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Meromorphic Solutions of Generalized Algebraic Differential Equations

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Abstract: The rate of growth of meromorphic functions f, which are solutions of algebraic differential equations whose coefficients $a(z)$ are arbitrary meromorphic functions, is investigated. By a method based on Nevanlinna's theory of meromorphic functions, it has been shown that if f'/f has ∞ as its Nevanlinna exceptional value, then the ratio $T(r, f'/f)/T(r, a(z))$, as $r \to \infty$ outside a set of r values of finite measure, is bounded for at least one of the coefficients $a(z)$.

INTRODUCTION

We shall study the growth rate of certain meromorphic solutions of an algebraic differential equation of arbitrary order whose coefficients are meromorphic functions. However, along this line, by methods based on the Wiman-Valiron theory [1,2,3] which applies only to entire functions, some interesting results in certain classes of equations, especially the first-order algebraic differential equations (*i.e.*, equations of the form $P(z,w,w') = 0$, where P is a polynomial in z, $w(z)$, and $w'(z)$), have been obtained. We mention here the following several results.

Valiron [4] proved that any transcendental entire function which satisfies a first-order algebraic differential equation must be of finite order. (In fact, it must be of regular growth.) Later on, Bank [5] made a further study and extended the case to its coefficients which are entire functions of finite order.

Another result, shown by Wittich [6; pp. 64-65], can be stated as follows:

THEOREM A. *Let $P(z,w,w_1,\ldots w_l) \equiv a_{n_0 n_1 \ldots n_l}(z) w^{n_0} w_1^{n_1} \ldots w_l^{n_l} = 0$ be an algebraic differential equation where the $a_{n_0 n_1 \ldots n_l}(z)$ are polynomials and $w^i(z) = w^{(i)}(z)$ is the ith derivative of w. Let $N = n_0 + n_1 + \ldots + n_l$ denote the dimension of the term $w^{n_0} w_1^{n_1} \ldots w_l^{n_l}$. Then the above equation has no transcendental entire solution if only one term appears in the equation with a maximal dimension.*

Regarding meromorphic solutions, A. A. Goldberg (see [7]) proved the following theorem.

THEOREM B. *Let $P(z,w,w') = 0$ be a first-order differential equation with all the coefficients which are polynomials; then all meromorphic solutions of $P(z,w,w') = 0$ are of finite order.*

Later on, by using different methods, Bank [7] obtained a stronger result as follows:

THEOREM C. *Let $P(z,w,w') \equiv \Sigma a_{kj}(z) w^k (w')^j = 0$ be a first-order differential equation with dimension N_0 (i.e., $N_0 = max\{k+j;\ a_{kj} \neq 0\}$, where a_{kj} is allowed to be an entire function of finite order if $k + j < N_0$, while if $k + j = N_0$, $a_{kj}(z)$ is required to be a polynomial). Then a meromorphic solution of such an equation cannot be written as the quotient of two entire functions f/g, where f is of infinite order and g is of finite order. In particular, any meromorphic solution whose sequence of poles has a finite exponent of convergence (see, e.g., [8]) must be of finite order.*

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All the above results indicated that for certain special classes of algebraic differential equation there exist close relations among (a) the growth rate of the coefficients of the equation, (b) the density of the zeros or poles of a meromorphic solution of the equation, and (c) the growth rate of the meromorphic solution of the equation.

The purpose of this report is to strengthen this indication for certain algebraic differential equations of any order. To be more specific, we shall show that if the densities of both the zeros and the poles of a meromorphic solution $f(z)$ of a generalized algebraic differential equation (the definition of which will be stated in the next section) is much smaller than the growth rate of f'/f, then the ratio $T(r, f'/f)/T(r, a(z))$, as $r \to \infty$ outside a set of r values of finite measure, is bounded for at least one coefficient $a(z)$ appearing in the differential equation. In particular, if all the coefficients $a(z)$ in $P(z,w,w_1,\dots w_l)$ are of finite order and no greater than α, then $T(r,f) = O(1)e^{r^{\alpha+\epsilon}}$ as $r \to \infty$ for any given positive constant ϵ. The proof of the above result will be based on Nevanlinna's fundamental theorems of meromorphic functions [9; chaps. 1, 2, and 3].

DEFINITIONS, TERMINOLOGY, AND NOTATION

Throughout the rest of the report when we say that a function is meromorphic, we mean that the function is meromorphic in the entire finite plane.

Definition 2-1: A generalized algebraic differential equation means a differential equation of the form

$$P(a_1(z), a_2(z), \dots a_n(z), w(z), w_1(z), \dots w_l(z)) = 0 \tag{1}$$

where P is a polynomial in $w(z)$, $w_1(z), \dots w_l(z)$, and all the coefficients $a_i(z)$ are meromorphic functions.

In order to give a quantitative measure relating the growth of a meromorphic function $f(z)$ and the density of its zeros or poles, the notation of Nevanlinna characteristic $T(r,f)$ and the counting function $N(r,f)$ are employed. It is therefore assumed that the reader is familiar with the fundamental concept of Nevanlinna's theory of meromorphic functions and its standard symbols such as $T(r,f)$, $N(r,f)$, $m(r,f)$, *etc.** In what follows, the notation f will always denote a nonconstant meromorphic function. The Nevanlinna deficiency of a value c for f is defined as

$$\delta(c,f) = \varliminf_{r \to \infty} \frac{m\left(r, \frac{1}{f-c}\right)}{T(r,f)} = 1 - \varlimsup_{r \to \infty} \frac{N\left(r, \frac{1}{f-c}\right)}{T(r,f)}.$$

We note that $0 \leq \delta(c,f) \leq 1$.

Definition 2-2: We shall call a number c in the extended complex plane a Nevanlinna exceptional value of f if $\delta(c,f) = 1$. We shall denote by $S(r,f)$ any quantity satisfying

$$S(r,f) = o\{T(r,f)\}$$

as $r \to \infty$ outside a set E of finite measure.

Definition 2-3: A differential polynomial $P_n(f)$ is used to denote a polynomial in f and its derivatives of degree, at most, n whose coefficients $a(z)$ are meromorphic functions satisfying

$$T(r,a(z)) = S(r,f). \tag{2}$$

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The general term of $P_n(f)$ has the form $a(z)(f)^{n_0}(f_1)^{n_1}\ldots(f_\ell)^{n_\ell}$ where f_i is the ith derivative of f.

PRELIMINARY LEMMAS

Our main results will be the easy consequences after we quote and establish the following lemmas:

LEMMA 1. *Let* $P_n(f) = f_n(z) + P_{n-1}(f)$; *then* $m(r,P_n(f)) \geqslant n\ m(r,f) + S(r,f)$.

Proof. We may write

$$\begin{aligned} P_n(f) &= f^n(z) + P_{n-1}(f) \\ &= f^n(z)\left(1 + \frac{a_1(z)}{f} + \frac{a_2(z)}{f^2} + \ldots + \frac{a_n(z)}{f^n(z)}\right), \end{aligned}$$

where

$$a_i(z) = a_{n_0\ldots n_\ell}(z)\left(\frac{f}{f}\right)^{n_0}\left(\frac{f_1}{f}\right)^{n_1}\ldots\left(\frac{f_\ell}{f}\right)^{n_\ell} \tag{3}$$

and the $a_{n_0\ldots n_\ell}(z)$ are coefficients of $P_{n-1}(f)$ with $n_0 + n_1 + \ldots n_\ell = n - i$. By conditions (3) and a result of Milloux's [see 9; p. 55] we conclude that

$$m(r,a_i(z)) = S(r,f),\ i = 1, 2,\ldots n. \tag{4}$$

Now on the circle $|z| = r$ we set

$$A(z) = \mathrm{Max}\{|a_i(z)|^{1/i}\},\ \mathrm{i} = 1, 2,\ldots n. \tag{5}$$

Let E_1 be the set of θ in the interval $[0,2\pi]$ for which $|f(re^{i\theta})| \geqslant 2A(re^{i\theta})$, and let E_2 be the complementary set. On E_1 we have

$$\begin{aligned} |P_n(f)| &= \left|f^n(z)\left(1 + \frac{a_1(z)}{f(z)} + \frac{a_2(z)}{f^2(z)} + \ldots + \frac{a_n(z)}{f^n(z)}\right)\right| \\ &\geqslant |f^n(z)|\left(1 - \left|\frac{a_1(z)}{f(z)}\right| - \left|\frac{a_2(z)}{f^2(z)}\right|\ldots - \left|\frac{a_n(z)}{f^n(z)}\right|\right) \\ &\geqslant |f^n(z)|\left(1 - \frac{A(z)}{2A(z)} - \frac{A^2(z)}{(2A(z))^2}\ldots - \frac{A^n(z)}{(2A(z))^n}\right) = \frac{|f^n(z)|}{2^n}. \end{aligned} \tag{6}$$

Thus

$$\begin{aligned} n\ m(r,f) &= m(r,f^n) \\ &= \frac{1}{2\pi}\int_{E_1} \log^+\left|f^n(re^{i\theta})\right| d\theta + \frac{1}{2\pi}\int_{E_2} \log^+\left|f^n(re^{i\theta})\right| d\theta \\ &\leqslant \frac{1}{2\pi}\int_0^{2\pi} \log^+\left|2^n P_n(f)\right| d\theta + \frac{1}{2\pi}\int_0^{2\pi} \log^+\left|2^n A_n(re^{i\theta})\right| d\theta \end{aligned}$$

$$= m(r,P_n(f)) + n \log 2 + S(r,f). \tag{7}$$

It follows that

$$m(r,P_n(f)) \geqslant n\, m(r,f) + S(r,f).$$

This completes the proof of Lemma 1.

We shall need the following definition before we state Lemma 2.

Definition 3-1: To each function $\lambda(r)$, positive, continuous, and nondecreasing on $0 < r < \infty$, we associate the class F_λ of functions f satisfying the conditions that

(a) f is meromorphic in the whole finite plane
(b) $T(r,f) = O\{\lambda(r)\}$; as $r \to \infty$.

In order to deal with our problem, we ask that condition (b) holds for any set values of r of infinite measure.

From this we immediately have that a meromorphic function $f \notin F_\lambda$, and then it is necessary $\lambda(r) = S(r,f)$.

It has been shown [11] that F_λ is algebraically closed in the field of all meromorphic functions.

With all the above definitions, one can obtain the following:

LEMMA 2. *Let $f(z)$ be a meromorphic solution of the equation*

$$f^n(z) + P_{n-1}(f) = 0 \tag{8}$$

where all the coefficients of $P_{n-1}(f)$ belong to F_λ. Furthermore, assume that $\overline{N}(r,f) = S(r,f)$ holds; then f also belongs to F_λ.

Proof. Assume that f is a meromorphic solution of Eq. (8) and that $f \notin F$; then we would have $T(r,b(z)) = S(r,f)$ for all the coefficients $b(z)$ in $P_{n-1}(f)$. Adding a nonzero constant c to both sides of the equation we obtain

$$f^n + P_{n-1}(f) + c = c.$$

Now we note that the Tumura-Clunie Theorem [see 9; p. 69] still holds when the condition: $N(r,f) = S(r,f)$ is replaced by $\overline{N}(r,f) = S(r,f)$. Thus by applying the theorem, we conclude that

$$(f + a(z))^n = c$$

where $a(z)$ is a meromorphic function satisfying $T(r,a(z)) = S(r,f)$.

This will lead to a contradition, and the lemma is thus proved.

LEMMA 3 [9; p. 73]. *Suppose $f(z)$ is meromorphic in a domain D and let $F(z) = f'(z)/f(z)$; then for $n \geqslant 1$, we have*

$$\frac{f^{(n)}}{f} = F^n + \frac{n(n-1)}{2} F^{n-2}F' + a_n F^{n-3}F'' + b_n F^{n-4}F'^2 + P_{n-3}(F).$$

$$= F^n + P_{n-1}(F)$$

where $a_n = (1/6)\, n(n-1)(n-2)$, $b_n = (1/8)\, n(n-1)(n-2)(n-3)$, and $P_{n-1}(F)$ is a differential polynomial in F with constant coefficients which vanishes identically for $n = 3$ and had degree $n-3$ when $n > 3$. (Remark: The original proof of Lemma 3 was due to Tumura [12].)

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MAIN RESULTS

THEOREM 1. *Let* $P(a_1(z), a_2(z), \dots a_n(z), w(z), w_1(z), \dots w_t(z)) = 0$ *be a generalized algebraic differential equation of dimension* N_0 *(where* $N_0 \geqslant 1$*). Assume that P has the form*

$$P \equiv w^n(z) + P_{n-1}(a_1(z), \dots a_n(z), w(z), \dots w_t(z)) = 0. \tag{9}$$

Suppose that $f(z)$ *is a transcendental meromorphic solution of Eq. (9) such that the*

$$T(r, a_i(z)) = S(r, f) \tag{10}$$

are satisfied for all the coefficients $a_i(z)$ *in* P_{n-1}. *Then* $\delta(\infty, f) = 0$. *In particular, we obtain*

COROLLARY 1. *Given a generalized algebraic differential equation which has the form*

$$w^n(z) + P_{n-1}(a_1(z), a_2(z), \dots a_n(z), w(z), w_1(z), \dots w_t(z)) = 0. \tag{11}$$

Then if a transcendental entire function $f(z)$ *be a solution of Eq. (11), it is necessary that at least one of the conditions (10) fails to hold.*

Proof of the theorem. Assume that f is a meromorphic solution of Eq. (11); then for a constant c ($c \neq 0$) we have

$$f^n + P_{n-1}(a_1(z), a_2(z), \dots a_n(z), f(z), f_1(z), \dots f_t(z)) + c = c. \tag{12}$$

By Lemma 1, we get

$$m(r, f^n + P_{n-1} + c) \geqslant nm(r, f) + S(r, f). \tag{13}$$

However, from Eq. (12) we should have

$$\begin{aligned} m(r, f^n + P_{n-1} + c) &= m(r, c) = O(1) \\ &= S(r, f). \end{aligned} \tag{14}$$

Therefore Eqs. (13) and (14) yield

$$m(r, f) = S(r, f).$$

It follows that $\delta(\infty, f) = 0$.

THEOREM 2. *Let* $P(z, w, w_1, \dots w_t) = 0$ *be a generalized algebraic differential equation of the form*

$$P \equiv w^n(z) + P_{n-1}(z, w, w_1, \dots w_t) = 0. \tag{15}$$

Assume that P_{n-1} *contains a term* $a(z)$ *which is of zero dimension and is not identically zero. Then* $\delta(0, f) = 0$ *for any transcendental meromorphic function* $f(z)$ *which is a solution of Eq. (15) such that the conditions (10) are satisfied.*

Proof. Simply by setting $f(z) = 1/g(z)$ and substituting into Eq. (15), we get

$$g^n(z) + Q_{n-1}(z, g, g_1, \dots g_t) = 0.$$

According to Theorem 1, we conclude that $\delta(\infty, g) = 0$, which is equivalent to $\delta(0, f) = 0$.

We now state and prove our main theorem.

THEOREM 3. *Let*

$$P(z, w, w_1, w_2 \dots w_t) = 0 \tag{16}$$

be a generalized algebraic differential equation of dimension N_0. The general term has the form $a_{n_0 n_1 \dots n_\ell}(z) w^{n_0}(w_1)^{n_1} \dots (w_\ell)^{n_\ell}$, where $a_{n_0 n_1 \dots n_\ell}(z)$ are meromorphic functions and belong to F_λ. Let us denote

$$ {}_{j_i}A_{N_i}(z) = \sum a_{n_0 n_1 \dots n_\ell}(z) \qquad (N_i \leqslant N_0,\ j_i \geqslant 0) \tag{17}$$

where $N_i = n_0 + n_1 + \dots + n_\ell$ and $j_i = n_1 + 2n_2 + \dots + \ell n_\ell = \operatorname{Max}\{n_1 + 2n_2 + \dots + \ell n_\ell, 0\}$, and assume that at least one of the ${}_{j_i}A_{N_i}(z)$ is not identically equal to zero.

Assume that $f(z)$ is a meromorphic function which is a solution of Eq. (16) such that the conditions

$$T(r, a_{n_0 \dots n_\ell}(z)) = S(r,f) \tag{18}$$

are satisfied for all the coefficients, and assume that the distinct zeros and poles of f satisfy $N(r,f'/f) = S(r,f'/f)$ which is equivalent to $\overline{N}(r,f) + \overline{N}(r,f/f) = S(r,f)$, then $T(r,f'/f) = O\{T(r,a_{n_0 \dots n_\ell}(z))\}$, as $r \to \infty$ outside a set finite measure for coefficient $a_{n_0 \dots n_\ell}(z)$ in $P(z,w,\dots w_\ell)$. In other words, f'/f also belongs to F_λ.

Corollary 2. In addition to the hypotheses of Theorem 3, if we assume that all the coefficients $a_{n_0 n_1 \dots n_\ell}(z)$ are meromorphic functions of finite order, then for meromorphic function f which is a solution of Eq. (16) with its zeros and poles satisfying $N(r,f'/f) = S(r,f'/f)$, and supposing that the multiplicities of all the zeros and poles of f are bounded, we have

$$T(r,f) = O(1)e^{r^{\alpha+\epsilon}}$$

where α is the maximal order among all the coefficients in $P(z,w,w_1,\dots w_\ell)$ and ϵ is an arbitrary positive constant.

Proof. According to Theorem 3 we have

$$T\left(r, \frac{f'}{f}\right) = O\{T(r, a_{n_0 \dots n_\ell}(z))\}, \tag{19}$$

as $r \to \infty$ outside a set of finite measure, for some coefficient

$$a_{n_0 \dots n_\ell}(z) \text{ in } P(z,w,\dots w_\ell).$$

From this and a result of Hellerstein's [13] we conclude that

$$T\left(r, \frac{f'}{f}\right) = O(1)r^{\alpha+\epsilon} \tag{20}$$

as $r \to \infty$ for any given positive constant. Combining this with the assumption on the multiplicities of the zeros and poles, and the results of [9; p. 26], we conclude that the exponent convergence of both the zeros and the poles of f are no greater than α. Hence we may write

$$f(z) = \frac{h_1(z)}{h_2(z)} e^{g(z)} \tag{21}$$

where $h_1(z)$ is the canonical product of genus $n_1 (\leqslant \alpha)$ formed with zeros of $f(z)$, and $h_2(z)$ is the canonical product of genus $n_2 (\leqslant \alpha)$ formed with the poles of $f(z)$; $g(z)$ is an entire function.

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By taking logarithmic derivative on both sides of Eq. (21) we obtain

$$\frac{f'(z)}{f(z)} = \frac{h_1'(z)}{h_1(z)} - \frac{h_2'(z)}{h_2(z)} + g'(z) \tag{22}$$

or

$$-g'(z) = \frac{h_1'(z)}{h_1(z)} - \frac{h_2'(z)}{h_2(z)} - \frac{f'(z)}{f(z)}.$$

Thus by a result of Whittaker's [see 14], we can readily obtain

$$\begin{aligned} T(r,g') &\leq T\left(r, \frac{h_1'}{h_1}\right) + T\left(r, \frac{h_2'}{h_2}\right) + T\left(r, \frac{f'}{f}\right) + O(1) \\ &= O(1)r^{\alpha+\epsilon}. \end{aligned}$$

Since ϵ can be artitrary small, we conclude that the order of g' (hence g) is no greater than α, and our assertion follows from this.

COROLLARY 3. *In addition to the hypotheses of Theorem 3, if we assume that all the coefficients are rational functions, then any possible meromorphic solution f, with $N(r,f'/f) = S(r, f'/f)$, is necessarily of finite order and has the form*

$$f(z) = \frac{p_1(z)}{p_2(z)} e^{p_3(z)}$$

where the $p_i(z)$ $(i = 1, 2, 3)$ are polynomials. (Thus it turns out that in this case the Nevanlinna exceptional value ∞ of f'/f is, in fact, Picard exceptional values of f.)

Proof. In this case we have $T(r,f'/f) = O\{\log r\}$. The results follows from this.

COROLLARY 4. *Let $f(z)$ be a meromorphic solution of Equation (16) whose coefficients are assumed to be rational functions. Suppose that for some rational function $R(z)$, we have $\overline{N}(r,f) + \overline{N}(r, 1/f-R) = S(r, f'-R'/f-R)$ then f must be of finite order and has the form $F(z) = R(z) + (p_1(z)/p_2(z))\, e^{p_3(z)}$.*

Proof. By considering the function $F(z) = f(z) - R(z)$, the result follows from this and Corollary 3. We omit the details here.

Before we proceed to prove Theorem 3 we shall give an example to show that the assumption on ${}_{j_i}A_{N_i}(z)$ (conditions (17)) are crucial.

Example [6; p. 70]: The equation

$$P(z,w,w_1 \ldots w_l) = (w''w)^2 - 2\, w''(w')^2\, w + (w')^4 + (w'w)^2 - w^4 = 0$$

has an entire solution $w(z) = e^{\sin z}$ which is of infinite order and satisfies $N(r, w'/w) = S(r, w'/w)$. However, the function w'/w is of finite order and nonconstant.

Proof of Theorem 3. By combining terms of the same dimenstion in $P(z,w,w_1, \ldots w_l)$ we obtain

$$\sum_{n_{01}+n_{02}+\cdots n_{0k_0}=N_0} a_{n_{01}n_{02}\cdots n_{0k_j}}(z)\,(w)^{n_{01}} \ldots (w_l)^{n_{0k_0}} + \ldots.$$

$$\ldots + \sum_{n_{i1}+n_{i2}+\ldots+n_{ik_i}=N_i} a_{n_{i1}\ldots n_{ik_i}}(z)\,(w)^{n_{i1}}(w_1)^{n_{i2}}\ldots(w)^{n_{ik_i}} + \ldots = 0 \tag{23}$$

where $N_0 \geqslant N_1 \geqslant N_2 \ldots$.

Now assume that f is a meromorphic solution of Eq. (16) which satisfies conditions (18) and $N(r, f'/f) = S(r, f'/f)$. By the fact that $T(r, f'/f) \leqslant T(r,f') + T(r,f) + O(1) \leqslant 2T(r,f) + T(r,f) + S(r,f)$ [9, p. 55], it follows that

$$\overline{N}(r,f) + \overline{N}\left(r, \frac{f'}{f}\right) = S\left(r, \frac{f'}{f}\right) = S(r,f). \tag{24}$$

Also by Nevanlinna's estimate on logarithmic derivative [10; p. 63[, we have

$$\begin{aligned} T\left(r, \frac{f'}{f}\right) &= m\left(r, \frac{f'}{f}\right) + N\left(r, \frac{f'}{f}\right) \\ &= S(r,f) + S(r,f) \\ &= S(r,f). \end{aligned} \tag{25}$$

Setting $f'(z)/f(z) = h(z)$ and using Lemma 3 we have

$$\frac{f^{(n)}(z)}{f(z)} = h^n(z) + P_{n-1}(h).$$

or

$$f^{(n)} = (h^n(z) + P_{n-1}(h))f(z), \quad n = 1, 2, 3 \ldots. \tag{26}$$

Substituting these into Eq. (16) we obtain

$$\begin{aligned} &({}_{j_0}A_{N_0}(z)\, h^{j_0} + Q_{j_0-1}(h))\, f^{N_0} + ({}_{j_1}A_{N_1}(z)\, h^{j_1} + Q_{j_1-1}(h))\, f^{N_1} \\ &\quad + \ldots + ({}_{j_i}A_{N_i}(z)\, h^{j_i} + Q_{j_i-1}(h))\, f^{N_i} + \ldots = 0 \end{aligned} \tag{27}$$

where $Q_{j_i}(h)$ are differential polynomials in h with coefficients $b_{n_{i0}n_{i1}\ldots n_{ik_i}}(z)$ which are linear combinations of the coefficients $a_{n_{i0}n_{i1}\ldots n_{ik_i}}(z)$ in $P(z,w,w_1,\ldots w_t)$.
Set

$${}_{j_i}A_{N_i}(z)\, h^{j_i} + Q_{j_i-1}(h) = B_i(z). \tag{28}$$

We note from Eq. (25) and Ref. 3, p. 55, that

$$T(r,h^{(n)}) = S(r,f).$$

Then

$$\begin{aligned} T(r,B_i(z)) \leqslant\ & T(r,\ {}_{j_i}A_{N_i}(z)) + O\{T(r,h)\} + \\ & \sum T(r,\ a_{n_{i0}n_{i1}\ldots n_{ik_i}}(z)) = S(r,f), \quad i = 1, 2 \ldots. \end{aligned}$$

Hence from conditions (24) and Lemma 2, we conclude that Eq. (27) holds only if

$$B_i(z) = 0 \tag{29}$$

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for all the i's. That is,

$$ {}_{j_i}A_{N_i}(z)\, h^{j_i} + Q_{j_i-1}(h) = 0, \quad i = 1, 2 \ldots . \tag{30} $$

Now since not all the ${}_{j_i}A_{N_i}$ are identically zero, at least one of the Eq. (29) is not identically zero unless, again by Lemma 2,

$$ T(r,h) = O\{T(r,\, b_{n_{i0}n_{i1}\ldots n_{ik_i}}(z))\} \tag{31} $$

as $r \to \infty$ outside a set of values of r of finite measure for some coefficient $b_{n_{i0}\ldots n_{ik_i}}(z)$ appearing through the Q's in Eq. (28). This implies that

$$ T(r,h) = O\{T(r,\, a_{n_{i0}n_{i1}\ldots n_{ik_i}}(z))\} \tag{32} $$

for some coefficient $a_{n_{i0}n_{i1}\ldots n_{ik_i}}(z)$ in $P(z,w,\ldots w_t)$ since the $b_{n_{i0}n_{i1}\ldots n_{ik_i}}(z)$ are linear combinations of $a_{n_{i0}\ldots n_{ik_i}}(z)$.

This completes the proof of our theorem.

Let us denote by H_N the homogeneous differential polynomial of degree N in Eq. (16). Assume that f is a meromorphic solution of Eq. (16), such that the conditions (18) and (24) are satisfied. Then in view of Eq. (29), f necessarily is a solution of $H_n = 0$ for all $N \leqslant N_0$.

Thus is H_N contains only one term for some $N \leqslant N_0$, then the solution of f of $H_N = 0$ will be a polynomial and hence the condition that $N(r,\, f'/f) = S(r,\, f'/f)$ will not be satisfied. So we have the following:

COROLLARY 5. *Let $P(z,w,w_1,\ldots w_t) = 0$ be an nth-order generalized algebraic differential equation with dimension N_0. Suppose that in P, for some N ($N \leqslant N_0$), H_N is a single term. Then to any f which is a meromorphic solution of $P = 0$, and non-constant we have either $\delta(0,\, f'/f) < 1$ or $\delta(\infty,\, f'/f) < 1$.*

Finally we wish to add that, after the present work had been undertaken, it was brought to the author's attention by Professor Steven Bank that he had done some work on the same problems. In fact, by using the same tool—Nevanlinna's theory—with a slightly different approach he [15] established a sufficient condition that a meromorphic solution of an nth-order algebraic differential equation, having arbitrary coefficients, necessarily be a solution of each homogeneous part of the equation.

Using this result and the main result of Ref. 5, he treated algebraic differential equations of *second* order and obtained a conclusion similar to what we have accomplished in Theorem 3. Nevertheless, the methods we used here can be applied to serve certain algebraic differential equations of any order.

In the meantime, we would like to point out that the assumption we made in Theorem 3, that some ${}_{j_i}A_{N_i}(z)$ is not identically zero, is very similar to the condition imposed by Wittich on the class of equations he treated in his analysis of entire solutions (see 6; pp. 71-72), and it also reveals why sometimes the growth rate of certain meromorphic solutions is too large in comparison with the coefficients of the equation.

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13. ABSTRACT

The rate of growth of meromorphic functions f, which are solutions of algebraic differential equations whose coefficients $a(z)$ are arbitrary meromorphic functions, is investigated. By a method based on Nevanlinna's theory of meromorphic functions, it has been shown that if f'/f has ∞ as its Nevanlinna exceptional values, then the ratio $T(r, f'/f)/T(r, a(z))$, as $r \to \infty$ outside a set of r values of finite measure, is bounded for at least one of the coefficients $a(z)$.

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