

# Numerical Calculation of Dispersion Relations for Internal Gravity Waves

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September 7, 1971



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## ABSTRACT

Numerical procedures have been developed to calculate the frequency wave number dispersion relations for internal gravity waves in arbitrary density gradients. The basic properties of the differential equation governing these waves have been considered in developing these procedures. These procedures have been used to develop a set of Fortran subroutines, capable of computing, printing in tabular form, and producing line printer plots of the dispersion relations. Source listings of these subroutines and full descriptions of their use are given. To illustrate the accuracy of the system and to display the results of violating certain restrictions involved in the numerical procedure, an extensive comparison is made between numerical results and an analytic solution for the dispersion relations. The analytical model corresponds to a hypothetical fluid system which closely approximates the average water properties measured during field tests in Bute Inlet, British Columbia, a fjord-type estuary with a density structure characterized by a strong gradient between two nearly homogeneous layers.

## PROBLEM STATUS

This is an interim report work on this problem is continuing.

## AUTHORIZATION

NRL Problem G01-06  
Project A37-370/F08-125-703

Manuscript submitted April 19, 1971.

# NUMERICAL CALCULATION OF DISPERSION RELATIONS FOR INTERNAL GRAVITY WAVES

UNCLASSIFIED

## INTRODUCTION

A numerical method has been developed for determining the relationships between frequency and wave number (dispersion relations) which must be satisfied for the free propagation of small-amplitude internal gravity waves. These internal waves, which rely on an increase of the density in a body of fluid with depth are a common feature of the oceans. They are characterized by maximum amplitudes in the interior of the fluid and produce little vertical displacements of the free surface. As a result, unlike the traditional surface water waves, internal waves are not directly observable. The presence of such waves, however, may be indicated by the organization of contaminant surface slicks into regular patterns (Ewing, 1950), by changes of the surface reflectance properties due to small slope changes in the ambient surface wave field (LaFond and Cox, 1962), or by other effects associated with the horizontal motions at the water surface resulting from an internal wave field (Ekman, 1906).

The dispersion relations represent conditions which must be imposed on free-wave solutions of the hydrodynamic equations of motion under certain general restrictions. They are required in any simple mathematical treatment of the waves produced by a localized disturbance in the fluid. Such treatments essentially describe how a complex waveform may be developed. These waveforms might be generated by the passage of a ship or by an explosion and are expressible mathematically as an infinite sum of basic free waves. A determination of the dispersion relations for internal waves in any given environment is thus a desirable computational step in the mathematical description of any wave field which might be expected in that environment. The purpose of this report is to describe one particular method for making that step.

## FORMULATION OF THE PROBLEM

The basic problem considered was finding solutions to a differential equation which satisfy certain boundary conditions. The equation and boundary conditions must describe conditions in the real world reasonably accurate in most situations of interest. A basic governing equation for two-dimensional, small-amplitude harmonic waves within a horizontally homogeneous, inviscid, incompressible fluid in which there are no mean velocity distributions may be arrived at through various cross differentiations and combinations of the basic equations of fluid motion (see, for example, Phillips, 1966, pp. 161-162). One such equation, for a nonrotating fluid in which the Boussinesq approximation\* is valid, is

$$\frac{d^2w}{dz^2} + \frac{\kappa^2}{\sigma^2} (N^2 - \sigma^2) w = 0 , \quad (1)$$

---

\* In the Boussinesq approximation, the density is considered as constant in the linearized momentum balance equations, except when it is multiplied by the acceleration due to gravity.

where the vertical velocity of the fluid is the real part of  $W(x, z, t) = w(z) \exp i(\kappa x - \sigma t)$ , the  $z$  axis being taken as vertically upward. The parameters  $\kappa$  and  $\sigma$  are the wave number and frequency, respectively, of the wave motion and are equal to  $2\pi$  times the inverses of the wavelength and the period of the motion. The speed of propagation of the waveform, or the phase speed  $c$ , is given by  $c = \sigma/\kappa$ . The function  $N(z) = \sqrt{(-g/\rho)(d\rho/dz)}$  is commonly called the Brunt-Väisälä frequency and is a measure of the stability of the fluid system,  $\rho(z)$  being the mean (undisturbed) density profile. From the equation of continuity for an incompressible fluid in two dimensions, the horizontal fluid velocity is given by the real part of  $U(x, z, t) = u(z) \exp i(\kappa x - \sigma t)$ , where  $u(z) = (i/\kappa)(dw/dz)$ . The vertical displacement  $\eta$  of a fluid parcel from its equilibrium position, which satisfies  $W = \partial\eta/\partial t$ , is given by  $\eta(x, z, t) = [w(z)/\sigma] \exp i[\kappa x - \sigma t + (1/2)\pi]$ .

The application of Eq. (1) to naturally occurring fluid systems such as the ocean necessarily imposes certain restrictions on the scales of motion over which the model may be expected to yield valid results. The assumptions of no-mean-velocity distributions and horizontal homogeneity imply that, in the undisturbed fluid system, the vertical velocity and vertical variations in the horizontal velocity are negligible when compared with  $w(z)$ , and that significant variations in the undisturbed density and horizontal velocity fields (and the fluid depth) may occur over only horizontal distances which are large compared with the wavelength of the disturbing waves. Two dimensionality is a convenient restriction which implies that the coordinate system is set up with the  $x$  axis in the direction of wave propagation so that wave properties don't vary in the  $y$  direction. If there is a nonsheared, basic horizontal current, the coordinate system must move with the component of flow in the direction of wave propagation.

The linearity of Eq. (1) is a result of the small-amplitude assumption and requires that terms like  $w(z)/c$  be small compared with unity. Compressibility effects on gravity waves may be neglected if  $g/N$  is very much smaller than the speed of sound in the fluid, and the neglect of viscosity required that a Reynolds number  $R = w/\kappa\nu$ , where  $\nu$  is some appropriate kinematic coefficient of viscosity, be large compared with one. If the frequency of the wave motion is large compared with the local inertial frequency ( $1.46 \times 10^{-4}$  sec  $\times$  the sine of the latitude), then neglecting the earth's rotation is justified. Also, if the ratio of the wavelength to the earth's circumference is small compared with unity, the earth may be considered as locally flat and the cartesian coordinates are appropriate. The Boussinesq approximation is quite valid when dealing with internal wave motions of this type in the oceans or other similar bodies of water (see Phillips, 1966, pp. 14 ff).

Although numerous, the restrictions listed above do not rule out the scales of motion which are of the greatest interest. When these restrictions are applied, the full nonlinear Navier-Stokes equations are condensed to a linear second-order ordinary differential equation involving two parameters (any combination from  $\sigma$ ,  $\kappa$ , and  $c$ ). The problem is determining the parameter values that solve Eq. (1) by satisfying the appropriate boundary conditions. The correct upper boundary condition is continuous pressure across the free surface. However, as described by Phillips (1966, pp. 164-165), this dynamic condition may be replaced with a condition appropriate for a rigid surface, that is, one for which the vertical velocity vanishes at  $z = 0$ . The condition that  $w(-H) = 0$ , where  $z = -H$  represents the rigid bottom, completes the specification of the boundary value problem.

## GENERAL PROPERTIES OF THE SOLUTION

The type of boundary value problem described above is fairly common in mathematical physics. It represents a Sturm-Liouville system in the parameter  $1/c^2 = \kappa^2/\sigma^2$  for any particular value of  $\kappa^2$ , provided that  $N^2(z) \geq 0$  everywhere in the fluid. Unfortunately, a general analytic solution of this type of problem is unknown. Certain general

properties of the solutions, however, which are of interest in the numerical calculation of the dispersion relations, are known. These general properties, listed below, are used to determine testing criteria and ranges of values for solving for the dispersion relations.

A. For a given value of  $\kappa^2 > 0$ , an infinite number of values of

$$1/c^2 (0 < 1/c_1^2 < 1/c_2^2 \dots)$$

and of positive values of  $\sigma^2 (\sigma_1^2 > \sigma_2^2 > \sigma_3^2 \dots)$  exist for which real solutions  $w_1(z)$ ,  $w_2(z)$ ,  $w_3(z)$ , ..., which satisfy the boundary conditions, exist. These solutions have the property that the function  $w_n(z)$  has  $n - 1$  zeros on the open interval  $0 < z < -H$  (Ince, 1956, section 10.61). The index  $n$  will be called the mode number of the solution, so that any particular modal solution or mode  $w_n(z)$  will be a function which vanishes at  $z = 0$ ,  $-H$ , and at  $n - 1$  intermediate depths.

B. The maximum value of  $N^2(z)$  is an upper bound on  $\sigma^2$  (Groen, 1948).

C. For any particular modal solution,  $\sigma$  is a monotonically increasing function of  $\kappa$  (Groen, 1948).

Normally, the distribution of the Brunt-Väisälä frequency will be recorded digitally, that is, as a series of values of  $N^2$  corresponding to a series of depths. In this case, the most straightforward method of solving Eq. (1) for  $\sigma$  when  $\kappa$  is given is to transform the differential equation (1) into an equation in finite differences of serial values of the vertical velocity function, together with the serial values of the Brunt-Väisälä frequency.\* The finite difference formulation may be considered as defining recursion relations among the serial values of  $w$ , from which the final values of  $w$  at the boundary  $z = -H$  may be determined for any particular wave number and frequency, if the initial value of  $w$  at the boundary  $z = 0$  and the serial values of  $N$  are given. For any given wave number, this system represents the final values of  $w$  as a function of frequency. The remaining problem is to determine the "zeros" of this function, that is, those values of  $\sigma$  for which a solution satisfying both boundary conditions exists for a given  $\kappa$ . If the finite difference formulation is assumed to represent a good enough approximation to the original differential equation so that the general properties of its solutions listed above remain valid, these properties may be used to determine an effective method of finding the "zeros." Determining these "zeros" for a range of wave number values will permit the construction graph of  $\sigma$  as a function of  $\kappa$  for each of the modes to be considered, that is, the construction of the dispersion curves.

## NUMERICAL METHODS

### Finite Difference Formulation

To represent the differential Eq. (1) as an equation in finite differences, some approximate form of  $d^2w/dz^2$  in terms of serial values of  $w$  must be determined. One particularly simple approximation is to assume a quadratic form for  $w(z)$  in each depth interval spanning two increments of depth. If  $w = a_0 + a_1z + a_2z^2$ , then  $w'' = 2a_2$ , where primes indicate derivatives with respect to  $z$ , so that we need only determine  $a_2$  to obtain an approximate value for the second derivative of  $w$ . Writing

\*Since  $\sigma$ ,  $\kappa$ , and  $N$  all appear in the governing equation as squared quantities, all subsequent references to these quantities will be to their positive square roots for convenience; thus, we will consider only waves which propagate in the positive  $x$  direction period. Dispersion relations corresponding to waves propagating in the negative  $x$  direction are obtained by changing the sign of  $\sigma$ .

$$w_{i-1} = a_{i0} + a_{i1}z_{i-1} + a_{i2}z_{i-1}^2 ,$$

$$w_i = a_{i0} + a_{i1}z_i + a_{i2}z_i^2 ,$$

and

$$w_{i+1} = a_{i0} + a_{i1}z_{i+1} + a_{i2}z_{i+1}^2 ,$$

we have a set of three linear equations in the three unknown coefficients of the assumed quadratic form for  $w$  in the interval  $z_{i-1} \geq z_i \geq z_{i+1}$ . Since the  $z$ 's are all different, and since  $z^0, z^1, z^2$  are linearly independent, the determinant of the efficient matrix is nonzero, and the coefficients can be determined by Cramer's Rule.

In matrix notation, we have

$$\begin{bmatrix} 1 & z_{i-1} & z_{i-1}^2 \\ 1 & z_i & z_i^2 \\ 1 & z_{i+1} & z_{i+1}^2 \end{bmatrix} \begin{bmatrix} a_{i0} \\ a_{i1} \\ a_{i2} \end{bmatrix} = \begin{bmatrix} w_{i-1} \\ w_i \\ w_{i+1} \end{bmatrix} .$$

Defining

$$\begin{aligned} \det X_i &= (z_{i+1}^2 z_i - z_i^2 z_{i+1}) - (z_{i-1} z_{i+1}^2 - z_{i-1}^2 z_{i+1}) + (z_{i-1} z_i^2 - z_{i-1}^2 z_i) \\ &= z_{i-1}^2 (z_{i+1} - z_i) + z_i^2 (z_{i-1} - z_{i+1}) + z_{i+1}^2 (z_i - z_{i-1}) , \end{aligned}$$

we have, by Cramer's Rule,

$$\begin{aligned} a_{i2} &= \frac{1}{\det X_i} \begin{vmatrix} 1 & z_{i-1} & w_{i-1} \\ 1 & z_i & w_i \\ 1 & z_{i+1} & w_{i+1} \end{vmatrix} \\ &= \frac{1}{\det X_i} \left[ w_{i-1}(z_{i+1} - z_i) - w_i(z_{i+1} - z_{i-1}) + w_{i+1}(z_i - z_{i-1}) \right] . \end{aligned}$$

So that, at the depth  $z = z_i$ , the differential Eq. (1) is approximated by the finite difference relation

$$\frac{2}{\det X} \left[ w_{i-1}(z_{i+1} - z_i) - w_i(z_{i+1} - z_{i-1}) + w_{i+1}(z_i - z_{i-1}) \right] + \frac{\kappa^2}{\sigma^2} (N_i^2 - \sigma^2) w_i = 0$$

or, defining  $\Delta_i = z_i - z_{i-1}$ ,

$$\Delta_{i+1} w_{i-1} + \left[ \frac{1}{2} \frac{\kappa^2}{\sigma^2} (N_i^2 - \sigma^2) \det X_i - \Delta_{i+1} - \Delta_i \right] w_i + \Delta_i w_{i+1} = 0 .$$

The condition at the boundary  $z_0 = 0$  requires that  $w_0 = 0$ . Since we are dealing with a homogeneous, linear differential equation with homogeneous boundary conditions, the solution  $w(z)$  will be ambiguous to within a multiplicative constant, and we may set  $w_1 = w(z_1) = 1$  without loss of generality. Thus, a recursion relation for the serial values of  $w$  is given by

$$w_{i+1} = (-1/\Delta_i) \left\{ \left[ \frac{1}{2} \frac{\kappa^2}{\sigma^2} (N_i^2 - \sigma^2) \det X_i - \Delta_{i+1} - \Delta_i \right] w_i + \Delta_{i+1} w_{i-1} \right\}, \quad (2)$$

where  $w_0 = 0$  and  $w_1 = 1$ . The basic problem is to determine the value of  $\sigma$  which will make  $w_{n+1} = w(z_{n+1}) = 0$ , where  $z_{n+1}$  represents the depth of the bottom, for any particular value of  $\kappa$ . Note that, in this formulation, the step lengths  $\Delta_i$  need not be equal. Although the use of unequal step lengths introduces some complexity into the finite difference equation over that corresponding to equal depth intervals, unequal steps permit the use of data in which observations of Brunt-Väisälä frequency may be clustered around depths at which  $N^2(z)$  is changing rapidly.

#### Calculation of the Frequency Corresponding to a Particular Modal Solution for a Given Wave Number

By means of the recursion relation (2), successive values of  $w$  may be calculated in series, down to  $z_{n+1} = -H$ , for any given set of  $\sigma$  and  $\kappa$ . If  $w_{n+1} = 0$ , then the set  $(\sigma, \kappa)$  allows for a particular modal solution and represents a point on the dispersion diagram. The mode number of the solution is one more than the number of times which the vertical velocity function  $w(z)$  passes through zero during the calculation. In general, however, an arbitrary choice of a pair  $(\sigma, \kappa)$  will not yield a solution with  $w(-H) = 0$ , so that some orderly method of determining that value of  $\sigma$  which will yield a particular modal solution for a given value of  $\kappa$  is needed.

From the general properties of the solution listed previously, we know, by properties A and B, that for any wave number  $\kappa$  the frequencies which will yield solutions satisfying the boundary conditions must lie in the interval  $0 = \sigma_{\min} < \sigma < \sigma_{\max} = N_{\max}$ , where  $N_{\max}$  is the maximum value of the Brunt-Väisälä frequency in the water column.\* As a first approximation to  $\sigma_m$ , the frequency corresponding to the  $m$ -th mode solution, the midpoint of the interval  $(\sigma_{\min}, \sigma_{\max})$  may be taken. If, during the recursive calculation based on this approximation to  $\sigma_m$ ,  $w(z)$  changes sign more than  $m - 1$  times, then by property A, we know that the approximation to  $\sigma_m$  is too low. The next approximation may be taken as the midpoint of the interval  $(\sigma_{\min}, \sigma_{\max})$ , where  $\sigma_{\min}$  is now the "too-low" approximation. If, on the other hand, the calculated value of  $w_{n+1}$  is nonzero and the number of zeros in  $w(z)$  in the open interval  $0 < z < z_{n+1} = -H$  is less than or equal to  $m - 1$ , the approximation to  $\sigma_m$  is too high, and the next approximation may be taken as the midpoint of the interval  $(\sigma_{\min}, \sigma_{\max})$ , where  $\sigma_{\max}$  is the "too-high" approximation to  $\sigma_m$ .

The procedure of dividing in half the frequency interval on which the proper value of  $\sigma_m$  is known to lie may be continued with successive approximations until an acceptable solution is found. An acceptable solution should have the frequency interval as a small fraction of the value of  $\sigma$  at the interval midpoint if a "true" solution is known to exist, that is, a solution for which  $w_{n+1} = 0$  and  $w(z)$  go through zero  $m - 1$  times, or for which  $|w_{n+1}|$  is small compared to 1.

\*By property A,  $\sigma_{\min}$  or  $\sigma_{\max}$  may be replaced by the frequency corresponding to a solution for the  $i$ -th mode ( $i$  either less than or greater than  $m$ , respectively, where  $m$  is the mode number of the desired solution for the given value of  $\kappa$ ) if it is known.

### Alternative Bottom Boundary Condition of Very Deep Water

In some applications, the significant stability information, that is, large values of the Brunt-Väisälä frequency, may be restricted to a surface layer which is a small fraction of the total depth of the body of water. In such a case, it would be convenient to terminate the calculation of  $w(z)$  at some intermediate depth; therefore, some alternative "pseudo-bottom" boundary condition is needed.

If it is assumed that the water is homogeneous below a depth  $-D$ , and that the total body of water is very deep compared to  $-D$ , then the simple solution of Eq. (1),  $w(z) = A \exp \kappa z$ , for homogeneous water of infinite depth may be compared with the numerical solution at  $z_{n+1} = -D$  in lieu of expending the computation to the actual bottom. In this formulation, the bottom boundary condition of  $w_{n+1} = 0$  would be replaced by the conditions that the numerical solution and its first derivative be equal to the analytic solution for homogeneous water and its first derivatives at the appropriate depth. These conditions reduce simply to the condition that  $w'_{n+1} = \kappa w_{n+1}$ . If the ratio of  $w'_{n+1}$  to  $w_{n+1}$  is less than  $\kappa$ , then the solution is not asymptotically approaching  $w(z) = 0$  fast enough, and, provided  $w(z)$  has passed through 0 exactly  $m - 1$  times, the value of  $\sigma$  used in the computation is too high. On the other hand, if the ratio is greater than  $\kappa$ , then the solution is approaching zero too fast, and  $\sigma$  is too low. In practice, a small range of values centered on  $(w'/w) = \kappa$  should be considered as yielding an acceptable solution.

The simplest approximation to  $w'$  is given by the first difference of two values of  $w$  at consecutive depths; that is,  $(w_{n+1} - w_n)/(z_{n+1} - z_n)$  is the first approximation to  $w'$  at a depth midway between  $z_n$  and  $z_{n+1}$ . The approximate value of  $w(z)$  at this depth is  $1/2 (w_{n+1} + w_n)$ , so that the ratio which should be compared with the wave number is  $2(w_{n+1} - w_n)/[(z_{n+1} - z_n)(w_{n+1} + w_n)]$ .

### Calculation of a Series of Points Along a Dispersion Curve

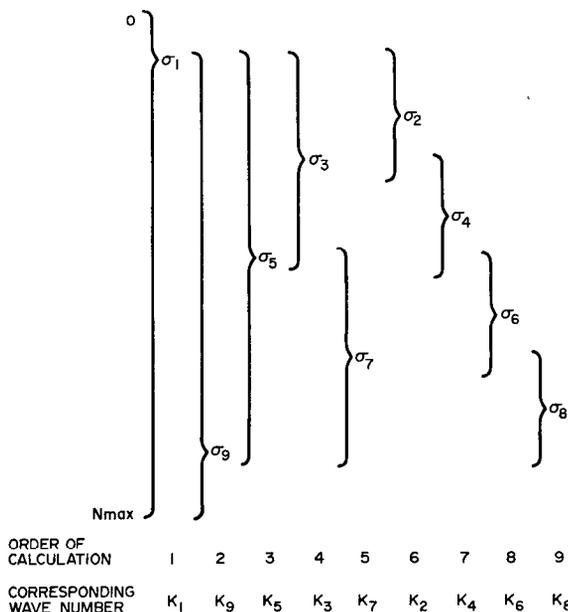
In constructing a dispersion curve, the frequency which allows for a particular modal solution must be calculated for various wave number values. If a large number of points along such a curve is to be calculated, then some systematic method of employing previously calculated information should be used. One simple method for this type of calculation relies on the choice of  $n = 2^p + 1$ , where  $p$  is an integer and  $n$  is the number of values of  $\kappa$  for which frequencies are to be calculated.\* If the series of wave numbers is arranged in ascending order and indexed ( $\kappa_1 < \kappa_2 < \dots < \kappa_n$ ,  $n = 2^p + 1$ ), then property C described previously may be used to calculate the corresponding values of  $\sigma(\sigma_1 < \sigma_2 < \dots < \sigma_n)$  for a particular mode. First,  $\sigma_1$  is calculated using a frequency interval ( $\sigma_{\min}, \sigma_{\max}$ ), as discussed in the previous subsection, where

$$\sigma_{\min} \leq \sigma_1 < \dots < \sigma_n \leq \sigma_{\max} .$$

Then,  $\sigma_n$  is calculated using the frequency interval  $\sigma_1 < \sigma_n \leq \sigma_{\max}$ . The third frequency to be calculated is  $\sigma_{(n+1)/2}$ , using the interval  $(\sigma_1, \sigma_n)$ . The frequencies corresponding to the median values of  $\kappa$  in the two subseries  $\kappa_1, \dots, \kappa_{(n+1)/2}$  and  $\kappa_{(n+1)/2}, \dots, \kappa_n$ , each of which contains one plus a power-of-two values of wave number, are then calculated, using the frequency intervals  $(\sigma_1, \sigma_{(n+1)/2})$  and  $(\sigma_{(n+1)/2}, \sigma_n)$ . This procedure of dividing up the wave number series into equal subseries continues until all  $n = 2^p + 1$  values of the frequency have been calculated. Figure 1 is a schematic diagram which

\*A wave number series containing an arbitrary number of points can always be "factored" into a number of subseries with overlapping endpoints, each of which contains one plus a power-of-two values of  $\kappa$ .

Fig. 1 - Schematic illustration of the order in which to calculate frequencies for a series of nine wave number ( $\kappa_i$ ) values for the case in which there is initially no previously calculated information about the frequencies, that is, it is only known that all frequencies lie between  $\sigma = 0$  and  $\sigma = N_{max}$ , the maximum value of the Brunt-Väisälä frequency. The order of calculation proceeds from left to right, and the brackets indicate the frequency interval  $\sigma_{min} < \sigma_i < \sigma_{max}$  which is to be used in calculating each  $\sigma_i$ .



illustrates the order of calculation of the  $\sigma$  values for a wave number series containing  $2^3 + 1 = 9$  values of  $\kappa$ . Starting with  $n = 2^p + 1$  values of  $\kappa$  insures that the procedure can be continued unaltered until the series of wave numbers has been exhausted.

### FORTRAN SUBROUTINES FOR THE CALCULATION OF DISPERSION RELATIONS

#### Subroutine Descriptions

A set of six Fortran subroutines (DISPER, DIFFER, SERIES, CALSIG, PRIDIS, and PLØDIS) were written for use on the NRL CDC 3800 computer. The subroutine DISPER, which relies on DIFFER, SERIES, and CALSIG for its operation, is based on the techniques discussed in the previous section and is designed to calculate the frequency/wave number relations for a set of mode numbers of free, progressive internal waves for an arbitrary distribution of Brunt-Väisälä frequency. PRIDIS and PLØDIS are "output" subroutines designed to print out or produce a line printer plot, respectively, of the calculated dispersion relations. The source language listings for these subroutines constitute Appendix A of this report.

#### Utilization of Subroutines

##### The Fortran Statement

CALL DISPER (NM, MN, NX, XØ, XF, M, Z, ZN, BB, EPS, RES, NER, L)

causes the array of frequencies  $S(I, J)$ ,  $I = 1, \dots, NM$ ;  $J = 1, \dots, NX$  to be calculated. This array is stored in common with DISPER in the common block SIGMA, which contains 2500 computer words; therefore, a COMMON declaration (COMMON/SIGMA/S(10,250)) is required in the calling program. The parameters involved in the call to DISPER are:

- NM - the number of modes for which the calculations are to be made ( $NM \leq 10$ ).
- MN - an integer array (MN(1), ..., MN(NM)) containing the mode numbers for which calculations are to be made. The mode numbers need not be consecutive but must be listed in ascending order.
- NX - the number of wave number values for which the calculations are to be made ( $1 \leq NX \leq 250$ ).
- XØ - the lowest wave number value for which the calculations are to be made.
- XF - the highest wave number value for which the calculations are to be made.
- M - the number of observations of Brunt-Väisälä frequency squared ( $M \leq 200$ ).
- Z - the array of depths (Z(1), ..., Z(M), positive numbers) to which the observations of Brunt-Väisälä frequency squared correspond. The depths must be listed in increasing order. The array of depth (Z) must be dimensioned to M + 1 in the calling program.
- ZN - the depth (positive number) at which the bottom boundary condition is to be evaluated ( $ZN > Z(M)$ ).
- BB - the array (BB(1), ..., BB(M)) of observations of Brunt-Väisälä frequency-squared corresponding to the depths Z(I).
- EPS - the allowable fractional error in the calculated frequencies.
- RES - the allowable fractional error in evaluating the bottom boundary condition in the calculations. A "valid" solution is one for which either the EPS or RES condition is satisfied.
- NER - an integer array (NER(1), ..., NER(NER(2) + 5)) of error returns. If NER(1)  $\neq$  0, an error in calculation has occurred. See the source listings of DISPER (Appendix A) for an explanation of this array. Dimension the array NER to 30 in the calling routine.
- L - an integer variable which determines the bottom boundary condition.
- If L = 0, ZN is the depth of a rigid bottom.
- If L = 1, ZN is the depth at which the calculated distribution of vertical velocity is to be compared with that for homogeneous water.

#### The Fortran Statement

```
CALL PRIDIS (IDENT, NUNITS, XØ, XF, NX, NM, MN)
```

causes the array of frequencies calculated by DISPER to be printed out in tabular form. The frequency array is stored in the common block SIGMA. The parameters are:

- IDENT - an integer array (IDENT(1), ..., IDENT(10)) containing 10 eight-character alphanumeric words, to be used as an identification field which will be printed above the table.

NUNITS - an integer variable composed of a six-character alphanumeric word, to be used to specify the units of depth (and thus of wave number), for example, FEET^^, CM^^^^, or METERS (^'s indicate blank spaces).

XØ, XF, NX, NM, and MN are the same as in DISPER.

The Fortran Statement

```
CALL PLØDIS (IDENT, NUNITS, XØ, XF, NX, NM, MN)
```

causes the array of frequencies calculated in DISPER (and stored in the common block SIGMA) to be graphed on a line printer plot. The symbols used in the plot are the appropriate mode numbers. If the separations between frequencies for various modes for a certain wave number are smaller than the size of the symbols, the lowest mode takes preference, and plotting of higher modes in the same location is not attempted. Only modes 1 through 9 can be plotted, since only one-character symbols are employed. If mode numbers higher than 9 are inadvertently specified, a comment to that effect is printed and only those mode numbers  $\leq 9$  are plotted. NX must be greater than one; otherwise, no restrictions are made on the number of points to be plotted. If, inadvertently NX = 1, a comment to that effect is printed, and no plot is produced. The parameters are the same as those in PRIDIS.

The number of lines output by this subroutine is determined by NX. If  $NX \leq 50$ , the number of plotted lines is  $\leq 100$ , and for  $NX > 50$ , the number of plotted lines is equal to NX.

A sample calling program which uses the above subroutines is included as Appendix B, along with the sample outputs produced by calls to PRIDIS and PLØDIS.

#### Storage Requirements (Exclusive of Computer System Library Functions)

##### DISPER

unique storage: 1272 octal (695 decimal) locations

common blocks (all real variables):

COMMON/SIGMA/S (10,250) length 4704 octal (2500 decimal) locations

COMMON/BLK/BB (200), Z (201), DEL (201), DET (201), W (202) length 1755 octal (1005 decimal) locations

##### DIFFER

unique storage: 321 octal (153 decimal) locations

##### SERIES

unique storage: 476 octal (318 decimal) locations

common blocks (all real variables):

COMMON/BLK/BB (200), Z (201), DEL (201), DET (201), W (202) length 1755 octal (1005 decimal) locations

##### CALSIG

unique storage: 474 octal (316 decimal) locations

common blocks (all real variables):

COMMON/BLK/BB (200), Z (201), DEL (201), DET (201), W (202) length 1755 octal (1005 decimal) locations

## PRIDIS

unique storage; 404 octal (260 decimal) locations

common blocks (all real variables):

CØMMØN/SIGMA/S (10,250) length 4704 octal (2500 decimal) locations

## PLØDIS

unique storage: 1465 octal (821 decimal) locations

common blocks (all real variables):

CØMMØN/SIGMA/S (10,250) length 4704 octal (2500 decimal) locations

## Timing

The test routine of Appendix B (together with the subroutines) required 42.87 seconds for compilation and assembly. The call to DISPER required 15.69 seconds, and the calls to PRIDIS and PLØDIS required a total of 2.14 seconds. Table 1 gives times for calls to DISPER under various conditions, using the Brunt-Väisälä distribution of Fig. 2.

Table 1  
Times Required for Calls to DISPER for Various Values of the  
Parameters NM, NX, M, EPS and RES\*

NM	NX	M	EPS	RES	Total Time (sec)	Time Per Dispersion Point (millisec)	Time Per Distribution Point Per Observation of B/V Freq. (millisec)
7	26	28	$10^{-6}$	$10^{-6}$	15.69	75.29	2.688
3	10	28	$10^{-3}$	$10^{-3}$	1.144	38.13	1.362
3	30	28	$10^{-3}$	$10^{-3}$	2.932	32.58	1.163
3	10	200	$10^{-3}$	$10^{-3}$	6.308	210.3	1.051
3	30	200	$10^{-3}$	$10^{-3}$	16.025	178.1	0.890

\*Times are based on programs run on the NRL CDC 3800 computer.

#### COMPARISON BETWEEN ANALYTICAL AND NUMERICAL RESULTS; SOURCES OF ERROR

The numerical procedure for calculating dispersion relations described in the preceding sections was compared with analytic solutions of Eq. (1) for the case in which  $N^2(z) = 0.0032825 \exp(-0.341642 z)$  for  $0 \geq z \geq -10$  and  $N^2(z) = 1.6/(z + 6)^2$  for  $-10 \geq z \geq -\infty$  (Fig. 2). There are two reasons for the choice of this particular distribution of Brunt-Väisälä frequency as a test case. The first reason is that it closely approximates the average conditions at a location in Bute Inlet, British Columbia ( $50^\circ 35.5'N$ ,

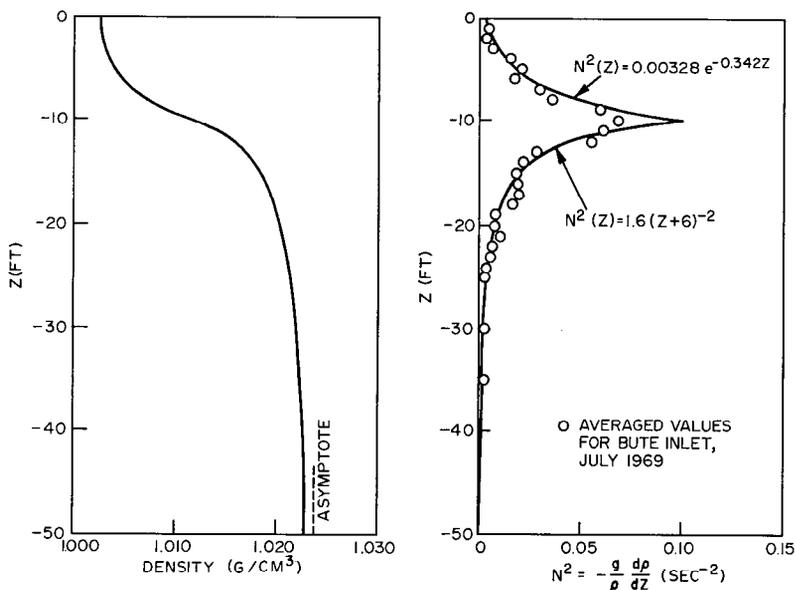


Fig. 2 - Distribution of density and Brunt-Väisälä frequency used in test solution

125°53'W; actual depth, 2000 ft)\* where field exercises were held in the summer of 1969. The numerical procedure was originally developed for making estimates of internal wave properties in these waters. The second reason is that it is an excellent distribution for testing the accuracy of the numerical calculations in that it provides a test for errors introduced into the numerical solution by the use of the pseudo-bottom boundary conditions at low frequencies and by the strong curvature in the distributions of the vertical velocity function with depth at frequencies approaching the maximum Brunt-Väisälä frequency.

Use of the pseudo-bottom boundary condition is strictly justified only when both the square of the ratio of the wave frequency to the Brunt-Väisälä frequency below a certain depth and the product of the wave number and the remaining fluid depth are large compared to unity. If either of these conditions are violated, calculated dispersion points may be expected to deviate from their true values. The wave number restriction is essentially the "deep-water" surface gravity wave assumption and is simply a statement that  $\tanh \kappa h' \approx 1$ , where  $h'$  is the depth of water below the pseudo bottom. A violation of this wave number assumption will, in general, be accompanied by a violation of the frequency assumption, but not vice versa. For this reason, only errors introduced by violations of the frequency assumption are explored in the comparison testing.

The dispersion points calculated numerically and those obtained from the evaluation of analytic solutions of Eq. (1)† are compared in Figs. 3 and 4. Dispersion points for two different numerical solutions are shown. The "coarse-iteration" solution was obtained using values of the Brunt-Väisälä frequency at 1-ft intervals from 0 to 25 ft, and at 30,

\*The particular functional forms for  $N(z)$  represent an optimum choice for "solvability" and reproduction of the form of the observed average distribution. The choice of the numerical values of the parameters in the functions involved optimizing the reproduction of the observed average distribution of Brunt-Väisälä frequency and the total density variation over the depth of the inlet.

†See Appendix C for a development of the analytic solution.

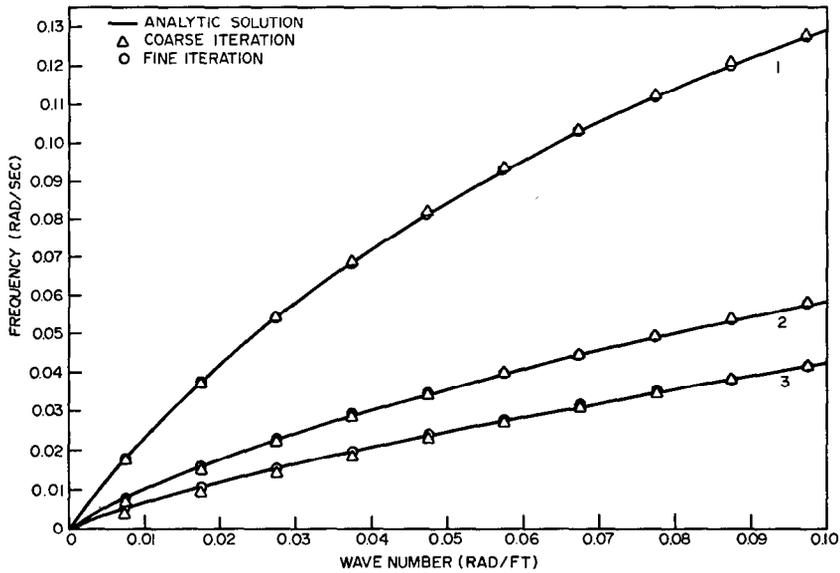


Fig. 3 - Dispersion relations for low wave numbers for the first three internal modes, obtained from analytical and numerical solutions of Eq. (1) for the Brunt-Väisälä frequency distribution of Fig. 2. Points were calculated to a nominal 0.1% accuracy.

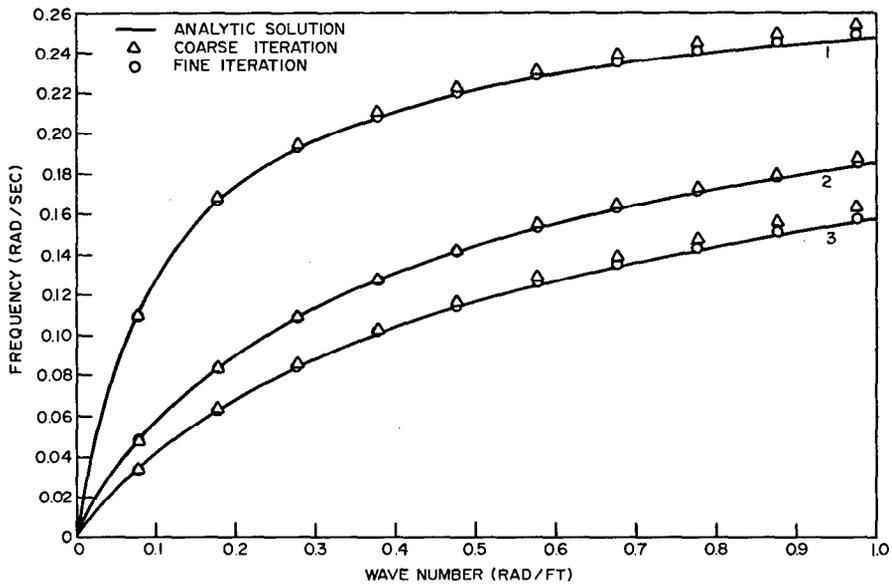


Fig. 4 - Dispersion relations for high wave numbers for the first three internal modes, obtained from analytical and numerical solutions of Eq. (1) for the Brunt-Väisälä frequency distribution of Fig. 2. Points were calculated to a nominal 0.1% accuracy.

35, and 40 ft, with the pseudo-bottom condition imposed at 45 ft. This sequence of depths was used for one of the comparisons because it corresponds to the sequence of depths at which the bulk of observations of water properties were made by Bute Inlet. The "fine-iteration" solution involves values of  $N^2$  at half-foot intervals from 0 to 200 ft, with the "pseudo bottom" at 200.5 ft, representing a finer recursion step interval and a lower Brunt-Väisälä frequency at the pseudo bottom than in the "coarse iteration."

The comparison was made over two ranges of wave numbers (low, 0 to 0.1 rad/ft and high, 0 to 1.0 rad/ft) to illustrate the systematic deviation of the numerical dispersion points from those of the analytic solution for the coarse iteration. At low wave numbers (and hence low frequencies), the numerically determined points fall below the analytic solution curve as a result of the use of the pseudo-bottom boundary condition at 45 feet in violation of its underlying high-frequency assumption. For high wave numbers, the deviation is reversed as a result of the inability of the finite difference formulation for  $w(z)$  to closely approximate the strong curvatures in the vertical velocity function. The result at low wave numbers of extending the numerical solution to a greater depth (200 feet, where  $N^2(200)$  is 3% of  $N^2(40)$ ) and at high wave numbers of employing a finer recursion interval (1/2 ft instead of 1 ft in the area of importance) is illustrated by the close adherence of the fine iteration points to the analytical solution curve. Percent deviations of the numerically calculated frequencies from those obtained from the analytic solution are shown as a function of wave number in Figs. 5 and 6 for the fine-iteration and coarse-iteration solutions, respectively. The effect or misuse of the pseudo-bottom boundary condition is well illustrated, and for purposes of comparison, the wave numbers for which the wave frequency is equal to the Brunt-Väisälä frequency at the last iteration step are marked on the wave number scale for each mode.

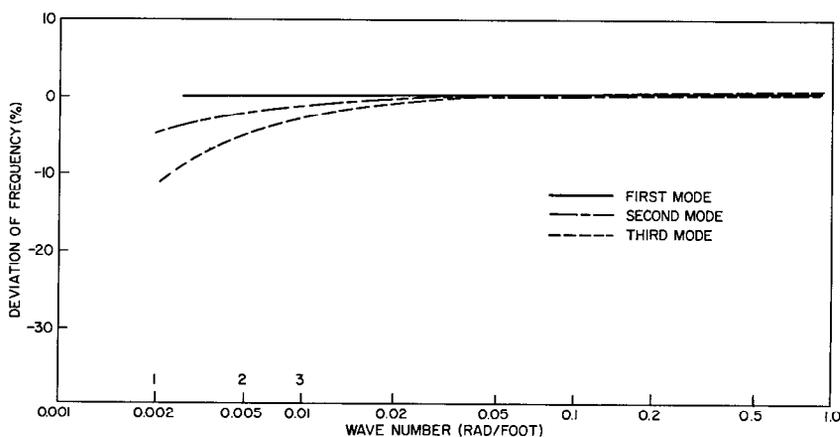


Fig. 5 - Percent deviation of fine-iteration dispersion points from the analytic curves of Figs. 3 and 4. Wave numbers above which the wave frequency is greater than the Brunt-Väisälä frequency at the pseudo bottom are indicated on the wave number scale for three modes.

The reason for the low-frequency deviations associated with the misuse of the pseudo-bottom boundary condition is illustrated in Fig. 7. This figure is a comparison between the analytical vertical velocity function and the function calculated in the coarse-iteration solution at wave numbers for which the wave frequency is less than the Brunt-Väisälä frequency at the last iteration step ( $N(40) = 0.03720$  rad/sec). The pseudo-bottom boundary condition requires that the numerical solution appear to approach the  $w(z) = 0$  axis

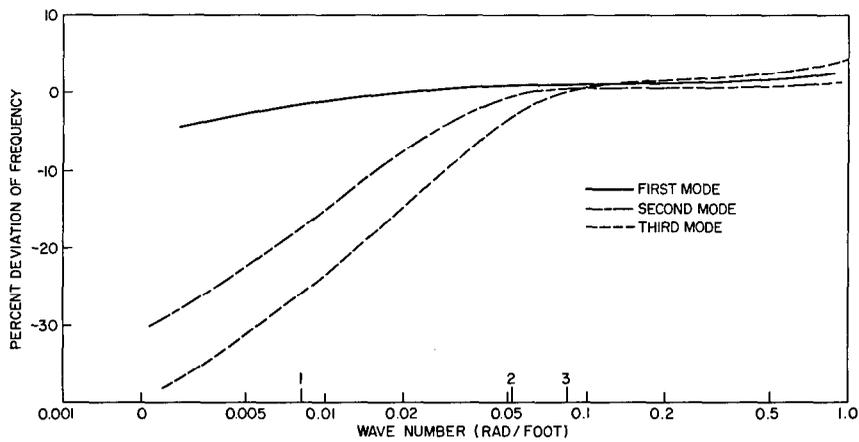


Fig. 6 - Percent deviation of coarse-iteration dispersion points from the analytic curves of Figs. 3 and 4. Wave numbers above which the wave frequency is greater than the Brunt-Väisälä frequency at the pseudo bottom are indicated on the scale for the three modes.

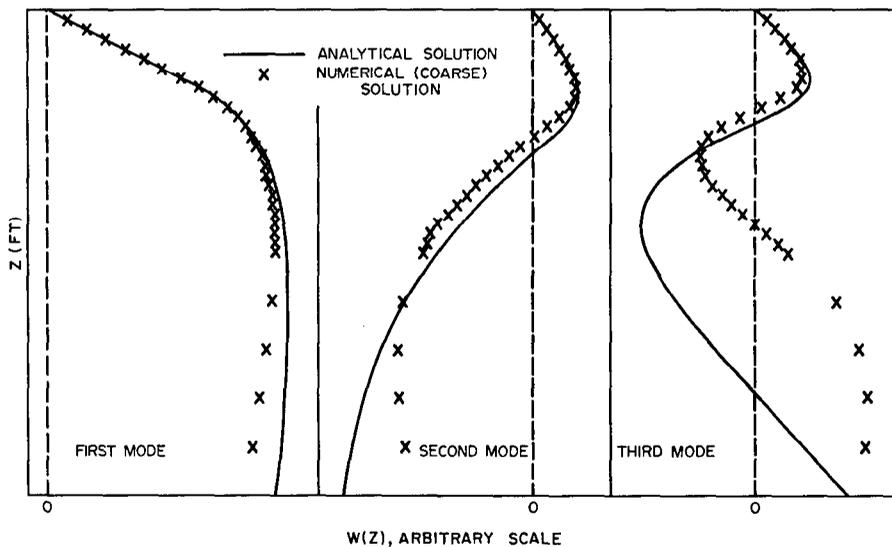


Fig. 7 - Comparison between analytical vertical velocity function and that calculated in coarse iteration solution at low wave numbers. Vertical velocity scale is arbitrary, and solutions are matched at 1 ft. First mode:  $\kappa = 0.007644$  rad/ft;  $\sigma = 0.01796$  rad/sec (analytic),  $0.01761$  rad/sec (numerical). Second mode:  $\kappa = 0.01101$  rad/ft;  $\sigma = 0.01123$  rad/sec (analytic),  $0.00964$  rad/sec (numerical). Third mode:  $\kappa = 0.009897$  rad/ft;  $\sigma = 0.00683$  rad/sec (analytic),  $0.00521$  rad/sec (numerical).

with an exponential decay, which requires that the vertical velocity and its second derivative be of the same sign. As can be seen from Eq. (1), if  $N^2 > \sigma^2$ , then the vertical velocity and its second derivative must be of opposite sign, resulting in a convex curvature rather than the concave curvature required by the pseudo-bottom boundary condition. This curvature results in a numerical solution which approaches the  $w(z) = 0$  axis too rapidly, that is, one which oscillates more rapidly than the true solution, and corresponds to a frequency which is lower than the correct wave frequency. In the extreme, misuse of the pseudo-bottom boundary condition may actually result in confusion between modes. For example, the analytical vertical velocity function for the third mode in Fig. 7 indicates that  $w'/w \approx 0.0099$  at  $z = -23$  ft; that is, a numerical solution involving a pseudo bottom at 23 ft would faithfully reproduce the vertical velocity function for the third mode at a wave number of 0.0099 rad/ft and a frequency of approximately 0.0068 rad/sec but would identify this frequency with the second mode of oscillation.

Comparisons at higher wave numbers for which the wave frequencies are greater than  $N(40)$  are shown in Fig. 8. The first- and second-mode solutions show a good agreement between numerical and analytical results. The third-mode solution, for which the ratio of wave frequency to Brunt-Väisälä frequency at 40 ft is 1.16, shows some of the "too-low" type deviation of Fig. 7. Of course, this "too-low" type low wave number deviation is a result of the misuse of the pseudo-bottom boundary condition and will not be present in situations in which the appropriate bottom boundary condition is that of a rigid plate.

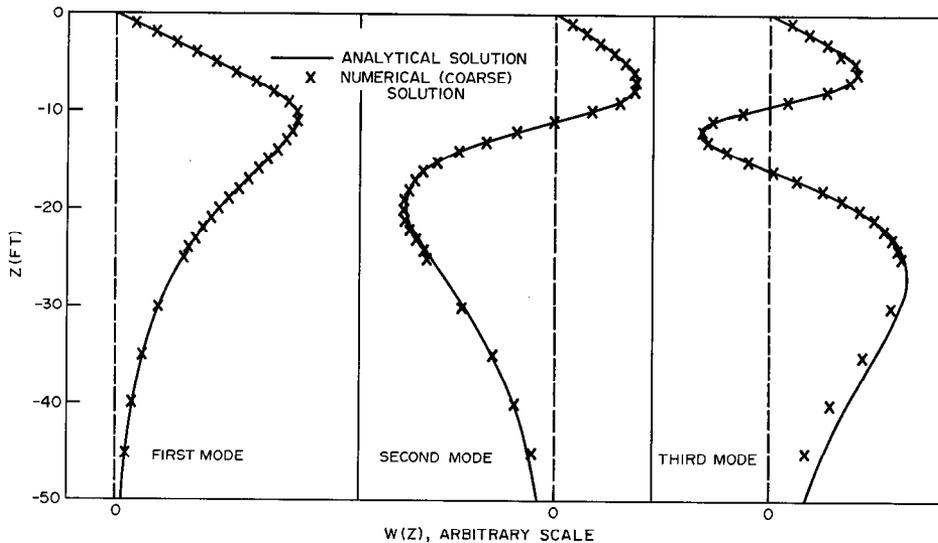


Fig. 8 - Comparison between analytical vertical velocity function and that calculated in coarse iteration solution at intermediate wave numbers. Vertical velocity scale is arbitrary, and solutions are matched at 1 ft. First mode:  $\kappa = 0.09645$  rad/ft;  $\sigma = 0.1263$  rad/sec (analytic), 0.1273 rad/sec (numerical). Second mode:  $\kappa = 0.1025$  rad/ft;  $\sigma = 0.05943$  rad/sec (analytic), 0.05975 rad/sec (numerical). Third mode:  $\kappa = 0.1003$  rad/ft;  $\sigma = 0.04214$  rad/sec (analytic), 0.04239 rad/sec (numerical).

The underlying problem in the high-wave-number deviations is shown in Fig. 9. At high wave numbers, the strong curvature of the vertical velocity function changes rapidly over distances comparable to the iteration step length. Because of this, the quadratic approximation of derived in the section on "Finite Difference Formulation" loses its

effectiveness, and incorrect frequencies are predicted. This effect is reduced by going to smaller-step intervals, as can be seen from a high-wave-number comparison of Figs. 5 and 6. The extreme condition would be realized if the recursion interval were greater than the distance between successive nodes in the vertical velocity function. If the step length used in calculations for the third mode in Fig. 9 were 4 ft, computations would be made at depths of 4, 8, 12, 16 ft, etc., and the negative portion of the actual vertical velocity function would be skipped over entirely, resulting in a search for zero crossings at other depths, which would tend to make for large deviations from the correct third-mode frequency in the computed value. Since numerical solutions involving this type of high-wave-number deviation oscillate less rapidly than their theoretical counterparts, the numerically determined frequencies will generally be higher than those corresponding to the correct solutions.

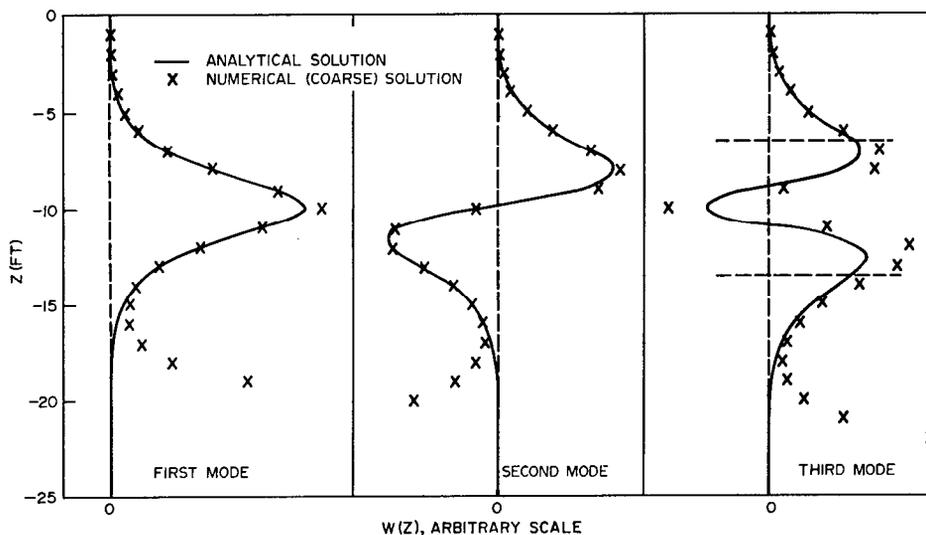


Fig. 9 - Comparison between analytical vertical velocity function and that calculated in coarse iteration solution at high wave numbers. Vertical velocity scale is arbitrary, and solutions are matched at 1 ft. The divergence of the numerical points below 15 ft is a general high-frequency result and contributes less than 1.0% to the frequency deviation. First mode:  $\kappa = 0.9084$  rad/ft;  $\sigma = 0.2444$  rad/sec (analytic), 0.2500 rad/sec (numerical). Second mode:  $\kappa = 0.9315$  rad/ft;  $\sigma = 0.1807$  rad/sec (analytic), 0.1826 rad/sec (numerical). Third mode:  $\kappa = 0.9802$  rad/ft;  $\sigma = 0.1559$  rad/sec (analytic), 0.1624 rad/sec (numerical).

The divergence of the points in the bottom portion of Fig. 9 is a general tendency which is enhanced by the high wave numbers involved. In this region, the wave frequency is much higher than the Brunt-Väisälä frequency at the last iteration step (for the first mode,  $\sigma^2/N^2(40) = 43.6$ ), and the frequency condition for the applicability of the pseudo-bottom boundary condition is satisfied quite well. The problem is that two linearly independent solutions exist for  $\sigma^2/N^2 \gg 1$ , one is the desired exponential decay, and the other is an exponential increase. Only for an exactly correct frequency will the exponential increase be eliminated, so that, when calculating to any finite accuracy, the exponential increase will enter into the calculated vertical velocity function at some depth. The frequencies in Fig. 9 were calculated to 0.1% accuracy, and a higher accuracy in calculation would have suppressed the exponential increase to a greater depth. Since the

solutions vary as  $\exp(\pm \kappa z)$ , this effect is more pronounced at the higher wave numbers and, hence, is not observed in Figs. 7 and 8. This divergence has no adverse effect on the accuracy of computed frequencies and in fact requires that frequencies be calculated to a higher degree of accuracy than for lower wave numbers.

## SUMMARY

Relatively fast (nominal 1 millisecond per point per observation of Brunt-Väisälä frequency) and accurate calculations of the dispersion relations for internal gravity waves for arbitrary distributions of Brunt-Väisälä frequency can be made using a set of Fortran subroutines which has been developed, provided that certain restrictions are not violated in the numerical procedure. In the numerical procedure, the basic integration of the governing equation for small-amplitude internal gravity waves is accomplished through a finite difference recursion formulation, and efficient calculation of a number of points on the dispersion curves is accomplished by making use of previously calculated information at each step in the computation. The accuracy of the calculations depends on the degree to which two basic assumptions are satisfied. One condition is involved only in calculations for fluid environments with an effective infinite depth, that is, those in which the significant stability information (large values of the Brunt-Väisälä frequency) is confined to a surface layer whose depth is small compared to the total depth of the fluid. This condition requires that the wave frequency be large compared to the Brunt-Väisälä frequency at the bottom of the stability layer, where the numerical integration is terminated. Violation of this assumption results in computed frequencies which are lower than the correct wave frequency. For situations in which the computation extends to the rigid bottom of the fluid system, this condition is irrelevant.

The other condition is that the changes in curvature of the vertical velocity function be small over distances comparable to the iteration step length involved in the finite difference procedure. Violation of this condition results in computed frequencies which are higher than the correct wave frequencies. A convenient rule of thumb for this condition might be that the quantity  $\kappa d_m$  be small compared with unity, where  $\kappa$  is the wave number,  $d$  is the iteration step length in the region of maximum Brunt-Väisälä frequency, and  $m$  is the mode number.

## REFERENCES

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Appendix A

SOURCE LISTINGS OF SUBROUTINES

The source language listings of the subroutines DISPER, DIFFER, SERIES, CALSIG, PRIDIS, and PLØDIS as produced by the NRL CDC 3800 computer on compilation are reproduced on the following pages.

FTN5.5A

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```

SUBROUTINE DISPER(NM,MN,NX,XO,XF,M,Y,ZN,BV,EPS,RES,NER,L)
COMMON/BLK/BB(200),Z(201),DEL(201),DET(201),W(202)
COMMON/SIGMA/S(10,250)
DIMENSION A(250),MN(NM),NEX(25),NER(30),Y(M),BV(M)
C THIS SUBROUTINE CALCULATES THE FREQUENCIES (S(I,J)) CORRESPONDING TO
C THE NX EQUALLY SPACED WAVE NUMBERS X WHICH LIE ON THE CLOSED INTERVAL
C XO...XF FOR THE NM MODES MN(1)...MN(NM) WHICH ARE ARRANGED IN
C INCREASING ORDER. I = 1,...,NM AND J = 1,...,NX.
C THE FREQUENCY ARRAY IS STORED IN COMMON WITH THE CALLING ROUTINE
C IN THE COMMON BLOCK SIGMA, WHICH CONTAINS 2500 COMPUTER WORDS.
C THE SUBROUTINES SERIES AND CALSIG ARE USED BY THIS SUBROUTINE.
C M IS THE NUMBER OF (INPUT) VALUES OF BRUNT-VAISALA FREQUENCY
C SQUARED (BB(I)) FOR THE DEPTHS Z(I), ZN BEING THE DEPTH AT WHICH
C THE BOTTOM BOUNDARY CONDITION IS TO BE EVALUATED. THE INPUT Z(I) AND
C ZN ARE POSITIVE NUMBERS, BEING THE MAGNITUDE OF THE VERTICAL POSITION
C RELATIVE TO THE NAVIFACE.
C THE FORMAL PARAMETER LIST CONTAINS THE VARIABLES Y AND BV INSTEAD
C OF Z AND BB, IN ORDER THAT THE COMMON DECLARATION MAY BE USED. THE
C VARIABLE NAMES ARE SET STRAIGHT IN THE DO LOOP TERMINATING WITH
C STATEMENT 10.
C THE SUBROUTINE DIFFER IS CALLED TO CONVERT THE Z(I) INTO TRUE
C POSITION RELATIVE TO THE NAVIFACE, +Z BEING TAKEN AS VERTICALLY UP,
C AND TO FORM THE ARRAYS DEL AND DET OF FINITE DIFFERENCE OPERATIONS
C IN THE Z(I).
C DEL AND DET ARE ARRAYS OF FINITE DIFFERENCE OPERATIONS WITH THE
C Z(I) AND ARE USED IN THE CALCULATION OF THE VERTICAL VELOCITIES OF
C THE WAVES (THE W(I+1)) IN CALSIG.
C BB,Z,DEL,DET, AND W ARE SHARED IN COMMON WITH THE TWO OTHER SUB-
C ROUTINES WHICH DISPER RELYS ON (SERIES AND CALSIG) IN THE COMMON
C BLOCK BLK, WHICH CONTAINS 500 COMPUTER WORDS.
C A VALID SOLUTION IS ONE FOR WHICH
C A. THE INTERVAL SL...SM WITHIN WHICH S IS KNOWN TO LIE HAS BEEN
C REDUCED TO 100*EPS PERCENT OF S, OR
C B. IF L=0 (REAL RIGID BOTTOM) THE VERTICAL VELOCITY W AT THE
C BOTTOM IS WITHIN RES OF ZERO
C IF L=1 (PSEUDO BOTTOM) THE RATIO OF DW/DZ TO W AT THE PSEUDO
C BOTTOM IS WITHIN 100*RES PERCENT OF THE WAVE NUMBER X, THAT
C IS, THE SOLUTION BELOW THE DEPTH Z(N) CLOSELY APPROXIMATES
C THAT FOR A HOMOGENEOUS FLUID.
C NER IS AN ARRAY OF ERROR RETURNS, DESCRIBED BELOW.
C
C
C DO 10 I=1,M
C Z(I)=Y(I)
10 BB(I)=BV(I)
C NER(I)=0
C N=M+1
C CALL DIFFER(N,Z,ZN,DEL,DET)
C SET THE INTERVAL SL...SM WITHIN WHICH THE S ARE KNOWN TO LIE. SM IS
C EQUAL TO THE MAXIMUM BRUNT-VAISALA FREQUENCY.
C SL=0.
C SM=0.
C DO 15 I=1,M
15 IF (BB(I).GT.SM) SM=BB(I)
C SM=SQRT(SM)
C BREAK THE SERIES OF NX EQUALLY SPACED VALUES OF X INTO NS SUB-SERIES

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C OF EQUALLY SPACED (SPACING DELX) VALUES OF X, EACH CONTAINING A POWER
C OF TWO (NEX(J), J=1,NS) PLUS 1 VALUE OF X (THE ENDPOINTS OF THE SUB-
C SERIES ARE ALL SHARED EXCEPT FOR THE ORIGINAL X0 AND XF SPECIFIED IN
C THE CALLING PROGRAM).
C NXP IS AN INTERMEDIATE VALUE OF NX, AND REPRESENTS THE NUMBER OF
C ELEMENTS REMAINING AFTER EACH SUCCESSIVE SUB-SERIES HAS BEEN DETERMIN-
C ED. THE PROGRAM IS SET TO RETURN AN ERROR IF MORE THAN 25 SUB-SERIES
C ARE REQUIRED TO MAKE UP NX (THAT IS, NER(1) SET TO -1).
  NS=1
  NXP=NX
  XFP=XF
  IF (NX.EQ.1) GO TO 35
  DELX=(XF-X0)/(NX-1)
  DO 27 NS=1,25
17     I=0
20     IF(NXP.LT.(2**I + 1)) GO TO 25
        I=I+1
        GO TO 20
25     NEX(NS)=I-1
        NXP=NXP-2**(I-1)
27     IF (NXP.EQ.1) GO TO 30
        NER(1)=-1
  RETURN
C THE FREQUENCIES (S(I,J), I=1,...,NM, J=1,...,2**NEX(1)+1) FOR THE
C FIRST SUB-SERIES ARE NOW CALCULATED BY CALLING SERIES FOR EACH MODE.
C NXP IS THE NUMBER OF ELEMENTS IN THE SUB-SERIES, NEX(1) BEING ITS
C POWER OF TWO. XFP IS THE UPPER LIMIT OF X FOR THE SUB-SERIES, AND SMP
C IS AN INTERMEDIATE VALUE OF SM, WHICH DECREASES AS THE SET OF FREQUEN-
C CIES FOR EACH MODE IS CALCULATED. A(J) IS AN ARRAY TO RETURN THE
C S(I,J) FROM SERIES. NERR IS AN ERROR RETURN FROM SERIES.
30     NXP=2**NEX(1)+1
        XFP=X0+(NXP-1)*DELX
35     SMP=SM
        K=1
        DO 40 I=1,NM
            CALL SERIES(MN(I),X0,XFP,NXP,NEX(1),1,N,SL,SMP,EPS,RES,A,NERR,L)
            IF (NERR.NE.0) GO TO 200
            DO 38 J=1,NXP
38             S(I,J)=A(J)
40             SMP=A(NXP)
        IF (NS.EQ.1) RETURN
C THE FREQUENCIES FOR THE REMAINING NS-1 SUB-SERIES ARE NOW CALCULATED,
C USING THE LAST FREQUENCY OF THE PREVIOUS SUB-SERIES FOR SL, AND XFP OF
C THE PREVIOUS SUB-SERIES FOR XOP. NXX IS THE TOTAL NUMBER OF PREVIOUSLY
C CALCULATED FREQUENCIES FOR EACH MODE. THE CALCULATED FREQUENCIES ARE
C TRANSFERRED FROM THE ARRAY A TO S(I,J), J= NXX+1,...,NXX+NXP.
  NXX=NXP
  DO 100 K=2,NS
    NXP=2**NEX(K)+1
    XOP=XFP
    XFP=XOP+(NXP-1)*DELX
    SMP=SM
    DO 60 I=1,NM
      CALL SERIES(MN(I),XOP,XFP,NXP,NEX(K),0,N,S(I,NXX),SMP,EPS,
1      RES,A,NERR,L)
      IF (NERR.NE.0) GO TO 200

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```
      DO 50 J=2,NXP
50      S(I,(NXX+J-1))=A(J)
60      SMP=A(NXP)
100     NXX=NXX+NXP-1
      RETURN
C      THE ERROR RETURN SEQUENCE
C      NER(I), I = 1,...,NS+5 IS AN INTEGER ARRAY WHICH CONTAINS ERROR INFOR-
C      MATION TO BE RETURNED TO THE CALLING PROGRAM. IF AN ERROR IN THE
C      CALCULATION OF A FREQUENCY (SEE CALSIG) HAS OCCURRED, SERIES RETURNS
C      NERR, WHERE A(NERR) IS THE FREQUENCY CALCULATION WHICH ENCOUNTERED
C      DIFFICULTY. NER(1) = THE SUB-SERIES NUMBER WITHIN WHICH THE DIFFICULTY
C      OCCURRED (EXCEPT IN THE CASE NER(1) = -1, SEE STATEMENT NUMBER 27 OF
C      THIS SUBROUTINE), NER(2) IS THE TOTAL NUMBER OF SUB-SERIES, NER(3) IS
C      THE NUMBER OF ELEMENTS IN THE SUB-SERIES, NER(4), THE MODE NUMBER,
C      NER(5), THE INDEX WITHIN THE SUB-SERIES (NERR), AND NER(6)...NER(5+NS)
C      ARE THE EXPONENTS CORRESPONDING TO EACH SUB-SERIES WHICH HAS BEEN SET
C      UP IN THIS SUBROUTINE.
200     NER(1)=K
      NER(2)=NS
      NER(3)=NXP
      NER(4)=MN(I)
      NER(5)=NERR
      DO 220 L=1,NS
220     NER(5+L)=NEX(L)
      RETURN
      END
```

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```
      SUBROUTINE DIFFER(N,Z,ZN,DEL,DET)
      DIMENSION Z(N),DEL(N),DET(N)
C     THIS SUBROUTINE CONVERTS THE DEPTHS (THE Z(I) AND ZN, POSITIVE
C     NUMBERS) INTO TRUE VERTICAL DISTANCE RELATIVE TO THE NAVIFACE, +Z
C     BEING TAKEN AS VERTICALLY UP, AND SETS UP THE ARRAYS DEL AND DET OF
C     FINITE DIFFERENCE OPERATIONS IN THE Z(I). N IS THE TOTAL NUMBER OF
C     DEPTHS, INCLUDING ZN WHICH IS RETURNED AS Z(N).
      M=N-1
      DO 5 I=1,M
      Z(I)=-Z(I)
      Z(N)=-ZN
      DET(1)=Z(1)*Z(2)*(Z(2)-Z(1))*0.5
      DEL(1)=Z(1)
      DEL(2)=Z(2)-Z(1)
      DO 10 I=2,M
      DEL(I+1)=Z(I+1)-Z(I)
      DET(I)=0.5*(Z(I-1)*Z(I-1)*DEL(I+1)+Z(I)*Z(I)*(Z(I-1)-Z(I+1))+
1     Z(I+1)*Z(I+1)*DEL(I))
      RETURN
      END
```

UNCLASSIFIED

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SUBROUTINE SERIES(MN,XO,XF,NX,NEX,M,N,SL,SM,EPS,RES,S,NERR,L)
COMMON/BLK/BB(200),Z(201),DEL(201),DET(201),W(202)
DIMENSION S(NX)
C THIS SUBROUTINE SYSTEMATICALLY CALCULATES THE NX FREQUENCIES
C (S(1)...S(NX)) WHICH CORRESPOND TO THE NX EQUALLY SPACED WAVE NUM-
C BERS WHICH ARE CONTAINED IN THE CLOSED INTERVAL XO...XF FOR THE MN-TH
C MODE. NX MUST BE A POWER OF TWO PLUS 1, THAT IS, NX = 2**NEX + 1.
C DELX IS THE EQUAL SPACING BETWEEN THE WAVE NUMBERS. THE ACTUAL
C CALCULATIONS ARE PERFORMED BY CALLING THE SUBROUTINE CALSIG.
C N-1 IS THE NUMBER OF OBSERVATIONS OF BRUNT-VAISALA FREQUENCY
C SQUARED (BB(I)) AS A FUNCTION OF DEPTH (Z(I)). Z(N) IS THE DEPTH AT
C WHICH THE BOTTOM BOUNDARY CONDITION IS TO BE EVALUATED. THE Z(I) ARE
C ACTUALLY NEGATIVE NUMBERS, +Z BEING TAKEN AS VERTICALLY UP. DEL AND
C DET ARE ARRAYS OF DIFFERENCE OPERATIONS WITH THE Z(I), AND MUST BE
C SET UP IN THE CALLING PROGRAM. THE W(I+1) ARE THE VERTICAL VELOCITIES
C CALCULATED BY THE SUBROUTINE FOR THE DEPTHS Z(I). BB, Z, DEL, DET,
C AND W ARE ALL SHARED IN COMMON WITH THE CALLING PROGRAM IN THE COMMON
C BLOCK BLK, WHICH CONTAINS 500 COMPUTER WORDS.
C THE FREQUENCIES S(I) ARE KNOWN TO BE GREATER THAN SL AND LESS
C THAN SM. INITIAL VALUES OF SL AND SM MUST BE INPUT.
C M = 0 SIGNIFIES THAT THE FIRST ELEMENT OF THE SERIES S(1)...S(NX)
C IS EQUAL TO SL, AND THAT ONLY NX-1 FREQUENCIES NEED BE CALCULATED.
C A VALID SOLUTION IS ONE FOR WHICH
C A. THE INTERVAL SL...SM WITHIN WHICH S IS KNOWN TO LIE HAS BEEN
C REDUCED TO 100*EPS PERCENT OF S, OR
C B. IF L=0 (REAL RIGID BOTTOM) THE VERTICAL VELOCITY W AT THE
C BOTTOM IS WITHIN RES OF ZERO
C IF L=1 (PSEUDO BOTTOM) THE RATIO OF DW/DZ TO W AT THE PSEUDO
C BOTTOM IS WITHIN 100*RES PERCENT OF THE WAVE NUMBER X, THAT
C IS, THE SOLUTION BELOW THE DEPTH Z(N) CLOSELY APPROXIMATES
C THAT FOR A HOMOGENEOUS FLUID.
C NERR IS AN ERROR RETURN WHICH IS SET EQUAL TO THE INDEX (I) CORRE-
C SPONDING TO THE FIRST FREQUENCY (S(I)) CALCULATION WHICH IS IN ERROR
C (SEE CALSIG).
C THE SYSTEMATIC CALCULATION IS ACCOMPLISHED BY BREAKING THE SERIES
C S(I) OF FREQUENCIES TO BE CALCULATED INTO NEX GROUPS, EACH CONTAINING
C SUCCESSIVELY HIGHER POWERS OF TWO ELEMENTS. THE ELEMENTS OF A GROUP
C (I.E. THE S(I) WITHIN THE GROUP) ARE CALCULATED USING LIMITS (SL
C AND SM) BASED ON THE ELEMENTS OF GROUPS WHICH HAVE BEEN PREVIOUSLY
C CALCULATED, AS IS ILLUSTRATED BELOW FOR THE CASE NX=9, NEX=3. THE
C NUMERALS INDICATE THE ORDER IN WHICH THE ELEMENTS (THE S(I)) ARE
C CALCULATED. THE SI-S IN PARENTHESES ARE THE PREVIOUSLY CALCULATED
C ELEMENTS WHICH ARE USED AS SL AND SM.
C
C S(1) 1. (S1) (S1) (S1) (S1)
C
C S(2) 6.
C
C S(3) 4. (S3)
C
C S(4) 7.
C
C S(5) 3. (S5) (S5)
C
C S(6) 8.
C

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C S(7)          5. (S7)
C
C S(8)          9.
C
C S(9)          2. (S9) (S9) (S9)
C
C GROUP          1   2   3   NUMBER OF GROUPS = NEX
C NO.EL. (NE)   1   2   4   NUMBER OF ELEMENTS DOUBLES FOR EACH
C SUCCESSIVE GROUP
C INDEX JUMP BETWEEN SUCCESSIVE ELEMENTS OF THE
C GROUP (IJ)    (8)  4   2
C INDEX OF FIRST MEMBER OF PREVIOUS
C GROUP (INDP)  (9)  5   3   INDP IS USED TO DESIGNATE SM FOR THE
C CALCULATIONS ON THE FIRST MEMBER OF A GROUP
C S(IL) AND S(IM) ARE THE FREQUENCIES CORRESPONDING TO SL AND SM FOR
C ANY PARTICULAR CALCULATION
  IF (M.EQ.0) 10,20
10  S(1)=SL
  GO TO 30
20  CALL CALSIG(MN,XO,N,SL,SM,EPS,RES,S(1),NERR,L)
  IF (NERR.EQ.1) GO TO 200
30  IF (NX.EQ.1) RETURN
  CALL CALSIG(MN,XF,N,S(1),SM,EPS,RES,S(NX),NERR,L)
  IF (NERR.EQ.1) GO TO 210
  IF (NX.EQ.2) RETURN
  DELX=(XF-XO)/(NX-1)
  NE=1
  IJ=NX-1
  INDP=NX
  DO 100 I=1,NEX
    X=XO+IJ*DELX/2.
    IND=INDP-IJ/2
    IL=1
    IM=INDP
    INDP=IND
    DO 50 J=1,NE
      CALL CALSIG(MN,X,N,S(IL),S(IM),EPS,RES,S(IND),NERR,L)
      IF (NERR.EQ.1) GO TO 220
      IL=IM
      IM=IM+IJ
      IND=IND+IJ
50  X=X+IJ*DELX
  IJ=IJ/2
100  NE=NE*2
  RETURN
200  NERR=1
  RETURN
210  NERR=NX
  RETURN
220  NERR=IND
  RETURN
  END

```

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```

SUBROUTINE CALSIG(MN,X,N,SL,SM,EPS,RES,S,NERR,L)
COMMON/BLK/BB(200),Z(201),DEL(201),DET(201),W(202)
C THIS SUBROUTINE CALCULATES THE FREQUENCY S CORRESPONDING TO THE
C MN-TH MODE OF OSCILLATION FOR THE WAVE NUMBER X.
C N=1 IS THE NUMBER OF OBSERVATIONS OF BRUNT-VAISALA FREQUENCY
C SQUARED (BB(I)) AS A FUNCTION OF DEPTH (Z(I)). Z(N) IS THE DEPTH AT
C WHICH THE BOTTOM BOUNDARY CONDITION IS TO BE EVALUATED. THE Z(I) ARE
C ACTUALLY NEGATIVE NUMBERS, +Z BEING TAKEN AS VERTICALLY UP. DEL AND
C DET ARE ARRAYS OF DIFFERENCE OPERATIONS WITH THE Z(I), AND MUST BE
C SET UP IN THE CALLING PROGRAM. THE W(I+1) ARE THE VERTICAL VELOCITIES
C CALCULATED BY THE SUBROUTINE FOR THE DEPTHS Z(I). BB, Z, DEL, DET,
C AND W ARE ALL SHARED IN COMMON WITH THE CALLING PROGRAM IN THE COMMON
C BLOCK BLK, WHICH CONTAINS 500 COMPUTER WORDS.
C THE FREQUENCY S IS KNOWN TO BE GREATER THAN SL AND LESS THAN SM.
C INITIAL VALUES FOR SL AND SM MUST BE INPUT. A VALID SOLUTION IS ONE
C FOR WHICH EITHER
C   A. THE INTERVAL SL...SM WITHIN WHICH S IS KNOWN TO LIE HAS BEEN
C   REDUCED TO 100*EPS PERCENT OF S, OR
C   B. IF L=0 (REAL RIGID BOTTOM) THE VERTICAL VELOCITY W AT THE
C   BOTTOM IS WITHIN RES OF ZERO
C   IF L=1 (PSEUDO BOTTOM) THE RATIO OF DW/DZ TO W AT THE PSEUDO
C   BOTTOM IS WITHIN 100*RES PERCENT OF THE WAVE NUMBER X, THAT
C   IS, THE SOLUTION BELOW THE DEPTH Z(N) CLOSELY APPROXIMATES
C   THAT FOR A HOMOGENEOUS FLUID.
C THE INTEGER VARIABLE J KEEPS TRACK OF THE NUMBER OF ZERO CROSSINGS
C IN THE DEPTH SERIES OF W. NERR IS AN ERROR RETURN WHICH IS EMPLOYED
C IF A SOLUTION IS NOT OBTAINED AFTER 50 INTEGRATIONS (NERR=1).
C THE INITIAL VALUES OF SL AND SM ARE RETURNED TO THE CALLING
C PROGRAM.
  NERR=0
  IF (X*Z(N).LT.-0.00628) GO TO 10
  S=0.
  RETURN
10  A=SL
   B=SM
   XX=X*X
   W(1)=0.
   W(2)=1.
   DO 400 K=1,50
   S=0.5*(SM+SL)
   IF((SM-SL).LE.(EPS*S)) GO TO 500
   SS=S*S
   J=0
   DO 100 I=2,N
   II=I
   W(I+1)=(-1./DEL(I-1))*((XX*DET(I-1))*(BB(I-1)/SS-1.)-
1  DEL(I) - DEL(I-1))*W(I) + DEL(I)*W(I-1))
   IF(((-1)**J*W(I+1)).GT.0.) GO TO 55
   IF (J.GE.(MN-1)) GO TO 115
   J=J+1
95  CONTINUE
100 CONTINUE
   IF (J.LT.(MN-1)) GO TO 105
   IF(L.EQ.1) GO TO 110
   IF (ABSF(W(N+1)).LE.RES) GO TO 500
105  SM=S

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      GO TO 390
110  IF(((-1)**J*W(N)).LE.0.) GO TO 130
      BBC=2.*(W(N+1)-W(N))/(DEL(N)*(W(N+1)+W(N)))-X
      IF (BBC.GT.(RES*X)) GO TO 125
      IF (-BBC.GT.(RES*X)) GO TO 130
      GO TO 500
115  IF(L.EQ.1.OR.II.NE.N) GO TO 125
      IF (ABSF(W(N+1)).LE.RES) GO TO 500
125  SL=S
      GO TO 390
130  SM=S
390  CONTINUE
400  CONTINUE
      NERR=1
500  SL=A
      SM=B
      RETURN
      END
```

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```

SUBROUTINE PRIDIS(IDENT,NUNITS,X0,XF,NX,NM,MN)
COMMON/SIGMA/S(10,250)
DIMENSION IDENT(10),MN(NM)
C THIS SUBROUTINE PRINTS THE CALCULATED FREQUENCIES CORRESPONDING TO
C THE NX EQUALLY SPACED WAVE NUMBERS (X0,...,XF) FOR THE NM MODES (MN(I)
C (NM MUST BE LESS THAN OR EQUAL TO 10) IN TABULAR FORM. IDENT IS AN
C 80 CHARACTER IDENTIFICATION, NUNITS IS A SIX CHARACTER WORD DES-
C IGNATING THE UNITS OF LENGTH.
C THE FREQUENCIES ARE SHARED IN COMMON BLOCK SIGMA WITH THE CALLING
C ROUTINE.
PRINT 1000,IDENT
1000 FORMAT (1H1,10A8)
PRINT 2000,NUNITS,(MN(I),I=1,NM)
2000 FORMAT (81H0DISPERSION RELATIONS = FREQUENCIES (RAD/SEC) CORRESPON-
1000DING TO WAVE NUMBERS (RAD/,A6,25H) FOR THE SPECIFIED MODES//1X,
211HMODE NUMBER,9(5X,I2,5X))
PRINT 3000
3000 FORMAT (20(4X,1H*)/1X,12HWAVE NUMBER*)
IF (NX.GT.1) GO TO 100
PRINT 4000, X0,(S(J,1),J=1,NM)
RETURN
100 DELX=(XF-X0)/(NX-1)
X=X0-DELX
DO 200 I=1,NX
X=X+DELX
200 PRINT 4000,X,(S(J,I),J=1,NM)
4000 FORMAT (1X,E10.3,2H *,9(E10.3,2X))
RETURN
END

```

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```

SUBROUTINE PLODIS (IDENT, NUNITS, X0, XF, NX, NMM, MN)
COMMON/SIGMA/S(10,250)
DATA ((IS(I), I=1,9)=1H1,1H2,1H3,1H4,1H5,1H6,1H7,1H8,1H9),
1 (IBLK=1H )
DIMENSION IDENT(10), MN(NMM), ISIG(9), LINE(101), IS(9), SCALE(11)
C THIS SUBROUTINE WILL PRODUCE A LINE PRINTER PLOT TO THE DISPERSION
C CURVES, FREQUENCY (S(I,J), I = MODE NUMBER, J = THE INDEX CORRE-
C SPONDING TO WAVE NUMBER. THE ARRAY S(I,J) IS STORED IN COMMON WITH
C THE CALLING PROGRAM IN THE COMMON BLOCK SIGMA, WHICH CONTAINS
C 2500 COMPUTER WORDS) AS A FUNCTION OF WAVE NUMBER (X, NX EQUALLY
C SPACED VALUES ON THE CLOSED INTERVAL X0...XF) FOR THE NMM MODES
C MN(I). THE SYMBOL FOR A PLOTTED POINT IS THE MODE NUMBER.
C IDENT IS AN 80 CHARACTER IDENTIFICATION FIELD, WHICH IS BROKEN UP
C 10 8-CHARACTER ALPHANUMERIC WORDS. NUNITS IS A SIX CHARACTER WORD
C SPECIFYING THE UNITS OF LENGTH.
NM=NMM
PRINT 500, IDENT
500 FORMAT (1H1,10A8)
IF (NX.GT.1) GO TO 100
PRINT 1000
1000 FORMAT (63H0ONLY ONE WAVE NUMBER SPECIFIED, THEREFORE NO PLOT WILL
1 RE MADE)
RETURN
100 IF (MN(NM).LE.9) GO TO 125
NM=NM-1
IF (NM.GT.0) GO TO 100
PRINT 1050
1050 FORMAT (77H0ALL MODE NUMBERS CONTAIN 2 OR MORE DIGITS AND CANNOT BE
1E PRINTED ON THE GRAPH)
RETURN
125 IF ((NMM-NM).EQ.0) GO TO 150
I=NMM-NM
PRINT 1100, I
1100 FORMAT (1H0,12,73H MODE NUMBERS CONTAIN 2 OR MORE DIGITS AND CANNOT
1T BE PRINTED ON THE GRAPH)
150 SM=S(1,NX)
A=1.
X=1.
NE=0
IF (SM=X) 201,208,202
201 X=0.1*X
NE=NE+1
IF (SM=X) 201,208,204
202 X=10.*X
NE=NE-1
IF (SM=X) 203,208,202
203 X=0.1*X
NE=NE+1
204 A=2.
IF (SM=2.*X) 208,208,205
205 A=4.
IF (SM=4.*X) 208,208,206
206 A=5.
IF (SM=5.*X) 208,208,207
207 NE=NE-1
A=1.0

```

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```

208  A=0.1*A
      SCALE(1)=0.0
      DO 300 I=2,11
300   SCALE(I)=SCALE(I-1)*A
      PRINT 2000,NE,NUNITS,(MN(I),I=1,NM)
2000  FORMAT(57H0DISPERSION CURVES - FREQUENCY (RAD/SEC) TIMES 10 TO THE
1     ,I3,41H POWER AS A FUNCTION OF WAVE NUMBER (RAD/,A6,1H)/11H FOR M
20DES ,9(I2,1H,))
      PRINT 3000,SCALE
3000  FORMAT(1H0,16X,11(F3.1,7X)/18X,10(1H+,9(1H.)),1H+)
      C=10**(NE+1)/A
      MX=10
      IF(NX.LE.10) GO TO 350
      MX=4
      IF(NX.LE.25) GO TO 350
      MX=2
      IF(NX.LE.50) GO TO 350
      MX=1
350   DELX=(XF-X0)/(MX*(NX-1))
      B=X0
      JJ=0
      DO 375 I=1,MX
      IF(B.LE.(0.5*DELX)) GO TO 400
375   B=B-DELX
      PRINT 4000
4000  FORMAT(17X,1H/,4(/17X,1H/))
      LP=0
      IQ=0
      KP=0
      B=X0
      GO TO 450
400   B=0.0
      PRINT 4100
4100  FORMAT(6X,5H0.000,6X,2H+1,100X,1H+)
      KP=1
      LP=1
      IQ=0
      IF (I.GT.1) GO TO 410
      JJ=MX
      I=MX+1
      IQ=1
410   IL=I-2
      DO 420 J=1,IL
      LP=LP+1
420   PRINT 4200
4200  FORMAT(17X,1H.,101X,1H.)
450   NNX=MX*(NX-1)-JJ+LP
      DELX=10*DELX
      DO 500 I=1,101
500   LINE(I)=IBLK
      DO 600 I=LP,NNX,MX
      IQ=IQ+1
      DO 510 KI=1,NM
      IKI=NM-KI+1
      ISIG(KI)=C*S(IKI,IQ)+1
      IF (ISIG(KI).GT.101) ISIG(KI)=101

```

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```
510     LINE(ISIG(KI))=IS(MN(IKI))
      IF((I/10-KP).LT.0) GO TO 515
      KP=KP+1
      R=B+DELX
      PRINT 5100,B,LINE
5100    FORMAT(5X,E10.3,2X,1H+,101A1,1H+)
      GO TO 520
515    PRINT 5200,LINE
5200    FORMAT(17X,1H.,101A1,1H.)
520    DO 517 J=1,101
517    LINE(J)=I8LK
      JJ=I+MX-2
      DO 540 J=I,JJ
      IF((J/10-KP).LT.0) GO TO 535
      KP=KP+1
      R=B+DELX
      PRINT 5300,B
5300    FORMAT(5X,E10.3,2X,1H+,101X,1H+)
      GO TO 538
535    PRINT 5350
5350    FORMAT(17X,1H.,101X,1H.)
538    CONTINUE
540    CONTINUE
600    CONTINUE
      PRINT 6000,SCALE
6000    FORMAT(18X,10(1H+,9(1H.)),1H+//17X,11(F3.1,7X))
      RETURN
      END
```

## Appendix B

### SAMPLE CALLING PROGRAM, DATA, AND OUTPUT

To illustrate the use of the subroutine set for computing internal wave dispersion relations, the source listing of a sample program which employs calls to DISPER, PRIDIS, and PLØDIS is reproduced on page 33. Sample input data and the resultant outputs from PRIDIS and PLØDIS are included as Figs. B1, B2, and B3, respectively.

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```

PROGRAM TESTIT
DIMENSION IDENT(10),MN(10),NER(30),Z(201),BV(200)
COMMON/SIGMA/S(10,250)
READ 1000, (IDENT(I),I=1,10)
1000 FORMAT(10A8)
READ 2000,NUNITS,EPS,RES,NX,X0,XF
2000 FORMAT (A6,2F10,5,13,2F10,5)
READ 3000, NM,(MN(I),I=1,NM)
3000 FORMAT(11I3)
READ 4000,M,ZN,(Z(I),BV(I),I=1,M)
4000 FORMAT(I3,F10,5/(5(F6,1,F10,5)))
CALL DISPER(NM,MN,NX,X0,XF,M,Z,ZN,BV,EPS,RES,NER,1)
IF(NER(1),NE.0) GO TO 900
CALL PRIDIS(IDENT,NUNITS,X0,XF,NX,NM,MN)
CALL PL0DIS(IDENT,NUNITS,X0,XF,NX,NM,MN)
GO TO 999
900 IF(NER(1),EQ.-1) GO TO 950
    IJ=NER(2)+5
    PRINT 5000, (NER(I),I=1,IJ)
5000 FORMAT(19H ERROR RETURN, NER=,30I3)
GO TO 999
950 PRINT 6000
6000 FORMAT (15H ERROR IN SETUP)
999 CONTINUE
END

```

CARD NO. 1 -  
 TEST SET EXP. AND INV. SQUARE. RES=DEL=10\*\* -6 (7 JAN 71)

CARD NO. 2 -  
 FEET 0.000001 0.000001 26 0.00 0.10

CARD NO. 3 -  
 7 1 2 3 4 5 6 8

CARD NO. 4 -  
 28 45.0

CARD NO. 5 -  
 1.0 0.004619 2.0 0.006501 3.0 0.009148 4.0 0.01287 5.0 0.01812

CARD NO. 6 -  
 6.0 0.02549 7.0 0.03588 8.0 0.05049 9.0 0.07105 10.0 0.10000

CARD NO. 7 -  
 11.0 0.06400 12.0 0.04444 13.0 0.03265 14.0 0.02500 15.0 0.01975

CARD NO. 8 -  
 16.0 0.01600 17.0 0.01322 18.0 0.01111 19.0 0.009467 20.0 0.008163

CARD NO. 9 -  
 21.0 0.007111 22.0 0.006250 23.0 0.005536 24.0 0.004938 25.0 0.004432

CARD NO. 10 -  
 30.0 0.002778 35.0 0.001902 40.0 0.001384

Fig. B1 - Input data for sample calling program

TEST SET EXP, AND INV, SQUARE, RES=DEL=10\*\*\*-6 (7 JAN 71)

DISPERSION RELATIONS - FREQUENCIES (RAD/SEC) CORRESPONDING TO WAVE NUMBERS (RAD/FEET ) FOR THE SPECIFIED MODES

MODE NUMBER	1	2	3	4	5	6	8
0,000+000	0,000+000	0,000+000	0,000+000	0,000+000	0,000+000	0,000+000	0,000+000
4,000-003	9,511+003	3,649+003	2,130+003	1,588+003	1,260+003	1,020+003	7,719+004
8,000-003	1,837+002	7,131+003	4,226+003	3,163+003	2,513+003	2,038+003	1,542+003
1,200-002	2,664+002	1,045+002	6,284+003	4,723+003	3,758+003	3,052+003	2,310+003
1,600-002	3,439+002	1,362+002	8,304+003	6,268+003	4,994+003	4,062+003	3,075+003
2,000-002	4,167+002	1,665+002	1,029+002	7,796+003	6,221+003	5,068+003	3,838+003
2,400-002	4,852+002	1,954+002	1,223+002	9,306+003	7,437+003	6,070+003	4,598+003
2,800-002	5,498+002	2,231+002	1,413+002	1,080+002	8,642+003	7,068+003	5,355+003
3,200-002	6,109+002	2,497+002	1,599+002	1,227+002	9,835+003	8,060+003	6,109+003
3,600-002	6,686+002	2,752+002	1,781+002	1,372+002	1,101+002	9,047+003	6,859+003
4,000-002	7,234+002	2,996+002	1,959+002	1,515+002	1,218+002	1,003+002	7,607+003
4,400-002	7,754+002	3,232+002	2,134+002	1,655+002	1,333+002	1,100+002	8,351+003
4,800-002	8,249+002	3,459+002	2,304+002	1,794+002	1,447+002	1,197+002	9,092+003
5,200-002	8,719+002	3,679+002	2,471+002	1,930+002	1,559+002	1,293+002	9,829+003
5,600-002	9,168+002	3,891+002	2,634+002	2,064+002	1,670+002	1,389+002	1,056+002
6,000-002	9,596+002	4,096+002	2,794+002	2,195+002	1,779+002	1,483+002	1,129+002
6,400-002	1,000+001	4,296+002	2,950+002	2,324+002	1,886+002	1,577+002	1,202+002
6,800-002	1,039+001	4,489+002	3,104+002	2,451+002	1,991+002	1,670+002	1,274+002
7,200-002	1,077+001	4,678+002	3,254+002	2,576+002	2,095+002	1,762+002	1,346+002
7,600-002	1,113+001	4,861+002	3,401+002	2,699+002	2,197+002	1,853+002	1,418+002
8,000-002	1,147+001	5,040+002	3,545+002	2,819+002	2,297+002	1,943+002	1,489+002
8,400-002	1,180+001	5,215+002	3,687+002	2,937+002	2,396+002	2,032+002	1,560+002
8,800-002	1,211+001	5,385+002	3,826+002	3,054+002	2,492+002	2,120+002	1,630+002
9,200-002	1,241+001	5,552+002	3,963+002	3,168+002	2,588+002	2,207+002	1,700+002
9,600-002	1,270+001	5,715+002	4,097+002	3,280+002	2,681+002	2,293+002	1,770+002
1,000-001	1,298+001	5,875+002	4,229+002	3,391+002	2,773+002	2,378+002	1,839+002

Fig. B2 - Output produced by call to subroutine PRIDIS

TEST SET EXP, AND INV,SQUARE, RES=DEL=10\*\*\*6 (7 JAN 71)

DISPERSION CURVES = FREQUENCY (RAD/SEC) TIMES 10 TO THE 1 POWER AS A FUNCTION OF WAVE NUMBER (RAD/FEET )  
FOR MODES 1, 2, 3, 4, 5, 6, 8,

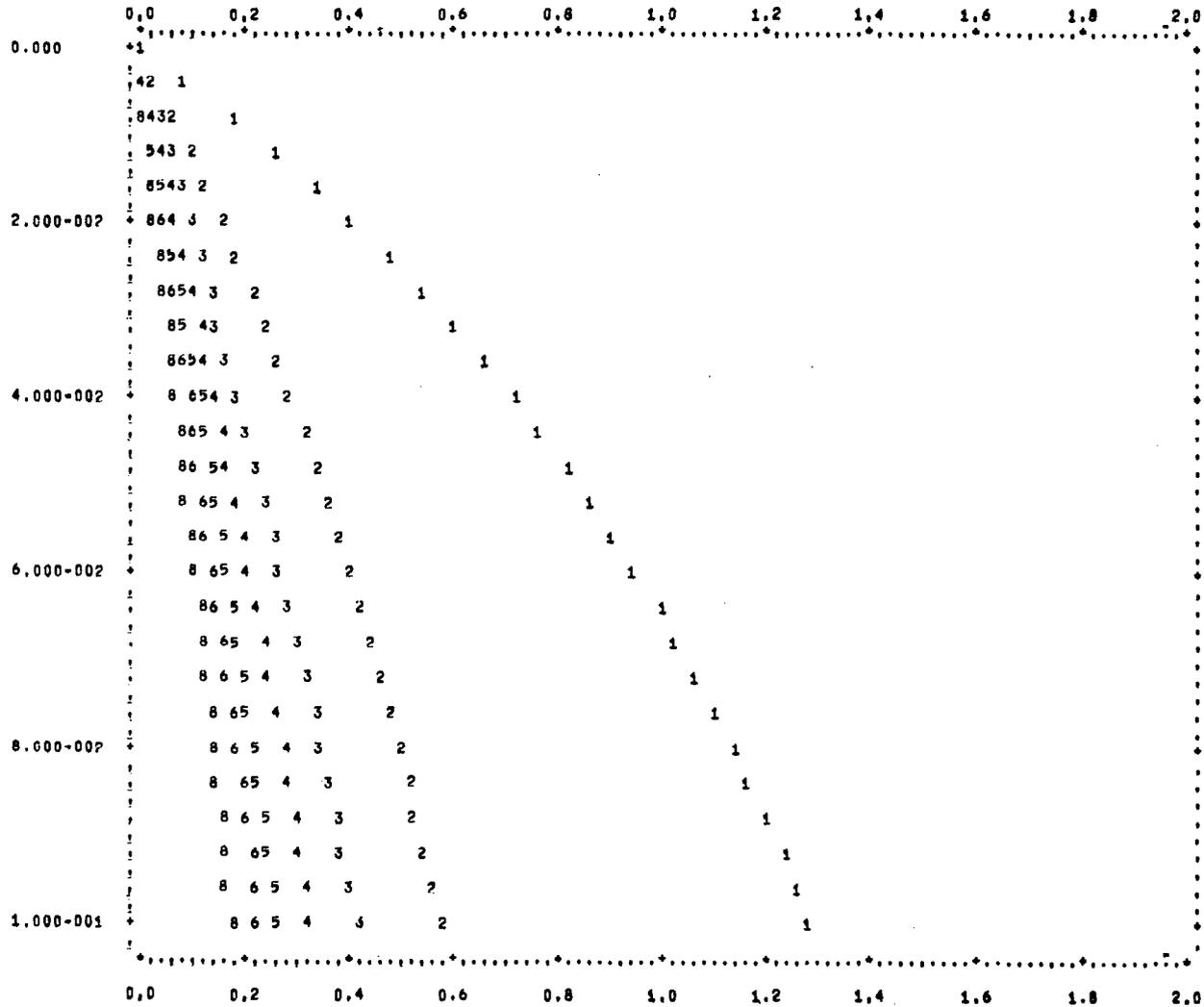


Fig. B3 - Output produced by call to subroutine PLØDIS

## Appendix C

## ANALYTICAL SOLUTION FOR THE TEST CASE

The test distribution of the Brunt-Väisälä frequency is divided into two regions

$$N^2(z) = 0.0032825 e^{-0.341642z} ; \quad -10 \leq z \leq 0 \quad (\text{C1a})$$

and

$$N^2(z) = 1.6/(z+6)^2 ; \quad -\infty < z \leq -10 . \quad (\text{C1b})$$

The analytical solution of Eq. (1) for such a distribution is obtained by determining solutions in each of the regions ( $w_E(z)$  corresponding to the exponential distribution and  $w_S(z)$  for the inverse square distribution) which will satisfy the conditions

$$w_E(0) = 0 \quad (\text{C2a})$$

$$w_E(-10) = w_S(-10) \quad (\text{C2b})$$

$$w_E'(-10) = w_S'(-10) \quad (\text{C2c})$$

$$\lim_{z \rightarrow -\infty} w_S(z) = 0 , \quad (\text{C2d})$$

where primes indicate derivatives with respect to  $z$ .

In the upper layer (Eq. (C1a)), Eq. (1) has the form

$$w_E'' + \frac{\kappa^2}{\sigma^2} (A^2 e^{-2\alpha z} - \sigma^2) w_E = 0 , \quad (\text{C3})$$

where  $A = 0.05729$  and  $\alpha = 0.170821$ . If a new independent variable  $\zeta = (\kappa A / \alpha \sigma) \exp(-\alpha z)$  is introduced, we have, defining  $X(\zeta) = w_E(z)$ ,

$$w_E'(z) = \frac{d\zeta}{dz} X'(\zeta)$$

$$w_E''(z) = \frac{d^2\zeta}{dz^2} X'(\zeta) + \left(\frac{d\zeta}{dz}\right)^2 X''(\zeta)$$

$$= \frac{\alpha \kappa A}{\sigma} e^{-\alpha z} X'(\zeta) + \frac{\kappa^2 A^2}{\sigma^2} e^{-2\alpha z} X''(\zeta)$$

where primes indicate derivatives with respect to the appropriate independent variable. The governing equation becomes

$$\frac{\kappa^2 A^2}{\sigma^2} e^{-2\alpha z} X'' + \frac{\alpha \kappa A}{\sigma} e^{-\alpha z} X' + \frac{\kappa^2}{\sigma^2} (A^2 e^{-2\alpha z} - \sigma^2) X = 0$$

or, simply

$$\zeta^2 X'' + \zeta X' + [\zeta^2 - (\kappa/\alpha)^2] X = 0, \quad (C4)$$

which is a standard form of Bessel's equation. Two linearly independent solutions of Eq. (C4) are the Bessel functions of the first and second kind of order  $\nu = \kappa/\alpha$ ,  $J_\nu(\zeta)$ , and  $Y_\nu(\zeta)$ . Using the notation of Abramowitz and Stegun (1964), any linear combination of these two solutions will be designated by  $\mathcal{C}_\nu(\zeta)$ . Thus, in the upper region, the vertical velocity function is given by

$$w_E(z) = \mathcal{C}_{\kappa/\alpha} \left( \frac{\kappa A}{\alpha \sigma} e^{-\alpha z} \right), \quad (C5)$$

which is subject to the appropriate boundary conditions. Since the first derivative of  $w_E(z)$  is involved in the matching condition at  $z = -10$ , we note that, since

$$\mathcal{C}'_\nu(\zeta) = -\mathcal{C}_{\nu+1}(\zeta) + \frac{\nu}{\zeta} \mathcal{C}_\nu(\zeta)$$

(Abramowitz and Stegun, 1964, p. 361),

$$\frac{dw_E}{dz} = \frac{d\zeta}{dz} \frac{dX}{d\zeta} = -\frac{\kappa A}{\sigma} e^{-\alpha z} \left[ -\mathcal{C}_{\frac{\kappa}{\alpha}+1} \left( \frac{\kappa A}{\alpha \sigma} e^{-\alpha z} \right) + \frac{\kappa/\alpha}{\frac{\kappa A}{\alpha \sigma} e^{-\alpha z}} \mathcal{C}_{\frac{\kappa}{\alpha}} \left( \frac{\kappa A}{\alpha \sigma} e^{-\alpha z} \right) \right]$$

or

$$w'_E = \frac{\kappa A}{\sigma} e^{-\alpha z} \mathcal{C}_{\frac{\kappa}{\alpha}+1} \left( \frac{\kappa A}{\alpha \sigma} e^{-\alpha z} \right) - \kappa \mathcal{C}_{\frac{\kappa}{\alpha}} \left( \frac{\kappa A}{\alpha \sigma} e^{-\alpha z} \right). \quad (C6)$$

In the lower layer, the governing equation is

$$w_S'' + \frac{\kappa^2}{\sigma^2} \left[ \frac{A^2}{(z - z_0)^2} - \sigma^2 \right] w_S = 0, \quad (C7)$$

where  $A^2 = 1.6$  and  $z_0 = -6$ . Making the changes of variable  $\zeta = \kappa(z_0 - z)$  and  $w_S(z) = (\zeta/\kappa)^{1/2} X(\zeta)$ , we have that

$$\begin{aligned} w'_S(z) &= \frac{d\zeta}{dz} \frac{d}{d\zeta} \left[ \left( \frac{\zeta}{\kappa} \right)^{1/2} X(\zeta) \right] = -\frac{1}{2} \left( \frac{\kappa}{\zeta} \right)^{1/2} X(\zeta) - (\kappa \zeta)^{1/2} X'(\zeta) \\ w''_S(z) &= \frac{d^2 \zeta}{dz^2} \frac{d}{d\zeta} \left[ \left( \frac{\zeta}{\kappa} \right)^{1/2} X(\zeta) \right] + \left( \frac{d\zeta}{dz} \right)^2 \frac{d^2}{d\zeta^2} \left[ \left( \frac{\zeta}{\kappa} \right)^{1/2} X(\zeta) \right] \\ &= \kappa^2 \frac{d}{d\zeta} \left[ \frac{1}{2} \left( \frac{1}{\kappa \zeta} \right)^{1/2} X(\zeta) + \left( \frac{\zeta}{\kappa} \right)^{1/2} X'(\zeta) \right] \\ &= \kappa^{3/2} \zeta^{1/2} \left[ X''(\zeta) + \frac{1}{\zeta} X'(\zeta) - \frac{1}{4\zeta^2} X(\zeta) \right], \end{aligned}$$

Therefore, the governing Eq. (C7) is transformed to

$$\kappa^{3/2} \zeta^{1/2} \left( X'' + \frac{1}{\zeta} X' - \frac{1}{4\zeta^2} X \right) + \frac{\kappa^2}{\sigma^2} \left( \frac{A^2 \kappa^2}{\zeta^2} - \sigma^2 \right) \left( \frac{\zeta}{\kappa} \right)^{1/2} X = 0$$

or simply

$$\zeta^2 X'' + \zeta X' - \left[ \zeta^2 + \left( \frac{1}{4} - \frac{\kappa^2 A^2}{\sigma^2} \right) \right] X = 0, \tag{C8}$$

which is again identifiable as a form of Bessel's equation, whose solutions are the modified Bessel functions of pure imaginary order  $\nu = i\mu = i\sqrt{(\kappa^2 A^2 / \sigma^2) - (1/4)}$ . The modified Bessel function of the second type,  $K_{i\mu}(\zeta)$ , is the appropriate solution, since it approaches 0 as  $\exp(-\zeta)$  for large  $\zeta$ , as is evidenced by the asymptotic expansion (Abramowitz and Stegun, 1964, p. 378)

$$K_{i\mu}(\zeta) \approx \sqrt{\frac{\pi}{2\zeta}} e^{-\zeta} \left[ 1 - \frac{1 + 4\mu^2}{8\zeta} + \frac{(1 + 4\mu^2)(9 + 4\mu^2)}{2!(8\zeta)^2} - \dots + \dots \right]. \tag{C9}$$

There is a problem in the solution, however, in that  $K_{i\mu}(\zeta)$  must be evaluated for small values of  $\zeta$ . Formally, we would express  $K_{i\mu}(\zeta)$  in terms of the modified Bessel functions of the first kind (Abramowitz and Stegun, 1964, p. 375)

$$K_{i\mu}(\zeta) = \frac{1}{2} \pi \left( \frac{I_{-i\mu}(\zeta) - I_{i\mu}(\zeta)}{\sin(i\mu\pi)} \right). \tag{C10}$$

However, the standard series expansion for  $I_\nu(\zeta)$  is not well suited for considering purely imaginary orders and yields complex-valued functions when  $\nu = i\mu$ . For this reason, we employ the series expansion for solutions of Eq. (A8), which was suggested by Boole (1884, p. 238). The basic solution is of the form

$$X(\zeta) = \text{Bo}_\mu(\zeta) \equiv \phi_\mu(\zeta) \cos(\mu \ln \zeta) + \psi_\mu(\zeta) \sin(\mu \ln \zeta), \tag{C11a}$$

where

$$\phi_\mu(\zeta) = \sum_{n=0}^{\infty} a_n \zeta^{2n}; \quad a_n = \frac{na_{n-1} - \mu b_{n-1}}{4n(n^2 + \mu^2)} \tag{C11b}$$

$$\psi_\mu(\zeta) = \sum_{n=0}^{\infty} b_n \zeta^{2n}; \quad b_n = \frac{nb_{n-1} + \mu a_{n-1}}{4n(n^2 + \mu^2)}. \tag{C11c}$$

The second linearly independent solution is simply

$$\begin{aligned} X(\zeta) &= \text{Bo}_{-\mu}(\zeta) = \phi_{-\mu}(\zeta) \cos(-\mu \ln \zeta) + \psi_{-\mu}(\zeta) \sin(-\mu \ln \zeta) \\ &= \psi_\mu(\zeta) \cos(\mu \ln \zeta) - \phi_\mu(\zeta) \sin(\mu \ln \zeta). \end{aligned} \tag{C11d}$$

To express  $K_{i\mu}(\zeta)$  in terms of these real-valued functions, we must determine some relationship between the real-valued  $\text{Bo}_{\pm\mu}(\zeta)$  and the complex-valued  $I_{\pm i\mu}(\zeta)$ . We note that the  $I_{\pm i\mu}(\zeta)$  must have the same real part, since  $K_{i\mu}(\zeta)$  is a real-valued function. The standard series expansions for  $I_{\pm i\mu}(\zeta)$  are given by (Abramowitz and Stegun, 1964, p. 375)

$$I_{\pm i\mu}(\zeta) = \left(\frac{1}{2}\zeta\right)^{\pm i\mu} \sum_{n=0}^{\infty} \frac{\left(\frac{1}{4}\zeta^2\right)^n}{n! \Gamma(n+1 \pm i\mu)}. \quad (\text{C12})$$

Since  $\Gamma(n+z) = (n-1+z)(n-2+z) \dots (1+z) \Gamma(1+z)$  (Abramowitz and Stegun, 1964, p. 256), Eq. (C12) may be written in the alternative form

$$I_{\pm i\mu}(\zeta) = \frac{\cos\left(\mu \ln \frac{1}{2}\right) \pm i \sin\left(\mu \ln \frac{1}{2}\right)}{\Gamma(1 \pm i\mu)} [\cos(\mu \ln \zeta) \pm i \sin(\mu \ln \zeta)] S_{\pm i\mu}(\zeta), \quad (\text{C13})$$

where

$$S_{\pm i\mu}(\zeta) = \left[ 1 + \frac{\zeta^2}{1! 2^2(1 \pm i\mu)} + \frac{\zeta^3}{2! 2^4(1 \pm i\mu)(2 \pm i\mu)} + \dots \right]. \quad (\text{C14})$$

Multiplying out the terms in Eq. (C13), we obtain

$$I_{\pm i\mu}(\zeta) = \frac{S_{\pm i\mu}(\zeta)}{\Gamma(1 + \mu^2)} [\alpha_{\mu} \cos(\mu \ln \zeta) + \beta_{\mu} \sin(\mu \ln \zeta)] \\ \mp \frac{i S_{\pm i\mu}(\zeta)}{\Gamma(1 + \mu^2)} [\beta_{\mu} \cos(\mu \ln \zeta) - \alpha_{\mu} \sin(\mu \ln \zeta)], \quad (\text{C15})$$

where

$$\alpha_{\mu} \equiv \cos\left(\mu \ln \frac{1}{2}\right) \text{Re} [\Gamma(1 + i\mu)] + \sin\left(\mu \ln \frac{1}{2}\right) \text{Im} [\Gamma(1 + i\mu)] \quad (\text{C16a})$$

$$\beta_{\mu} \equiv \sin\left(\mu \ln \frac{1}{2}\right) \text{Re} [\Gamma(1 + i\mu)] - \cos\left(\mu \ln \frac{1}{2}\right) \text{Im} [\Gamma(1 + i\mu)] \quad (\text{C16b})$$

with the symbols Re and Im denoting real and imaginary parts, respectively. Term-by-term comparison of the series expansions for  $I_{\pm i\mu}(\zeta)$  and  $\text{Bo}_{\pm\mu}(\zeta)$  yields the result that

$$I_{\pm i\mu}(\zeta) = \frac{1}{2} \{ [(\alpha_{\mu} - \beta_{\mu}) \pm i(\alpha_{\mu} + \beta_{\mu})] \text{Bo}_{+\mu}(\zeta) + [(\alpha_{\mu} + \beta_{\mu}) \mp i(\alpha_{\mu} - \beta_{\mu})] \text{Bo}_{-\mu}(\zeta) \}, \quad (\text{C17})$$

so that the series expansion for  $K_{i\mu}(\zeta)$  may be expressed by

$$K_{i\mu}(\zeta) = \frac{1}{2\mu} [(\alpha_{\mu} - \beta_{\mu}) \text{Bo}_{-\mu}(\zeta) - (\alpha_{\mu} + \beta_{\mu}) \text{Bo}_{\mu}(\zeta)]. \quad (\text{C18})$$

Thus, the solution in the lower region is given by

$$w_s(z) = \frac{(z_0 - z)^{1/2}}{2\mu} \{ (\alpha_{\mu} - \beta_{\mu}) \text{Bo}_{-\mu}[\kappa(z_0 - z)] - (\alpha_{\mu} + \beta_{\mu}) \text{Bo}_{\mu}[\kappa(z_0 - z)] \}, \quad (\text{C19})$$

where  $\mu = \sqrt{(\kappa^2 A^2 / \sigma^2) - (1/4)}$ .

Since our expression for  $K_{\nu}(\zeta)$  is designed for purely imaginary orders  $\nu = i\mu$ , the standard cylinder function recurrence relations among derivatives of functions and functions whose orders differ by  $\pm 1$ , one of which was used in obtaining an expression for

$dw_E(z)/dz$ , are not applicable if one desires to retain purely imaginary orders in the expressions for the functions and their derivatives. The expression for  $dw_S(z)/dz$  is therefore obtained by formally taking the derivative of Eq. (C19). We have that

$$\begin{aligned} \frac{dw_S(z)}{dz} &= \frac{-1}{2(z_0 - z)} w_S(z) + \frac{(z_0 - z)^{1/2}}{2\mu} \frac{d\zeta}{dz} \frac{d}{d\zeta} [(\alpha_\mu - \beta_\mu) Bo_{-\mu}(\zeta) - (\alpha_\mu + \beta_\mu) Bo_\mu(\zeta)] \\ &= \frac{-w_S(z)}{2(z_0 - z)} - \frac{\kappa(z_0 - z)^{1/2}}{2\mu} \left[ (\alpha_\mu - \beta_\mu) Bo'_{-\mu}(\zeta) - (\alpha_\mu + \beta_\mu) Bo'_\mu(\zeta) \right] \end{aligned} \quad (C20)$$

where

$$\frac{dBo_\mu(\zeta)}{d\zeta} = \frac{\mu}{\zeta} Bo_{-\mu}(\zeta) + \phi'_\mu(\zeta) \cos(\mu \ln \zeta) + \psi'_\mu(\zeta) \sin(\mu \ln \zeta) \quad (C21a)$$

$$\frac{dBo_{-\mu}(\zeta)}{d\zeta} = -\frac{\mu}{\zeta} Bo_\mu(\zeta) + \psi'_\mu(\zeta) \cos(\mu \ln \zeta) - \phi'_\mu(\zeta) \sin(\mu \ln \zeta) \quad (C21b)$$

with

$$\phi'_\mu(\zeta) = \sum_{n=1}^{\infty} 2na_n \zeta^{2n-1} = \frac{2}{\zeta} \sum_{n=0}^{\infty} na_n \zeta^{2n} \quad (C22a)$$

$$\psi'_\mu(\zeta) = \sum_{n=1}^{\infty} 2nb_n \zeta^{2n-1} = \sum_{n=0}^{\infty} nb_n \zeta^{2n} \quad (C22b)$$

Since  $w_S$  is constructed to asymptotically approach 0 at large depths, the conditions which must be satisfied for the complete solution are

$$w_E(0) = C_1 J_{\kappa/\alpha} \left( \frac{\kappa A}{\alpha \sigma} \right) + C_2 Y_{\kappa/\alpha} \left( \frac{\kappa A}{\alpha \sigma} \right) = 0 \quad (C23a)$$

$$w_E(-10) = C_1 J_{\kappa/\alpha} \left( \frac{\kappa A}{\alpha \sigma} e^{-10\alpha} \right) + C_2 Y_{\kappa/\alpha} \left( \frac{\kappa A}{\alpha \sigma} e^{-10\alpha} \right) = w_S(-10) \quad (C23b)$$

$$w'_E(-10) = \frac{\kappa A}{\sigma} e^{-10\alpha} \left\{ C_1 J_{\frac{\kappa}{\alpha}+1} \left( \frac{\kappa A}{\alpha \sigma} e^{-10\alpha} \right) + C_2 Y_{\frac{\kappa}{\alpha}+1} \left( \frac{\kappa A}{\alpha \sigma} e^{-10\alpha} \right) \right\} - \kappa w_E(-10) = w'_S(-10), \quad (C23c)$$

where  $C_1$  and  $C_2$  are arbitrary parameters. If the conditions (C23a) and (C23b) are used to evaluate  $C_1$  and  $C_2$ , Eq. (C23c) will be satisfied only if the proper choices of  $\kappa$  and  $\sigma$  are made. The conditions (C23i) represent, of course, a highly transcendental equation system for determining the dispersion relations and must be evaluated numerically. The error involved in a numerical evaluation will depend on the accuracy to which the various functions are computed. In computing the dispersion relations which correspond to this analytic solution, standard program library subroutines which have a high degree of accuracy were used to evaluate the gamma and regular Bessel functions. The modified Bessel functions of imaginary order were evaluated using the series expansions for  $Bo_{\pm\mu}(\zeta)$  and  $Bo'_{\pm\mu}(\zeta)$  described in the text and were, in general, computed to a theoretical accuracy of much better than 0.01% based on the convergence of the series. In practice, the analytical dispersion curves were computed for the wave number  $\kappa$  as a function of phase speed  $c = \sigma/\kappa$  by specifying a phase speed  $c$  and determining the values of  $\kappa$  for which Eq. (C23c) was satisfied, with  $C_1$  and  $C_2$  determined by (C23a) and (C23b).

Security Classification		DOCUMENT CONTROL DATA - R & D	
<i>(Security classification of title, body of abstract and indexing annotation must be entered when the overall report is classified)</i>			
1. ORIGINATING ACTIVITY (Corporate author)		2a. REPORT SECURITY CLASSIFICATION	
Naval Research Laboratory Washington, D.C. 20390		Unclassified	
		2b. GROUP	
3. REPORT TITLE			
NUMERICAL CALCULATION OF DISPERSION RELATIONS FOR INTERNAL GRAVITY WAVES			
4. DESCRIPTIVE NOTES (Type of report and inclusive dates)			
An interim report on a continuing NRL problem.			
5. AUTHOR(S) (First name, middle initial, last name)			
Thomas H. Bell, Jr.			
6. REPORT DATE		7a. TOTAL NO. OF PAGES	7b. NO. OF REFS
September 7, 1971		46	8
8a. CONTRACT OR GRANT NO.		9a. ORIGINATOR'S REPORT NUMBER(S)	
NRL Problem G01-06		NRL Report 7294	
b. PROJECT NO.		9b. OTHER REPORT NO(S) (Any other numbers that may be assigned this report)	
A37-370/F08-125-703			
c.			
d.			
10. DISTRIBUTION STATEMENT			
Approved for public release; distribution unlimited.			
11. SUPPLEMENTARY NOTES		12. SPONSORING MILITARY ACTIVITY	
		Department of the Navy (Naval Air Systems Command) Washington, D.C. 20360	
13. ABSTRACT			
<p>Numerical procedures have been developed to calculate the frequency/wave number dispersion relations for internal gravity waves in arbitrary density gradients. The basic properties of the differential equation governing these waves have been considered in developing these procedures. These procedures have been used to develop a set of Fortran subroutines, capable of computing, printing in tabular form, and producing line printer plots of the dispersion relations. Source listings of these subroutines and full descriptions of their use are given. To illustrate the accuracy of the system and to display the results of violating certain restrictions involved in the numerical procedure, an extensive comparison is made between numerical results and an analytic solution for the dispersion relations. The analytical model corresponds to a hypothetical fluid system which closely approximates the average water properties measured during field tests in Bute Inlet, British Columbia, a fjord-type estuary with a density structure characterized by a strong gradient between two nearly homogeneous layers.</p>			

14. KEY WORDS	LINK A		LINK B		LINK C	
	ROLE	WT	ROLE	WT	ROLE	WT
Internal waves Dispersion relations Numerical methods Bessel functions (imaginary order) Strum-Liouville problem						