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On Periodic Left Factors of Meromorphic Functions

FRED GROSS

Mathematics Research Center
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Sincerely yours,

Paul B. Richards

PAUL B. RICHARDS
Superintendent
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INTRODUCTION

A function $F(z)$, meromorphic in the plane and periodic, may or may not be expressible as

$$F = f(g), \tag{1}$$

where f is meromorphic and g is entire, nonlinear, and not periodic. For example, if $f(u) = \cos \sqrt{u}$ and $g(z) = z^2$ or $f(u) = e^u$ and $g(z) = h(z) + z$, then $F(z)$ will be periodic, where h is entire and $h(z + 2\pi i) = h(z)$. General factorizations (1) of F are investigated in Refs. [1-10]. When F has the factorization (1), f is called a left factor of F (see [2] for definitions).

In the present paper we shall be concerned with the problem of determining whether certain classes of meromorphic functions F have or do not have any left periodic (or more generally periodic modulo a function g) factors f. In particular we shall generalize certain results already proved in [1].

PRELIMINARIES

In this section we state some definitions and lemmas that are needed in the sequel.

Our definition of periodic modulo g is motivated by an earlier definition of Whittaker ([11], p. 84).

Definition 1 (Whittaker). A meromorphic function F is said to be asymptotic periodic with asymptotic period β if and only if $\rho(F(z+\beta) - F(z)) < \rho(F(z))$. Here and in the sequel $\rho(f)$ denotes the order of f.

Definition 2. A meromorphic function F is said to be periodic modulo a meromorphic function g with period β if and only if $F(z + \beta) - F(z) = g(z)$.

LEMMA 1 [12]. *Let $n(r)$ denote the number of zeros of a nonconstant entire function f. The lower order $\lambda(f)$ of f satisfies*

$$\lambda(f) > \underset{r \to \infty}{\underline{\lim}} \frac{\log n(r)}{\log r}. \tag{2}$$

Note: Dr. Gross holds joint appointments at the NRL Mathematics Research Center and the University of Maryland.

NRL Problem B01-11; Project RR 003-02-41-6153. This is an interim report on the problem; work is continuing. Manuscript submitted July 16, 1969.

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Lemma 1 [12]. *Let $n(r)$ denote the number of zeros of a nonconstant entire function f. The lower order $\lambda(f)$ of f satisfies*

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When $z = -(1/2)\ (n\theta + k/n) = z_{kn}$ (say), the two arguments on the left side of (6) are equal, and consequently each point z_{kn} is either a pole of $f(z^2)$ or a zero of $g^*(z+n\theta) - g^*(z)$.

Differentiating (6), we get for all integers n_0 and k_0

$$2(z - n_0\theta)f'((z - n_0\theta)^2 + k_0\theta) - 2zf'(z^2) = g^{*\prime}(z - n_0\theta) - g^{*\prime}(z), \tag{7}$$

where $g^{*\prime}$ is the derivative of g^*. For any fixed integer c, let us consider the points z_{kn} of the form $-(1/2)\ (2n\theta + c)\ (n=0,1,2,\ldots)$. We shall show with the aid of (7) that there does not exist an infinite sequence of n for which these points are poles of $f(z^2)$. Suppose, on the contrary, that for an infinite sequence of integers $n_i\ (i=1,2,\ldots)$ approaching infinity,

$$\xi_i^2 = \left(\frac{1}{2}\ (2n_i\theta + c)\right)^2$$

are poles of $f'(z)$. To arrive at a contradiction, it suffices to prove that the set S of z such that $(z - n_0\theta)^2 + k_0\theta$ is one of the points ξ_i^2 for appropriate integers n_0 and k_0, has a finite limit point. For (7) implies that the points of S must be poles of $f'(z^2)$ and this, of course, is impossible.

Let us set

$$(z - n_0\theta)^2 + k_0\theta = \left(\frac{1}{2}\ (2n_i\theta + c)\right)^2.$$

Solving for z, we get

$$z = n_0\theta \pm \left(\left(\frac{1}{2}\ (2n_i\theta + c)\right)^2 - k_0\theta\right)^{1/2}.$$

We consider in particular the roots

$$z = n_0\theta - \left(\left(n_i\theta + \frac{c}{2}\right)^2 - k_0\theta\right)^{1/2} = n_0\theta - \left(n_i^2\theta^2 + n_i\theta c + \frac{c^2}{4} - k_0\theta\right)^{1/2}.$$

When $k_0 = cn_o = cn_i$, we obtain

$$z = n_i\theta - \left(n_i^2\theta^2 + \frac{c^2}{4}\right)^{1/2}$$

and these values clearly approach zero as i approaches infinity.

Thus, for any fixed integer c, we may assume that $g^*(z + n\theta) - g^*(z)$ vanishes at $z = -n\theta - (c/2)$ for sufficiently large n. That is

$$g^*\left(-\frac{c}{2}\right) = g^*\left(-n\theta - \frac{c}{2}\right) \tag{8}$$

for sufficiently large n. Equation (8), however, implies that the function

$$G(z) = g^*\left(z - \frac{c}{2}\right) - g^*\left(-\frac{c}{2}\right)$$

has the zeros $-n\theta$ for sufficiently large n. Thus, by Lemma 1 either $G(z)$ is zero or $\rho(G) > \lambda(G) \geqslant 1$. Since the latter contradicts our hypotheses, $G(z) \equiv 0$, and $g^*(z)$ must be constant.

It follows that $g(z)$ in (3) is zero and that $f(z)$ and $f(z^2)$ are both periodic. This, however, is impossible as one can easily verify by an argument almost identical to the one used in [1] to prove Theorem 7. Thus, Theorem 1 follows.

More generally we prove

THEOREM 2. *Let f be meromorphic and periodic modulo an entire function g. If $\rho(f) \leqslant 1/2$, $\rho(g) < 1/2$, and p is a polynomial of degree 2, then $f(p)$ cannot be periodic modulo any entire function of order less than 1.*

Proof. Assume that the theorem is false. Again we may replace $p(z)$ by z^2 and assume that the periods of $f(z)$ and $f(p(z))$ are the same. Thus, for some $\theta \neq 0$ we may write

$$f(z+\theta) = f(z) + g(z), \qquad g \text{ entire and } \rho(g) < 1/2$$

$$f((z+\theta)^2) = f(z^2) + h(z), \qquad h \text{ entire and } \rho(h) < 1.$$

By Lemma 3 we may write

$$f(z) = H_1(z) + g^*(z), \qquad g^* \text{ entire and } \rho(g^*) = \rho(g) \tag{9}$$

$$f(z^2) = H_2(z) + h^*(z), \qquad h^* \text{ entire and } \rho(h^*) = \rho(h), \tag{10}$$

where $H_i(z)$ are meromorphic and periodic with period θ for $i = 1, 2$.

From (9) and (10) we obtain

$$H_1(z^2) = H_2(z) + h^*(z) - g^*(z^2).$$

Hence, $H_1(z)$ is periodic and $H_1(z^2)$ is periodic modulo $h^*(z) - g^*(z^2)$ which is of order less than 1. Our conclusion now follows from Theorem 1.

It is natural to ask if Theorem 2 can be generalized to polynomials of degree greater than 2. This question seems to be considerably more difficult. It is known, however, that

I. For any polynomial $p(z)$ of degree greater than 2 and any entire function f, $f(p)$ is not periodic modulo a polynomial (see [9]).

It is also known that

II. For any entire f, $f(z^n)$ for $n > 2$ is not periodic mod any entire g with $\rho(g) < \rho(f)$ (see [15]).

We now ask more generally what can be said about left factors of meromorphic functions periodic modulo entire functions, when no assumptions are made about the right factors. This question is partially answered by Theorems 4 and 5 below. We first prove the following preliminary result.

THEOREM 3. *Let f be meromorphic and let p be a polynomial of degree k, then*

$$\rho(f(p)) = k\rho(f).$$

Proof. By Lemma 4

$$\int^{\infty} \frac{T(r,f)}{r^{\lambda+1}}\, dr$$

converges for all $\lambda > \rho(f)$ and diverges for all $\lambda < \rho(f)$. By Lemma 5, for all a with at most two exceptions, the order of convergence of f-a is $\rho(f)$. It is easy to verify for each of these a's that the zeros of $f(p(z)) - a$ have order of convergence $k\rho(f)$. Thus, $\rho(f(p)) \geqslant k\rho(f)$. If inequality actually holds, then for some λ with

$$\rho(f(p)) > \lambda > k\rho(f),$$

$$\int^{\infty} \frac{T(r,f(p))}{r^{\lambda+1}}\, dr \tag{11}$$

must diverge. Since, however

$$\sum_i (r_i(a))^{-\lambda}$$

converges for three distinct a's (where $r_i(a)$ are the moduli of the a-points of $f(p)$), it follows that (11) must also converge and we get a contradiction. The proof is thus complete.

With the aid of Theorem 3 we can prove

THEOREM 4. *Let f be a nonconstant meromorphic function which is periodic modulo g_1 with period τ_1 and periodic modulo g_2 with period τ_2 (imaginary part of $\tau_1/\tau_2 > 0$), where g_1 and g_2 are entire. (Such a function will be called double mod periodic.) If $\rho(g_1) < 1$ and $\rho(g_2) < 1$, then f cannot have a periodic left factor.*

Proof. We may write

$$f(z+\tau_1) = f(z) + g_1(z)$$

$$f(z+\tau_2) = f(z) + g_2(z).$$

By Lemma 3, one can find entire $g_i^*(z)$ with $\rho(g_i^*) = \rho(g_i)$ and meromorphic $H_i(z)$, periodic with period τ_i $(i = 1,2)$ such that

$$f(z) = H_i(z) + g_i^*(z). \tag{12}$$

Hence,

$$H_1(z) = H_2(z) + g_2^*(z) - g_1^*(z),$$

or

$$H_1(z) = H_2(z) + g^*(z), \tag{13}$$

where $g^*(z) = g_2^*(z) - g_1^*(z)$. Clearly $\rho(g^*) < 1$.

From (13), we have

$$H_1(z+\tau_2) - H_1(z) = g^*(z+\tau_2) - g^*(z).$$

Also for any integer m

$$H_1(z+m\tau_1+\tau_2) - H_1(z+m\tau_1) = g^*(z+m\tau_1+\tau_2) - g^*(z+m\tau_1).$$

Thus, from the periodicity of H_1

$$g^*(z + m\tau_1 + \tau_2) - g^*(z + m\tau_1) = g^*(z + \tau_2) - g^*(z).$$

Letting $z = 0$, we have

$$g^*(m\tau_1 + \tau_2) - g^*(m\tau_1) = g^*(\tau_2) - g^*(0).$$

Hence,

$$g^*(w + \tau_2) - g^*(w) - \text{constant}$$

has the zeros $m\tau_1$, so that by Lemma 1, $g^*(w + \tau_2) - g^*(w)$ must be at least or order 1 unless it is a constant. Since the former is contrary to our hypotheses, it follows that $g^*(w+\tau_2) - g^*(w)$ is a constant or $(g^{*\prime}(w))$ is periodic. Since $\rho(g^{*\prime}) = \rho(g^*) < 1$, $g^{*\prime}$ must be a constant and g^* must be linear. Say

$$g^*(z) = Az + B, \tag{14}$$

where A and B are constants. Equations (13) and (14) yield

$$H_1(z) = H_2(z) + Az + B. \tag{15}$$

It follows from (15) that unless $H_1(z)$ is linear, $H_1'(z)$ is an elliptic function and $\rho(H_1') = 2$. Hence, by (12), f is either of order 2 or of order less than 1.

Suppose that $f(z) = h(g(z))$, where h is meromorphic and periodic and where g is entire. By Lemma 2, g must be a polynomial since $\rho(h) \geqslant 1$ (h being periodic). By Theorem 3, p can be at most of degree 2, but by Theorem 1, this is impossible. This concludes the proof.

COROLLARY: *A nonconstant elliptic function cannot have a periodic left factor.*

It is worth noting that the intermediate result (15) established in the proof of the preceding theorem is best possible. For the Weierstrass zeta function, ζ, is not elliptic and for some w_1, w_2 with imaginary part of $w_1/w_2 > 0$ it satisfies

$$\zeta(z + 2w_1) = \zeta(z) + 2\eta_1$$

$$\zeta(z + 2w_2) = \zeta(z) + 2\eta_2,$$

where η_1 and η_2 satisfy

$$\eta_1 w_2 - \eta_2 w_1 = \frac{1}{2}\tau i,$$

and in general $\eta_1 \neq \eta_2$.

One can write

$$\zeta(z) = H_i(z) + \frac{\eta_i}{w_i} z,$$

where $H_i(z + 2w_i) = H_i(z) \quad (i = 1,2)$. Thus,

$$H_1(z) = H_2(z) + Az,$$

where

$$A = \left(\frac{\eta_2}{w_2} - \frac{\eta_1}{w_1}\right) \neq 0.$$

Theorem 4 can also be proved somewhat more directly by the method used to prove Theorem 9 in [3].

As our final result we prove the following generalization of Theorem 4.

THEOREM 5. *Let f be as in the previous theorem. Then f cannot have a left factor which is periodic modulo an entire function of order* $< 1/2$.

Proof. Assume that $f(z) = h(g(z))$, where h is meromorphic and periodic modulo an entire function $t(z)$ with $\rho(t) < 1/2$. Again, as in the proof of Theorem 4, it is clear that g must be a polynomial of degree at most 2. As usual, we may write

$$h(z) = H(z) + t^*(z), \tag{16}$$

where $H(z)$ is periodic and $\rho(t^*) = \rho(t)$.

From (16), we have

$$h(g) = H(g) + t^*(g).$$

Since $\rho(t^*(g)) < 1$ and $h(g)$ is doubly mod periodic (mod functions of order less than 1), it follows that $H(g)$ has the same property. We can now apply Theorem 4 to arrive at the desired conclusion.

As an immediate corollary we have an extension of II mentioned earlier.

COROLLARY. *If h is meromorphic and periodic modulo an entire function of order less than 1/2, then for* $k > 2$, $h(z^k)$ *cannot be periodic modulo any function of order less than 1.*

Proof. Suppose that $h(z^k)$ is periodic mod (t) with period τ, where t is entire and $\rho(t) < 1$. Then we have for any k-th root of unity, ξ, and some entire t^* with $\rho(t^*) < 1$

$$h((\xi z + \xi\tau)^k) = h((\xi z)^k) + t^*(\xi z).$$

Setting $z' = \xi z$, we get

$$h((z' + \xi\tau)^k) = h((z')^k) + t^*(z').$$

Thus, $f(z)$ must be doubly mod periodic and our assertion follows from Theorem 5.

It would be interesting to know whether any of the preceding theorems can be extended to meromorphic functions which are periodic modulo meromorphic functions of small growth.

REFERENCES

1. Gross, F., "On the Periodicity of Compositions of Entire Functions," *Can. J. Math.* **18**:724 (1966)
2. Gross, F., "On Factorization of Meromorphic Functions," *Transactions of the Amer. Math. Soc.* **131**:215 (1968)
3. Gross, F., "On Factorization of Elliptic Functions," *Can. J. Math.* **20**:486 (1968)
4. Gross, F., "On the Periodicity of $f(g)$," *American Math Monthly* **75**:279 (1968)
5. Gross, F., "On the Periodicity of Compositions of Entire Functions II," *Can. J. Math.* (1968)
6. Gross, F., "Some Theorems on Factorization of Meromorphic Functions," *Bull. of Amer. Math. Soc.* **74**:649 (1968)
7. Baker, I.N., "On Some Results of A. Rényi and C. Rényi Concerning Periodic Entire Functions," *Acta Sci. Math.* (Szeged) **27**:197 (1966)

8. Baker, I.N., and Gross, F., "On Factorizing Entire Functions," *Proc. Lond. Math. Soc.* **18**:69 (1968)
9. Baker, I.N., and Gross, F., "Further Results on Factorization of Meromorphic Functions," *The Proc. of Symposia in Pure Math.* **11**:30, American Mathematical Society, 1968
10. Rényi, A., and Tényi, C., "Some Remarks on Periodic Entire Functions," *J. Analyse. Math.* **14**:303 (1965)
11. Whittaker, J.M., "Interpolatory Function Theory," *Cambridge Tracks in Math. and Math. Phys.*, No. 33, Cambridge:Cambridge University Press 1935
12. Boas, R.P., "Entire Functions," New York:Academic Press, p. 17, 1954
13. Edrei, A., and Fuchs, W.H.J., "On the Zeros of $f(g(z))$ Where f and g are Entire Functions," *Jour. Analyse Math.* **12**:243 (1964)
14. Nevanlinna, R., "Le Théorème de Picard-Borel et la Théorie des Fonctions Méromorphes," Paris:Gauthier-Villars pp. 31-36, 1929
15. Whittaker, J.M., "On the Asymptotic Periods of Integral Functions," *Proc. Edinburgh Math. Soc.* **3**:241 (1933)

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