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Calculation of the Retarded Electric Field for an Arbitrary Charge and Current Distribution

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CONTENTS

ABSTRACT

In the Lorentz gauge, the solutions to the scalar and vector potential equations (free space) are the retarded potentials, which are integrals of charge and current distributions over all space, subject to the retardation condition. It is shown that if the charge and current are "turned on" in the finite past ($t' = 0$), integrals over a finite region result, in particular, over an expanding sphere of radius ct, centered at the field point. The limits of integration, which are time and space dependent, have been obtained. The corresponding expressions for $\mathbf{A}$ and Φ, have been used to derive $\mathbf{E} = -\nabla\phi - \partial\mathbf{A}/\partial t$ for an arbitrary charge and current distribution. This involves the partial derivatives of the densities in volume integrals, surface integrals which vary as $1/ct$, and other integrals which vanish due to the geometry. The formulation is valid for times $t \leq (r \sin\theta)/c$.

PROBLEM STATUS

This is an interim report; work on the problem continues.

AUTHORIZATION

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CALCULATION OF THE RETARDED ELECTRIC FIELD FOR AN ARBITRARY CHARGE AND CURRENT DISTRIBUTION

INTRODUCTION

Maxwell's equations for the fields E and B give rise to equations for the potentials A and Φ, which are in general coupled and unsymmetric. Imposing the Lorentz gauge (in free space)

$$\nabla \cdot \mathbf{A} + \frac{1}{c^2}\frac{\partial \Phi}{\partial t} = 0$$

uncouples the equations and makes them symmetric; they become the wave equations, whose solutions (1) are the well-known "retarded potentials":

$$\mathbf{A}(\mathbf{r}, t) = \frac{\mu_0}{4\pi} \int_{\text{all space}} \frac{\mathbf{J}\left(\mathbf{r}', t - \frac{|\mathbf{r} - \mathbf{r}'|}{c}\, dv'\right)}{|\mathbf{r} - \mathbf{r}'|} \tag{1}$$

and

$$\Phi(\mathbf{r}, \mathbf{t}) = \frac{1}{4\pi\epsilon_0} \int_{\text{all space}} \frac{\rho\left(\mathbf{r}', t - \frac{|\mathbf{r} - \mathbf{r}'|}{c}\right)}{|\mathbf{r} - \mathbf{r}'|}\, dv' , \tag{2}$$

where, according to the retardation prescription, the charge density ρ and the current density J are evaluated at the point $\mathbf{r}'$ at the retarded time $t' = t - (|r - r'|/c)$ as indicated. The field point is $\mathbf{r}$, and the variable of integration is $\mathbf{r}'$. The electric field is then given by

$$\mathbf{E} = -\nabla\Phi - \frac{\partial \mathbf{A}}{\partial t}. \tag{3}$$

Formally, Eqs. (1) and (2) are integrals over all space (as indicated), so that Eq. (3) becomes, formally,

$$\mathbf{E}(\mathbf{r}, t) = \frac{-1}{4\pi\epsilon_0} \int_{\text{all space}} \nabla\left[\frac{\rho\left(\mathbf{r}', t - \frac{|\mathbf{r} - \mathbf{r}'|}{c}\right)}{|\mathbf{r} - \mathbf{r}'|}\right] dv'$$

$$- \frac{\mu_0}{4\pi} \int_{\text{all space}} \frac{\dfrac{\partial \mathbf{J}\left(\mathbf{r}', t - \frac{|\mathbf{r} - \mathbf{r}'|}{c}\right)}{\partial t}}{|\mathbf{r} - \mathbf{r}'|}\, dv' . \tag{4}$$

However, these integrals over all space can be replaced by integrals over a *finite* volume, assuming that the current and charge were turned on in the finite past, say, at $t' = t_0 \neq -\infty$. This fact leads to a somewhat different form for $\mathbf{E}$; instead of an integral over all space of a partial derivative, one gets an integral over a *finite* volume of a partial derivative, plus a surface integral which varies as $1/(t - t_0)$. This applies to both the $\nabla\Phi$ and the $\partial\mathbf{A}/\partial t$ integrals. The volume integrals are performed over a sphere of radius ct, centered at the field point $\mathbf{r}$. The surface integrals are performed over the surface of this sphere. If either t approaches $+\infty$ or t_0 approaches $-\infty$ (charge and current created in the infinite past), the volume integrals will extend over all space and the surface integrals will vanish, leaving only terms like those in Eq. (4).

These results will be derived in this report, along with the explicit geometry involved, including the limits of integration, and some remarks on the applicability of the results. For simplicity it will be assumed that $t_0 = 0$ (charge and current are turned on at time $t' = 0$) and the electric field due to such arbitrary sources will be computed for times $t \geq 0$. Before proceeding, the notation should be made clear; $(\mathbf{r}, t)$ represents the field point and the time t when we wish to calculate a quantity at the field point. These are parameters in relation to the integration over $(\mathbf{r}', t'_{ret})$, which represents the location of a volume element and the retarded time $t'_{ret} = t - (|r - r'|/c)$ at which the contribution of charge or current from this volume element is added to the integral (in accordance with the retardation prescription).

CALCULATION OF $\partial\mathbf{A}(\mathbf{r}, t)/\partial t$ AND $\nabla\Phi(\mathbf{r}, t)$

The retarded vector potential is given by Eq. (1), which includes the retardation condition

$$|\mathbf{r} - \mathbf{r}'| = c(t - t') . \tag{5}$$

The condition in Eq. (5) is applied to $\rho(\mathbf{r}', t')$ and $\mathbf{J}(\mathbf{r}', t')$ and dictates that the contribution from charge or current all around the spherical shell of thickness $\delta r'$ and radius $|\mathbf{r} - \mathbf{r}'|$ given by Eq. (5) is all added to the integrals for Φ and $\mathbf{A}$ at the time t' in the interval $\delta t'$. By Eq. (5), the shell of maximum radius, therefore, contains charge or current which existed at the earliest t'. Therefore, since the sources were turned on at $t' = 0$, the integration is carried on within a spherical volume centered at $\mathbf{r}$, with the radius given by Eq. (5), setting $t' = 0$:

$$|\mathbf{r} - \mathbf{r}'| = ct . \tag{6}$$

This is the finite volume of integration alluded to earlier. If one now calculates $\mathbf{A}(\mathbf{r})$ at the times t and $t + \Delta t$, it will have changed during the interval Δt due to two effects:

1. From a given shell of radius $|\mathbf{r} - \mathbf{r}'|$, $\mathbf{J}(\mathbf{r}')$ must be calculated at the slightly later time $t' + \Delta t = t + \Delta t - (|\mathbf{r} - \mathbf{r}'|/c)$, instead of at $t' = t - (|\mathbf{r} - \mathbf{r}'|/c)$. So $\mathbf{J}$ in each shell is different.

2. The contribution from the shell of radius $|\mathbf{r} - \mathbf{r}'| = c(t + \Delta t)$ must be added; $\mathbf{J}$ is calculated for $t' = 0$.

This means that $\partial\mathbf{A}(\mathbf{r}, t)/\partial t$ has two parts:

1. A finite volume integral

$$\frac{\mu_0}{4\pi} \int \frac{\frac{\partial \mathbf{J}}{\partial t}}{|\mathbf{r} - \mathbf{r}'|} \, dv' .$$

2. What amounts to a surface integral, something like

$$\frac{\mu_0}{4\pi}\left(\frac{d}{dt}\,|\mathbf{r}-\mathbf{r}'|_{\substack{\text{on the}\\ \text{surface}}}\right)\int \frac{\mathbf{J}\ ds}{ct}\ ,$$

where $d|\mathbf{r}-\mathbf{r}'|/dt_{\text{on the surface}}$ is the rate of expansion of the spherical* volume (over which integration is allowed) and $\int \mathbf{J}\,ds/ct$ is the "volume" integral of the extra shell of thickness $d|\mathbf{r}-\mathbf{r}'|$ that is added during dt (note that $|\mathbf{r}-\mathbf{r}'|_{\text{on the surface}}$ is equal to ct. It is important to take note of exactly how the integration will be done: the spherical volume centered at the field point $\mathbf{r}$ with radius ct will be integrated over by intersecting it with a succession of spherical shells centered at a different point, namely, the origin of coordinates.† These shells have nothing to do with the "shells" alluded to earlier, which were introduced only to show the plausibility of the appearance of a surface integral in $\mathbf{E}(\mathbf{r},t)$.

The precise formulation is as follows: $\mathbf{A}(\mathbf{r},t)$ is a triple integral given by

$$\mathbf{A}(\mathbf{r},t) = \frac{\mu_0}{4\pi}\int_{g_3(r,t)}^{h_3(r,t)} r'^2\,dr' \int_{g_2(r',r,t)}^{h_2(r',r,t)} \sin\theta'\,d\theta' \int_{g_1(r',\theta',r,t)}^{h_1(r',\theta',r,t)} \mathbf{F}(\phi',\theta',r',t,\theta,\phi)\,d\phi\ ,$$

where

$$\mathbf{F} \equiv \frac{\mathbf{J}\left(\mathbf{r}',t-\dfrac{|\mathbf{r}-\mathbf{r}'|}{c}\right)}{|\mathbf{r}-\mathbf{r}'|}.$$

For convenience, define

$$Q \equiv \int_{g_1}^{h_1} d\phi'\,\mathbf{F}\ , \tag{7}$$

so that $\mathbf{A}(\mathbf{r},t)$ is of the form

$$\mathbf{A}(\mathbf{r},t) = \frac{\mu_0}{4\pi}\int_{g_3}^{h_3} r'^2\,dr' \int_{g_2}^{h_2} \sin\theta'\,d\theta' Q\ . \tag{8}$$

Now using the appropriate formulas for the derivative of an integral with variable limits, one obtains

---

*The idea of contributions from different shells at different times is explained in a somewhat different connection in Ref. 1.

†The situation of the field point with respect to the origin is shown in Fig. 1. The explicit limits of integration are obtained in Appendix A with reference to Figs. 1, 2, and 3.

$$
\begin{aligned}
\frac{\partial \mathbf{A}(\mathbf{r},t)}{\partial t} = & \int_{g_3(r,t)}^{h_3(r,t)} r'^2\, dr' \int_{g_2(r',r,t)}^{h_2(r',r,t)} \sin\theta'\, d\theta'\, \frac{\partial Q}{\partial t} + \int_{g_3(r,t)}^{h_3(r,t)} r'^2\, dr' \left[\frac{\partial h_2(r',r,t)}{\partial t} Q\right]_{\theta'=h_2} \\
& - \int_{g_3(r,t)}^{h_3(r,t)} r'^2\, dr' \left[\frac{\partial g_2(r',r,t)}{\partial t} Q\right]_{\theta'=g_2} + \frac{\partial h_3(r,t)}{\partial t} \int_{g_2[h_3(t),r,t]}^{h_2[h_3(t),r,t]} Q \sin\theta'\, d\theta' \\
& - \frac{\partial g_3(r,t)}{\partial t} \int_{g_2[h_3(t),r,t]}^{h_2[h_3(t),r,t]} Q \sin\theta'\, d\theta' \,. \qquad (9)
\end{aligned}
$$

Rather fortuitously, the last four integrals are zero (Appendix B). Recalling also from Appendix A the fact that all the limits (on r', θ', and ϕ') are situated symmetrically with respect to the field coordinates r, θ, and ϕ, Q becomes

$$
Q = \int_{\phi-\Delta\phi'}^{\phi+\Delta\phi'} \frac{\mathbf{J}_{\text{retarded}}}{|\mathbf{r}-\mathbf{r}'|}\, d\phi' \,,
$$

and again applying the appropriate formula for the derivative of an integral with variable limits,

$$
\begin{aligned}
\frac{\partial Q}{\partial t} = & \int_{\phi-\Delta\phi'}^{\phi+\Delta\phi'} \frac{\dfrac{\partial \mathbf{J}_{\text{retarded}}}{\partial t}}{|\mathbf{r}-\mathbf{r}'|}\, d\phi' + \left(\frac{\partial}{\partial t}\Delta\phi'\right)\left(\frac{\mathbf{J}_{\text{retarded}}}{|\mathbf{r}-\mathbf{r}'|}\right)_{\phi'=\phi+\Delta\phi'} \\
& - \left(\frac{\partial}{\partial t}\Delta\phi'\right)\left(\frac{\mathbf{J}_{\text{retarded}}}{|\mathbf{r}-\mathbf{r}'|}\right)_{\phi'=\phi-\Delta\phi'} . \qquad (10)
\end{aligned}
$$

Then, since at the limits of ϕ', $|\mathbf{r}-\mathbf{r}'| = ct$,* one obtains

$$
\frac{\partial \mathbf{A}(\mathbf{r},t)}{\partial t} = \frac{\mu_0}{4\pi} \iiint_{\substack{\text{spherical} \\ \text{volume}}} \frac{\dfrac{\partial \mathbf{J}_{\text{retarded}}}{\partial t}}{|\mathbf{r}-\mathbf{r}'|}\, dv'
$$

$$
+ \frac{\mu_0}{4\pi} \int_{\substack{\text{spherical} \\ \text{surface}}} r'^2\, dr' \int \sin\theta'\, d\theta' \left(\frac{\partial}{\partial t}\Delta\phi'\right)\left[\frac{J(r',\theta',t'=0)\big|^{\phi+\Delta\phi'} + J(r',\theta',t'=0)\big|^{\phi-\Delta\phi'}}{ct}\right].
$$

A similar procedure for $\nabla\Phi(\mathbf{r},t)$ gives†

*The limits of ϕ' correspond to the surface of the sphere of integration, whence $|\mathbf{r}-\mathbf{r}'| = ct$ there. Note that $\mathbf{J}$ is also evaluated on the surface in this term.

†In the case of $\nabla\Phi$, the surface integral arises because the derivative operation of the volume element with respect to the field point coordinates involves the difference between a spherical volume integral centered at the field point $\mathbf{r}$ and one centered at $\mathbf{r}+\delta\mathbf{r}$. The difference between these two volume integrals is a surface integral.

$$\nabla\Phi(\mathbf{r},t) = \frac{1}{4\pi\epsilon_0} \iiint\limits_{\substack{\text{spherical}\\ \text{volume}}} \nabla\left(\frac{\rho_{\text{retarded}}}{|\mathbf{r}-\mathbf{r}'|}\right) dv'$$

$$+ \frac{1}{4\pi\epsilon_0} \int\limits_{\substack{\text{spherical}\\ \text{surface}}} r'^2\, dr' \int \sin\theta'\, d\theta'\, \nabla(\Delta\phi') \frac{\left[\rho(r',\theta',\overset{\phi+\Delta\phi'}{,}t'=0) + \rho(r',\theta',\overset{\phi-\Delta\phi'}{,}t'=0)\right]}{ct}$$

This latest result for $\nabla\Phi$ is valid only if ∇ is taken in spherical operator coordinates. On the other hand, the equation giving $\partial\mathbf{A}/\partial t$ is valid only if $\mathbf{A}$ is computed in rectangular cartesian coordinates. Therefore, the components of $\mathbf{E}$ are not obtained by simply adding $-\nabla\Phi$ and $-\partial\mathbf{A}/\partial t$ components as they are directly computed from the integrals. The integrals for $\partial\mathbf{A}/\partial t$ give cartesian components, while the integrals for $\nabla\Phi$ give spherical components. The choice is to transform one or the other of these so that they can be added to give either the cartesian or the spherical components of $\mathbf{E}$. The reason for this is that, on the one hand, it is the spherical coordinates which appear in the limits of Φ (thus each spherical component of ∇ has a corresponding coordinate in the limits of Φ), so that each component of ∇ on Φ is analogous to $\partial/\partial t$ on $\mathbf{A}$ and gives the same simple integral form. It is not clear what integral form the direct cartesian operation of ∇ on Φ would give. On the other hand, only in cartesian coordinates does the equation $\Box^2\mathbf{A} = \mu_0\mathbf{J}/4\pi$ split up into three equations (2) whose solutions are

$$\mathbf{A} = \frac{\mu_0}{4\pi}\int \frac{\mathbf{J}_{\text{ret}}}{|\mathbf{r}-\mathbf{r}'|}\, dv'$$

Thus, $\partial\mathbf{A}/\partial t$ is of necessity a vector computed in cartesian components. With these facts in mind, the integrals for $-\nabla\Phi$ and $-\partial\mathbf{A}/\partial t$ can be "added" to give the total electric field.

Putting in the explicit limits of integration, $\mathbf{E}$ then takes the form

$$\mathbf{E} = -\int_{r-ct}^{r+ct} r'^2\, dr' \int_{\theta-k}^{\theta+k} \sin\theta'\, d\theta' \int_{\phi-\Delta\phi'}^{\phi+\Delta\phi'} d\phi' \left[\frac{1}{4\pi\epsilon_0}\nabla \frac{\rho\left(r',\theta',t-\frac{|\mathbf{r}-\mathbf{r}'|}{c}\right)}{|\mathbf{r}-\mathbf{r}'|}\right.$$

$$\left. + \frac{\mu_0}{4\pi} \frac{\frac{\partial}{\partial t}\mathbf{J}\left(r',\theta',t-\frac{|\mathbf{r}-\mathbf{r}'|}{c}\right)}{|\mathbf{r}-\mathbf{r}'|}\right]$$

$$-\frac{1}{ct}\int_{r-ct}^{r+ct} r'^2\, dr' \int_{\theta-k}^{\theta+k} \sin\theta'\, d\theta' \left\{\frac{1}{4\pi\epsilon_0}[\nabla(\Delta\phi')][\rho(r',\theta',\phi+\Delta\phi',t'=0)\right.$$

$$+ \rho(r',\theta',\phi-\Delta\phi',t'=0)]$$

$$\left. + \frac{\mu_0}{4\pi}\left[\frac{\partial}{\partial t}(\Delta\phi')\right][\mathbf{J}(r',\theta',\phi+\Delta\phi',t'=0) + \mathbf{J}(r',\theta',\phi-\Delta\phi',t'=0)]\right\}, \tag{11}$$

where

$$\Delta\phi' = \sin^{-1} \sqrt{\sin^2 k - \cos^2 k \tan^2 (\theta' - \theta)}$$

and

$$k = \cos^{-1} \frac{r^2 + r^{12} - (ct)^2}{2rr'}$$

Equation (11) gives the retarded electric field due to an arbitrary charge and current distribution which are turned on at time $t' = 0$. The first two terms comprise an integral over the finite sphere centered at $\mathbf{r}$ with radius ct. The second two terms comprise an integral over the surface of this finite sphere and vary as $(ct)^{-1}$. They are, therefore, less significant as time goes on.

REMARKS

This formulation is valid only for times $t \leq (r \sin \theta)/c$, the reason being that for greater times, the allowed sphere of integration will engulf a part of the Z axis. Once this happens, the spherical volume of radius ct centered at $\mathbf{r}$ can no longer be integrated by intersecting it with spherical shell volume elements centered at the origin. The volume integration becomes much more involved, and an extremely complicated set of limits would have to be found. Figure 4 depicts such a situation, where the limits used in this report apply only in the shaded region, which does not cover the whole sphere. The limits used in this report correspond to Figs. 1, 2, and 3, in which such a condition has not been reached. Since the allowed sphere of integration must not reach the Z axis, it is probably desirable to put the origin and the field point on opposite sides of the charge distribution (if the charge distribution is localized to permit this), so that the effects of the entire charge distribution can be accounted for in computing $\mathbf{E}$. On the other hand, if it is necessary to take into account some inherent symmetry of the charge distribution by situating the origin in the midst of the charge, then it must be remembered that any region of charge which cannot be contained within a sphere centered at the field point that does not engulf part of the Z axis cannot be taken into account by this formulation. Since the size of the sphere gives the upper limit t for which $\mathbf{E}(\mathbf{r}, t)$ can be accurately calculated, this amounts to a possible tradeoff between symmetry and time interval of validity: to make the calculation valid for the longest time the origin should be taken as far away from the field point as possible. To take advantage of symmetry of the charge distribution, the origin might have to be placed in a position fairly close to the field point or in the midst of the distribution, which would limit the time of validity of the result $\mathbf{E}(\mathbf{r}, t)$.

It should be reemphasized that the turning on of the distribution in the finite past is what allowed the volume integral over all space to reduce to a finite volume for $\mathbf{A}$ and ϕ, and this in turn introduced surface terms in $\mathbf{E}$ which vary inversely with ct.

CONCLUSIONS

The solution to the equations for $\mathbf{A}$ and ϕ in free space and with the Lorentz gauge imposed are the retarded potentials, which are, by the nature of the solution to the differential equations, integrals over all space. The assumption that the charge and current sources were turned on in the finite past (at time $t' = 0$), allowed (due to retardation condition) integration over a finite sphere centered at the field point, and this in turn led to both volume and surface integral contributions to the electric field $\mathbf{E}$. The specific geometry involved and the limits of integration were obtained. The surface integrals were seen to vary inversely with ct and, hence, are less important as time goes on.

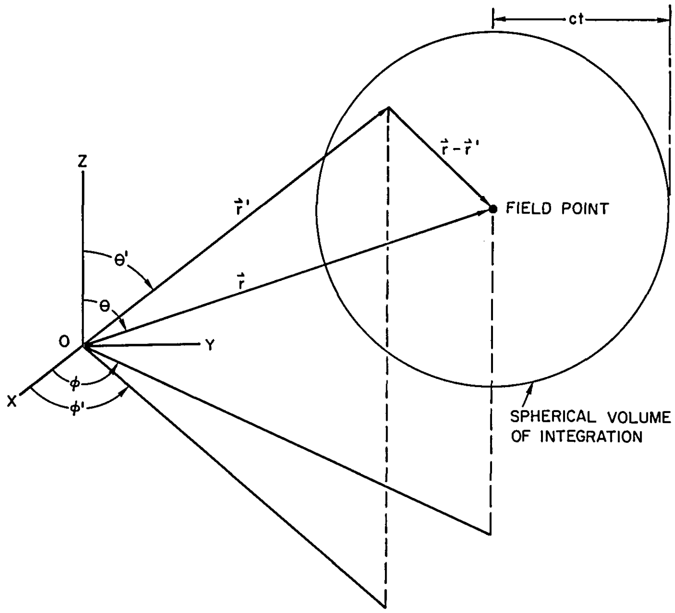

Fig. 1 - Geometry for the calculation of the regarded potentials, showing the field point **r** and the expanding spherical volume of integration. The variables r, θ, and ϕ of the field point are parameters in the integration. The vector **r**$'$ is a representative point in the volume, and its coordinates are r', θ', and ϕ'. The origin is at 0.

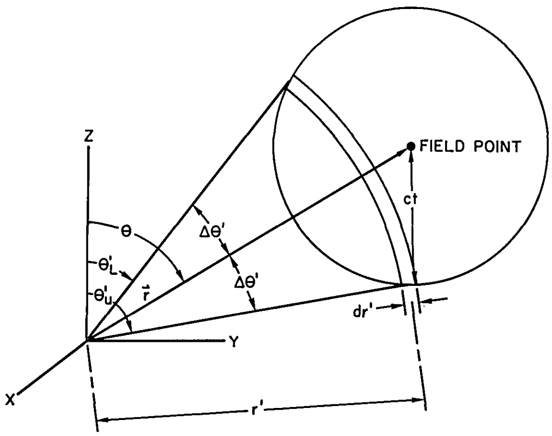

Fig. 2 - The limits on θ'. The lower limit is θ_L', and the upper limit is θ_u'. They are symmetrically situated at an angle $\Delta\theta'$ on either side of θ (the field point angle). From the figure and the cosine law it is clear that $\cos\Delta\theta' = [r^2 + r'^2 - (ct)^2]/2rr'$, $\theta_u = \theta + \Delta\theta'$, and $\theta_L = \theta - \Delta\theta'$. A spherical shell, centered at the origin, is depicted intersecting the spherical volume centered at the field point, **r**. Such shells of successive radii can cover the entire sphere, until the sphere intersects the **Z** axis.

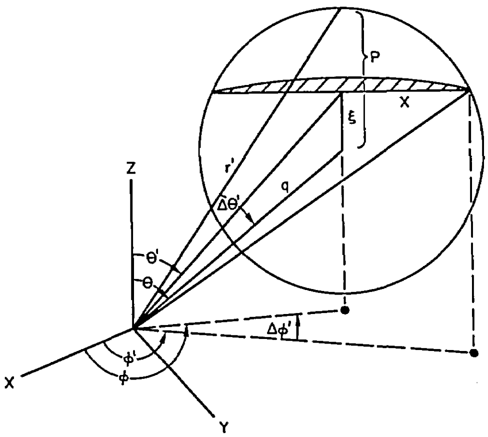

Fig. 3 - Limits on ϕ'. The center of this circle lies on the radius vector **r**. A measure of the limit on ϕ' is x, a measure of θ' is ζ, and a measure of the limit on θ' is P. The circle is the intersection of a typical spherical shell of radius **r**$'$ centered at 0 with the spherical volume of integration of radius ct centered at **r**.

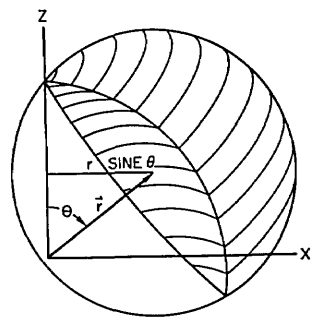

Fig. 4 - The condition of the "bubble" of integration for times $t < (r \sin\theta)/c$. In the shaded region, the limits for ϕ' have the same functional dependence on θ', r', r, and t as before. In the unshaded region, a completely different set of limits applies.

A singularity in these when t approaches 0 occurs only if there is charge present at the field point when the distribution is turned on, and such singularity conditions must be avoided in any case.* It was shown that the formulation is valid for times $t \leq (r \sin\theta)/c$ and that the origin of coordinates must be chosen with this restriction in mind. In the calculation of $\nabla\phi$ and $\partial\mathbf{A}/\partial t$ the appropriate formulas were applied for taking the derivative of integrals with variable limits. This led to a host of integrals, most of which, however, were seen to vanish. The remaining integrals should be programmable on a computer without serious difficulty.

REFERENCES

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2. Jackson, J.D., "Classical Electrodynamics," New York:Wiley, p. 141, 1962

*One always has to be cautious about calculating the field at a point where charge is present. The necessary precautions are not peculiar to this formalism. The singularity in the formula obtained here when t approaches 0 is no more and no less than a singularity due to any charge present at the field point and must be dealt with accordingly.

Appendix A

DETERMINATION OF THE LIMITS OF INTEGRATION

The volume of integration is a sphere of radius $|\mathbf{r} - \mathbf{r}'| = ct$, whose center is at the field point $\mathbf{r}$ (Fig. 1). The spherical volume is integrated by intersecting it with a series of spherical shells centered at the origin 0 with radii r'. Each such shell intersects the sphere in a circle, which determines the limits on θ' and ϕ'. The limits on θ' depend on r', and the limits on ϕ' depend on θ'. It is clear that the limits on r', θ', and ϕ' are situated symmetrically with respect to $\mathbf{r} = (r, \theta, \phi)$. Therefore, the upper and lower limits on these variables may be written, respectively, as

$$r_L' = r - \Delta r' \qquad \theta_L' = \theta - \Delta\theta' \qquad \phi_L' = \phi - \Delta\phi'$$

$$r_u' = r + \Delta r' \qquad \theta_u' = \theta + \Delta\theta' \qquad \phi_u' = \phi + \Delta\phi'$$

with u and L representing the "upper limit" and "lower limit," respectively.

The variables $\Delta r'$, $\Delta\theta'$, and $\Delta\phi'$ can be determined as follows:

1. From inspection of Fig. 2,

$$\Delta r' = ct \ . \tag{A1}$$

2. $\Delta\theta'$ is calculated from the cosine rule (Fig. 2)

$$c^2 t^2 = r^2 + r'^2 - 2rr' \cos \Delta\theta'$$

and (A2)

$$\Delta\theta' = \cos^{-1} \frac{r^2 + r'^2 - (ct)^2}{2rr'}$$

3. $\Delta\phi'$ is calculated from the relationships in Fig. 3:

$$x = r' \sin \Delta\phi' ,$$

$$x^2 + \zeta^2 = p^2 ,$$

$$\zeta = q \tan (\theta' - \theta) ,$$

and

$$q = r' \cos \Delta\theta' \ .$$

Therefore,

$$\sin \Delta\phi' = \frac{x}{r'} \quad \frac{\sqrt{p^2 - \zeta^2}}{r'} = \sqrt{\sin^2 \Delta\theta' - \cos^2 \Delta\theta' \tan^2 (\theta' - \theta)}$$

and (A3)

$$\Delta\phi' = \sin^{-1} \sqrt{\sin^2 \Delta\theta' - \cos^2 \Delta\theta' \tan^2 (\theta' - \theta)}$$

Appendix B

VANISHING INTEGRALS

The last four integrals in Eq. (9) all vanish for essentially the same reason, which is related to the particular way in which the limits on ϕ' depend on θ' and the limits on θ' depend on r' and to the symmetry of these limits. In the first two of these integrals, one must first evaluate Q at the upper and lower limits of θ'. From the geometry of Fig. 3 or from Eq. (A3) it is seen that at both of these limits $(\theta' - \theta = \pm\Delta\theta)$, $\Delta\phi' = 0$. This means that at both of these θ' limits, Q is given by

$$Q = \int_{\phi}^{\phi} d\phi' F = 0 \ .$$

In the second two of these integrals, one must integrate Q between the lower and upper limits of θ' subject to the condition that r' takes on its upper or lower limit, respectively. From the geometry of Fig. 2 or from Eq. (A2) it is seen that at both of these limits $(r' - r = \pm ct)$, $\Delta\theta' = 0$. This means that at both of these r' limits the integrals of Q are

$$\int_{\theta}^{\theta} Q \sin\theta' \, d\theta' = 0 \ .$$

Q.E.D.

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13. ABSTRACT

In the Lorentz gauge, the solutions to the scalar and vector potential equations (free space) are the retarded potentials, which are integrals of charge and current distributions over all space, subject to the retardation condition. It is shown that if the charge and current are "turned on" in the finite past ($t' = 0$), integrals over a finite region result, in particular, over an expanding sphere of radius ct, centered at the field point. The limits of integration, which are time and space dependent, have been obtained. The corresponding expressions for $\mathbf{A}$ and Φ, have been used to derive $\mathbf{E} = -\nabla\phi - \partial\mathbf{A}/\partial t$ for an arbitrary charge and current distribution. This involves the partial derivatives of the densities in volume integrals, surface integrals which vary as $1/ct$, and other integrals which vanish due to the geometry. The formulation is valid for times $t \leq (r \sin\theta)/c$.

14. KEY WORDS	LINK A		LINK B		LINK C	
	ROLE	WT	ROLE	WT	ROLE	WT
Retarded electric field Arbitrary charge Arbitrary current						