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ABSTRACT

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NONLINEAR ENERGY TRANSFER IN GRAVITY-CAPILLARY WAVE SPECTRA

INTRODUCTION

The nonlinear interactions of surface gravity waves are weak (of third order), and their dynamics have been investigated by Phillips (1960), Longuet-Higgins (1962), and Benney (1962) among others. The energy transfer in gravity wave spectra is of fourth order and has been obtained by Hasselmann (1962, 1963a, and 1963b) for a homogeneous and stationary Gaussian sea by means of a fifth-order perturbation analysis. The energy transfer principally redistributes the energy from intermediate-wavenumber gravity waves toward low- and high-wavenumber gravity waves.

For gravity-capillary waves the $\omega = \omega(k)$ dispersion relation connecting the radian frequency ω and the wavenumber k becomes concave when surface tension becomes important, at which time the resonant interactions among gravity-capillary waves can occur at second order. Energy is exchanged among a triad of waves; the dynamics of these interactions have been investigated in detail by McGoldrick (1965, 1970) and Simmons (1969) by the variational method.

Wave-tank measurements of slope spectra by Cox (1958) and Wright and Keller (1971), and open seas height-spectrum measurements by Valenzuela et al. (1971) for light winds, have shown a dip in the spectrum for waves in the neighborhood of 1.7 cm, which have the minimum phase velocity. To investigate whether this dip in the spectrum is a result of the dynamics of nonlinear wave-wave interactions, we have applied Hasselmann's (1962) inviscid formulation to obtain the energy transfer in gravity-capillary wave spectra by introducing surface-tension effects. To simplify the mathematics we have not included viscosity, which should be important for participating waves in the capillary region. Fortunately McGoldrick (1965) has shown that viscous dissipation may not be significant for a moderate range of wavenumbers.

The formulation preserves the assumption that the linear approximation for the random process (undisturbed free surface) is homogeneous, stationary, and Gaussian. These conditions may be a great deal more difficult to justify for gravity-capillary waves than for gravity waves; for gravity-capillary waves the interactions are stronger, which would tend to destroy these properties faster if they ever existed. However, if we restrict the energy transfer expression obtained with these assumptions to apply for gravity-capillary waves of small slopes, the results may shed light into the energy flux of gravity-capillary waves in the initial stage of development of the spectrum.

The analysis here is basically quite similar to Hasselmann's (1962) and uses a number of results given in detail in his original paper; in particular the present report uses the asymptotic formulas for the solutions of harmonic differential equations which are encountered in the development.

My understanding of the dynamics of nonlinear wave-wave interactions has been greatly helped by conversations with Dr. K. Hasselmann, to whom I am very grateful.

PERTURBATION ANALYSIS

The analysis that follows is similar to the original formalism given by Hasselmann (1962). For convenience we will adopt the same notation and we will tend to emphasize the points in the analysis which are different. The analysis applies for irrotational motion for a horizontally unbounded fluid of infinite depth with a free surface at $z = \zeta(x, y; t)$ where x, y, and z denote Cartesian coordinates, with the z axis directed vertically upward.

The velocity potential $\phi(x, y, z; t)$ and surface deviation ζ are determined by the following nonlinear system of equations, and initially we include surface tension.

- The continuity equation:

$$\nabla^2\phi = 0, \text{ for } z < \zeta. \tag{1}$$

- The equation for the kinematical boundary condition at the free surface:

$$\frac{\partial\zeta}{\partial t} - \frac{\partial\phi}{\partial z} + \tilde{\nabla}\zeta \cdot \tilde{\nabla}\phi = 0, \tag{2}$$

where

$$\tilde{\nabla} = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}\right).$$

- The equation for the dynamical boundary condition at the free surface:

$$\frac{\partial\phi}{\partial t} + g\zeta + \frac{1}{2}(\nabla\phi)^2 - \frac{T'}{\rho}\left(\frac{\nabla \cdot \nabla\zeta}{N} - \frac{\nabla\zeta \cdot \nabla\nabla\zeta \cdot \nabla\zeta}{N^3}\right) = 0, \tag{3}$$

where T' is the surface tension, ρ is the water density, g is the acceleration of gravity,

$$\nabla = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right), \text{ and } N = \left[1 + \left(\frac{\partial\zeta}{\partial x}\right)^2 + \left(\frac{\partial\zeta}{\partial y}\right)^2\right]^{1/2}.$$

The appropriate initial conditions for ϕ and ζ at $t = 0$ were given by Hasselmann (1962).

As customary, ϕ and ζ are expanded in perturbation series,

$$\phi = {}_1\phi + {}_2\phi + {}_3\phi + \ldots \tag{4}$$

and

$$\zeta = {}_1\zeta + {}_2\zeta + {}_3\zeta + \ldots, \tag{5}$$

where the index numbers indicate the order of the perturbation terms. Also, the boundary conditions are expanded in a Taylor series about $\underset{\sim}{z} = 0$; and ζ can be eliminated from one of the equations by operating on Eq. (2) with $-(g - \widetilde{\nabla}^2)$ and by taking $\partial/\partial t$ from Eq. (3) and adding the resulting equations. We find ${}_n\phi$ satisfies

$$\left(\frac{\partial^2}{\partial t^2} + \omega_{\mathbf{k}}^2\right) {}_n\phi = - \sum_{\boldsymbol{\nu}} \left\{ \frac{\partial}{\partial t} \left[\frac{{}_{\nu_1}\zeta_{\nu_2}\zeta \cdots {}_{\nu_{p-1}}\zeta}{(p-1)!} \frac{\partial^p}{\partial z^{p-1}\partial t} {}_{\nu_p}\phi \right.\right.$$

$$\left. + \frac{{}_{\nu_1}\zeta_{\nu_2}\zeta \cdots {}_{\nu_{p-2}}\zeta}{2(p-2)!} \frac{\partial^{p-2}}{\partial z^{p-2}} \left(\nabla_{\nu_{p-1}}\phi \cdot \nabla_{\nu_p}\phi\right) \right]$$

$$+ (g - T\widetilde{\nabla}^2) \left[\frac{{}_{\nu_1}\zeta_{\nu_2}\zeta \cdots {}_{\nu_{p-1}}\zeta}{(p-1)!} \frac{\partial^p}{\partial z^p} {}_{\nu_p}\phi \right.$$

$$\left.\left. - \frac{{}_{\nu_1}\zeta_{\nu_2}\zeta \cdots {}_{\nu_{p-2}}\zeta}{(p-2)!} \left(\widetilde{\nabla}_{\nu_{p-1}}\zeta\right) \cdot \frac{\partial^{p-2}}{\partial z^{p-2}} \left(\widetilde{\nabla}_{\nu_p}\phi\right) \right]\right\}$$

$$- T \sum_{\boldsymbol{\nu}} \frac{\partial}{\partial t} [O({}_{\nu_1}\zeta \cdots {}_{\nu_p}\zeta)] \qquad \text{at } z = 0, \tag{6}$$

where $T = T'/\rho$ and $\omega_{\mathbf{k}}^2 = gk + Tk^3$, and we find ${}_n\zeta$ satisfies

$$(g + Tk^2)_n\zeta + \frac{\partial}{\partial t}{}_n\phi = - \sum_{\boldsymbol{\nu}} \left[\frac{{}_{\nu_1}\zeta_{\nu_2}\zeta \cdots {}_{\nu_{p-1}}\zeta}{(p-1)!} \frac{\partial^p}{\partial z^{p-1}\partial t} {}_{\nu_p}\phi \right.$$

$$\left. + \frac{{}_{\nu_1}\zeta_{\nu_2}\zeta \cdots {}_{\nu_{p-2}}\zeta}{2(p-2)!} \frac{\partial^{p-2}}{\partial z^{p-2}} \left(\nabla_{\nu_{p-1}}\phi \cdot \nabla_{\nu_p}\phi\right) \right]$$

$$- T \sum_{\boldsymbol{\nu}} O({}_{\nu_1}\zeta \cdots {}_{\nu_p}\zeta) \qquad \text{at } z = 0. \tag{7}$$

The first summations in Eqs. (6) and (7) are taken over all combinations of index groups $\boldsymbol{\nu} = (\nu_1, \nu_2, \ldots, \nu_p)$ with $2 \leqslant p \leqslant n$ and $\Sigma_{j=1}^{p} \nu_j = n$; the second sums in these equations are terms of third order and higher, and since these terms will not contribute to the energy transfer in gravity-capillary waves, they have been noted only in symbolic form.

Since the ${}_n\phi$ must satisfy the linear Laplace equation $\nabla^2{}_n\phi = 0$ for homogeneous initial random wave fields, ${}_n\phi$ and ${}_n\zeta$ are expected also to be random and homogeneous and thus expandable in approximating Fourier sums:

$${}_n\phi = \sum_{\mathbf{k}} {}_n\phi_{\mathbf{k}}(t) e^{kz} e^{i\mathbf{k}\cdot\mathbf{x}} \tag{8}$$

and

$$_n\zeta = \sum_{\mathbf{k}} {}_nZ_{\mathbf{k}}(t)e^{i\mathbf{k}\cdot\mathbf{x}}, \tag{9}$$

where the $_n\phi_{\mathbf{k}}$ and $_nZ_{\mathbf{k}}$ are statistically independent random variables for different wavenumbers. Furthermore, since $_n\phi$ and $_n\zeta$ are real,

$$_n\phi_{\mathbf{k}} = ({}_n\phi_{-\mathbf{k}})^* \tag{10}$$

and

$$_nZ_{\mathbf{k}} = ({}_nZ_{-\mathbf{k}})^*, \tag{11}$$

where the asterisk means the complex conjugate value.

For $n = 1$ we have the familiar differential equations

$$\left(\frac{\partial^2}{\partial t^2} + \omega_{\mathbf{k}}^2\right) {}_1\phi = 0 \tag{12}$$

and

$$(g + Tk^2)_1\zeta + \frac{\partial_1\phi}{\partial t} = 0. \tag{13}$$

Substituting Eqs. (8) and (9) into Eqs. (12) and (13), we find that the solution of these equations for each $\mathbf{k}$ are given by

$$_1\phi_{\mathbf{k}} = {}_1\Phi_{\mathbf{k}}^+ e^{-i\omega_{\mathbf{k}}t} + {}_1\Phi_{\mathbf{k}}^- e^{i\omega_{\mathbf{k}}t} \tag{14}$$

and

$$_1Z_{\mathbf{k}} = {}_1Z_{\mathbf{k}}^+ e^{-i\omega_{\mathbf{k}}t} + {}_1Z_{\mathbf{k}}^- e^{i\omega_{\mathbf{k}}t}, \tag{15}$$

where $_1Z_{\mathbf{k}}^{\pm}$ and $_1\Phi_{\mathbf{k}}^{\pm}$ are related according to

$$_1Z_{\mathbf{k}}^{\pm} = \frac{\pm i\omega_{\mathbf{k}}}{g + Tk^2} {}_1\Phi_{\mathbf{k}}^{\pm}.$$

At second order ($n = 2$) we find that $_2\phi_{\mathbf{k}}$ and $_2Z_{\mathbf{k}}$ must satisfy the differential equations

$$\left(\frac{\partial^2}{\partial t^2} + \omega_{\mathbf{k}}^2\right) {}_2\phi_{\mathbf{k}} = \sum_{\substack{\mathbf{k}_1+\mathbf{k}_2=\mathbf{k} \\ s_1,s_2}} D_{\mathbf{k}_1,\mathbf{k}_2}^{s_1,s_2}\, {}_1\Phi_{\mathbf{k}_1}^{s_1}\, {}_1\Phi_{\mathbf{k}_2}^{s_2}\, e^{-(s_1\omega_{\mathbf{k}_1}+s_2\omega_{\mathbf{k}_2})t} \tag{16}$$

and

$$(g + Tk^2)_2Z_{\mathbf{k}} = -\frac{\partial}{\partial t} {}_2\phi_{\mathbf{k}} + \sum_{\substack{\mathbf{k}_1+\mathbf{k}_2=\mathbf{k} \\ s_1,s_2}} E_{\mathbf{k}_1,\mathbf{k}_2}^{s_1,s_2}\, {}_1\Phi_{\mathbf{k}_1}^{s_1}\, {}_1\Phi_{\mathbf{k}_2}^{s_2}\, e^{-i(s_1\omega_{\mathbf{k}_1}+s_2\omega_{\mathbf{k}_2})t}, \tag{17}$$

where

$$D^{s_1,s_2}_{\mathbf{k}_1,\mathbf{k}_2} = \frac{i}{2}\left\{(\omega_1+\omega_2)(k_1k_2-\mathbf{k}_1\cdot\mathbf{k}_2)+\omega_1\omega_2(\omega_1+\omega_2)\left(\frac{k_1}{g+Tk_1^2}+\frac{k_2}{g+Tk_2^2}\right)\right.$$
$$\left.-(g+Tk^2)\left[\frac{\omega_1(k_2^2+\mathbf{k}_1\cdot\mathbf{k}_2)}{(g+Tk_1^2)}+\frac{\omega_2(k_1^2+\mathbf{k}_1\cdot\mathbf{k}_2)}{(g+Tk_2^2)}\right]\right\} \tag{18}$$

and

$$E^{s_1,s_2}_{\mathbf{k}_1,\mathbf{k}_2} = \frac{1}{2}\left[\mathbf{k}_1\cdot\mathbf{k}_2 - \frac{\omega_1\omega_2(\omega_1^2+\omega_2^2+\omega_1\omega_2)}{(g+Tk_1^2)(g+Tk_2^2)}\right], \tag{19}$$

in which $\omega_1 = s_1\omega_{\mathbf{k}_1}$, $\omega_2 = s_2\omega_{\mathbf{k}_2}$ and $k = |\mathbf{k}_1+\mathbf{k}_2|$.

Gravity-capillary waves, unlike gravity waves, can have resonant interactions at second order, so that $\mathbf{k}_1+\mathbf{k}_2=\mathbf{k}$ and $\omega_1+\omega_2=\omega$ are satisfied simultaneously, which is treated in detail by McGoldrick (1965) and Simmons (1969). Consequently the solution of Eq. (16) in Hasselmann's (1962) notation is

$${}_2\phi_{\mathbf{k}} = \sum_{\substack{\mathbf{k}_1+\mathbf{k}_2=\mathbf{k}\\ s_1,s_2}} D^{s_1,s_2}_{\mathbf{k}_1,\mathbf{k}_2}\,{}_1\Phi^{s_1}_{\mathbf{k}_1}\,{}_1\Phi^{s_2}_{\mathbf{k}_2}\,\mathcal{I}_1(\omega_{\mathbf{k}_1+\mathbf{k}_2}, -s_1\omega_{\mathbf{k}_1}-s_2\omega_{\mathbf{k}_2}; t), \tag{20}$$

where $\mathcal{I}_1(\omega,\omega';t)$ is the solution of the differential equation

$$\frac{d^2\psi}{dt^2}+\omega^2\psi = e^{i\omega' t}$$

for $\psi = d\psi/dt = 0$ at $t=0$.

Similarly at third order, we find that the potential must satisfy the differential equation

$$\left(\frac{\partial^2}{\partial t^2}+\omega_{\mathbf{k}}^2\right){}_3\phi_{\mathbf{k}}$$
$$= \sum_{\substack{\mathbf{k}_1+\mathbf{k}_2+\mathbf{k}_3=\mathbf{k}\\ s_1,s_2,s_3}} D^{s_1,s_2,s_3}_{\mathbf{k}_1,\mathbf{k}_2,\mathbf{k}_3}\,{}_1\Phi^{s_1}_{\mathbf{k}_1}\,{}_1\Phi^{s_2}_{\mathbf{k}_2}\,{}_1\Phi^{s_3}_{\mathbf{k}_3}\,\mathcal{I}_1(\omega_{\mathbf{k}_2+\mathbf{k}_3}, -s_2\omega_{\mathbf{k}_2}-s_3\omega_{\mathbf{k}_3}; t)e^{-is_1\omega_{\mathbf{k}_1}t}$$
$$+ \text{noncontributing terms,} \tag{21}$$

where

$$D^{s_1,s_2,s_3}_{\mathbf{k}_1,\mathbf{k}_2,\mathbf{k}_3} = i\left\{(\omega_1+\omega_2+\omega_3)\left[\frac{\omega_1(\omega_2+\omega_3)\omega^2_{\mathbf{k}_2+\mathbf{k}_3}}{(g+Tk_1^2)(g+T|\mathbf{k}_2+\mathbf{k}_3|^2)}+\frac{\omega_1^2\omega^2_{\mathbf{k}_2+\mathbf{k}_3}}{(g+Tk_1^2)(g+T|\mathbf{k}_2+\mathbf{k}_3|^2)}\right.\right.$$
$$\left.-\mathbf{k}_1\cdot(\mathbf{k}_2+\mathbf{k}_3)+\frac{\omega_1^3(\omega_2+\omega_3)}{(g+Tk_1^2)(g+T|\mathbf{k}_2+\mathbf{k}_3|^2)}\right]-(g+Tk^2)\left[\frac{\omega_1|\mathbf{k}_2+\mathbf{k}_3|^2}{(g+Tk_1^2)}\right.$$
$$\left.\left.+\frac{(\omega_2+\omega_3)k_1^2}{(g+T|\mathbf{k}_2+\mathbf{k}_3|^2)}+\frac{\omega_1\mathbf{k}_1\cdot(\mathbf{k}_2+\mathbf{k}_3)}{(g+Tk_1^2)}+\frac{(\omega_2+\omega_3)\,\mathbf{k}_1\cdot(\mathbf{k}_2+\mathbf{k}_3)}{(g+T|\mathbf{k}_2+\mathbf{k}_3|^2)}\right]\right\}D^{s_2,s_3}_{\mathbf{k}_2,\mathbf{k}_3} \tag{22}$$

in which $k^2 = |\mathbf{k}_1 + \mathbf{k}_2 + \mathbf{k}_3|^2$.

Since resonances occur for $\mathbf{k}_1 + \mathbf{k}_2 + \mathbf{k}_3 = \mathbf{k}$ and $\omega_1 + \omega_2 + \omega_3 = \omega$, the solution of Eq. (21) expressed again in Hasselmann's notation is

$$_3\phi_{\mathbf{k}} = \sum_{\substack{\mathbf{k}_1+\mathbf{k}_2+\mathbf{k}_3=\mathbf{k}\\ s_1,s_2,s_3}} D^{s_1,s_2,s_3}_{\mathbf{k}_1,\mathbf{k}_2,\mathbf{k}_3}\, {}_1\Phi^{s_1}_{\mathbf{k}_1}\, {}_1\Phi^{s_2}_{\mathbf{k}_2}\, {}_1\Phi^{s_3}_{\mathbf{k}_3}$$
$$\mathcal{J}_2(\omega_{\mathbf{k}}, -s_1\omega_{\mathbf{k}_1}, \omega_{\mathbf{k}_2+\mathbf{k}_3}, -s_2\omega_{\mathbf{k}_2} - s_3\omega_{\mathbf{k}_3}; t)$$
$$+ \text{noncontributing terms,} \tag{23}$$

where $\mathcal{J}_2(\omega, \omega', \omega'', \omega'''; t)$ is the solution of the differential equation

$$\frac{d^2\psi}{dt^2} + \omega^2\psi = e^{i\omega' t}\,\mathcal{J}_1(\omega'', \omega'''; t)$$

for initial values $\psi = d\psi/dt = 0$ for $t = 0$.

It is worthwhile to note at this point that the coefficient given by Eq. (22) can be expressed in the form

$$D^{s_1,s_2,s_3}_{\mathbf{k}_1,\mathbf{k}_2,\mathbf{k}_3} = 2D^{\omega_1,\omega_2+\omega_3}_{\mathbf{k}_1,\mathbf{k}_2+\mathbf{k}_3}\, D^{s_2,s_3}_{\mathbf{k}_2,\mathbf{k}_3}. \tag{24}$$

THE ENERGY TRANSFER

For gravity-capillary waves McGoldrick (1965) has shown that the average energy per unit projected area in a wave system is given by

$$E = \frac{1}{2}\rho \overline{\int_{-\infty}^{\zeta} (\nabla\phi)^2\, dz} + \frac{1}{2}\rho g\overline{\zeta^2} + T'\overline{[(1 + \tilde{\nabla}\zeta \cdot \tilde{\nabla}\zeta)^{1/2} - 1]}, \tag{25}$$

where the bars denote ensemble means. For homogeneous random wave systems and using Leibnitz' theorem, Eq. (25) can be transformed into

$$E = \frac{1}{2}\rho \overline{\left[\phi\left(\frac{\partial\phi}{\partial z} - \tilde{\nabla}\phi \cdot \tilde{\nabla}\zeta\right)\right]_{z=\zeta}} + \frac{1}{2}\rho g\overline{\zeta^2} + T'\overline{[(1 + \tilde{\nabla}\zeta \cdot \tilde{\nabla}\zeta)^{1/2} - 1]}. \tag{26}$$

Substituting for ϕ and ζ the perturbation expansions given by Eqs. (4) and (5) respectively, expanding Eq. (26) in a Taylor series about $z = 0$, and collecting terms of the same order to obtain

$$E = {}_2E + {}_3E + {}_4E + {}_5E + \ldots, \tag{27}$$

we get ${}_3E = {}_5E = \ldots = 0$ because of the Gaussian assumption and

$${}_2E = \frac{\rho}{2}\overline{\left({}_1\phi \frac{\partial_1\phi}{\partial z}\right)} + \frac{\rho}{2} g\overline{{}_1\zeta^2} + \frac{T'}{2}\overline{(\tilde{\nabla}_1\zeta \cdot \tilde{\nabla}_1\zeta)} \tag{28}$$

and

$${}_4E = \frac{\rho}{2}\overline{\left({}_2\phi \frac{\partial_2\phi}{\partial z}\right)} + \frac{\rho}{2}\overline{\left({}_1\phi \frac{\partial_3\phi}{\partial z}\right)} + \frac{\rho}{2}\overline{\left({}_3\phi \frac{\partial_1\phi}{\partial z}\right)}$$

$$+ \frac{\rho}{2} g\overline{{}_2\zeta^2} + \rho g\overline{{}_1\zeta_3\zeta} + \frac{T'}{2}\overline{(\tilde{\nabla}_2\zeta \cdot \tilde{\nabla}_2\zeta)} + T'\overline{(\tilde{\nabla}_1\zeta \cdot \tilde{\nabla}_3\zeta)}. \tag{29}$$

Substituting in Eq. (29) the Fourier approximating expansions for ${}_n\phi$ and ${}_n\zeta$, we find that

$${}_2E = 2\rho \sum_{\mathbf{k}} k\overline{|{}_1\Phi_{\mathbf{k}}^{+}|^2} = 2\rho \sum_{\mathbf{k}} (g + Tk^2)\overline{|{}_1Z_{\mathbf{k}}^{+}|^2} \tag{30}$$

and

$${}_4E = \frac{\rho}{2}\sum_{\mathbf{k}} \Big\{ k[\overline{|{}_2\phi_{\mathbf{k}}|^2} + 2Re\overline{({}_1\phi_{-\mathbf{k}}\ {}_3\phi_{\mathbf{k}})}]$$

$$+ (g + Tk^2)[\overline{|{}_2Z_{\mathbf{k}}|^2} + 2Re\overline{({}_1Z_{-\mathbf{k}}\ {}_3Z_{\mathbf{k}})}]\Big\}. \tag{31}$$

As in the case for gravity waves, which was treated by Hasselmann (1962), the nonstationary contributions to the energy components from the terms involving the potentials are identical to the nonstationary contributions coming from terms involving the displacement amplitudes; thus in obtaining the energy transfer we only need to form the covariance products for the potentials.

From Eq. (20) and by means of symmetry considerations in forming the contributions by two-conjugate pairs, we find that

$$\overline{|{}_2\phi_k|^2} = 2 \sum_{\substack{k_1+k_2=k \\ s_1,s_2}} \left| D^{s_1,s_2}_{k_1,k_2} \right|^2 \overline{\left|{}_1\Phi^{s_1}_{k_1}\right|^2}\ \overline{\left|{}_1\Phi^{s_2}_{k_2}\right|^2}$$

$$\times\, \mathcal{J}_1(\omega_k, -s_1\omega_{k_1} - s_2\omega_{k_2}; t)\, \mathcal{J}_1(\omega_k, s_1\omega_{k_1} + s_2\omega_{k_2}; t)$$

$$+ \text{ stationary terms.} \tag{32}$$

Separating the nonsteady components according to their direction of propagation and using Hasselmann's (1962) asymptotic identity, given by his Eq. (3.5), we find that

$$\overline{\left|{}_2\phi^+_k\right|^2} = t \sum_{\substack{k_1+k_2=k \\ s_1,s_2}} \overline{\left|{}_1\Phi^{s_1}_{k_1}\right|^2}\ \overline{\left|{}_1\Phi^{s_2}_{k_2}\right|^2}\ \frac{\pi}{\omega_k^2} \left|D^{s_1,s_2}_{k_1,k_2}\right|^2 \delta(\omega_k + s_1\omega_{k_1} + s_2\omega_{k_2}). \tag{33}$$

The other covariance product contributing to ${}_4E$ can be obtained from Eq. (23), and again invoking symmetry considerations we find that

$$2Re\overline{({}_1\phi_{-k}\ {}_3\phi_k)} = 8Re\left[\sum_{\substack{k_1 \\ s_1,s}} D^{-s_1\omega_{k_1}, s\omega_k + s_1\omega_{k_1}}_{-k_1,k+k_1} D^{s,s_1}_{k,k_1} \overline{\left|{}_1\Phi^s_k\right|^2}\ \overline{\left|{}_1\Phi^{s_1}_{k_1}\right|^2} \right.$$

$$\left. \times\, e^{is\omega_k t}\, \mathcal{J}_2(\omega_k, s_1\omega_{k_1}, \omega_{k+k_1}, -s\omega_k - s_1\omega_{k_1}; t)\right]$$

$$+ \text{ noncontributing terms.} \tag{34}$$

Using Hasselmann's (1962) asymptotic identity given by his Eq. (3.14) and separating components according to their direction of propagation, we find

$$2Re\overline{({}_1\phi^-_{-k}\ {}_3\phi^+_k)} = -t \sum_{\substack{k_1 \\ s_1}} \overline{\left|{}_1\Phi^+_k\right|^2}\ \overline{\left|{}_1\Phi^{s_1}_{k_1}\right|^2}$$

$$\left[D^{-s_1,+}_{-k_1,k+k_1} D^{+,s_1}_{k,k_1} \frac{2\pi\delta(\omega_{k+k_1} - \omega_k - s_1\omega_{k_1})}{\omega_k(\omega_k + s_1\omega_{k_1})}\right.$$

$$\left. + D^{-s_1,-}_{-k_1,k+k_1} D^{+,s_1}_{k,k_1} \frac{2\pi\delta(\omega_{k+k_1} + \omega_k + s_1\omega_{k_1})}{\omega_k(\omega_k + s_1\omega_{k_1})}\right]. \tag{35}$$

The energy of the sea can be expressed in terms of the two-dimensional energy spectrum, $\hat{F}(\mathbf{k})$:

$$E = \iint_{-\infty}^{\infty} \hat{F}(\mathbf{k})\, d^2\mathbf{k} + \Delta E, \tag{36}$$

where ΔE contains stationary and noncontributing terms to the energy transfer.

Without going into all the detail given by Hasselmann (1962), we expand $\hat{F}(\mathbf{k})$ into a perturbation series and we separate components according to their direction of propagation. Then keeping only terms $F(\mathbf{k})$ propagating in the positive direction, we find that (for the linear approximation)

$$\iint_{-\infty}^{\infty} {}_2F(\mathbf{k})\, d^2\mathbf{k} = 2\rho \sum_{\mathbf{k}} k\overline{|{}_1\phi_{\mathbf{k}}^{+}|^2}, \tag{37}$$

and from the second approximation ${}_4E$ we find

$$\iint_{-\infty}^{\infty} {}_4F(\mathbf{k})\, d^2\mathbf{k} = 2\rho \sum_{\mathbf{k}} k[\overline{|{}_2\phi_{\mathbf{k}}^{+}|^2} + 2Re\overline{({}_1\phi_{-\mathbf{k}}^{-}\ {}_3\phi_{\mathbf{k}}^{+})}]. \tag{38}$$

Substituting Eqs. (33) and (35) into Eq. (38) and using Eq. (37) as the definition for the linear energy spectrum, we find that

$$\begin{aligned} {}_4F(\mathbf{k}) = t & \iint_{-\infty}^{\infty} \sum_{s',s''} \frac{\pi}{2\rho} \frac{(g + Tk'^2)(g + Tk''^2)}{(g + Tk^2)\omega_{\mathbf{k}'}^2\omega_{\mathbf{k}''}^2} \left| D_{\mathbf{k}',\mathbf{k}''}^{s',s''} \right|^2 \\ & \times \delta(\omega_{\mathbf{k}} + s'\omega_{\mathbf{k}'} + s''\omega_{\mathbf{k}''})\, {}_2F(s'\mathbf{k}')\, {}_2F(s''\mathbf{k}'')\, dk_x'\, dk_y' \\ - t & \iint_{-\infty}^{\infty} \sum_{s',s''} \frac{\pi}{\rho} \frac{(g + Tk'^2)}{\omega_{\mathbf{k}'}^2(\omega_{\mathbf{k}} + s'\omega_{\mathbf{k}'})\omega_{\mathbf{k}}} D_{-\mathbf{k}',\mathbf{k}+\mathbf{k}'}^{s',s''} D_{\mathbf{k},\mathbf{k}'}^{+,s'} \\ & \times \delta(\omega_{\mathbf{k}} + s'\omega_{\mathbf{k}'} - s''\omega_{\mathbf{k}+\mathbf{k}'})\, {}_2F(\mathbf{k})\, {}_2F(s'\mathbf{k}')\, dk_x'\, dk_y', \end{aligned} \tag{39}$$

where $F(\mathbf{k})$ includes only contributions from waves propagating in the positive $\mathbf{k}$ direction.

For gravity-capillary waves resonant interactions are possible for the case in which $\mathbf{k} = \mathbf{k}_1 + \mathbf{k}_2$ and $\omega = \omega_1 + \omega_2$ are both satisfied. Then the energy transfer in gravity-capillary spectra can be expressed in the form (at this point we will drop the indices in the spectra, keeping in mind that this is a second-order energy transfer).

$$\frac{\partial F(\mathbf{k})}{\partial t} = \iint_{-\infty}^{\infty} F(\mathbf{k}')\,F(\mathbf{k}'')\,T_1(\mathbf{k}',\mathbf{k}'')\,dk'_x dk'_y$$

$$- F(\mathbf{k}) \iint_{-\infty}^{\infty} F(\mathbf{k}')\,T_2(\mathbf{k},\mathbf{k}')\,dk'_x dk'_y, \tag{40}$$

where

$$T_1(\mathbf{k}_1,\mathbf{k}_2) = \frac{\pi}{2\rho}\frac{(g+Tk_1^2)(g+Tk_2^2)}{(g+Tk^2)\,\omega_{\mathbf{k}1}^2\,\omega_{\mathbf{k}2}^2}\left[\left|D^{+,+}_{\mathbf{k}1,\mathbf{k}2}\right|^2 \delta(\omega_{\mathbf{k}1+\mathbf{k}2} - \omega_{\mathbf{k}1} - \omega_{\mathbf{k}2})\right.$$

$$\left. + \left|D^{+,-}_{\mathbf{k}1,-\mathbf{k}2}\right|^2 \delta(\omega_{\mathbf{k}1-\mathbf{k}2} + \omega_{\mathbf{k}1} - \omega_{\mathbf{k}2})\right] \tag{41}$$

and

$$T_2(\mathbf{k},\mathbf{k}_1) = \frac{\pi}{\rho}\frac{(g+Tk_1^2)}{\omega_{\mathbf{k}1}^2}\left[\frac{D^{+,-}_{\mathbf{k},-\mathbf{k}1}\,D^{+,+}_{\mathbf{k}-\mathbf{k}1,\mathbf{k}1}}{\omega_{\mathbf{k}}(\omega_{\mathbf{k}} - \omega_{\mathbf{k}1})}\,\delta(\omega_{\mathbf{k}} - \omega_{\mathbf{k}1} - \omega_{\mathbf{k}-\mathbf{k}1})\right.$$

$$\left. + \frac{D^{+,+}_{\mathbf{k},\mathbf{k}1}\,D^{+,-}_{\mathbf{k}+\mathbf{k}1,-\mathbf{k}1}}{\omega_{\mathbf{k}}(\omega_{\mathbf{k}} + \omega_{\mathbf{k}1})}\,\delta(\omega_{\mathbf{k}} + \omega_{\mathbf{k}1} - \omega_{\mathbf{k}+\mathbf{k}1})\right]. \tag{42}$$

Next we find that the conservation of energy condition, namely,

$$\iint_{-\infty}^{\infty} \frac{\partial}{\partial t} F(\mathbf{k}_3)\,d^2\mathbf{k}_3 = 0,$$

is satisfied for the case in which $\mathbf{k}_1 + \mathbf{k}_2 = \mathbf{k}_3$ and $\omega_1 + \omega_2 = \omega_3$ (where now ω_1, ω_2, and ω_3 are taken to be positive) when

$$\frac{k_3}{\omega_3} D^{-,-}_{-\mathbf{k}1,-\mathbf{k}2} = \frac{k_2}{\omega_2} D^{+,-}_{\mathbf{k}3,-\mathbf{k}1} + \frac{k_1}{\omega_1} D^{+,-}_{\mathbf{k}3,-\mathbf{k}2}; \tag{43}$$

and we find that the conservation of momentum conditions, namely,

$$\iint_{-\infty}^{\infty} \frac{\partial}{\partial t} F(\mathbf{k}_3)\,\frac{\mathbf{k}_3}{\omega_3}\,d^2\mathbf{k}_3 = 0,$$

requires that

$$\frac{\mathbf{k}_3}{\omega_3}\frac{k_3}{\omega_3} D^{-,-}_{-\mathbf{k}1,-\mathbf{k}2} = \frac{\mathbf{k}_2}{\omega_2}\frac{k_2}{\omega_2} D^{+,-}_{\mathbf{k}3,-\mathbf{k}1} + \frac{\mathbf{k}_1}{\omega_1}\frac{k_1}{\omega_1} D^{+,-}_{\mathbf{k}3,-\mathbf{k}2}. \tag{44}$$

Equations (43) and (44) are three homogeneous equations with three unknowns, so that only the ratios of the coefficients are known; these ratios are given by

$$\frac{k_3}{\omega_3^2} D^{-,-}_{-\mathbf{k}_1,-\mathbf{k}_2} = \frac{k_2}{\omega_2^2} D^{+,-}_{\mathbf{k}_3,-\mathbf{k}_1} = \frac{k_1}{\omega_1^2} D^{+,-}_{\mathbf{k}_3,-\mathbf{k}_2}, \tag{45}$$

which can be checked directly using Eq. (18). Expressions similar to Eqs. (43) and (44) can be derived for difference interactions $\mathbf{k}_2 - \mathbf{k}_1 = \mathbf{k}_3$ and $\omega_2 - \omega_1 = \omega_3$; these expressions are related to Eqs. (43) and (44) by an exchange of the indices only.

The final form for the energy transfer expression is given by

$$\frac{\partial F(\mathbf{k}_3)}{\partial t} = \frac{\pi k_3}{2\rho\omega_3^4} \iint_{-\infty}^{\infty} \frac{|D^{+,+}_{\mathbf{k}_1,\mathbf{k}_2}|^2}{k_1 k_2} \omega_3 [\omega_3 F(\mathbf{k}_1)F(\mathbf{k}_2) - \omega_2 F(\mathbf{k}_1)F(\mathbf{k}_3)$$

$$- \omega_1 F(\mathbf{k}_2)F(\mathbf{k}_3)] \; \delta(\omega_{\mathbf{k}3} - \omega_{\mathbf{k}2} - \omega_{\mathbf{k}1}) \, dkx_1 dky_1$$

$$+ \frac{\pi k_3}{2\rho\omega_3^4} \iint_{-\infty}^{\infty} \frac{|D^{+,-}_{\mathbf{k}_1,-\mathbf{k}_2}|^2}{k_1 k_2} \omega_3 [\omega_3 F(\mathbf{k}_1)F(\mathbf{k}_2) - \omega_2 F(\mathbf{k}_1)F(\mathbf{k}_3)$$

$$+ \omega_1 F(\mathbf{k}_2)F(\mathbf{k}_3)] \; \delta(\omega_{\mathbf{k}3} - \omega_{\mathbf{k}2} + \omega_{\mathbf{k}1}) \, dkx_1 dky_1, \tag{46}$$

where the first two-dimensional integral is the contribution from the sum interactions to the energy transfer in which $\mathbf{k}_1 + \mathbf{k}_2 = \mathbf{k}_3$ and $\omega_1 + \omega_2 = \omega_3$ and the second two-dimensional integral is the contribution from the difference interactions in which $\mathbf{k}_2 - \mathbf{k}_1 = \mathbf{k}_3$ and $\omega_2 - \omega_1 = \omega_3$.

INTERPRETATION AND DISCUSSION

The main result of this investigation is Eq. (46), which yields the energy transfer in gravity-capillary wave spectra; and as expected from earlier investigations energy is transferred from two active components to a third passive wave component. Two types of interactions participate: sum resonant interactions in which $\mathbf{k}_1 + \mathbf{k}_2 = \mathbf{k}_3$ and $\omega_1 + \omega_2 = \omega_3$ are simultaneously satisfied and difference resonant interactions in which $\mathbf{k}_2 - \mathbf{k}_1 = \mathbf{k}_3$ and $\omega_2 - \omega_1 = \omega_3$ are simultaneously satisfied. Our result is an inviscid result; thus viscous dissipation, which should damp the components in the capillary region, must be considered in a more exact formulation.

As obtained by Hasselmann (1962) for gravity waves, the energy transfer also vanishes for gravity-capillary waves when the linear approximation for the spectrum is white and isotropic.

Our result for the energy transfer in gravity-capillary wave spectra should shed light on the initial growth of a wave spectrum. McGoldrick (1965) has pointed out that immediately after the initial development of waves by a turbulent wind starting with the prominent ripples at a 1.7-cm wavelength as described in Phillips' (1957) wind-wave resonance theory, the nonlinear wave-wave interaction mechanism in transferring this wind-induced energy to other waves might contribute more energy to those waves than the wind contributes. A dip in the spectrum in the neighborhood of 1.7-cm waves found by Cox (1958) and Wright and Keller (1971) may be evidence supporting McGoldrick's statement. For example, in Wright and Keller's spectra the dip is present for wind speeds smaller than about 6 to 7 m/sec, and above this wind speed the dip disappears abruptly. Thus, if the nonlinear wave-wave interaction is the explanation for this dip in the spectrum for wind speeds below 6 to 7 m/sec, the energy removed from the 1.7-cm waves is greater than the energy gained from the wind, and when the winds are above 6 to 7 m/sec, there should be somehow an energy balance between the energy flux from these two processes.

A more detailed study of this point will be made in a forthcoming publication in which numerical evaluation of the energy transfer expression for gravity-capillary wave spectra will be given.

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