \lambda)]$ is identically zero.*
Proof. $p(x_i)$ can be expressed as

$$p(x_i) = a\phi + \theta,$$

where ϕ and θ are, in general, multilinear polynomials, neither of which contains the variable a.

Assume the theorem is false. Then

$$\left[p(x_i)(a + \lambda)\right] \equiv \left[(a\phi + \theta)(a + \lambda)\right] \equiv \left[a([\lambda\phi] + \theta) + \phi + [\lambda\theta]\right] \equiv 0.$$

For this to be true, the two conditions

$$(1)\ [\lambda\phi] + \theta \equiv 0 \text{ and } (2)\ [\lambda\theta] + \phi \equiv 0$$

must be satisfied simultaneously. Substituting (1) into $p(x_i)$ we have $[p(x_i)] \equiv [a\phi] + [\lambda\phi] \equiv [a\phi + 2\lambda\phi] \equiv [\phi(a + 2\lambda)]$. Hence, $p(x_i)$ contains the linear factor $a + 2\lambda$, which contradicts the given conditions. Q.E.D.

Comment. If the second condition is substituted into the result, we have $[p(x_i)] \equiv [2[\lambda\theta]\ (a + 2\lambda)] \equiv [2\lambda\theta(a + 2\lambda)]$. Hence λ is a factor of $p(x_i)$ also.

May 29, 1964

Theorem. *Let $p(x_i)$ be a multilinear polynomial with three or more variables such that it cannot be reduced; then, there exists no linear polynomial $a + \mu$ that $[p(x_i)\ (a + \mu)]$ is identically zero, where a is any variable.*
Proof. Write $p(x_i)$ in the form $p(x_i) = a\phi + \theta$. Note that if $[\phi] \equiv 0$, then the theorem is true trivially. Hence assume $[\phi] \not\equiv 0$. Then

$$[p(x_i)\ (a + \mu)] \equiv [(a\phi + \theta)\ (a + \mu)] \equiv [a(\theta + [\mu\phi]) + (\phi + [\mu\theta])].$$

Now assume the theorem is false, i.e., that

$$[a(\theta + [\mu\phi]) + (\phi + [\mu\theta])] \equiv 0.$$

This can be true if, and only if, both expressions in parentheses are zero simultaneously. Hence two conditions must be satisfied: (1) $[\theta] \equiv 2[\mu\phi]$ and (2) $[\phi] \equiv 2[\mu\theta]$.

Put (1) into $p(x_i)$, giving

$$[p(x_i)] \equiv [a\phi + 2[\mu\phi]] \equiv [\phi(a + 2\mu)].$$

Now put (2) into this expression, giving

$$[p(x_i)] \equiv [2[\mu\theta]\ (a + 2\mu)] \equiv [2\mu\theta(a + 2\mu)].$$

But this shows that $p(x_i)$ was reducible, which violates the given conditions. Q.E.D.

May 30, 1964

Technique of Reduction.

Given $p(x_i)$ as a multilinear polynomial, we first define a *necessary reduction.* But this operation is based upon the notion of a *necessary factor,* which can be explained as follows. Let a be any variable in $p(x_i)$ and write $p(x_i) = a\phi + \theta$, where neither ϕ nor θ contains the variable a. If there exists a linear polynomial λ which does not contain the variable a such that $[\theta] \equiv [\lambda\phi]$, then $[p(x_i)] \equiv [a\phi] + [\lambda\phi] = [a\phi + \lambda\phi] = [(a + \lambda)\phi]$ and $p(x_i)$ is said to contain the *necessary factor* $(a + \lambda)$. In a complete generality $p(x_i)$ must be tested for each variable to be certain there are no necessary factors.

Now assume that $p(x_i)$ has the necessary factor $a + \lambda$ where $\lambda = b + \mu$, b being a second variable, so that $[p(x_i)] \equiv [(a + b + \mu)\phi]$. Now write ϕ as $\phi = b\delta + r$. If ϕ has a necessary factor $b + \mu$, i.e., $r = [\mu\delta]$, then ϕ becomes $[\phi] \not\equiv [(b + \mu)\delta]$. $p(x_i)$ becomes $[p(x_i)] \equiv [(a + \lambda)\ [(b + \mu)\delta]] \equiv [(a + \lambda)\ (b + \mu)\delta] \equiv [(a + b + \mu)\ (b + \mu)\delta]$. In such a case $p(x_i)$ is said to have a *necessary reduction.*

A necessary reduction is made as follows. The condition of a nonzero product requires that $a = b + \mu$. This relation can be solved for any variable which appears in it. Suppose it is solved for x_j. Then one forms the product $[x_j p(x_i)]$ and substitutes the values of x_j, say $x_j = \beta$, where β is a linear factor. This operation will reduce the number of variables by one and the number of factors by one, for either $x_j = a$ or $x_j = b$ (b is any term in λ). If $x_j = a$, we have $[ap(x_i)]_{a=\lambda} \equiv [\lambda(a + \lambda)\lambda\delta]_{a=\lambda} \equiv [\lambda^3\delta] \equiv [\lambda\delta]$. If $x_j = b$, then $b = a + 2\mu$ and $[bp(x_i)] \equiv [(a + 2\mu)(2a)(a)\delta] \equiv [(a + 2\mu)\delta]$.

If $p(x_i)$ has no necessary reduction, then one can select any variable x_j and set it equal to β, where β is arbitrary but usually set to be a constant times another variable for reasons of simplicity. Then one forms $x_j p(x_i)_{x_j=\beta}$. This will reduce the number of variables by one, but in general increases the number of factors by one. The new polynomial most generally has a necessary reduction.

July 8, 1964

Theorem. *Let $\phi(x_i)$, $i = 1, 2, \ldots, p$, be a nonreducible polynomial derived from the product of $k(k < p)$ linear independent factors. Then $\phi(x_i)$ has no equivalent factorization into k dependent factors.*
Proof. The polynomial specified has at least one term involving k variables. Hence any factorization must involve k linear factors.

Now assume that a factorization in dependent factors exists. Then there exists an expression of the form $c_1\lambda_1(x_i) + c_2\lambda_2(x_i) + \ldots + c_k\lambda_k(x_i) \equiv 0$. This being the case, the expression can be solved for some variable, say x_k, in terms of the other variables. This expression for x_k can be substituted into $\phi(x_i)$, yielding a polynomial $\theta(x_i)$, $x = 1, 2, \ldots, k-1, k+1, \ldots, p$ with only $p - 1$ variables. But this implies that $\phi(x_i)$ is reducible, which is contrary to the stated conditions.

Theorem. *Let $\phi(x_i)$ $i = 1, 2, \ldots, q$ be a polynomial that is partially reducible (i.e., there exist $q - p$ relations of the form $x_s = c_ix_i$ $i = 1, 2, \ldots, p$, which will eliminate $q - p$ variables) terminating in the polynomial $\theta(x_i)$, $i = 1, 2, \ldots, p$, which is nonreducible. Then there exists no linear factor $\psi(x_i)$ such that $[\psi(x_i)\phi(x_i)] \equiv 0$, unless the substitution of the $q - p$ relation into $\psi(x_i)$ makes it null.*
Proof. If the substitution of the linear factors makes $\psi(x_i)$ null, the exception is obvious. Otherwise, the conditions stated with the linear factors substituted violate the conditions for a zero product with a linear factor.

Supplement to Lemma 25

Definition. The polynomial $[c(x_i)]$ is said to be symmetrical in the variable x_i and x_j if an interchange of x_i for x_j and vice versa leaves the polynomial unchanged.

Comment. If $[c(x_i)]$ is symmetrical in the variables x_i and x_j, and the most general form that $[c(x_i)]$ can take is $[c(x_i)] = \lambda x_ix_j + \mu(x_i + x_j) + \omega$, where λ, μ, and ω are polynomials, none of which contains either x_i or x_j.

Lemma A. *If the polynomial $[c(x_i)]$ is symmetrical in the variables x_i and x_j so that $[c(x_i)] = \lambda x_ix_j + \mu(x_i + x_j) + \omega$, then $[c(x_i)]$ can have $x_i + 2x_j$ as a factor if, and only if, the three conditions $[\lambda] \not\equiv 0$, $[\omega] \equiv [2\lambda]$, and $[\mu] \equiv 0$ are satisfied.*
Proof. Substitution of the conditions yields

$$[\lambda(x_ix_j + 2\lambda] \equiv [\lambda(x_ix_j + 2)] \equiv [\lambda x_j(x_i + 2x_j)].$$

On the other hand, assume that $(x_i + 2x_j)$ is a factor of $[c(x_i)]$. Then the substitution $x_j = x_i$ must yield a zero result. Thus $[\lambda + 2\mu x_i + \omega] \equiv 0$. Since λ, μ, and ω do not contain x_i or x_j, either $[\lambda] \equiv [\mu] \equiv [\omega] \equiv 0$, which is a trivial result, or $[\lambda + \omega] \equiv 0$, with $[\lambda] \not\equiv 0$ and $[\mu] \equiv 0$. These are the conditions stated.

Let dependent variables be chosen for Fig. 4b in the following fashion. Consistency relations for vertices $\overline{v}$, v_1, and v_5 will not be invoked. Let dependent variables be selected by the following scheme: $\overline{x}_4$ with a consistency at v_2, $\overline{x}_5$ with consistency at v_3, $\overline{x}_6$ with consistency at v_4, $\overline{x}_7$ with consistency at v', x_8 with v_1 and v_5 as two of its vertices (not shown in Fig. 4b) with consistency at some new vertex v_6, and the remainder to be chosen so that no variable will become dependent if it is associated with a region incident on v_2, v_3, or v_4. It follows from the nature of the selection that

$\overline{x}_6 = 2\overline{x}_1 + 2\overline{x}_2 + 2\phi_1$, where ϕ_1 involves variables associated with regions incident on v_4;

$\overline{x}_5 = \overline{x}_1 + \overline{x}_2 + \phi_2$, where ϕ_2 involves variables associated with regions incident on v_3 and all but one variable from ϕ_1;

$\overline{x}_4 = 2\overline{x}_1 + 2\overline{x}_2 + 2\phi_3$, where ϕ_3 involves variables associated with regions incident on v_2 and some variables from ϕ_1 and ϕ_2;

$\overline{x}_7 = 2\overline{x}_1 + 2\overline{x}_2 + 2\overline{x}_3$, with consistency at v'.

Let ψ be the polynomial obtained from the product of all dependent variables other than $\overline{x}_4$, $\overline{x}_5$, and $\overline{x}_6$. Then

$$[c(x_i)] \equiv [\{(2\overline{x}_1 + 2\overline{x}_2 + 2\phi_1)(\overline{x}_1 + \overline{x}_2 + \phi_2)(2\overline{x}_1 + 2\overline{x}_2 + 2\phi_3)\psi\}(2\overline{x}_1 + 2\overline{x}_2 + 2\overline{x}_3)] \not\equiv 0.$$

Now consider the polynomial with the last factor deleted:

$$[c_1(x_i)] \equiv [\{(2\overline{x}_1\overline{x}_2(\phi + \phi_2 + \phi_3) + (\overline{x}_1 + \overline{x}_2)(1 + \phi_1\phi_2 + \phi_1\phi_3 + \phi_2\phi_3) + (2\phi_1 + 2\phi_2 + 2\phi_3 + \phi_1\phi_2\phi_3)\}\psi] \not\equiv 0.$$

Note that this expression is symmetrical in $\overline{x}_1$ and $\overline{x}_2$.

By Lemma 15, the edges in the path $v_3, \overline{v}, v'$, and v_5 can lie in a simple closed path with all edges labeled alike. Thus there exist solutions in which $\overline{x}_2 = 2\overline{x}_1$. Hence $(\overline{x}_1 + \overline{x}_2)$ is not a factor of $[c_1(x_i)]$. We seek to show that $(\overline{x}_1 + 2\overline{x}_2)$ is also not a factor.

Since $(\overline{x}_1 + \overline{x}_2)$ is not a factor, the substitution $\overline{x}_2 = 2\overline{x}_1$ can be made without yielding a zero result. Therefore,

$$[c_1(x_i)]_{\overline{x}_2 = 2\overline{x}_1} \equiv [\{(\phi_1 + \phi_2 + \phi_3) + (2\phi_1 + 2\phi_2 + 2\phi_3 + \phi_1\phi_2\phi_3)\}\psi] \equiv [\phi_1\phi_2\phi_3\psi] \not\equiv \phi.$$

By Lemma A, $(\overline{x}_1 + 2\overline{x}_2)$ can be a factor only if the three conditions

$$(1)\quad [(2\phi_1 + 2\phi_2 + 2\phi_3 + \phi_1\phi_2\phi_3)\psi] \not\equiv 0$$
$$(2)\quad [\phi_1 + \phi_2 + \phi_3 + \phi_1\phi_2\phi_3)\psi] \equiv 0$$
$$(3)\quad [(1 + \phi_1\phi_2 + \phi_1\phi_3 + \phi_2\phi_3)\psi] \equiv 0$$

are satisfied simultaneously.

By Lemma A, $(\overline{x}_1 + 2\overline{x}_2)$ can be a factor only if the three conditions,

$$[(2\phi_1 + 2\phi_2 + 2\phi_3 + \phi_1\phi_2\phi_3)\psi] \not\equiv 0,$$
$$[(\phi_1\phi_2\phi_3)\psi] \not\equiv 0,$$

and

$$[(1 + \phi_1\phi_2 + \phi_1\phi_3 + \phi_2\phi_3)\psi] \equiv 0,$$

are met. The fact that $(\overline{x}_1 + \overline{x}_2)$ is not a factor means that the substitution $\overline{x}_1 = 2\overline{x}_2$ does not yield a zero result. Hence

$$[c_1(x_i)]_{x_2 = 2x_1} \equiv [\phi_1\phi_2\phi_3\psi] \not\equiv 0,$$

which shows that the second condition holds.

Let us turn to the last condition and note that $[\psi] \not\equiv 0$. Hence the condition can be satisfied only if the expression in the parenthesis is zero. But for this to be true, $[\phi_1\phi_2 + \phi_1\phi_3 + \phi_2\phi_3]$ must consist of only squares of the variable. We now show that this cannot happen.

Since the addition is ordered, there must be at least three regions outside of P incident on each vertex. Consider vertex v_2. There must be a region with one of its vertices on v_2 and none on v_3 or v_4. Let the variable associated with this region be x_q. Then x_q occurs in ϕ_3 but not in ϕ_1 or ϕ_2. Likewise at v_3 there is a similar region with a variable x_s which does not appear in ϕ_1. It follows that in the polynomial $[\phi_1\phi_2 + \phi_1\phi_3 + \phi_2\phi_3]$ there exists a term $[x_q(\phi_1 + \phi_2)]$, but this cannot be zero for ϕ_2 contains the variable x_s while ϕ_1 does not. There are no other terms containing x_q. If follows that $[\phi_1\phi_2 + \phi_1\phi_3 + \phi_2\phi_3]$ cannot involve only squares of variables. It follows that the third condition cannot be satisfied, and hence $[c_1(x_i)]$ does not contain $(\overline{x}_1 + 2\overline{x}_2)$ as a factor.

ADDITIONAL NOTES

January 4, 1964

A New Statement of the Four-Color Problem

Consider Fig. B1, in which the nodal constraint is satisfied at v_4, but not at v_1, v_2, and v_3. Assume also that the nodal constraints are satisfied at every other node in the graph and that these constraints give rise to the multilinear polynomial θ. Let ϕ be the incidence on v_4 provided by the graph outside of v_1, v_2, v_3, v_4. The region v_1, v_2, v_4 is taken as dependent while $v_2, v_3, v_4 \approx$ a is independent. Then for the whole graph we have a multilinear polynomial, $\overline{2\theta(a+\phi)} \equiv \overline{2a\theta} + \overline{2\theta\phi}$. Assume that a valid solution exists for the graph. Then $2a\theta + 2\theta\phi$ must contain this solution; it may also contain other solutions which make the incidence on v_1, v_2, and v_3 nonzero. Note that $\overline{2\theta\phi}$ does not contain a, and that $\overline{\theta} \not\equiv 0$. Also $\overline{\phi} \not\equiv 0$ if four or more regions are incident on v_4. Assume further, without loss of generality, that the incidence on $v_2 \geqslant 4$.

Now consider Fig. B2, which is the graph of Fig. B1 with the (v_2, v_4) edge rotated to (v_1, v_3). For this graph, with v_1, v_2, and v_3 again not having their constraints satisfied, we have the multilinear polynomial $\overline{2\theta\phi}$, which is exactly the second term for Fig. B1. In Fig. B2, the variable a does not appear in the polynomial. Hence the polynomial of Fig. B2 can be obtained from Fig. B1 by setting a = 0. It follows that such an edge can be rotated, giving a valid solution, provided $\overline{\theta\phi} \not\equiv 0$ for the initial graph, Fig. B1. Proof of this fact would in effect prove the four-color conjecture.

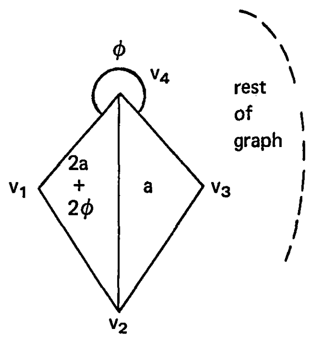

Fig. B1.

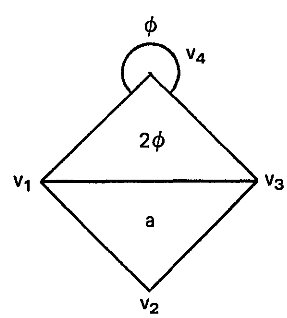

Fig. B2.

August 7, 1964

A New Throught On The Four-Color Problem

Suppose we seek to add a vertex of degree five to G, which is known to have a coloring. Thus we seek to add a vertex to a region and obtain a coloring for the Enlarged graph G′ as in Fig. B3.

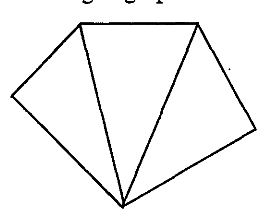

Fig. B3

It is easy to show that the only situation which will not give an immediate coloring for G′ is the one in Fig. B4,

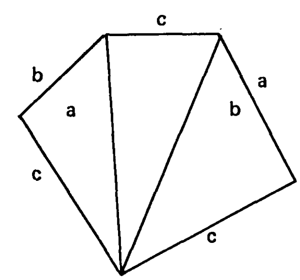

Fig. B4.

with the additional requirement that both interior edges are contained in meshes, all edges of which are of the same coloring. Let us arrive at a graph G_1 as follows.

1. Exchange to the right of interior edge b, giving the edge labeling shown in Fig. B5. If the interior a is still in a similarly colored mesh, a solution to G′ is immediate. So assume a is not in such a mesh.

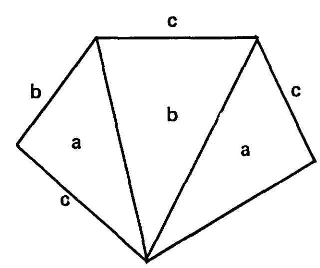

Fig. B5.

2. Add a vertex of degree four on the interior b, giving the result shown in Fig. B6. In this graph all interior a's and b's are *not* in colored meshes.

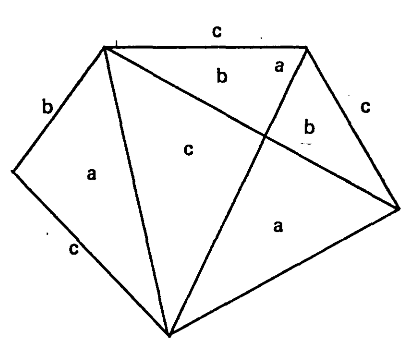

Fig. B6.

3. Add a vertex of degree four on the interior a to the left, giving the graph G_1, as in Fig. B7.

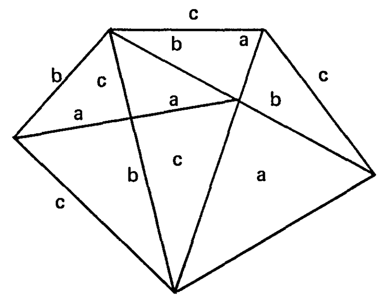

Fig. B7.

The point is that G_1 has a coloring—hence there exists a nonzero polynomial for G_1. Let the dependent regions for G_1 be chosen according to the scheme in Fig. B8.

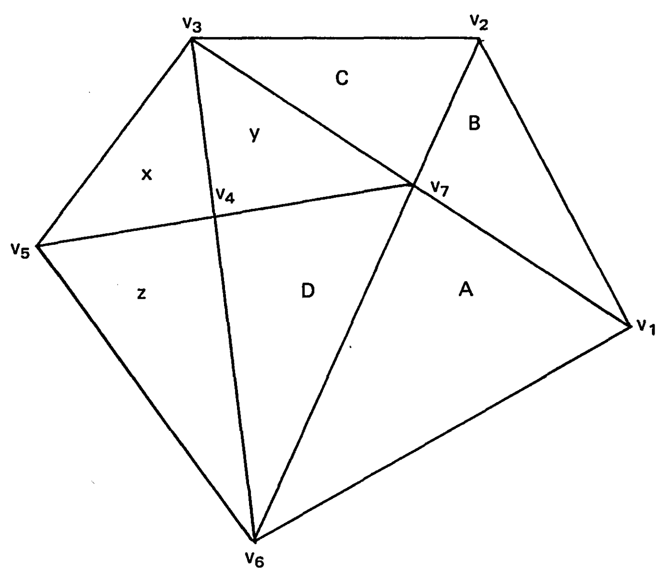

Fig. B8.

The selections are made in the order:

a. Region A with zero sum on v_1,
b. Region B with zero sum on v_2,
c. Region C with zero sum on v_3,
d. Region D with zero sum on v_4,
e. Exterior selections none of which have zero sum on v_5, v_6, or v_7.

Thus all vertices are included except v_5, v_6, v_7; i.e., three vertices on two adjacent edges. These selections give

$D = 2[(x + y) + z]$,
$C = 2(x + y) + 2\phi_1$ where ϕ_1 is the rest of the regions on v_3,
$B = (x + y) + \phi_1 + 2\phi_2$ where ϕ_2 is the rest of the regions on v_2,
$A = 2(x + y) + 2\phi_1 + \phi_2 + 2\phi_3$ where ϕ_3 is the rest of the regions on v_1.

If ψ_1 is the multilinear polynomial for G_1, then it is of the form

$$\psi_1 = \left\{[2(x+y)+2\phi_1+\phi_2+2\phi_3]\ [(x+y)+\phi_1+2\phi_2]\ [2(x+y)+2\phi]\ [2(x+y)+2z]\theta\right\},$$

where θ is the product of factors from the remainder of G_1. The point is (a) that θ does not contain x, y, or z, and (b) that x and y always appear in the combination x + y.

Let ψ' be ψ_1 with the factor $[2(x+y)+2z]$ deleted. Then since ψ_1 is not zero (it has at least two possible solutions) so also is ψ'. If ψ' contains some necessary reductions and if the regions eliminated contain x and y, they must contain them in the form (x + y). After all such reductions are made on the fully reduced polynomial, say ψ_r', it must contain (x + y) in that combination, but not z. Since this fully reduced polynomial is nonzero, there exists at least one solution with x = y. Hence there exists a graph which is colorable in the form shown in Fig. B9.

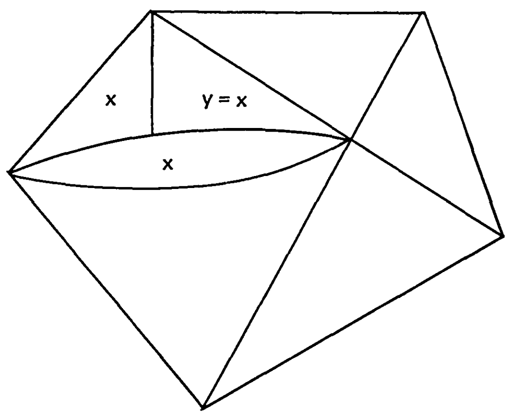

Fig. B9.

Removal of the vertex of incidence 3 gives the desired result.

There remains only the question of a possible gap in this argument. If no gap exists, the theorem is true and proved as of this date.

September 14, 1964

On an Aspect of the Four-Color Conjecture

Let G be an ST-graph and let Fig. B10 be a subgraph of G, let a polynomial be based upon v_2, v_3, v_4 with only one dependent region y being incident on v_1, let θ be the sum of the regions incident on v_1 other than y so that $y = 2\theta$, let ψ be the polynomial obtained from the remaining k-2 dependent region, and let a labeling exist. Then $[y\psi] \equiv [2\theta\psi] \not\equiv 0$.

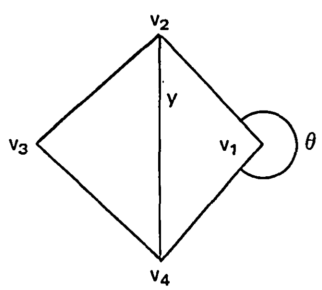

Fig. B10.

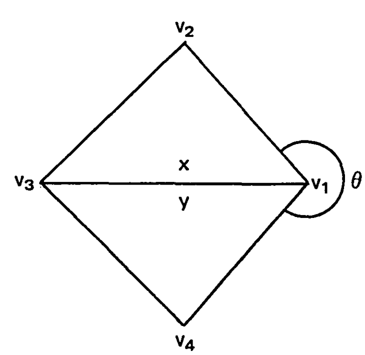

Fig. B11.

Now consider Fig. B11. Let y' be the dependent region and let this be the only change from Fig. B10. Then $y' = 2x + 2\theta$. The polynomial for this graph becomes $[y'\psi] \equiv [(2x + 2\theta)\psi] \equiv (2x\psi + 2\theta\psi] \not\equiv 0$, for ψ and θ do not contain x. In fact both terms are nonnull. Hence we know that solutions exist for the graph of Fig. B11 in which the consistency relations are satisfied at all vertices other than v_2, v_3, v_4. Furthermore, there are multiple solutions since the number of variables has increased by one.

In conclusion, if it can be shown that one of these multiple solutions satisfies the consistency relation at v_2 (or v_4), then the four-color problem will have been solved.

Note that x and y' are interchangeable without affecting any solution.

Notes of September 15, 1964

Lemma. *Let G be an ST-graph for which it is known that a nonzero polynomial exists that satisfies the consistency conditions at every vertex except three, v_i, v_j, v_k, where v_i is connected to v_j, and v_j is connected to v_k, but v_i is* **not** *connected to v_k. Then, either the polynomial for G will contain every solution that satisfies the consistency conditions at v_i, v_j, and v_k, or v_j may be split into v_j' and v_j'' with these two vertices connected so that the consistency conditions are satisfied at v_i, v_j', v_j'', and v_k.*

Proof. For any solution let R_i, R_j, and R_k be the sums of the incidences on the respective vertices. Since all regions have been used in establishing the characteristic polynomial, it follows that $R_i + R_j + R_k = 0$. There are three cases to consider.

1. If $R_i = R_j = R_k = 0$, then consistency exists at v_i, v_j, and v_k.

2. If $R_i = R_j = R_k \neq 0$. Let v_j be split, thus creating two new regions v_i and v_k, where v_i has an incidence on v_i, and v_k has an incidence on v_k. Both are incident on v_j' and v_j''. By the parity lemma, $R_j' = R_j'' = 2R_j$. Then let the variable associated with v_i be $x_i = 2R_i$. In a similar manner, let $x_k = 2R_k$. Thus at v_1 we have $R_i + 2R_i = 0$; at v_k, $R_k + 2R_k = 0$; at v_j' (or v_j'') we have $2R_j + 2R_i + 2R_k = 2(R_i + R_j + R_k) = 0$.

3. If $R_i = 2R_k$, $R_j = 0$. Let v_j be split and let $x_i = 2R_i$ and $x_k = 2R_k = R_i$. The conditions on parity require that $R_j' = R_j'' = 0$. Then at v_i we have $R_i + 2R_i = 0$; at v_k, $R_k + 2R_k = 0$; and at v_j' (or v_j'') $R_j' + R_i + 2R_i = 0$.

Notes of September 18, 1964

Lemma. *Let G be an ST-graph, let v_1 and v_3 be two connected vertices, let v_2 and v_4 be the other vertices incident on the regions having the edge (v_1, v_3) as a boundary and let there be at least one labeling for G. Then the vertex (v_1, v_3) can be rotated to (v_2, v_3) if the incidence on v_1 and v_3 is at least 4.*

Proof. Let the situation be as shown in Fig. B12. Then there exists a polynomial that satisfies the consistency relations at every vertex except v_2, v_3, and v_4. This polynomial will contain a solution that is consistent at these vertices; it will also contain other possibilities, in which some are not zero at all of the three vertices.

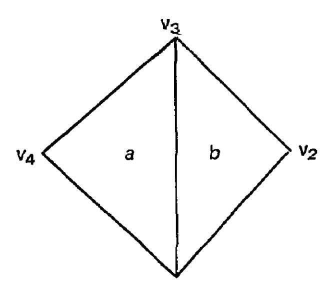

Fig. B12.

To form this polynomial, let dependent regions be selected as follows. Take a region having (v_2, v_3) as an edge but not b. Take a second region bounding the one just taken and having v_i on its boundary. Continue in this fashion until v_i is satisfied. Then starting at (v_3, v_4) select a region not a. Continue in this fashion but not using any regions incident on v_1 not used in the first sequence of selections.

Now every linear factor that contains a also contains b in the combination a + b. Hence these two variables are indistinguishable in the polynomial. The polynomial cannot be completely reducible, for it must contain at least three solutions (number of variables = k and number of factors = k − 1). Hence one arbitrary reduction is b = a. But this gives a possible rotation, as the following argument shows.

There are two cases.

Case 1. b = a and consistency holds at v_2, v_3, v_4. Then the situation is equivalent to the labeling of Fig. B13, in which case a rotation is immediate.

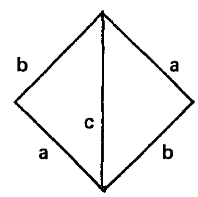

Fig. B13.

Case 2. b = a but consistency does not hold at v_2, v_3, v_4. Then the situation is equivalent to the labeling of Fig. B14. This gives an immediate solution, shown in Fig. B15, in which consistency now holds at v_2, v_3, and v_4.

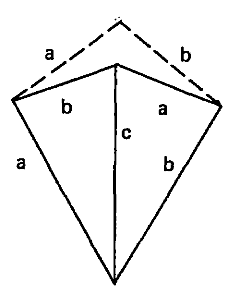

Fig. B14.

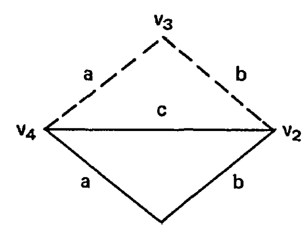

Fig. B15.

The conclusion is that if the reduction b = a is either forced or can be arbitrary, then a rotation is possible.

September 24, 1964

A Key Result in a Proof of the Four-Color Conjecture

Lemma: *Let the subgraph of Fig. B16 be a part of an ST-graph for which a labeling is known to exist. Assume further that there are at least two regions exterior to the subgraph which are incident on v_1, v_2, v_4, v_5, and v_6 in turn. Let P be a polynomial which assumes consistency at every vertex in the whole graph except at v_1, v_2, and v_3. We wish to show that $(x + 2y)$ is not a factor of this polynomial.*

Proof. Let dependent regions be selected in the following fashion:

1. The region $(v_2\ v_3\ v_4)$ for consistency at v_4,
2. The region $(v_3\ v_4\ v_5)$ for consistency at v_5,
3. The region $(v_3\ v_5\ v_6)$ for consistency at v_6,
4. The remaining to start with $(v_1\ v_2\ v_7)$ for consistency at v_7 and no choice to be based on v_4, v_5, or v_6.

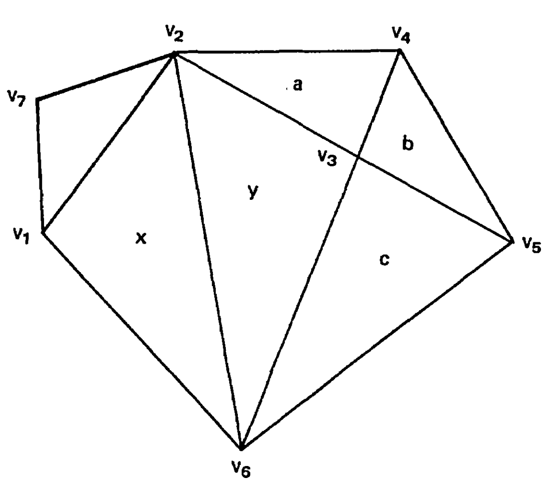

Fig. B16.

Let λ_1, $2\lambda_2$, and λ_3 be the linear factors which give the independent variables at v_4, v_5, and v_6 in order. Let ψ be the polynomial obtained from the product of the factors resulting from the choices in Step 4. For the remaining three factors, we have

$$x + y + c + \lambda_3 = 0, \text{ or } c = 2(x + y + \lambda_3) \text{ at } v_6;$$
$$b + c + 2\lambda_2 = 0, \text{ or } b = 2c + \lambda_2 = x + y + \lambda_2 + \lambda_3 \text{ at } v_5;$$

and

$$a + b + \lambda_1 = 0, \text{ or } a = 2b + 2\lambda_1 = 2(x + y + \lambda_1 + \lambda_2 + \lambda_3) \text{ at } v_4.$$

Then

$$\begin{aligned} P &= [abc\psi] \\ &= [(x + y + \lambda_1 + \lambda_2 + \lambda_3)(x + y + \lambda_2 + \lambda_3)(x + y + \lambda_3)\psi]. \end{aligned}$$

Let the first three factors be multiplied together, not making the heterogeneous substitution as yet, giving

$$P = \left[\left\{(x + y)^2 (\lambda_1 + 2\lambda_2) + (x + y)(1 + \lambda_2^2 + \lambda_1\lambda_2 + 2\lambda_1\lambda_3 + \lambda_2\lambda_3) + \lambda_3(\lambda_2^2 + \lambda_1\lambda_2 + \lambda_1\lambda_3 + 2\lambda_2\lambda_3 + \lambda_3^2)\right\}\psi\right].$$

Now consider two cases.

Case 1. The graph is known to have a labeling such that $x = y$. In this case, clearly $(x + 2y)$ cannot be a factor.

Case 2. The graph is known to have a labeling such that $x = 2y$. We seek to show that this selection is not forced in P. From the assumption, it follows that

$$P_{(x=2y)} = [\lambda_3(\lambda_2^2 + \lambda_1\lambda_2 + \lambda_1\lambda_3 + 2\lambda_2\lambda_3 + \lambda_3^2)\psi] \not\equiv 0.$$

It is also clear that $(x + 2y)$ cannot be a factor of P unless

$$[(1 + \lambda_2^2 + \lambda_1\lambda_2 + 2\lambda_1\lambda_3 + \lambda_2\lambda_3)\psi] \equiv 0,$$

for if $(x + 2y)$ is a factor, then the substitution $y = x$ into P must give a null result. Let this condition be substituted into the expression for P, giving, say,

$$\begin{aligned} P' &= \left[\left\{(x + y)^2 (\lambda_1 + 2\lambda_2) + \lambda_3(2 + 2\lambda_1\lambda_3 + \lambda_2\lambda_3 + \lambda_3^2)\right\}\psi\right] \\ &= \left[\left\{(x + y)^2 (\lambda_1 + 2\lambda_2) + (2\lambda_3 + 2\lambda_1\lambda_3^2 + \lambda_2\lambda_3^2 + \lambda_3)\right\}\psi\right] \\ &= \left[\left\{(x + y)^2 (\lambda_1 + 2\lambda_2) + \lambda_3^2(2\lambda_1 + \lambda_2)\right\}\psi\right] \\ &= \left[(\lambda_1 + 2\lambda_2)\left\{(x + y)^2 + 2\lambda_3^2\right\}\psi\right] \\ &= [(\lambda_1 + 2\lambda_2)(x + y + \lambda_3)(x + y + 2\lambda_3)\psi] \\ &\not\equiv 0 \text{ if } (x + 2y) \text{ is to be a factor.} \end{aligned}$$

It follows that $(x + 2y)$ is a factor only if $[\lambda_3^2] \equiv 1$ so that

$$\begin{aligned} P' &= \left[(\lambda_1 + 2\lambda_2)\left\{(x + y)^2 + 2\right\}\psi\right] \equiv [(\lambda_1 + 2\lambda_2)(2 + 2xy + 2)\psi] \\ &\equiv [(\lambda_1 + 2\lambda_2)(1 + 2xy)\psi] \equiv [x(\lambda_1 + 2\lambda_2)(x + 2y)\psi]. \end{aligned}$$

But λ_3 has at least two terms, so that $[\lambda_3^2] \not\equiv 1$ when the heterogeneous relations are substituted. Hence the conjecture is true. This result is the key one for proving the four-color conjecture, for it allows an ordered introduction of a five-vertex, as the following analysis shows.

The previous result shows that in P, $x = y$ is a possible solution. Now consider two cases which are exhaustive.

Case 1. The substitution $x = y$ satisfies the consistency relations at v_1, v_2, and v_3. In this case the edge $(v_2\ v_6)$ can be rotated, leaving consistency at every vertex. That is, the figure becomes as given in Fig. B17.

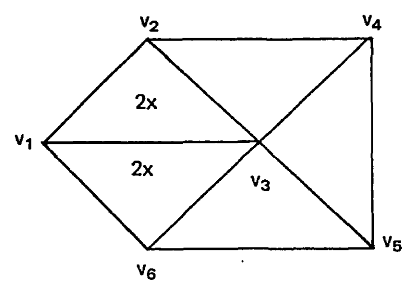

Fig. B17.

Case 2. The substitution $x = y$ does not satisfy the consistency relations at v_1, v_2, and v_3. The discrepancy at v_1, v_2, v_3, is, say, f at each vertex. Then $x = y = f$ and $x + y = 2f$. Let the edge $(v_2\ v_6)$ be rotated to give the graph of Fig. B18.

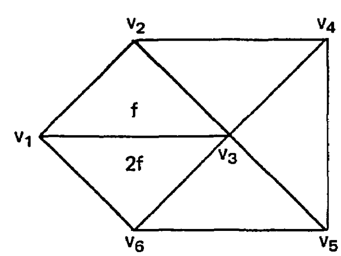

Fig. B18.

It follows that with this situation consistency has been established locally at v_1, v_2, and v_3, since the incidence has been reduced by f at each of these vertices.

Hence we have the result: If in an ST-graph, consistency holds at every vertex except v_1, v_2, and v_3 with v_1 connected to v_2 and v_2 connected to v_3; if v_1, v_2, and v_3 are a four-sided subgraph with two interior regions; and if the polynomial allows the two regions to be equal, then consistency holds everywhere when the common edge is rotated.

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13. ABSTRACT

According to Courant and Robbins, the four-color problem was probably proposed by Möbius in 1840. It was proposed again by De Morgan in 1850 and by Cayley in 1878. The problem may be stated as follows: Given any linear graph which will map on a plane (or sphere), it is possible to assign one of four colors to each face of the map so that no two faces colored alike have a common edge on their boundaries.

In 1879 Kempe published a proof of the conjecture. However, in 1890, Heawood showed that Kempe's proof was defective, by use of a counterexample that disproved one of Kempe's assertions. The portion of this manuscript from Comment 18 through Lemma 24 is designed to take care of the analog to Heawood's counterexample. In the same paper, Heawood showed that five colors were sufficient to color a planar map.

The problem has received much attention in the intervening years. Although a number of interesting results have been obtained since 1890, no one has been able to exhibit a planar graph that required more than four colors, nor has anyone been able to prove that four colors were sufficient in all cases. The situation up to now is summed up rather well by Courant and Robbins, "Despite the efforts of many famous mathematicians, the matter essentially rests with this more modest result: It has been proved that five colors suffice for all maps and it is *conjectured* that four will likewise suffice." This manuscript presents a new approach to and a partial proof of the conjecture.

Heawood's paper of 1890 included some other remarkable results. He was able to define upper and lower limits for the number of distinct colors needed to solve the coloring problem on a surface of any genus. Furthermore, for all surfaces of greater complexity than the sphere, he was able to show that the two limits were the same, provided it was possible to construct a map in which there were this many faces each of which touched all the others. In the case of the sphere, however, Heawood's upper limit was six and his lower limit was four. He used a separate argument to reduce the upper limit to five. In a real sense then, a solution to the four-color problem amounts to showing that Heawood's lower limit is also the upper limit. In this work, the concept of the stably triangulated graph (the ST-graph) is introduced, and its properties are derived. It is shown that the arbitrary planar graph can be converted to an ST-graph by the insertion of appropriate edges.

Next the notion of the valid edge labeling for an ST-graph is defined, and algebraic conditions sufficient to insure the existence of a valid edge labeling are obtained. It is shown that its existence implies the affirmative answer to the four-color conjecture. Finally, an inductive, but incomplete, proof is given that there exists a valid edge labeling for every finite graph.

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