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Stress Intensity and Crack-Opening Displacement for Coplanar Cracks in Thermoelasticity

W. S. Fu

Ocean Materials Criteria Branch
Ocean Technology Division

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NAVAL RESEARCH LABORATORY
Washington, D.C.

CONTENTS

ABSTRACT

The solution to the problem of two coplanar penny-shaped cracks in an infinite elastic medium is sought for the case where asymmetric heat flux or temperature difference is prescribed over the surfaces of the cracks. The problem is reduced to determining the solution to two sets of coupled Fredholm integral equations. An example is worked for a case where the temperature is prescribed over the crack faces and the heat flux normal to the plane containing the cracks is zero. The integral equations are then solved iteratively assuming that the spacing between the cracks is large relative to their radii. Physical quantities of interest, such as crack-opening displacement and stress intensity factor, are investigated.

PROBLEM STATUS

This is an interim report; work is continuing on the problem.

AUTHORIZATION

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STRESS INTENSITY AND CRACK-OPENING DISPLACEMENT FOR COPLANAR CRACKS IN THERMOELASTICITY

INTRODUCTION

Reflection of compressive waves at free boundaries often results in the creation of cracks associated with high temperature and heat flow in the material. The crack may then grow under these thermal loadings. After the transient stage, the stress distributions in the material may be obtained by considering boundary value problems in thermoelasticity.

This report is the study of a reduced boundary value problem in thermoeleasticity — the determination of the crack-opening displacement and the stress intensity factor for the extension of two coplanar penny-shaped cracks opened by heating with additional mechanical loadings. Only steady-state temperature fields are considered.

The problem of a single penny-shaped crack opened by axisymmetric thermal loading was first solved by Olesiak and Sneddon (1) using the method of integral transforms. Their solution to the equations of thermoelasticity is appropriate for the case of a crack, which has stress-free faces and produces zero shear stress on the plane containing the crack. Williams (2) subsequently showed that the displacement vector as described in Ref. 1 can be written in terms of two harmonic functions, one which is directly related to the temperature field, and indicated that the problem in Ref. 1 can be reduced to simple boundary value problems in potential theory.

The problem of two coplanar penny-shaped cracks, each under asymmetric loading, is formulated in terms of harmonic functions, and the method used is that of Williams (2) and Collins (3) who solved the isothermal problem. The solution to the problem is reduced to the determination of some auxiliary function for a set of Fredholm integral equations of the second kind, which may be solved by the method of successive approximation when the radii of cracks are small relative to the crack separation. The case where constant temperature is prescribed over the faces of the cracks is given as an illustrative example in which the crack-opening displacement, stress intensity factor, and potential energy decrease per crack are determined.

FORMULATION OF PROBLEM

The solution to certain problems involving combined mechanical and thermal loadings on solids which contain two coplanar penny-shaped cracks is sought by techniques appropriate to the classical theory of thermoelasticity. The two coplanar cracks Σ and $\bar{\Sigma}$ are each of radius a, their centers being a distance $2h$ apart. Their planes coincide with the plane $z = 0$, and two sets of cylindrical polar coordinates (r, θ, z) and $(\bar{r}, \bar{\theta}, \bar{z})$ are used, with the centers 0 and $\bar{0}$ of Σ and $\bar{\Sigma}$ as origins, $\bar{0}$ and 0 being the points $(2h, 0, 0)$ and $(2h, \pi, 0)$ in these respective coordinates. The regions Σ and $\bar{\Sigma}$ are thus given by

$$\Sigma: \; z = 0 \;\; (0 \le r \le a, \;\; 0 \le \theta \le 2\pi)$$

and

$$\bar{\Sigma}: \; \bar{z} = 0 \;\; (0 \le \bar{r} \le a, \;\; 0 \le \bar{\theta} \le 2\pi) \; .$$

A solution is required to satisfy the equilibrium field equations in thermoelasticity:

$$(1-2\nu)\,\Delta^2\underline{u} + \text{grad}\,[\Delta\cdot\underline{u} - 2(1+\nu)\,\alpha T] = 0\ , \tag{1}$$

and

$$\Delta^2 T = 0\ , \tag{2}$$

where Δ^2, ν, and α denote the Laplacian operator, Poisson's ratio, and the linear thermal expansion of the solid, respectively, and $\underline{u} = (u, v, w)$, which are displacements in the directions (r, θ, z) for the cylindrical system of coordinates associated with crack Σ. A similar set of equations can be written for the coordinate system $(\bar{r}, \bar{\theta}, \bar{z})$ associated with crack $\bar{\Sigma}$. It is easily verified that Eqs. (1) and (2) are satisfied by the following choice of $\underline{u}$ and T (Ref. 2):

$$\begin{aligned} 2G\underline{u} = {} & -\text{grad}\,\Phi + z\,\text{grad}\,\partial\Phi/\partial z + \partial\Phi/\partial z\,\underline{k} - (1-2\nu)\,\text{grad}\,K \\ & - z\,\text{grad}\,\partial K/\partial z + (3-4\nu)\,\partial K/\partial z\,\underline{k} \end{aligned} \tag{3}$$

and

$$2G\,T = [2(1-\nu)/\alpha(1+\nu)]\,\partial^2\Phi/\partial z^2\ , \tag{4}$$

where Φ and K are harmonic functions and G is the shear modulus. The shear stresses vanish automatically on the plane $z = 0$, and the normal stress σ_z and normal displacement w on this plane remain as

$$\sigma_z = \partial\Psi/\partial z - \partial^2\Phi/\partial z^2 \tag{5}$$

and

$$2Gw = 2(1-\nu)\,\Psi\ , \tag{6}$$

where

$$\Psi = \partial K/\partial z\ .$$

Since the problem possesses certain symmetry properties with respect to the plane $z = 0$, only the solution for $z \geq 0$ is considered. The functions Ψ and Φ are then represented by integrals involving Bessel functions of the first kind in the following form to ensure that the stresses do not have singularities of too high an order:

$$\begin{aligned} \Psi &= \psi(r,\theta,z) + \bar{\psi}(\bar{r},\bar{\theta},\bar{z}) \\ &= \sum_{m=0}^{\infty} (\pi/2)^{1/2} \cos m(\theta+\alpha_m) \int_0^a g_m(t) \int_0^\infty (\lambda t)^{1/2} J_{m+1/2}(\lambda t)\, J_m(\lambda r)\, e^{-\lambda z}\, d\lambda dt \\ &\quad + \sum_{m=0}^{\infty} (\pi/2)^{1/2} \cos m(\bar{\theta}+\bar{\alpha}_m) \int_0^a \bar{g}_m(t) \int_0^\infty (\lambda t)^{1/2} J_{m+1/2}(\lambda t)\, J_m(\lambda\bar{r})\, e^{-\lambda\bar{z}}\, d\lambda dt \end{aligned} \tag{7}$$

and

$$\partial^2\Phi/\partial z^2 = \partial^2\varphi(r,\theta,z)/\partial z^2 + \partial^2\bar{\varphi}(\bar{r},\bar{\theta},\bar{z})/\partial \bar{z}^2$$

$$= \sum_{m=0}^{\infty} (\pi/2)^{1/2} \cos m(\theta + \beta_m) \int_0^a j_m(t) \int_0^{\infty} (\lambda t)^{1/2} J_{m-1/2}(\lambda t)\, J_m(\lambda r)\, e^{-\lambda z}\, d\lambda dt$$

$$+ \sum_{m=0}^{\infty} (\pi/2)^{1/2} \cos m(\bar{\theta} + \bar{\beta}_m) \int_0^a \bar{j}_m(t) \int_0^{\infty} (\lambda t)^{1/2} J_{m-1/2}(\lambda t)\, J_m(\lambda \bar{r})\, e^{-\lambda \bar{z}}\, d\lambda dt\ , \tag{8}$$

where ψ, φ, $\bar{\psi}$, and $\bar{\varphi}$ are harmonic functions, and g_m, j_m, $\bar{g}_m$, and $\bar{j}_m$ are auxiliary functions to be determined by boundary conditions; the barred quantities are taken with respect to the coordinate system $(\bar{r},\bar{\theta},\bar{z})$, and the unbarred quantities are taken with respect to the coordinate system (r,θ,z).

It is assumed that the harmonic functions involved and all boundary data can be expanded into the form of a Fourier series, and a subscript m will refer to the mth term in the series. For example, boundary data may be given in the form

$$X = x(r,\theta,0) + \bar{x}(\bar{r},\bar{\theta},0)$$

$$= \sum_{m=0}^{\infty} \cos m(\theta + \eta_m)\, x_m(r)\ , \quad \text{on } \Sigma\ ,$$

$$\sum_{m=0}^{\infty} \cos m(\bar{\theta} + \bar{\eta}_m)\, \bar{x}_m(\bar{r})\ , \quad \text{on } \bar{\Sigma}\ .$$

The integral representations chosen for ψ, $\bar{\psi}$, $\partial^2\varphi/\partial z^2$, and $\partial\bar{\varphi}/\partial\bar{z}^2$ in Eqs. (7) and (8) possess certain interesting properties on the plane $z = 0$. As $z \to 0$, the functions ψ and φ yield

$$\psi(r,\theta,0) = \sum_{m=0}^{\infty} \cos m(\theta + \alpha_m)\, r^m \int_r^a t^{-m}(t^2 - r^2)^{-1/2} g_m(t)\, dt\ , \quad r < t\ , \tag{9}$$

$$= 0 \text{ for } r > t\ ;$$

therefore,

$$\partial\psi(r,\theta,0)/\partial z = -\sum_{m=0}^{\infty} \cos m(\theta + \alpha_m)\, r^{-(m+1)} \frac{d}{dr} \int_0^r t^{m+1}(r^2 - t^2)^{-1/2} g_m(t)\, dt\ , \tag{10}$$

$$\partial^2\varphi(r,\theta,0)/\partial z^2 = \sum_{m=0}^{\infty} \cos m(\theta + \beta_m)\, r^{-m} \int_0^r t^m(r^2 - t^2)^{-1/2} j_m(t)\, dt\ , \quad r > t\ , \tag{11}$$

and

$$\partial^3\varphi(r,\theta,0)/\partial z^3 = \sum_{m=0}^{\infty} \cos m(\theta + \beta_m)\, r^{m-1} \frac{d}{dr} \int_r^a t^{-m+1}(t^2 - r^2)^{-1/2} j_m(t)\, dt\ , \quad r < t\ , \tag{12}$$

$$= 0\ , \quad \text{for } r > t\ ,$$

where Eq. (7) in Ref. 4 is used. As $\bar{z} \to 0$, the functions $\bar{\psi}$ and $\bar{\varphi}$ will yield a similar set of relations as in Eqs. (9) through (12), except now corresponding quantities should have bars. From these values it is easily seen that ψ is zero at all points except those on Σ, and $\bar{\psi}$ is zero at all points except those on $\bar{\Sigma}$; $\partial^3\varphi/\partial z^3$ is zero in Σ, and $\partial^3\bar{\varphi}/\partial \bar{z}^3$ is zero in $\bar{\Sigma}$. Hence, the choice of Ψ and Φ as in Eqs. (7) and (8) leads to the satisfaction of Eqs. (1) and (2) and also to the result that Ψ is zero at all points except those in Σ and $\bar{\Sigma}$. It is also noted that $\psi_m(r) = 0$ in Eq. (9) ensures that $\partial\psi/\partial z$ will not have singularities of too high an order on the edges of Σ and $\bar{\Sigma}$.

BOUNDARY CONDITIONS AND SOLUTION

It is interesting to study two thermal loading conditions where cracks are opened by (a) the application of a prescribed temperature to their flat faces which are stress free and (b) the application of a prescribed heat flux to their faces. The boundary conditions imposed on the harmonic function Φ are, for case (a),

$$\partial^2\Phi/\partial z^2 = -\frac{\alpha(1+\nu)G}{(1-\nu)}T_0 \ \Sigma \cos m(\theta+\xi_m)\, p_m(r) \ , \quad \text{in } \Sigma \ , \tag{13}$$

$$= -\frac{\alpha(1+\nu)G}{(1-\nu)}T_0 \ \Sigma \cos m(\bar{\theta}+\bar{\xi}_m)\, \bar{p}_m(\bar{r}) \ , \quad \text{in } \bar{\Sigma} \ , \tag{14}$$

and

$$\partial^3\Phi/\partial z^3 = 0 \text{ at all other points on } z = 0 \ ; \tag{15}$$

for case (b),

$$\partial^3\Phi/\partial z^3 = \pm\frac{\alpha(1+\nu)}{(1-\nu)}G \sum_{m=0}^{\infty} \cos m(\theta+\eta_m)\, q_m(r) \ , \quad \text{in } \Sigma \ , \tag{16}$$

$$= \pm\frac{\alpha(1+\nu)}{(1-\nu)}G \sum_{m=0}^{\infty} \cos m(\bar{\theta}+\bar{\eta}_m)\, \bar{q}_m(\bar{r}) \ , \quad \text{in } \bar{\Sigma} \ , \tag{17}$$

and

$$\partial^3\Phi/\partial z^3 = 0 \text{ at all other points on } z = 0 \ , \tag{18}$$

where T_0 is a positive constant, p_m and q_m are the prescribed temperature and heat flux, respectively, and Eq. (4) is used. Let the cracks be opened by thermal loadings superposed on a uniform tension field at infinity, the elastic boundary conditions on $z = 0$ can be written as (Ref. 5)

$$\partial\Psi/\partial z - \partial^2\Phi/\partial z^2 = \sum_{m=0}^{\infty} \cos m(\theta+\zeta_m)\, f_m(r) \ , \quad \text{in } \Sigma \ , \tag{19}$$

$$= \sum_{m=0}^{\infty} \cos m(\bar{\theta}+\bar{\zeta}_m)\, \bar{f}_m(\bar{r}) \ , \quad \text{in } \bar{\Sigma} \ , \tag{20}$$

and

$$\Psi = 0 \text{ at all other points on } z = 0 \ , \tag{21}$$

where Eqs. (5) and (6) are used and f_m and $\bar{f}_m$ are related to the applied pressure on the crack faces. It should be noted here that the additional requirement that shear must vanish on plane $z = 0$ is satisfied automatically by the choice of $\underline{u}$ and T as in Eqs. (4) and (5) and hence imposes no condition on Φ and Ψ.

From Eqs. (9) through (12) it is easily seen that the conditions in Eqs. (15), (18), and (21) are identically satisfied. Substituting Eq. (11) and a similarly written $\partial^2\bar{\varphi}/\partial\bar{z}^2$ in Eq. (13) leads to a set of integral equations satisfied by j_m, $\bar{j}_m$, β_m, and $\bar{\beta}_m$ in Σ:

$$j_0(t) + \sum_{n=0}^{\infty} (-1)^n \cos n\bar{\beta}_n \int_0^a L_{n,0}(s,t)\, \bar{j}_n(s)\, ds = P_0(t) , \tag{22a}$$

$$j_m(t) \cos \beta_m + \sum_{n=0}^{\infty} (-1)^n \cos \beta_n \int_0^a \left[L_{n,m}(s,t) + (-1)^m L_{n,-m}(s,t)\right] \bar{j}_n(s)\, ds$$
$$= P_m(t) \cos m\xi_m , \tag{22b}$$

and

$$j_m(t) \sin \beta_m - \sum_{n=0}^{\infty} (-1)^n \sin n\bar{\beta}_n \int_0^a \left[L_{n,m}(s,t) - (-1)^m L_{n,-m}(s,t)\right] j_n(s)\, ds$$
$$= P_m(t) \sin m\xi_m , \tag{22c}$$

where

$$L_{n,\pm m}(s,t) = (st)^{1/2} \int_0^{\infty} \lambda J_{n-1/2}(\lambda s)\, J_{m-1/2}(\lambda t)\, J_{n\pm m}(2\lambda h)\, d\lambda \tag{23}$$

and

$$P_m(t) = -\left[2G\alpha T_0(1+\nu)/\pi(1-\nu)\, t^m\right] \frac{d}{dt} \int_0^t r^{m+1}(t^2 - r^2)^{-1/2}\, p_m(r)\, dr . \tag{24}$$

To obtain Eqs. (22), the addition theorem involving Bessel (6) functions

$$J_m(\lambda\bar{r}) \begin{matrix}\cos\\ \sin\end{matrix}\, m\theta^1 = \sum_{n=-\infty}^{\infty} J_{m+n}(2\lambda h)\, J_n(\lambda r) \begin{matrix}\cos\\ \sin\end{matrix}\, n\theta , \tag{25}$$

where $\theta^1 = \pi - \bar{\theta}$ must be used to transform all quantities in the $(\bar{r},\bar{\theta},\bar{z})$ system to the (r,θ,z) system. Writing quantities in $(\bar{r},\bar{\theta},\bar{z})$ and substituting Eq. (11) in Eq. (14) yield a set of integral equations similar to those of Eqs. (22) and can be obtained by replacing the barred and unbarred quantities with $\bar{P}_m$ defined as

$$\bar{P}_m = (-1)^{m+1} \left[2G\alpha T_0(1+\nu)/\pi(1-\nu)\, t^m\right] \frac{d}{dt} \int_0^t \bar{r}^{m+1}(t^2 - \bar{r}^2)^{-1/2}\, \bar{p}_m(\bar{r})\, d\bar{r} .$$

Considering j_m, $\bar{j}_m$, β_m, and $\bar{\beta}_m$ as known quantities and substituting Eqs. (10) and (11) in Eq. (19) leads to a set of integral equations satisfied by g_m, $\bar{g}_m$, α_m, and $\bar{\alpha}_m$ in Σ:

$$g_0(t) + \sum_{n=0}^{\infty} (-1)^n \cos n\bar{\alpha}_n \int_0^a M_{n,0}(s,t)\, \bar{g}_n(s)\, ds$$

$$= -F_0(t) - \int_0^t j_0(s)\, ds + \sum_{n=0}^{\infty} (-1)^n \cos n\bar{\beta}_n \int_0^a N_{n,0}(s,t)\, \bar{j}_n(s)\, ds \,, \tag{26a}$$

$$g_m(t) \cos m\alpha_m + \sum_{n=0}^{\infty} (-1)^n \cos n\bar{\alpha}_n \int_0^a [M_{n,m}(s,t) + (-1)\, M_{n,-m}(s,t)]\, \bar{g}_n(s)\, ds$$

$$= -F_m(t) \cos m\zeta_m - \cos m\beta_n \int_0^t (s/t)^m\, j_m(s)\, ds$$

$$+ \sum^{\infty} (-1)^n \cos n\bar{\beta}_n \int_0^a [N_{n,m}(s,t) - (-1)^m N_{n,-m}(s,t)]\, \bar{j}_n(s)\, ds \,, \tag{26b}$$

and

$$g_m(t) \sin m\alpha_m - \sum_{n=0}^{\infty} (-1)^n \sin n\bar{\alpha}_n \int_0^a [M_{n,m}(s,t) - (-1)^m M_{n,-m}(s,t)]\, \bar{g}_n(s)\, ds$$

$$= -F_m(t) \sin m\zeta_m - \sin m\beta_m \int_0^a (s/t)^m\, j_m(s)\, ds$$

$$- \sum_{n=0}^{\infty} (-1)^n \sin n\bar{\beta}_n \int_0^a [N_{n,m}(s,t) - (-1)^m N_{n,-m}(s,t)]\, \bar{j}_n(s)\, ds \,, \tag{26c}$$

where

$$M_{n,\pm m}(s,t) = (st)^{1/2} \int_0^{\infty} \lambda J_{n+1/2}(\lambda_s)\, J_{m+1/2}(\lambda t)\, J_{n\pm m}(2\lambda h)\, d\lambda \,, \tag{27}$$

$$N_{n,\pm m}(s,t) = (st)^{1/2} \int_0^{\infty} J_{n-1/2}(\lambda s)\, J_{m+1/2}(\lambda t)\, J_{n\pm m}(2\lambda h)\, d\lambda \,, \tag{28}$$

and

$$F_m = (2/\pi)\, \frac{d}{dt} \int_0^t r(t^2 - r^2)^{-1/2} f_m(r)\, dr \,. \tag{29}$$

When the temperature prescribed over the cracks are symmetric with respect to the plane which is the perpendicular bisector of $0\bar{0}$, namely, mirror symmetry is possessed by the prescribed temperature, it is seen that

$$p_m = \bar{p}_m \quad \text{and} \quad \xi_m = \pi - \bar{\xi}_m \ .$$

It follows that

$$j_m(t) = \bar{j}_m(t) \quad \text{and} \quad \beta_m = \pi - \bar{\beta}_m \ , \tag{30}$$

where $j_m(t)$ and the angles β_m satisfy a set of coupled Fredholm integral equations

$$j_0(t) + \sum_{n=0}^{\infty} \cos n\beta_n \int_0^a L_{n,0}(s,t)\, j_n(s)\, ds = P_0(t) \ , \tag{31a}$$

$$j_m(t) \cos m\beta_m + \sum_{n=0}^{\infty} \cos n\beta_n \int_0^a \left[L_{n,m}(s,t) + (-1)^m L_{n,-m}(s,t)\right] j_n(s)\, ds$$
$$= P_m(t) \cos m\xi_m \ , \tag{31b}$$

and

$$j_m(t) \sin m\beta_m + \sum_{n=0}^{\infty} \sin n\beta_n \int_0^a \left[L_{n,m}(s,t) - (-1)^m L_{n,-m}(s,t)\right] j_n(s)\, ds$$
$$= P_m(t) \sin m\xi_m \ . \tag{31c}$$

If the applied pressure also possesses the mirror symmetry with respect to a plane perpendicular to $0\bar{0}$, the following relations can be obtained:

$$f_m = \bar{f}_m \ , \quad \zeta_m = \pi - \bar{\zeta}_m \ ,$$

$$g_m = \bar{g}_m \ , \quad \alpha_m = \pi - \bar{\alpha}_m \ ,$$

and Eqs. (26) reduce to

$$g_0(t) + \sum_{n=0}^{\infty} \cos n\alpha_n \int_0^a M_{n,0}(s,t)\, g_n(s)\, ds$$

$$= -F_0(t) - \int_0^t j_0(s)\, ds + \sum_{n=0}^{\infty} \cos n\beta_n \int_0^a N_{n,0}(s,t)\, j_n(s)\, ds \ , \tag{32a}$$

$$g_m(t) \cos m\alpha_m + \sum_{n=0}^{\infty} \cos n\alpha_n \int_0^a \left[M_{n,m}(s,t) + (-1)^m M_{n,-m}(s,t)\right] g_n(s)\, ds$$

$$= -F_m(t) \cos m\zeta_m - \cos m\beta_m \int_0^t (s/t)^m j_m(s)\, ds$$

$$+ \sum_{n=0}^{\infty} \cos n\beta_m \int_0^a \left[N_{n,m}(s,t) + (-1)^m N_{n,-m}(s,t)\right] j_n(s)\, ds \ , \tag{32b}$$

and

$$g_m(t)\ \sin m\alpha_m - \sum_{n=0}^{\infty} \sin n\alpha_n \int_0^a [M_{n,m}(s,t) - (-1)^m M_{n,-m}(s,t)]\ g_n(s)\ ds$$

$$= -F_m(t)\ \sin m\zeta_m - \sin m\beta_m \int_0^t (s/t)^m\ j_m(s)\ ds$$

$$- \sum_{n=0}^{\infty} \sin n\beta_n \int_0^a [N_{n,m}(s,t) - (-1)^m N_{n,-m}(s,t)]\ j_n(s)\ ds\ . \tag{32c}$$

Substituting Eq. (12) and $\partial^3\varphi/\partial z^3$ in Eqs. (16) and (17) leads to a set of integral equations satisfied by j_m, $\bar{j}_m$, β_m, and $\bar{\beta}_m$ for case (b). If heat flux prescribed over the crack faces has mirror symmetry with respect to the plane perpendicular to $0\bar{0}$, the set of coupled integral equations satisfied by j_m and β_m can be obtained in a manner similar to that for case (a) as

$$j_0(t) + \sum_{n=0}^{\infty} \cos n\beta_n \int_0^a K_{n,0}(s,t)\ j_n(s)\ ds = \mp Q_0(t)\ , \tag{33a}$$

$$j_m(t)\ \cos m\beta_m + \sum_{n=0}^{\infty} \cos n\beta_n \int_0^a [K_{n,m}(s,t) + (-1)^m K_{n-m}(s,t)]\ j_n(s)\ ds$$

$$= \mp Q_m(t)\ \cos m\eta_m\ , \tag{33b}$$

and

$$j_m(t)\ \sin m\beta_m + \sum \sin n\beta_n \int_0^a [K_{n,m}(s,t) - (-1)^m K_{n,-m}(s,t)]\ j_n(s)\ ds$$

$$= \mp Q_m(t)\ \sin m\eta_m\ , \tag{33c}$$

where

$$Q_m(t) = [2G\alpha(1+\mu)/\pi(1-\mu)]\ \frac{d}{dt} \int_t^a r(r^2-t^2)^{-1/2} \int_0^r s^{-(m-1)}\, q_m(s)\ dsdr$$

and

$$K_{n,\pm m}(s,t) = t^{-2m+1}\ (st)^{1/2} \int_0^{\infty} \lambda J_{n-1/2}(\lambda s)\ J_{m-3/2}(\lambda t)\ J_{n\pm m}(2\lambda h)\ d\lambda\ .$$

The set of integral equations to be satisfied by g_m and α_m is identical to Eqs. (32). It should be noted that the relations

$$j_m = \bar{j}_m\ , \quad \beta_m = \pi - \bar{\beta}_m$$

and

$$g_m = \bar{g}_m \ , \quad \alpha_m = \pi - \bar{\alpha}_m$$

are used.

The solution to the boundary value problems is now reduced to the determination of the auxiliary functions j_m and g_m in Eqs. (31) and (32) for case (a) and Eqs. (32) and (33) for case (b). In general, the solution of these equations must be found numerically; however, when the crack radii a is small relative to the crack separation $2h$, and approximate solution can be developed in terms of a power series depending on the ratio $\alpha = a/2h$. This technique is demonstrated by an example where the temperature prescribed over the crack faces is a constant; the method of successive approximation is used.

Certain quantities of physical interest can be written directly in terms of the auxiliary functions. Manipulating Eqs. (6) and (7) gives the crack-opening displacement (C.O.D.) in Σ and $\bar{\Sigma}$ as

$$\text{C.O.D.} = (w^+ - w^-) = 2 \sum_{m=0}^{\infty} w_m(r) \cos m(\theta + h_m)$$

$$= [2(1-\nu)/G] \sum_{m=0}^{\infty} \cos m(\theta + \alpha_m)\, r^m \int_r^a t^{-m}(t^2 - r^2)^{-1/2} g_m(t)\, dt \ . \tag{34}$$

The stress intensity factor defined as

$$\sigma = \lim_{r \to a^+} (r^2 - a^2)^{1/2} \sigma_z(a^+, \theta, 0)$$

is given as

$$\sigma = \sum_{m=0}^{\infty} \sigma_m(r) \cos m(\theta + k_m)$$

$$= -\lim_{r \to a^+} (r^2 - a^2)^{1/2} \sum_{m=0}^{\infty} \cos m(\theta + \alpha_m)\, r^{-(m+1)} \frac{d}{dr} \int_a^r t^{m+1}(r^2 - a^2)^{1/2} g_m(t)\, dt \ , \tag{35}$$

where Eqs. (5), (10), and (11) are used. It is noted here that $\partial\bar{\psi}/\partial\bar{z}$ and $\partial^2\Phi/\partial\bar{z}^2$ contain no singularity as $r \to a^+$ and hence are not included in Eq. (35).

SOLUTION OF AN EXAMPLE PROBLEM

As an example, suppose the coplanar penny-shaped cracks are opened under the combined loading of a constant applied tension at infinity and a uniform heating of the solid. Using the standard procedure as described in Ref. 5, it is easily understood that solving the stated problem is equivalent to solving the problem where constant pressure and temperature are prescribed over the crack faces; at all other points on the plane $z = 0$ the displacement normal to the plane and heat flux across the plane $z = 0$ must vanish. The shearing stresses on $z = 0$ should be zero, as a point in the solid approaches infinity, the stresses, displacement, and temperature at that point must die off and approach zero. In other words, this is a thermal condition as in case (a). Let the constant applied pressure and constant temperature prescribed on the crack faces be f and k, respectively. Equations (31) and (32) are reduced, since $\alpha_m = \beta_m = \xi_m = 0$, to

$$j_0(t) + \sum_{n=0}^{\infty} \int_0^a L_{n,0}(s,t)\, j_n(s)\, ds = -[2G\alpha(1+\nu)\, T_0 k/\pi(1-\nu)] \tag{36a}$$

and

$$j_m(t) + \sum_{n=0}^{\infty} \int_0^a [L_{n,m}(s,t) + (-1)^m L_{n,-m}(s,t)]\, j_n(s)\, ds = 0 \; ; \tag{36b}$$

$$g_0(t) + \sum_{n=0}^{\infty} \int_0^a M_{n,0}(s,t)\, g_n(s)\, ds = -2f/\pi - \int_0^t j_0(s)\, ds$$

$$+ \sum_{n=0}^{\infty} \int_0^a N_{n,0}(s,t)\, j_n(s)\, ds \tag{37a}$$

and

$$g_m(t) + \sum_{n=0}^{\infty} \int_0^a [M_{n,m}(s,t) + (-1)^m M_{n,-m}(s,t)]\, g_n(s)\, ds$$

$$= -\int_0^t (s/t)^m j_m(s)\, ds + \sum_{n=0}^{\infty} \int_0^a [N_{n,m}(s,t) + (-1)^m N_{n,-m}(s,t)]\, j_n(s)\, ds \; . \tag{37b}$$

Since the isothermal case for cracks under constant pressure was discussed in detail by Collins (3), in what follows, f is taken as zero and the net effect of temperature on the crack-opening displacement and stress intensity factor is sought.

The method of successive approximation for the Fredholm (7) integral equation is used to obtain the interated values of $j_m(t)$ and $g_m(t)$, $m = 0, 1, 2,$ from Eqs. (36) and (37) when the ratio $a/2h$ is small and the kernels in Eqs. (34) and (35) are expressed in terms of the fourth type of Appell's hypergeometric function of two variables. The integrals $L_n, \pm m(s,t)$; $M_n, \pm m(s,t)$; and $N_n, \pm m(s,t)$ in Eqs. (23), (27), and (28) are written as

$$L_{n,m}(s,t) = \frac{2s^n t^m \Gamma\left(n + m + \frac{1}{2}\right)}{\pi^{1/2}(2h)^{n+m+1}\Gamma\left(n+\frac{1}{2}\right)\Gamma\left(m+\frac{1}{2}\right)}$$

$$F_4\left[\frac{1}{2},\ n+m+\frac{1}{2};\ n+\frac{1}{2},\ m+\frac{1}{2};\ \left(\frac{s}{2h}\right)^2,\ \left(\frac{t}{2h}\right)^2\right]$$

and

$$L_{n,-m}(s,t) = \frac{(-1)^m 2s^n t^m}{\pi(2h)^{n+m+1}} F_4\left[n+\frac{1}{2},\ m+\frac{1}{2};\ n+\frac{1}{2},\ m+\frac{1}{2};\ \left(\frac{s}{2h}\right)^2,\ \left(\frac{t}{2h}\right)^2\right].$$

$$M_{n,m}(s,t) = \frac{-s^{n+1}t^{m+1}\Gamma\left(n+m+\frac{3}{2}\right)}{\pi^{1/2}(2h)^{n+m+3}\Gamma\left(n+\frac{3}{2}\right)\Gamma\left(m+\frac{3}{2}\right)}$$

$$F_4\left[\frac{3}{2},\ n+m+\frac{3}{2};\ n+\frac{3}{2},\ m+\frac{3}{2};\ \left(\frac{s}{2h}\right)^2,\ \left(\frac{t}{2h}\right)^2\right]$$

and

$$M_{n,-m}(s,t) = \frac{(-1)^{m+1}2s^{n+1}t^{m+1}}{\pi(2h)^{n+m+3}} F_4\left[m+\frac{3}{2},\ n+\frac{3}{2};\ n+\frac{3}{2},\ m+\frac{3}{2};\ \left(\frac{s}{2h}\right)^2,\ \left(\frac{t}{2h}\right)^2\right].$$

$$N_{n,m}(s,t) = \frac{s^n t^{m+1}\Gamma\left(n+m+\frac{1}{2}\right)}{(2h)^{n+m+1}\Gamma\left(n+\frac{1}{2}\right)\Gamma\left(m+\frac{3}{2}\right)\pi^{1/2}}$$

$$F_4\left[\frac{1}{2},\ n+m+\frac{1}{2};\ n+\frac{1}{2},\ m+\frac{3}{2};\ \left(\frac{s}{2h}\right)^2,\ \left(\frac{t}{2h}\right)^2\right]$$

and

$$N_{n,-m}(s,t) = \frac{(-1)^m 2s^n t^{m+1}}{\pi(2m+1)(2h)^{n+m+1}} F_4\left[m+\frac{1}{2},\ n+\frac{1}{2};\ n+\frac{1}{2},\ m+\frac{1}{2};\ \left(\frac{s}{2h}\right)^2,\ \left(\frac{t}{2h}\right)^2\right],$$

where Bailey's (8) formula for integrals involving multiplication of Bessel functions is employed. It is readily observed from $L_{n,\pm m}$ that to obtain j_m correct to $0(h^{-5})$ the functions j_m for $m > 5$ may be neglected, and Eqs. (36) yield five simultaneous integral equations for the functions $j_m(t)$, where $m = 0, 1, \ldots, 4$. The solution to these equations is found by iteration as

$$\begin{aligned} j_0(\beta) = -K[&1 - 2\epsilon/\pi + 4\epsilon^2/\pi^2 \\ &- (1/3\pi + 8/\pi^3)\epsilon^3 - \epsilon\beta^2/\pi + (13/\pi^2 + 16/\pi^3)\epsilon^4 \\ &+ 2\epsilon^2\beta^2/\pi^2 - 3\epsilon^3\beta^2/2\pi - (3/4\pi + 4/\pi^3)\epsilon\beta^4 \\ &- (3/20\pi + 38/\pi^3)\epsilon^5], \end{aligned} \tag{38a}$$

$$\begin{aligned} j_1(\beta) = 4K\epsilon\beta/\pi[&1 - 2\epsilon/\pi + (1/2 + 4/\pi^2)\epsilon^2 \\ &+ 14\beta^2/\pi - (5/3\pi + 8/\pi^3)\epsilon^3 - 7\epsilon\beta^2/\pi^2], \end{aligned} \tag{38b}$$

$$j_2(\beta) = 4K\epsilon\beta^2/\pi[1 - 2\epsilon/\pi + 4\epsilon^2/\pi^2], \tag{38c}$$

$$j_3(\beta) = 4K\epsilon\beta^3/\pi\,[1 - 2\epsilon/\pi] \ , \tag{38d}$$

$$j_4(\beta) = 4K\epsilon\beta^4/\pi \ , \tag{38c}$$

where

$$\epsilon = a/2h \ ,$$

$$\beta = t/2h \ ,$$

and

$$K = 2G\,\alpha\,(1+\nu)\,T_0 k/\pi\,(1-\nu) \ .$$

Using a similar procedure and solving Eqs. (37) by iteration yield the values for $g_m(t)$, where $m = 0, 1, \ldots, 4$, correct to $0[h^{-5}]$ as

$$\begin{aligned} g_0(\beta) = {} & Kt\,[1 - 4\epsilon/\pi + 8\epsilon^2/\pi^2 - 16\epsilon^3/\pi^3 - 2\epsilon\beta^2/3\pi \\ & + (47/3\pi^2 + 16/\pi^3 + 16/\pi^4)\,\epsilon^4 + 2\epsilon^2\beta^2/3\pi^2 \\ & - (4/\pi^2 + 12/\pi^3)\,\epsilon^3\beta^2 - (3/20\pi + 4/5\pi^3)\,\alpha\beta^4 \\ & - (-3/10\pi + 1/\pi^2 + 86/3\pi^3 + 32/\pi^4)\,\epsilon^5] \ , \end{aligned} \tag{39a}$$

$$\begin{aligned} g_1(\beta) = {} & -[8Kt\epsilon\beta/3\pi]\,[1 - 2\epsilon/\pi + (-3 + 2/15\pi + 4/\pi^2)\,\epsilon^2 \\ & + (2/5 + 21/5\pi)\,\beta^2 - 21\epsilon\beta^2/10\pi^2 - (-13/12\pi + 8/\pi^2)\,\epsilon^3] \ , \end{aligned} \tag{39b}$$

$$g_2(\beta) = -[8Kt\epsilon\beta^2/5\pi]\,[1 - 2\epsilon/\pi + (-5/4 + 25/8\pi + 10/\pi^2)\,\epsilon^2 + \beta^2/7] \ , \tag{39c}$$

$$g_3(\beta) = -[8Kt\epsilon\beta^3/7\pi]\,[1 - 2\epsilon/\pi] \ , \tag{39d}$$

$$g_4(\beta) = -[8Kt\epsilon\beta^4/9\pi] \, . \tag{39e}$$

Substituting Eqs. (39) in Eq. (34) yields the Fourier coefficients for the crack-opening displacement for each of the coplanar cracks as

$$\begin{aligned} 2w_0 = {} & H(1-\rho^2)^{1/2}\,\{1 - 4\epsilon/\pi + 8\epsilon^2/\pi - (2\pi + 16/\pi^3)\,\epsilon^3 \\ & + (49/3\pi^2 + 16/\pi^3 + 16/\pi^4)\,\epsilon^4 \\ & - (-3/20\pi + 5/\pi^2 + 622/15\pi^3 + 32/\pi^4)\,\epsilon^5 \\ & + [4(1-\rho^2)/3]\,[\epsilon^3/3\pi - \epsilon^4/3\pi^2 + (3/20\pi + 2/\pi^2 + 34/\pi^3)\,\epsilon^5] \\ & - [8(1-\rho^2)^2/15]\,(3/20\pi + 4/5\pi^3)\} \ , \end{aligned} \tag{40a}$$

$$2w_1 = -[8H\epsilon^2\rho(1-\rho^2)^{1/2}/3\pi]\{1 - 2\epsilon/\pi$$

$$+ (-13/5 + 65/15\pi + 4/\pi^2)\epsilon^2 - (-13/12\pi + 101/\pi^2)\epsilon^3$$

$$- (1-\rho^2)[(2/15 + 7/5\pi)\epsilon^2 - 7\epsilon^3/10\pi^2]\} , \tag{40b}$$

$$2w_2 = -[8H\epsilon^3\rho^2(1-\rho^2)^{1/2}/5\pi][1 - 2\epsilon/\pi + (-31/28 + 25/8\pi + 10/\pi^2)\epsilon^2$$

$$- (1-\rho^2)\epsilon^2/3] , \tag{40c}$$

$$2w_3 = -[8H\epsilon^4\rho^3(1-\rho^2)^{1/2}/7\pi][1 - 2\epsilon/\pi] , \tag{40d}$$

$$2w_4 = -8H\epsilon^5\rho^4(1-\rho^2)^{1/2}/9\pi , \tag{40e}$$

where

$$\rho = r/a$$

and

$$H = 4\alpha(1+\nu)\,T_0 ka/\pi .$$

Substituting Eqs. (39) in Eq. (35) leads to the Fourier coefficients of the stress intensity factor:

$$\sigma_0 = -Ka\,[1 - 4\epsilon/\pi + 8\epsilon^2/\pi^2 - (2/3\pi + 16/\pi^3)\epsilon^3$$

$$+ (49/3\pi^2 + (6/\pi^3 + 16/\pi^4)\epsilon^4$$

$$- (-3/20\pi + 5/\pi^2 + 622/15\pi^3 + 32/\pi^4)\epsilon^5] , \tag{41a}$$

$$\sigma_1 = (8Ka\epsilon^2/3\pi)[1 - 2\epsilon/\pi + (-13/5 + 13/3\pi + 4/\pi^2)\epsilon^2$$

$$- (13/12\pi + 101/8\pi^2)\epsilon^3] , \tag{41b}$$

$$\sigma_2 = (8Ka\epsilon^3/5\pi)[1 - 2\epsilon/\pi + (-31/28 + 25/8\pi + 10/\pi^2)\epsilon^2] , \tag{41c}$$

$$\sigma_3 = (8Ka\epsilon^4/7\pi)(1 - 2\epsilon/\pi) , \tag{41d}$$

$$\sigma_4 = 8Ka\epsilon^5/9\pi . \tag{41e}$$

It is also a simple matter to show that the decrease in potential energy of deformation per crack due to the temperature field is

$$
\begin{aligned}
-\delta P &= \frac{1}{2}\int_{\Sigma} \sigma_z (w^+ - w^-)\, d\Sigma \\
&= [2\pi(1-\nu)\, f/G] \int_0^a t g_0(t)\, dt \\
&= [\,4\,(1+\nu)\,\alpha T_0 k f a^3/3]\,[\,1 - 4\epsilon/\pi \\
&\quad + 8\epsilon^2/\pi^2 - (2/5\pi - 16/\pi^3)\epsilon^3 \\
&\quad + (241/15\pi^2 + 16/\pi^3 + 16/\pi^4)\,\epsilon^4 \\
&\quad - (-33/14\pi + 17/5\pi^2 + 3802/105\pi^3 + 32/\pi^4)\,\epsilon^5]\ ,
\end{aligned} \tag{42}
$$

where f is the applied constant pressure on the crack faces. It is noted that as $2h \to \infty$ the quantities in Eqs. (40) through (42) reduce to the known result for those of a single crack.

DISCUSSION

In this analysis, the problem of two coplanar penny-shaped cracks in a heated, infinite elastic medium is formulated in terms of potential functions, and the solution is reduced to that of sets of coupled integral equations for the auxiliary functions j_m, $\bar{j}_m$, and g_m, $\bar{g}_m$. An assumption that the applied loading and temperature field possess mirror symmetry about the perpendicular bisector of $0\bar{0}$ leads to the simple relations that $j_m(t) = \bar{j}_m(t)$ and $g_m(t) = \bar{g}_m(t)$. The problem of two coplanar penny-shaped cracks being opened by constant uniaxial tension when the heating of the solid is steadily uniform is studied in detail. The magnitudes of auxiliary functions, crack-opening displacement, stress intensity factor, and decrease in potential energy per crack are given in Eqs. (38) through (42). It is noted that the effect of the presence of a second coplanar crack appears in the solution as a perturbation and when $2h \to 0$ the solution reduces to that of a single crack.

The method of solution in this report follows that of Collins (3). It is capable of immediate generalization to similar problems in which the cracks are of different radii. Further, by combining this method with that of Shail (9) the solution to the two coplanar cracks opened under tension in a heated thick elastic plate can be obtained easily.

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13. ABSTRACT

The solution to the problem of two coplanar penny-shaped cracks in an infinite elastic medium is sought for the case where asymmetric heat flux or temperature difference is prescribed over the surfaces of the cracks. The problem is reduced to determining the solution to two sets of coupled Fredholm integral equations. An example is worked for a case where the temperature is prescribed over the crack faces and the heat flux normal to the plane containing the cracks is zero. The integral equations are then solved iteratively assuming that the spacing between the cracks is large relative to their radii. Physical quantities of interest, such as crack-opening displacement and stress intensity factor, are investigated.

14. KEY WORDS	LINK A		LINK B		LINK C	
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