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13. ABSTRACT

To analyze the effect of an underwater explosion on submarine appendages, it is necessary to account for the presence of the surrounding fluid. One of the pressure loadings exerted by the fluid on the appendage will be due to its motion through the fluid. For most bodies, it is difficult to compute the resultant force due to this fluid loading.

The force exerted on an object moving in an acoustic medium can be related to its acceleration through a convolution integral; the kernel of this integral represents the impulsive force $\lambda(t)$ which results when the body is given a step velocity change. The sphere is one of the few geometries for which an analytic expression for this impulsive pressure exists. For more complicated bodies, various approximations for $\lambda(t)$ have been proposed, based on its known assymptotic early and late time behavior.

One possible method of generating these approximations is developed and applied to the special case of a rigid, oscillating sphere elastically connected to a fixed base. The sphere is subjected to an impulse and its resultant motion is computed using both the exact and approximate form of $\lambda(t)$. It is found that good agreement exists between the exact and approximate time histories of sphere motion for high frequencies at early times and for low frequencies at late times. The usual assumption of early time radiation gives poor results at low frequencies. Physical and mathematical interpretations of the effect of fluid pressure on oscillator motion is given.

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Fluid-structural interaction						
Fluid-structure decoupling approximation						
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Acoustic radiator						
Acoustic transients						
Impulsive pressure						
Added mass						

CONTENTS

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PROBLEM STATUS

This is an interim report on one phase of the problem; work is continuing on other phases.

AUTHORIZATION

NRL Problem F02-12.504
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Manuscript submitted March 30, 1973.

A STUDY OF FLUID-STRUCTURE INTERACTION AND DECOUPLING APPROXIMATIONS

INTRODUCTION

Recently there has been much interest in the problem of analyzing the effect of an underwater explosion on exposed submarine appendages such as rudders, sail, etc. To adequately model the physical problem, it is necessary to account for the effect of the fluid on the appendage.

When a submerged structure is subjected to an underwater explosion, it experiences three types of pressure loadings over its surface. There is a loading from the free-field pressure and one from the scattered pressure; in computing the scattered pressure, the structure is considered rigid and immovable. The third pressure, called the radiation pressure, results from motion of the structure through the fluid.

The radiation pressure loading is a consequence of fluid-structural interaction; for realistic structures, it is difficult to compute. In this report, we attempt to illustrate some of the main features of this interaction by a simple mathematical model, with the idea of generating some approximate methods to account for fluid-structural interaction for complicated structural geometries.

The pressure on the surface of a submerged, moving body can be related to body motion by an equation of the form*

$$P(\mathbf{r}, t) = \int_{\tau=0}^{t} z(\mathbf{r}, t-\tau)\, \ddot{x}(\tau)\, d\tau \,, \tag{1}$$

where

$\mathbf{r}$	represents a point on the surface of the body
$P(\mathbf{r}, t)$	is surface pressure
$w(\mathbf{r}, t)$	is displacement of body along outward normal
$\phi(\mathbf{r})$	is a deflection pattern
$x(t)$	is a normal coordinate

$w(\mathbf{r}, t) = x(t)\, \phi(\mathbf{r})$.

*Symbols will be defined as they appear in the text.

We note that $z(\mathbf{r}, t)$ is proportional to the surface pressure which results when the body is given a step velocity change in deflection pattern $\phi(\mathbf{r})$, i.e., $\dot{w}(\mathbf{r}, t) = \dot{v}_0 \phi(\mathbf{r}) J(t)$ where $J(t)$ is the Heaviside step function. Except for some simple geometries, we do not have analytic expressions for $z(\mathbf{r}, t)$; however, even for complicated geometries, we can generate approximations which give us early time radiation and late time added mass behavior. One possible way of constructing these approximations is outlined in Appendix A.

The oscillating rigid sphere is a geometry for which we have an exact solution for $z(\mathbf{r}, t)$.* Physically, this solution is a decaying sinusoid with decay constant and period proportional to the time for a sound wave to travel around the perimeter of the sphere. Similar results hold for more complicated bodies;* however, for these bodies, determining the exact form of $z(\mathbf{r}, t)$ involves the inversion of a complicated Fourier transform. Since $z(\mathbf{r}, t)$ for the sphere embodies the main physical features we would obtain for the more complicated geometries, we can gain some insight into the more general problem of an oscillating finite body by considering the solution for the sphere.

We will consider the following problem. A rigid sphere is connected to a fixed base by an elastic spring and set in motion by an impulse (see Fig. 1). The effect of fluid loading on the sphere is expressed via the convolution integral above. By generating an approximation to $z(\mathbf{r}, t)$, we obtain an approximate solution for motion of the sphere; this is to be compared with the exact solution. We discover that our particular approximation (one of many we could have chosen) gives us good results for early times at high frequencies and for late times at low frequencies.

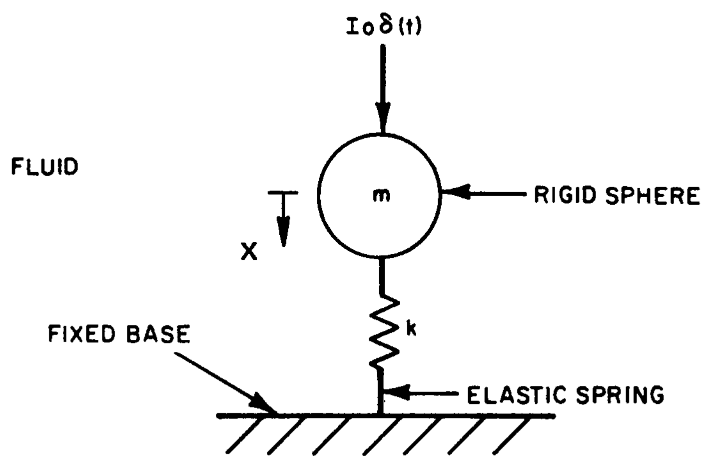

Fig. 1—Rigid sphere immersed in acoustic fluid and coupled to a fixed base through an elastic element

For the sphere, we find that high frequencies are those for which $\omega_1 << \omega_0$, with ω_1 being the frequency of $z(\mathbf{r}, t)$ and ω_0 the oscillator frequency in vacuo. Thus, a high-frequency, early time case is one in which the oscillator goes through many cycles while the impulsive pressure $z(\mathbf{r}, t)$ is essentially constant. The low-frequency, late time case corresponds to the oscillator slowly responding while the impulsive pressure is rapidly decaying, with decay constant and equal to ω_1, and $\omega_1 >> \omega_0$.

*M. C. Junger and W. Thompson, J. Acoust. Soc. Amer. 38, 978-986 (1965).

Interestingly enough, the form of $z(\mathbf{r}, t)$ for a sphere has been proposed as an approximation to the impulsive pressure for more complicated bodies. We also choose as our approximation to $z(\mathbf{r}, t)$ a one-dimensional version of an approximate impulsive pressure form proposed by Geers.*

EXACT SOLUTION

Consider a submerged sphere connected to a rigid base by means of a spring. The sphere is given an impulse at $t = 0$, and we determine its resulting motion from the equation

$$m\ddot{x} + kx = F_f + I_0\, \delta(t)\,, \tag{1}$$

where

m = mass of sphere

x = displacement of sphere

k = stiffness of spring

F_f = fluid force on sphere

I_0 = impulse

$\delta(t)$ = Dirac delta function

We assume zero initial conditions. Obviously, we need the value of F_f to specify the problem completely.

For an arbitrary body in motion through an acoustic medium, we know that the pressure field generated by the motion of the body satisfies the wave equation

$$\nabla^2 P = \frac{1}{c^2}\, \ddot{P}\,, \tag{2}$$

where c = sound speed in medium. Further, on the surface of the body, continuity and momentum considerations require that

$$\frac{\partial P}{\partial n} = -\rho \ddot{w}\,, \tag{3}$$

where

n = outward normal to body

ρ = fluid density

w = displacement of body along its outward normal.

*T. L. Geers, J. Acoust. Soc. Amer. 49, 1505-1510 (1971).

Taking the Laplace transform of the wave equation (for zero initial conditions) and using the boundary conditions enable us to form the Helmholtz integral equation

$$\frac{1}{2}\,\overline{P}(\mathbf{r},s) - \oiint_{\sigma_0} \frac{\overline{P}(\mathbf{r}_0,s)}{4\pi}\,\frac{\partial}{\partial n_0}\left(\frac{e^{(s/c)|\mathbf{r}-\mathbf{r}_0|}}{|\mathbf{r}-\mathbf{r}_0|}\right) d\sigma_0$$

$$= \rho s^2 \oiint_{\sigma_0} \frac{\overline{w}(\mathbf{r}_0,s)}{4\pi}\,\frac{e^{(s/c)|\mathbf{r}-\mathbf{r}_0|}}{|\mathbf{r}-\mathbf{r}_0|}\, d\sigma_0 \,. \tag{4}$$

Here $\mathbf{r}$ represents a field point on the surface of the body, $\mathbf{r}_0$ is a source point, and $\partial\sigma_0$ represents a surface area element, as shown in Fig. 2.

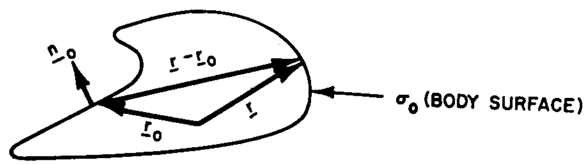

Fig. 2—Defines quantities used in Eq. (4)

The solution to this integral equation has the form

$$\overline{P}(\mathbf{r},s) = s^2 \oiint_{\sigma_0} \overline{H}(\mathbf{r},\mathbf{r}_0;s)\,\overline{w}(\mathbf{r}_0,s)\,d\sigma_0 \,. \tag{5}$$

Where $\overline{H}(\mathbf{r},\mathbf{r}_0;s)$ has units of mass/length^4. Taking the inverse transform via the convolution theorem yields

$$P(\mathbf{r},t) = \int_{\tau=0}^{t}\left[\oiint_{\sigma_0} H(\mathbf{r},\mathbf{r}_0;t-\tau)\,\ddot{w}(\mathbf{r}_0,\tau)\,d\sigma_0\right] d\tau \,. \tag{6}$$

For an oscillating rigid sphere, acceleration along the outward normal is given by $\ddot{w}(\mathbf{r},t) = \ddot{x}(t)\cos\theta$; here θ is the angle measured from the direction of motion, as shown in Fig. 3.

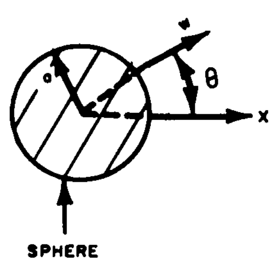

Fig. 3—Defines angle θ

Then we have

$$P(\mathbf{r}, t) = \int_{\tau=0}^{t} \ddot{x}(t-\tau) \left[\oiint_{\sigma_0} H(\mathbf{r}, \mathbf{r}_0; \tau) \cos\theta \, d\sigma_0 \right] d\tau . \tag{7}$$

Let us denote the surface integral by $z(\mathbf{r}, \tau)$. We claim that the quantity $z(\mathbf{r}, t)$ is proportional to the surface pressure when the sphere is given a step velocity change at $t = 0$; i.e., $\ddot{w}(\mathbf{r}, t) = \dot{U}_0 \, \delta(t) \cos\theta$. By direct substitution,

$$P_\delta(\mathbf{r}, t) = \int_{\tau=0}^{t} \dot{U}_0 \, \delta(t-\tau) \, z(\mathbf{r}, \tau) \, d\tau = \dot{U}_0 \, z(\mathbf{r}, t) , \tag{8}$$

or

$$z(\mathbf{r}, t) = \frac{P_\delta(\mathbf{r}, t)}{\dot{U}_0} . \tag{9}$$

The surface pressure for a velocity step has been calculated as*

$$P_\delta(\mathbf{r}, t) = \rho c \, \dot{U}_0 \, e^{-\omega_1 t} \cos \omega_1 t \cos\theta , \tag{10}$$

where $\omega_1 = c/a$, a = radius of sphere. Consequently, we have

$$z(\mathbf{r}, t) = \rho c \, e^{-\omega_1 t} \cos \omega_1 t \cos\theta . \tag{11}$$

The work done by the fluid acting over the surface of the sphere is expressed by

$$W = - \oiint_{\sigma} P(\mathbf{r}, t) \, w(\mathbf{r}, t) \, d\sigma , \tag{12}$$

with a corresponding generalized force

$$F_f(t) = \frac{\partial W}{\partial x} = - \oiint_{\sigma} P(\mathbf{r}, t) \cos\theta \, d\sigma , \tag{13}$$

since $w(\mathbf{r}, t) = x(t) \cos\theta$. It is found that

$$F_f(t) = - \int_{\tau=0}^{t} \lambda(t-\tau) \, \ddot{x}(\tau) \, d\tau , \tag{14}$$

where

$$\lambda(t) = \oiint_{\sigma} z(\mathbf{r}, t) \cos\theta \, d\sigma . \tag{15}$$

*M. C. Junger and W. J. Thompson, J. Acoust. Soc. Amer. 38, 978-986 (1965).

Direct computation gives us the value of $\lambda(t)$ as

$$\lambda(t) = \frac{4}{3}\,\pi a^2 \rho c\, e^{-\omega_1 t} \cos \omega_1 t\,. \tag{16}$$

As a consequence, we can write our equation of motion in the form

$$m\ddot{x} + kx = -\frac{4}{3}\,\pi a^2 \rho c \int_{\tau=0}^{t} \ddot{x}(t-\tau)\, e^{-\omega_1 \tau} \cos \omega_1 \tau d\tau + I_0 \delta(t)\,. \tag{17}$$

Taking the Laplace transform for zero initial conditions and using dimensionless variables gives us

$$(\tilde{s}^2 + 1)\,\tilde{x}(\tilde{s}) = -\frac{R\tilde{s}^2(\tilde{s} + \omega_1/\omega_0)\,\tilde{x}(\tilde{s})}{\left(\tilde{s} + \dfrac{\omega_1}{\omega_0}\right)^2 + (\omega_1/\omega_0)^2} + \tilde{I}_0\,, \tag{18}$$

with

$$\omega_0^2 = k/m$$

$$\tilde{s} = s/\omega_0$$

$$\tilde{x}(\tilde{s}) = \bar{x}(s)/a\omega_0$$

$$R = 4\pi a^2 \rho c/(3m\omega_0)$$

$$\tilde{I}_0 = I_0/(ma\omega_0)\,.$$

We note that R is a measure of the coupling between the sphere and fluid, and can be interpreted in several ways. If we note that ρc has the dimensions of impedance/area, then the quantity $(4/3)\pi a^2 \rho c$ is the impedance (for early times) of an unbaffled piston with total surface area equal to 2/3 of the surface area of the sphere. The quantity $m\omega_0$ is the impedance of the sphere and spring system in vacuo. Hence, R is the ratio of the fluid impedance to the oscillator impedance in vacuo.

We can also rewrite

$$R = \left(\frac{4\pi a^3 \rho}{3m}\right)\left(\frac{c/a}{\omega_0}\right) = \left(\frac{m_w}{m}\right)\left(\frac{\omega_1}{\omega_0}\right), \tag{19}$$

where m_w = mass of water displaced by sphere. We note that $a/c = 1/\omega_1 = (2\pi)^{-1} \times$ time for a "creeping wave" to progress around the sphere; $1/\omega_0 = (2\pi)^{-1} \times$ period of oscillation in vacuo. In another form,

$$R = \left(\frac{m_w}{m}\right)\left(\frac{T_0}{T_1}\right), \tag{20}$$

where

T_0 = period of oscillation in vacuo

T_1 = $2\pi/\omega_1$.

Returning to the transformed equation of motion, we put it in the form

$$\tilde{x}(\tilde{s}) = \frac{\tilde{I}_0[(\tilde{s}+\epsilon)^2 + \epsilon^2]}{\left[(\tilde{s}^2+1)[(\tilde{s}+\epsilon)^2 + \epsilon^2] + \tilde{s}^2\left(\frac{m_w}{m}\right)\epsilon(\tilde{s}+\epsilon)\right]}, \tag{21}$$

where $\epsilon = \omega_1/\omega_0$ and $R = (m_w/m)\epsilon$.

The above equation can be rewritten as

$$\tilde{x}(\tilde{s}) = \frac{\tilde{I}_0[(\tilde{s}+\epsilon)^2 + \epsilon^2]}{(\tilde{s}-\tilde{s}_1)(\tilde{s}-\tilde{s}_2)(\tilde{s}-\tilde{s}_3)(\tilde{s}-\tilde{s}_4)}, \tag{22}$$

where $\tilde{s}_1$ through $\tilde{s}_4$ are roots of the denominator; i.e., they are roots of

$$(\tilde{s}^2+1)[(\tilde{s}+\epsilon)^2 + \epsilon^2] + \tilde{s}^2\left(\frac{m_w}{m}\right)\epsilon(\tilde{s}+\epsilon) = 0. \tag{23}$$

In the $\tilde{s}$ domain, large values of the transform variable $\tilde{s}$ correspond to solutions valid for the initial stages of oscillator motion, and small values of $\tilde{s}$ correspond to much later times. Solutions associated with large $\tilde{s}$ will be called early time solutions, or X_e; those associated with small $\tilde{s}$ will be called late time solutions, or X_l. It is insufficient to say that $\tilde{s}$ is "large" or "small"; we must compare it with some dimensionless quantity. The asymptotic values of $\tilde{s}$ will depend upon the frequency ratio ϵ as shown in Table 1. This table defines early and late times depending on frequency ratio.

Table 1
Dependency of Early and Late Times on Frequency

Time	Low Frequencies ($\epsilon >> 1$, or $\omega_1 >> \omega_0$)	High Frequencies ($\epsilon << 1$, or $\omega_1 << \omega_0$)
Early	$\tilde{s} >> 1; \tilde{x}_e >> \tilde{x}_l$	$\tilde{s} >> \epsilon; \tilde{x}_e >> \tilde{x}_l$
Late	$\tilde{s} << \epsilon; \tilde{x}_e << \tilde{x}_l$	$\tilde{s} << 1; \tilde{x}_e << \tilde{x}_l$

Suppose we look at the low-frequency case where $\epsilon >> 1$ and consider late times; i.e., $\tilde{s} << \epsilon$. Then the quartic Eq. (23) above has the approximate form

$$(\tilde{s}^2+1)(2\epsilon^2) + \tilde{s}^2\epsilon^2\left(\frac{m_w}{m}\right) = 0, \tag{24}$$

which gives us the roots

$$\tilde{s}_{3,4} = \pm i\left(\frac{2}{2 + \frac{m_w}{m}}\right)^{1/2} \quad ; \quad i = \sqrt{-1} \,.$$

Physically, this represents an undamped sinusoidal vibration for low frequencies at late times. The ratio of this submerged frequency to the in-vacuo frequency is

$$\frac{\omega_{\text{sub}}}{\omega_0} = \left(\frac{2}{2 + \frac{m_w}{m}}\right)^{1/2} ,$$

which shows how the added mass effect lowers the frequency.

Now let us consider the low-frequency case at early times; i.e., $\epsilon >> 1$ and $\tilde{s} >> 1$. The quartic becomes approximately

$$\left[\tilde{s}^2 + \left(2 + \frac{m_w}{m}\right) \epsilon\tilde{s} + \left(2 + \frac{m_w}{m}\right) \epsilon^2\right]\tilde{s}^2 = 0 \,, \tag{27}$$

which has roots*

$$\tilde{s}_{1,2} = -\frac{\epsilon}{2}\left\{\left(2 + \frac{m_w}{m}\right) \pm \left[\left(2 + \frac{m_w}{m}\right)^2 - 4\left(2 + \frac{m_w}{m}\right)\right]^{1/2}\right\} . \tag{28}$$

If the mass of displaced water m_w is greater than $2m$, then $\tilde{s}_{1,2}$ will be real; this gives exponentially decaying solutions to the equation of motion at early times. If $m_w < 2m$, then the sphere will experience damped vibration for early times.

Consider the high-frequency case $\epsilon << 1$, and let us look at late time solutions; i.e., $\tilde{s} << 1$. The quartic reduces to†

$$\tilde{s}^2 + 2\epsilon\tilde{s} + 2\epsilon^2 = 0 \,, \tag{29}$$

or

$$\tilde{s}_{3,4} = -\epsilon \pm i\epsilon \,;$$

we see that late-time solutions correspond to lightly damped, slow oscillations. For early time solutions, we have (for $\epsilon << 1$ and $\tilde{s} >> \epsilon$)

$$\left[\tilde{s}^2 + \epsilon s\left(\frac{m_w}{m}\right) + 1\right]\tilde{s}^2 = 0 \,, \tag{30}$$

which has roots

*The two spurious roots $\tilde{s} = 0, 0$ correspond to a trivial solution.
†We assume that $\tilde{s}\epsilon[(m_w/m)] << 1$, $\epsilon^2[(m_w/m)] << 1$.

$$\tilde{s}_{1,2} = -\frac{1}{2}\left[\epsilon\left(\frac{m_w}{m}\right) \pm \left[\epsilon^2\left(\frac{m_w}{m}\right)^2 - 4\right]^{1/2}\right]. \tag{31*}$$

Depending on the sign of the discriminant, we can have underdamped, critically damped, or overdamped solutions. However, since we have stipulated that $\epsilon << 1$, it will take a large mass ratio m_w/m to give us an overdamped solution, even for early times.

Roots for the specific case of $m_w/m = 2.5$ are given in Table B1 of Appendix B. The early time roots are given as s_1 and s_2; late time roots are $s_{3,4}$. These are in good agreement with the asymptotic roots listed above.

For the high-frequency cases $\epsilon = 0.01$ and $\epsilon = 0.1$, we have a solution in the form of two decaying sinusoids:

$$\tilde{x}(\tilde{t}) = \tilde{x}_e(\tilde{t}) + x_l(\tilde{t})$$

$$= \left(\frac{\tilde{I}_0 e^{a_1\tilde{t}}}{b_1}\right)[Q_1 \sin b_1\tilde{t} + Q_2 \cos b_1\tilde{t}] + \frac{\tilde{I}_0 e^{a_3\tilde{t}}}{b_3}[Q_3 \sin b_3\tilde{t} + Q_4 \cos b_3\tilde{t}] \tag{32}$$

$$\tilde{x} = x/a, \ \tilde{t} = \omega_0 t$$

where we have set

$$\tilde{s}_1 = a_1 + ib_1 \ ; \ \tilde{s}_2 = a_1 - ib_1$$

$$\tilde{s}_3 = a_3 + ib_3 \ ; \ \tilde{s}_4 = a_3 - ib_3 \ ,$$

and the constants Q_1 through Q_4 are defined in Table B3. Note that a_1 and a_3 are negative. Here $\tilde{x}_e(\tilde{t})$ is the early solution corresponding to roots s_1, s_2; $\tilde{x}_l(\tilde{t})$ is the late solution. This can be verified by noting that the period of $\tilde{x}_e$ is much less than that of $\tilde{x}_l$.

From Table B1, we have the result that as ϵ decreases, the decay constants a_1 and a_3 decrease also; at high frequencies, the oscillator impedance dominates the fluid impedance so that fluid effects become less with decreasing ϵ. Further evidence of decreasing fluid effect is seen from the fact that the oscillator period approaches the in-vacuo period as ϵ becomes small.

For the intermediate- ($\epsilon = 1$) and low-frequency cases, we have

$$\tilde{x}(\tilde{t}) = \tilde{x}_e(\tilde{t}) + \tilde{x}_l(\tilde{t})$$

$$= \frac{\tilde{I}_0[\alpha_1{}^2 + \epsilon^2]\, e^{a_1\tilde{t}}}{(a_1 - a_2)(A_1{}^2 + b_3{}^2)} + \frac{\tilde{I}_0[\alpha_2{}^2 + \epsilon^2]\, e^{a_2\tilde{t}}}{(a_2 - a_1)(A_2 + b_3{}^2)}$$

$$+ \frac{\tilde{I}_0\, e^{a_3\tilde{t}}}{b_3}[q \sin b_3\tilde{t} + h \cos b_3\tilde{t}] \ . \tag{33}$$

The coefficients are defined in Table B4. The decaying exponentials in Eq. (33) represent the early time solution; the late time solution is given by the decaying sinusoids. For increasing ϵ, fluid effects increase the decay constants in the early time solution, shortening the early time domain ($\tilde{x}_e >> \tilde{x}_l$). Increasing ϵ causes the period of the late time sinusoids to increase due to added mass effects.

Plots of the (dimensionless) displacement $\tilde{x}$ as a function of $\tilde{t}$ are shown in Figs. 4-8 for $\tilde{I}_0 = 1$. In Figs. 9 and 10, we have plotted the exponentially decaying early time solution $\tilde{x}_e$ (for $\epsilon = 10$ and 100) over the time domain in which it reaches its maximum. On the same graphs is plotted the oscillatory late time solution $\tilde{x}_l$.* Note the following:

1. The oscillator displacement $\tilde{x}$ is negligible over this time domain (as compared with its maximum value at late times).

2. The late time solution $\tilde{x}_l$ soon becomes greater than the early time solution $\tilde{x}_e$.

3. Any theory which attempts to use early time radiation damping over this time domain and neglects added mass effects will be seriously in error.

4. The amount of energy lost to the fluid through radiation damping is negligible, and it is sufficient to consider only added mass effects for $\epsilon >> 1$. Whether this statement can be made for geometries other than the sphere remains to be demonstrated.

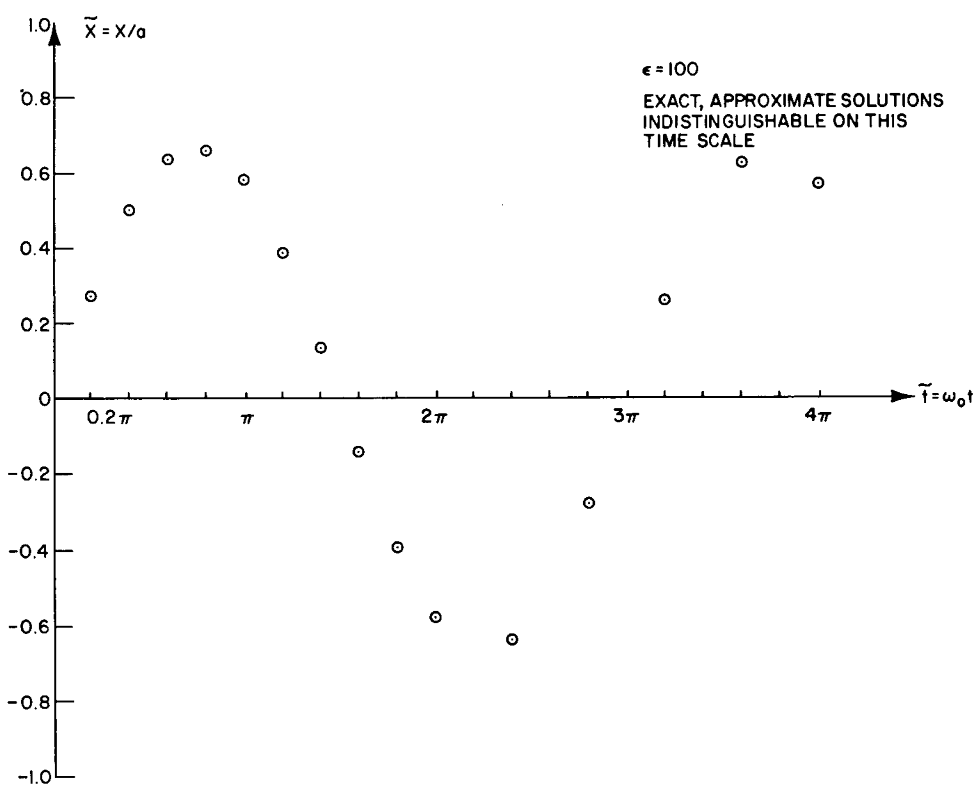

Fig. 4—Oscillator excursions for exact ⊙ and approximate ⊙ solutions ($\epsilon = 100$). (Exact and approximate solutions are indistinguishable for this value of ϵ.)

*For simplicity, the cosine term is omitted.

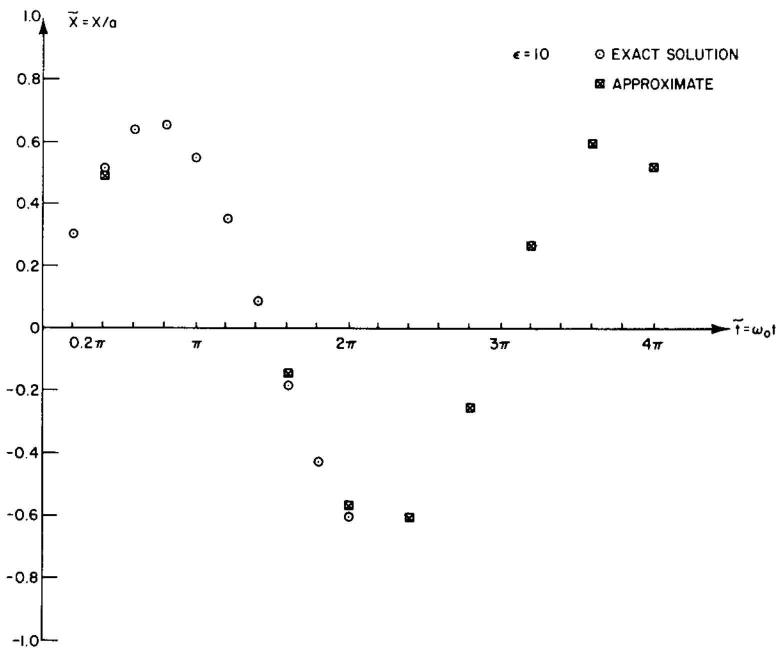

Fig. 5—Oscillator excursions for exact ⊙ and approximate ⊠ solutions (ϵ = 10)

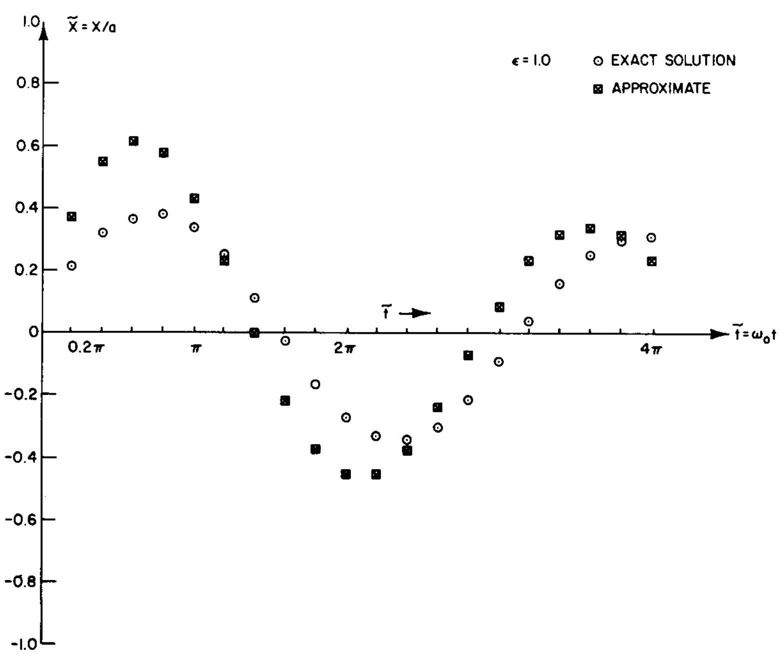

Fig. 6—Oscillator excursions for exact ⊙ and approximate ⊠ solutions (ϵ = 1.0)

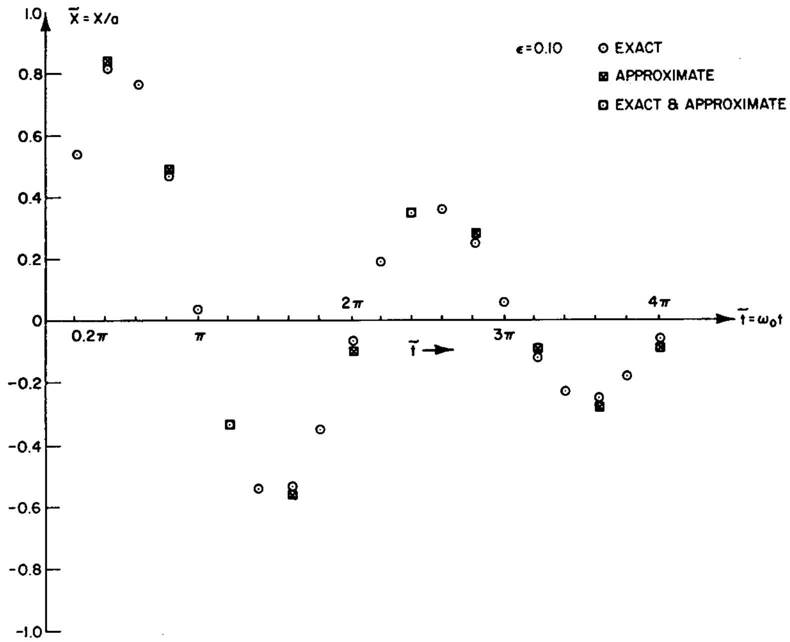

Fig. 7—Oscillator excursions for exact ⊙ and approximate ⊠ solutions (ϵ = 0.10)

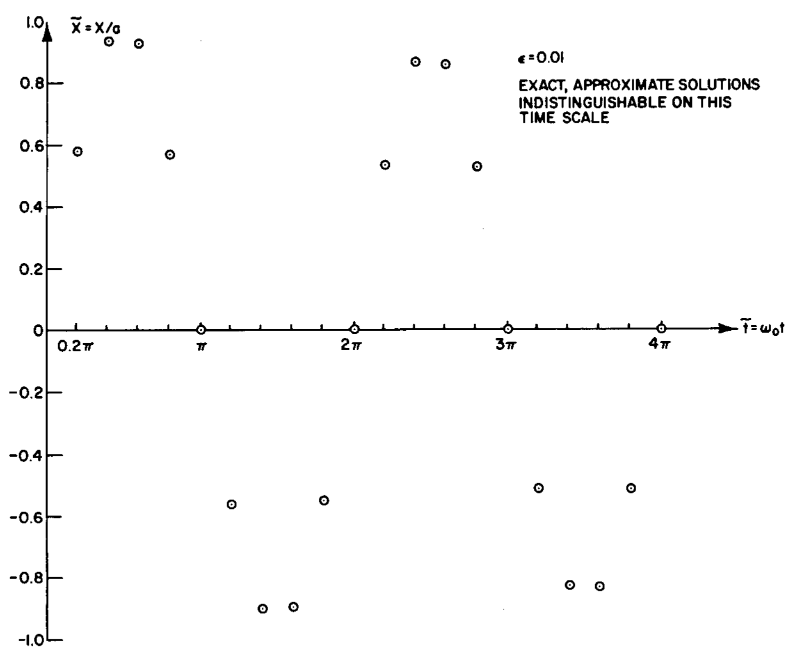

Fig. 8—Oscillator excursions for exact ⊙ and approximate ⊙ solutions (ϵ = 0.01)

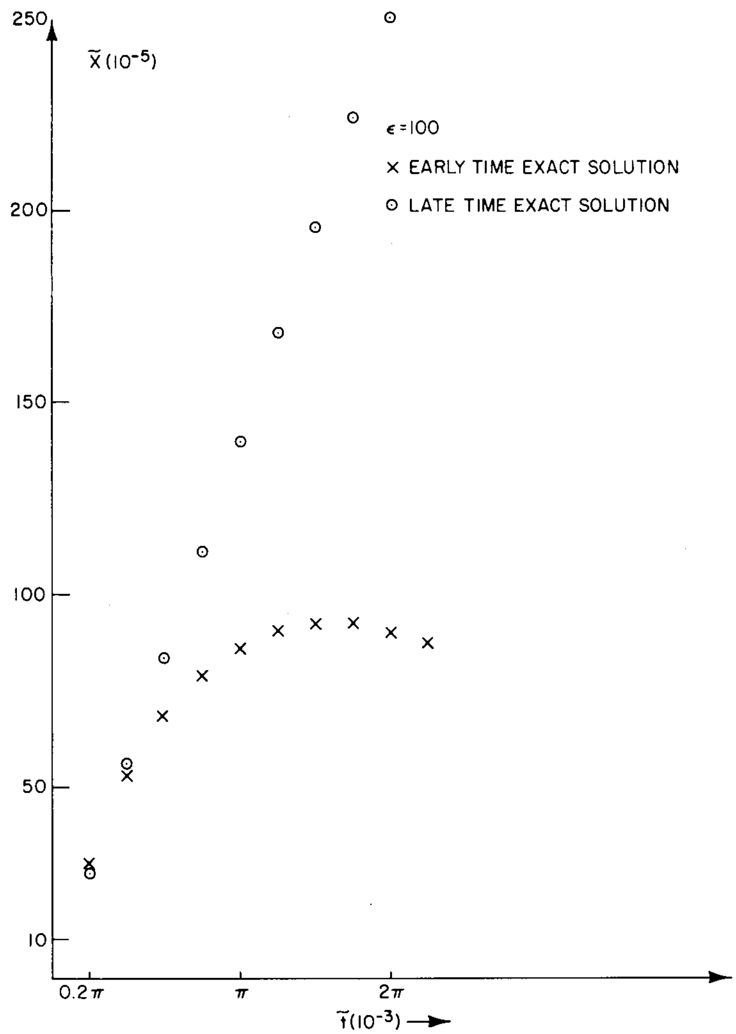

Fig. 9—Early and late time exact solutions for $\epsilon = 100$

APPROXIMATE SOLUTION

The sphere is one of the few geometries for which we have an exact solution for the hydrodynamic function $\lambda(t)$. For more complicated geometries, we can construct approximations to $\lambda(t)$ based upon its asymptotic early and late time properties. It is instructive to examine the quality of some of these approximations for the sphere by comparing the exact with the approximate solutions.

We require that $\overline{\lambda}(s)$ possess the following properties:*

$$\lim_{s \to 0} \overline{\lambda}(s) = \int_{\tau=0}^{\infty} \lambda(\tau)\, d\tau = m_a , \tag{35}$$

where m_a is the added mass coefficient for the rigid sphere in translation, and

*For the reasons behind this, we refer to Appendix A.

$$\lim_{s \to \infty} s\bar{\lambda}(s) = \oint\!\!\oint_{\sigma} \rho c \, \phi^2(\mathbf{r}) \, d\sigma \, , \tag{36}$$

where

$$w(\mathbf{r}, t) = x(t) \phi(\mathbf{r}) \, .$$

For the oscillating sphere, $\phi(\mathbf{r}) = \cos\theta$.

In the time domain, these conditions are equivalent to saying that

$$\lim_{t \to \infty} \lambda(t) = m_a \, \delta(t) \, ; \quad \lim_{t \to 0} \lambda(t) = J(t) \oint\!\!\oint_{\sigma} \rho c \, \phi^2(\mathbf{r}) \, d\sigma \tag{37}$$

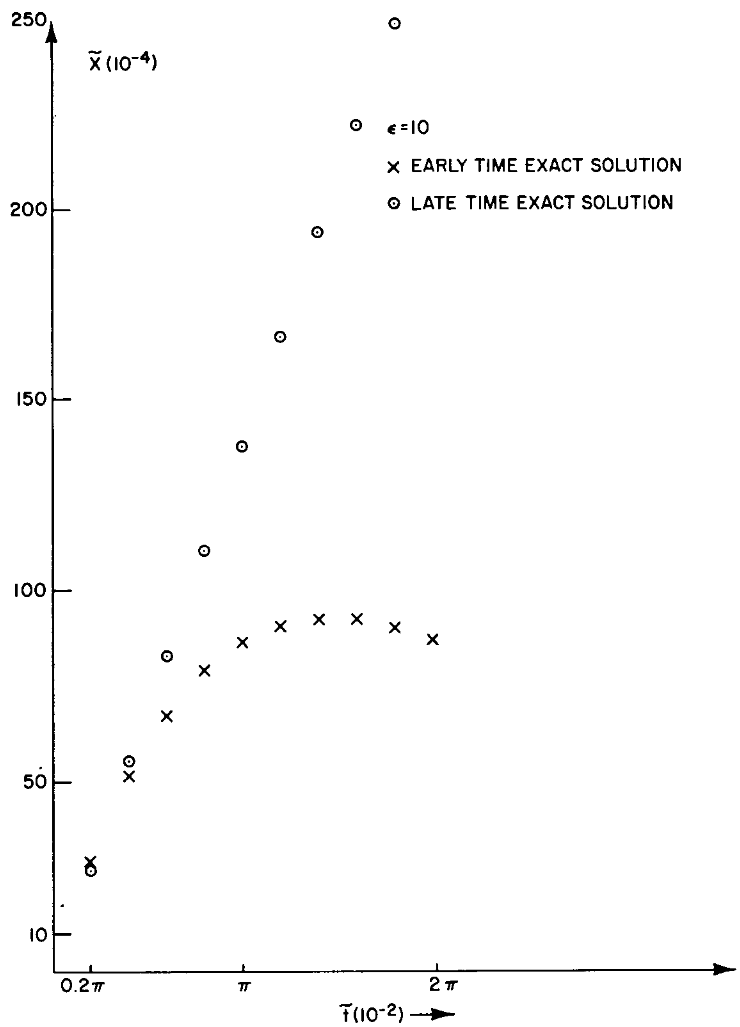

Fig. 10—Early and late time exact solutions for $\epsilon = 10$

where $J(t)$ is the Heaviside step function. We choose

$$\lambda_a(t) = \alpha e^{-t/\beta} , \tag{38}$$

and satisfying our conditions above will give us the unknown α and β. As a result, we have

$$\lambda_a(t) = \frac{\rho c A}{3} e^{-t/t'} , \tag{39}$$

where $t' = 3\, m_a/(\rho c A)$ and A = surface area of sphere.

This value of $\lambda_a(t)$ is substituted into the equation of motion;

$$m\ddot{x} + kx = -\int_{\tau=0}^{t} \lambda_a(t-\tau)\,\ddot{x}(\tau)\,d\tau + I_0\,\delta(t) . \tag{40}$$

We take the Laplace transform and cast the equation in the following dimensionless form:

$$\tilde{x}(\tilde{s}) = \frac{\tilde{I}_0(\tilde{s} + 1/\tilde{t}')}{(\tilde{s}^2 + 1)(\tilde{s} + 1/\tilde{t}') + \tilde{s}^2 \left(\frac{m_w}{m}\right) \epsilon} , \tag{41}$$

where $\tilde{t}' = \omega_0 t' = [3 m_a \omega_0/\rho c A]$. We can rewrite this in a more familiar form if we recall that $m_a = 2/3\pi a^3 \rho$; consequently, $\tilde{t}' = 0.5/\epsilon$, and

$$\tilde{x}(\tilde{s}) = \frac{\tilde{I}_0(\tilde{s} + 2\epsilon)}{(\tilde{s}^2 + 1)(\tilde{s} + 2\epsilon) + \tilde{s}^2 \left(\frac{m_w}{m}\right) \epsilon} = \frac{\tilde{I}_0(\tilde{s} + 2\epsilon)}{(\tilde{s} - \tilde{s}_1)(\tilde{s} - \tilde{s}_2)(\tilde{s} - \tilde{s}_3)} . \tag{42}$$

We note in passing that the solution (in the time domain) depends upon the roots of a cubic in the approximate case, as compared with a quartic in the exact solution. Hence, using the exponential approximation $\lambda_a(t) = \alpha \exp(-t/t')$ generates a third-order equation of motion, as compared to a fourth-order equation for the exact case, $\lambda(t) = \alpha \exp(-\omega_1 t) x \cos \omega_1 t$.

Let us examine the roots of the cubic for the asymptotic case $\epsilon >> 1$; i.e., for low frequencies. The late time solution $\tilde{s} << \epsilon$ is given by

$$\tilde{s}_{2,3} = \pm i \left[\frac{2}{2 + \frac{m_w}{m}}\right]^{1/2} , \tag{43}$$

which is exactly what we obtained for the late time, low-frequency case in the exact solution. For the early time solution, we have $\tilde{s} >> 1$:

$$\tilde{s}_1 = -\left(2 + \frac{m_w}{m}\right) \epsilon ; \tag{44}$$

the corresponding roots for the exact solution are

$$\tilde{s}_{1,2} = -\frac{\epsilon}{2}\left[\left(2 + \frac{m_w}{m}\right) \pm \left[\left(2 + \frac{m_w}{m}\right)^2 - 4\left(2 + \frac{m_w}{m}\right)\right]^{1/2}\right]. \tag{45}$$

For the high-frequency case $\epsilon << 1$ and early times $\tilde{s} >> \epsilon$, the cubic can be rewritten as

$$\tilde{s}^2 + \epsilon\tilde{s}\left(\frac{m_w}{m}\right) + 1 = 0\,, \tag{46}$$

so that

$$\tilde{s}_{1,2} = \frac{1}{2}\left[-\epsilon\left(\frac{m_w}{m}\right) \pm \left[\epsilon^2\left(\frac{m_w}{m}\right)^2 - 4\right]^{1/2}\right]. \tag{47}^*$$

This is equal to the corresponding set of roots for the exact solution. For the late time case, we have $\tilde{s} << 1$ and a root $\tilde{s}_3 = -2\epsilon$. The corresponding root for the exact solution is given by $\tilde{s}_{3,4} = -\epsilon \pm i\epsilon$.

From the preceding discussion we see that we would expect good agreement between the exact and approximate solutions for (a) the late time, low-frequency cases, and (b) the early time, high-frequency cases. We can easily demonstrate the reason for this if we look at the (dimensionless) forms of $\tilde{\lambda}(\tilde{s})$ and $\tilde{\lambda}_a(\tilde{s})$:

$$\tilde{\lambda}(\tilde{s}) = \left(\frac{m_w}{m}\right)\epsilon\left(\frac{\tilde{s} + \epsilon}{(\tilde{s} + \epsilon)^2 + \epsilon^2}\right) \tag{48}$$

$$\tilde{\lambda}_a(\tilde{s}) = \left(\frac{m_w}{m}\right)\left(\frac{\epsilon}{\tilde{s} + 2\epsilon}\right). \tag{49}$$

It is easy to verify that

$$\lim_{\substack{\epsilon\to\infty \\ \tilde{s}<<\epsilon}} \tilde{\lambda}(\tilde{s}) = \lim_{\substack{\epsilon\to\infty \\ \tilde{s}<<\epsilon}} \tilde{\lambda}_a(\tilde{s}) \quad \text{(low-frequency, late time case)} \tag{50}$$

and

$$\lim_{\substack{\epsilon\to 0 \\ \tilde{s}>>\epsilon}} \tilde{\lambda}(\tilde{s}) = \lim_{\substack{\epsilon\to 0 \\ \tilde{s}>>\epsilon}} \tilde{\lambda}_a(\tilde{s}) \quad \text{(high-frequency, early time case)} \tag{51}$$

COMPARISON OF EXACT AND APPROXIMATE SOLUTIONS

For the approximate solution, we have, in the low-frequency case

*Valid as long as $\epsilon^2(m_w/m)^2 << 4$.

$$\tilde{x}_a(\tilde{t}) = \tilde{x}_e(\tilde{t}) + \tilde{x}_l(\tilde{t}) = \frac{\tilde{I}_0 e^{a_1 \tilde{t}}}{A_1^2 + b_2^2}(a_1 + 2\epsilon) =$$

$$-\frac{\tilde{I}_0 e^{a_2 \tilde{t}}}{A_1^2 + b_2^2}\left[\frac{(\mu_2 A_1' - b_2^2)}{b_2} \sin b_2 \tilde{t} + (A_1' + \mu_2) \cos b_2 \tilde{t}\right], \tag{52}$$

where

$$\tilde{s}_1 = a_1$$

$$\tilde{s}_2 = a_2 + i b_2$$

$$\tilde{s}_3 = a_2 - i b_2$$

$$\mu_2 = a_2 + 2\epsilon$$

$$A_1' = a_1 - a_2.$$

The closeness of this approximation to the true solution is indicated in Figs. 4 and 5 over two periods of the oscillator. It can be seen that the late time portions are in good agreement. We note that $\tilde{x}_a$ has an early time exponentially decaying part as well as a decaying oscillatory part.

The approximate solution for high frequencies is

$$\tilde{x}_a(\tilde{t}) = \tilde{x}_e(\tilde{t}) + \tilde{x}_l(\tilde{t})$$

$$-\frac{\tilde{I}_0 e^{a_1 \tilde{t}}}{A_3^2 + b_1^2}\left[\frac{(\mu_1 A_3 - b_1^2)}{b_1} \sin b_1 \tilde{t} + (A_3 + \mu_1) \cos b_1 \tilde{t}\right]$$

$$+\frac{\tilde{I}_0 e^{a_3 \tilde{t}}(a_3 + 2\epsilon)}{A_3^2 + b_1^2}, \tag{53}$$

where

$$\tilde{s}_1 = a_1 + i b_1$$

$$\tilde{s}_2 = a_1 - i b_1$$

$$\tilde{s}_3 = a_3$$

$$\mu_1 = a_1 + 2\epsilon$$

$$A_3 = a_3 - a_1 .$$

From Figs. 7 and 8 we see that there is good agreement between $\tilde{x}$ and $\tilde{x}_a$ over the time frame $0 \leqslant \tilde{t} \leqslant 4\pi$; this corresponds to early times for the high-frequency case. There is less than 7% difference between $\tilde{x}$ and $\tilde{x}_a$ in this span of time. We recall that the roots of the cubic and quartic are given by

$$\tilde{s}^2 + \tilde{s}\left(\frac{m_w}{m}\right)\epsilon + 1 = 0 \tag{54}$$

for $\epsilon << 1$ and $\tilde{s} >> \epsilon$. This equation is the same one we would obtain using an infinite-plate model to approximate the fluid loading on the sphere.* Thus, the plate model is adequate for early times in the high-frequency case but fails to give a good representation for early times in the low-frequency case.

We note that the added mass effect will be negligible for $\epsilon(m_w/m) << 1$ and the frequency will be close to that of the oscillator in vacuo. Furthermore, the decay constant will decrease with increasing frequency, as we see from

$$\tilde{s}_{1,2} = \frac{1}{2}\left[-\left(\frac{m_w}{m}\right)\epsilon \pm \left\{\left(\frac{m_w}{m}\right)^2 \epsilon^2 - 4\right\}^{1/2}\right]. \tag{55}$$

In effect, the impedance of the oscillator in vacuo is much greater than that due to the presence of the fluid, so that the oscillator doesn't "see" the fluid as we increase the frequency.

ASYMPTOTIC FORMS OF CONVOLUTION INTEGRAL

Now, Junger and Thompson (p. 2) have pointed out that, for a body of finite size, the surface pressure due to impulsive motion has a period proportional to the time it takes a creeping wave to pass around the body; the surface pressure is damped by the radiation of sound energy away from the body. In the case of the sphere, the decay constant for this damping equals the frequency of the impulsive pressure.

For the low-frequency ($\epsilon >> 1$), late time case, the pressure wave due to impulsive motion has been attenuated greatly by the time the oscillator begins to respond, so that the oscillator "sees" a kind of impulse at $t = 0$ due to the surface pressure. We can illustrate this graphically, as shown in Fig. 11. Since both $\lambda(t)$ and $\lambda_a(t)$ have the impulse behavior, the fine details of the difference between them are immaterial to a low-frequency oscillator, so that we have

$$\int_{\tau=0}^{t} \lambda(t-\tau)\,\ddot{x}(\tau)\,d\tau = \int_{\tau=0}^{t} \lambda_a(t-\tau)\,\ddot{x}(\tau)\,d\tau\,, \qquad (\omega_1 >> \omega_0) \tag{56}$$

and so $\tilde{x}_a(\tilde{t}) = \tilde{x}(\tilde{t})$ for late times.

For the high-frequency case ($\omega_1 << \omega_0$), the pressure wave has not been attenuated significantly at early times, and $\cos \omega_1 t \cong 1$; hence, $\lambda(t)$ looks like a Heaviside step function to the oscillator, as shown in Fig. 12. We can say that, for early times,

*True, providing we set the plate area equal to 4/3 the projected area of the sphere.

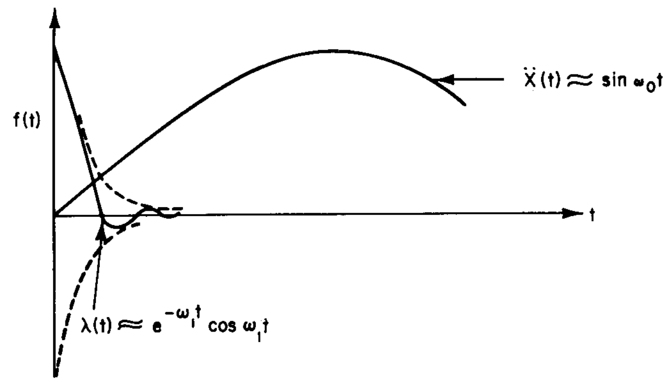

Fig. 11—Values of $\lambda(t)$ and $\ddot{x}(t)$ for the low-frequency, late time case ($\omega_1 >> \omega_0 ; \tilde{x}_\ell >> \tilde{x}_e$) (not to scale)

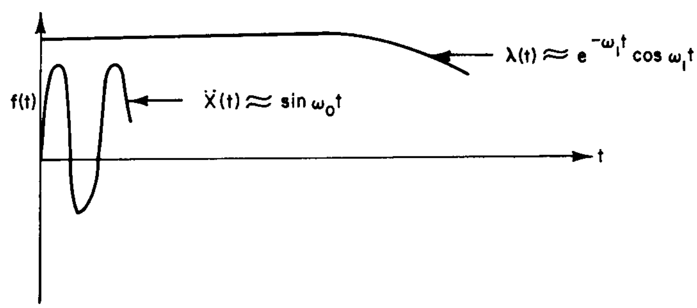

Fig. 12—Values of $\lambda(t)$ and $\ddot{x}(t)$ for the high-frequency, early time case $\omega_1 << \omega_0 ; x_e >> x_l$) (not to scale)

$$\int_{\tau=0}^{t} \lambda(t-\tau)\, \ddot{x}(\tau)\, d\tau = \int_{\tau=0}^{t} \lambda_a(t-\tau)\, \ddot{x}(\tau)\, d\tau = \frac{\rho c A}{3}\, \dot{x}(t) , \tag{57}$$

which will give us a viscous damping term in our equation of motion.

DIFFERENTIAL EQUATION APPROACH

We can easily derive further interesting properties of $\lambda(t)$ and $\lambda_a(t)$. Let us recall that

$$\tilde{F}_f(\tilde{s}) = \tilde{s}^2 \tilde{x}(\tilde{s})\, \tilde{\lambda}(\tilde{s}) ,$$

and, since

$$\tilde{\lambda}(\tilde{s}) = \frac{m_w}{m}\, \epsilon \left[\frac{\tilde{s} + \epsilon}{(\tilde{s} + \epsilon)^2 + \epsilon^2} \right] ,$$

we rewrite this as

$$(\tilde{s}^2 + 2\epsilon\tilde{s} + 2\epsilon^2)\, \tilde{F}_f(\tilde{s}) = \left[(\tilde{s}^3 + \epsilon\tilde{s}^2)\, \tilde{x}(\tilde{s}) \left(\frac{m_w}{m}\right) \right] \epsilon , \tag{58}$$

which is equivalent to the differential equation

$$\left(\frac{1}{\epsilon}\right)\left(\frac{m}{m_w}\right)\left(\ddot{\tilde{F}}(\tilde{t}) + 2\epsilon\dot{\tilde{F}}(\tilde{t}) + 2\epsilon^2\tilde{F}_f(\tilde{t})\right) = \dddot{\tilde{x}}(\tilde{t}) + \epsilon\ddot{\tilde{x}}(\tilde{t})\,. \tag{59}$$

This is one of the approximations which has been proposed for fluid-structural decoupling. We find that this approximate equation for determining the fluid force $\tilde{F}_f$ is the exact solution for an oscillating sphere.

If we now use the approximate form $\tilde{\lambda}_a(\tilde{s})$, we have

$$(\tilde{s} + 2\epsilon)\tilde{F}_f(\tilde{s}) = \left(\frac{m_w}{m}\right)\ \epsilon\tilde{s}^2\tilde{x}(\tilde{s}) \tag{60}$$

or

$$\dot{\tilde{F}}_f(\tilde{t}) + 2\epsilon\tilde{F}_f(\tilde{t}) = \left(\frac{m_w}{m}\right)\ \epsilon\ddot{\tilde{x}}(\tilde{t})\,. \tag{61}$$

This turns out to be the exact solution for a breathing sphere; Geers (p. 3) proposed a three-dimensional extension of this to decouple the fluid-structure interaction.

CONCLUSIONS

The exact and approximate solutions $\tilde{x}(\tilde{t})$ and $\tilde{x}_a(\tilde{t})$ are derived from roots of the quartic (Eq. (23)) and cubic (Eq. (42)) equations, respectively. For high and low frequencies, good approximations to these roots can be obtained by using asymptotic values of $\tilde{s}$ as defined in Table 1. This table gives a systematic way of defining the magnitude of $\tilde{s}$, rather than saying that $\tilde{s}$ is "small" or "large."

For high frequencies ($\epsilon << 1$), the exact solution consists of two decaying sinusoids. As ϵ decreases, the effect of the fluid on the oscillator becomes less, as evidenced by decreasing decay constants; the period of the oscillator approaches that in vacuo. Decreasing ϵ increases the early time domain; maximum oscillator excursions occur there.

The intermediate- ($\epsilon = 1$) and low-frequency ($\epsilon >> 1$) cases have exact solutions consisting of early time decaying exponentials and a late time damped sinusoid. For ϵ increasing, fluid effects increase, causing increasing decay of the early time solution; increasing ϵ also increases added mass effects, which increases the period of the late time solution and decreases its damping. Little energy is lost to the fluid through radiation damping, and maximum oscillator excursions occur at late times. Theories which attempt to predict early time behavior on the basis of radiation damping (flat-plate model) will be seriously in error. Increasing ϵ causes the early time domain to shrink.

In Appendix A, we show how to generate an approximation to $\lambda(t)$; this is used to derive an approximate equation of motion. The approximate solution $\tilde{x}_a(\tilde{t})$ is found to be in good agreement with the exact solution for (a) the high-frequency case at early times, (b) the low-frequency case at late times. This can be explained by noting that $\lambda(s)$ and $\lambda_a(s)$ have the same asymptotic form in these cases. For case (a), $\lambda(t)$ and $\lambda_a(t)$ are effectively Heaviside step functions; for case (b), they act as delta functions.

The impulsive pressure $\lambda(t)$ can be related to the fluid force and various derivatives of $\tilde{x}$ via a differential equation approach; some of the differential equations have been proposed in the literature as fluid-structural decoupling approximations.

It is anticipated that we can define an "equivalent sphere" for more complicated exposed submarine appendages. Should this be the case, we have the impulse response for the submerged appendage accounting for radiation loading. From this impulse response, we can, in theory, construct the appendage response to any kind of mechanical or fluid loading.

Appendix A

APPROXIMATION TO HYDRODYNAMIC FUNCTION $\lambda(t)$

We wish to show the reasons we must demand that

$$\lim_{s\to 0} \lambda(s) = m_a \; ; \qquad \lim_{s\to\infty} s\lambda(s) = \rho c \oiint_{\sigma} \phi^2(\mathbf{r})\, d\sigma \,,$$

with $w(\mathbf{r}, t) = x(t)\, \phi(\mathbf{r})$. To prove the first assertion, we return to the Helmholtz integral equation

$$\frac{1}{2}\, \bar{P}(\mathbf{r}, s) - \oiint_{\sigma_0} \frac{\bar{P}(\mathbf{r}_0, s)}{4\pi} \frac{\partial}{\partial n_0} \left[\frac{\exp\left(\frac{s}{c}\right) |\mathbf{r} - \mathbf{r}_0|}{|\mathbf{r} - \mathbf{r}_0|} \right] \partial\sigma_0 =$$

$$\rho s^2 \oiint_{\sigma_0} \frac{\bar{w}(\mathbf{r}_0, s)}{4\pi} \frac{\exp\left(\frac{s}{c}\right) |\mathbf{r} - \mathbf{r}_0|}{|\mathbf{r} - \mathbf{r}_0|} \partial\sigma_0$$

and set $s \to 0$. Then we have a solution to the integral equation which states that

$$\bar{P}_{s\to 0}(\mathbf{r}, s) = \oiint_{\sigma_0} \bar{H}(\mathbf{r}, \mathbf{r}_0 ; 0)\, s^2 \bar{w}(\mathbf{r}_0, s)\, d\sigma_0 \,. \tag{A1}$$

We note that $\bar{H}(\mathbf{r}, \mathbf{r}_0 ; 0)$ has dimensions of mass/(length)^4. If our fluid were incompressible, or if we had low-frequency oscillations, then

$$P(\mathbf{r}, t) = \oiint_{\sigma_0} \bar{H}(\mathbf{r}, \mathbf{r}_0, 0)\, \ddot{x}(t)\, \phi(\mathbf{r}_0)\ \partial\sigma_0 \,, \tag{A2}$$

which can be rewritten as

$$P(\mathbf{r}, t) = \bar{z}(\mathbf{r}, 0)\, \ddot{x}(t) \,. \tag{A3}$$

This gives rise to a force on the oscillator body

$$F_f(t) = - \oiint_{\sigma} P(\mathbf{r}, t)\, \phi(\mathbf{r})\, \partial\sigma = -m_a \ddot{x}(t) \,, \tag{A4}$$

where

$$m_a = \iint_\sigma \bar{\bar{z}}(\mathbf{r}, 0)\, \phi(\mathbf{r})\, \partial\sigma$$

is the added mass coefficient for the oscillator vibrating in an incompressible fluid with normal displacement $w(\mathbf{r}, t) = x(t)\, \phi(\mathbf{r})$.

In the most general case, we have

$$F_f(t) = -\int_{\tau=0}^{t} \lambda(t-\tau)\, \ddot{x}(\tau)\, \partial\tau\,, \tag{A5}$$

or $\bar{\bar{F}}_f(s) = -s^2 \bar{\bar{x}}(s)\, \bar{\bar{\lambda}}(s)$. Taking the Laplace transform of Eq. (A4) and comparing gives

$$\lim_{s \to 0} \bar{\bar{\lambda}}(s) = m_a\,. \tag{A6}$$

We now wish to demonstrate that

$$\lim_{s \to \infty} s\bar{\bar{\lambda}}(s) = \rho c \iint_\sigma \phi^2(\mathbf{r})\, \partial\sigma\,.$$

We know that $z(\mathbf{r}, t) = P_\delta(\mathbf{r}, t)/\dot{U}_0$, where $P_\delta(\mathbf{r}, t)$ is the pressure distribution when the oscillator is given a step velocity change, $\dot{w}(\mathbf{r}, t) = \dot{U}_0 J(t)\, \phi(\mathbf{r})$. We know that, at early times, each element of the oscillator acts as a pistonlike radiator of sound, i.e.,

$$\underset{s \to \infty}{\bar{\bar{P}}_\delta(\mathbf{r}, s)} = \rho c\, \dot{U}_0\, \phi(\mathbf{r})/s, \tag{A7}$$

so that

$$\underset{s \to \infty}{\bar{\bar{z}}(\mathbf{r}, s)} = \underset{s \to \infty}{\bar{\bar{P}}_\delta(\mathbf{r}, s)}/s\, \dot{U}_0 = \rho c \phi(\mathbf{r})/s, \tag{A8}$$

which says that at early times $z(\mathbf{r}, t)$ equals the characteristic impedance of the fluid times the deflection pattern $\phi(\mathbf{r})$. Since

$$\bar{\bar{\lambda}}(s) = \iint_\sigma \bar{\bar{z}}(\mathbf{r}, s)\, \phi(\mathbf{r})\, \partial\sigma\,, \tag{A9}$$

it is immediately obvious that

$$\underset{s \to \infty}{s\bar{\bar{\lambda}}(s)} = \rho c \iint_\sigma \phi^2(\mathbf{r})\, \partial\sigma\,. \tag{A10}$$

Appendix B

TABLES OF CONSTANTS USED IN EXACT AND APPROXIMATE SOLUTIONS

Table B1
Roots of Quartic Equation (m_w/m = 2.5)

Value of ϵ	Roots
0.01	$\tilde{s}_{1,2} = -0.0125 \pm 0.9998i$ $\tilde{s}_{3,4} = -0.0100 \pm 0.0100i$
0.10	$\tilde{s}_{1,2} = -0.125 \pm 0.979i$ $\tilde{s}_{3,4} = -0.100 \pm 0.1026i$
1.0	$\tilde{s}_1 = -2.0$; $s_2 = -2.462$ $\tilde{s}_{3,4} = -0.0191 \pm 0.637i$
10.0	$\tilde{s}_1 = -15.037$, $s_2 = -29.963$ $\tilde{s}_{3,4} = -0.000 \pm 0.667i$
100.0	$\tilde{s}_1 = -150$, $s_2 = -300$ $\tilde{s}_{3,4} = -0.000 \pm 0.667i$

Table B2
Roots of Cubic Equation (m_w/m = 2.5)

Value of ϵ	Roots
0.01	$\tilde{s}_{1,2} = -0.012 \pm 1.00i$ $\tilde{s}_3 = -0.020$
0.10	$\tilde{s}_{1,2} = -0.120 \pm 0.967i$ $\tilde{s}_3 = -0.211$
1.0	$\tilde{s}_1 = -4.376$ $\tilde{s}_{2,3} = -0.062 \pm 0.637i$
10.0	$\tilde{s}_1 = -44.998$ $\tilde{s}_{2,3} = -0.006 \pm 0.667i$
100.0	$\tilde{s}_1 = -450.0$ $\tilde{s}_{2,3} = -0.001 \pm 0.667i$

Table B3
Constants for High-Frequency Solution

$$
\begin{aligned}
s_1 &= a_1 + ib_1\,; \quad s_2 = a_1 - ib_1 \\
s_3 &= a_3 + ib_3\,; \quad s_4 = a_3 - ib_3 \\[1ex]
\alpha_1 &= a_1 + \epsilon \\
F_1 &= \alpha_1^2 + \epsilon^2 - b_1^2 \\
G_1 &= 2\alpha_1 b_1 \\
A_3 &= a_3 - a_1 \\
B_3 &= b_3 + b_1 \\
\beta_3 &= b_1 - b_3 \\
H_3 &= A_3^2 - B_3\beta_3 \\
J_3 &= B_3 + \beta_3 \\
M_3 &= (A_3^2 + \beta_3^2)(A_3^2 + B_3^2) \\
Q_1 &= (F_1H_3 - A_3G_1J_3)/M_3 \\
Q_2 &= (G_1H_3 + F_1A_3J_3) \\[1ex]
\alpha_3 &= a_3 + \epsilon \\
F_3 &= \alpha_3^2 + \epsilon^2 - b_3^2 \\
G_3 &= 2\alpha_3 b_3 \\
A_1 &= a_1 - a_3 \\
B_1 &= b_1 + b_3 \\
\beta_1 &= b_3 - b_1 \\
H_1 &= A_1^2 - B_1\beta_1 \\
J_1 &= B_1 + \beta_1 \\
M_1 &= (A_1^2 + \beta_1^2)(A_1^2 + B_1^2) \\
Q_3 &= (F_3H_1 - A_1G_3J_1)/M_1 \qquad Q_4 = (G_3H_1 + A_1F_3J_1)
\end{aligned}
$$

Table B4
Coefficients for Intermediate- and
Low-Frequency Solutions

$$
\begin{aligned}
s_1 &= a_1 \\
s_2 &= a_2 \\
s_3 &= a_3 + ib_3 \\
s_4 &= a_3 - ib_3 \\[1ex]
H_{12} &= A_1A_2 - b_3^2 \\
A_{12} &= A_1 + A_2 \\
q &= (F_3H_{12} - G_3b_3A_{12})/(A_1^2 + b_3^2)(A_2^2 + b_3^2) \\
h &= (G_3H_{12} + b_3A_{12}F_3)/(A_1^2 + b_3^2)(A_2^2 + b_3^2)
\end{aligned}
$$

(For definitions of F_3, G_3, A_1, A_2, etc., see Table B3.)

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