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Detection of Signals in Non-Gaussian  
Correlated Noise Derived from Cauchy  
Processes

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<p>A new processor for detecting a radar target in correlated, non-Gaussian noise is obtained. When this processor and a matched filter are excited with this noise, performance is improved over that of the matched filter alone. The processor is obtained by developing an approximate, bivariate, probability density for the noise and constructing a Neyman-Pearson test and then using an approximation to the likelihood ratio obtained from the Neyman-Pearson test. The bivariate density was constructed from an underlying Cauchy process and matches the true bivariate density only in the marginals and first two moments.</p>				
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# DETECTION OF SIGNALS IN NON-GAUSSIAN CORRELATED NOISE DERIVED FROM CAUCHY PROCESSES

## INTRODUCTION

The detection of additive signals in correlated noise or radar clutter is a problem that is of interest and has been extensively studied. The optimal detector when the correlated noise is Gaussian-distributed is the Wiener filter, or matched filter, after prewhitening followed by a threshold. This detector can be obtained by using the Neyman-Pearson procedure that maximizes the probability of detection for a given probability of false alarm for a binary hypothesis. In applying this procedure to non-Gaussian, correlated noise, three problems are encountered. First, we seldom know or can easily measure the required multivariate probability density of the noise; second, often there are unknown parameters that must be accounted for in some way; and third, the likelihood ratio obtained in the test sometimes is difficult to simplify. All three of these problems are addressed in this study for a given situation to be described.

The most difficult problem encountered is obtaining the multivariate probability density of the noise. A procedure for constructing an approximate representation of the multivariate probability density is described by Martinez, Swaszek, and Thomas [1]. The procedure constructs the desired multivariate density from one that can be analytically represented, such as a Gaussian one, by using a nonlinear transform to map the one into the other. The mapping is adjusted so that the marginal distributions and the first two moments of the constructed multivariate distributions are correct. Often these are the only properties of the clutter that can be measured easily. In Ref. 2, these results are extended to include complex numbers to represent radar baseband signals, to provide a suitable transformation, and to give the closed-form, multivariate probability density for both correlated Weibull and log-normal clutter that is correct in the marginals and covariance matrices. The purpose of this report is to repeat the procedure outlined for Ref. 2 except using an underlying Cauchy process rather than an underlying Gaussian process. The examples in Ref. 2 use a bivariate distribution, and consequently that is what is developed here. The two processes are identical in the marginals and the first two moments but differ in other respects.

Once the bivariate distributions are known, the optimal detector for additive signal can be found in closed form by using the Neyman-Pearson test. In Ref. 2, an approximation to the Neyman-Pearson test was found that simplified the test and removed the need for knowing the signal amplitude and phase. This detector was shown to perform better than the classical matched filter after prewhitening, which is optimal only for Gaussian noise.

We describe a means of generating correlated Cauchy distributed samples, then develop an expression for the bivariate density function for the Cauchy process. This is followed by the development of the Weibull and log-normal bivariate probability densities. The Neyman-Pearson test is constructed and simplified, and an example is given. Further work will involve comparing detection results of the detectors described in Ref. 2, and those developed here by using each other's noise models.

**CONSTRUCTING THE BIVARIATE CAUCHY PROCESS**

Given that one can easily construct independent Gaussian random variables, Cauchy variables can be constructed from them through a nonlinear transformation

$$z_c^r(l) = h^{-1}(z_g^r(l))$$

and

$$z_c^i(l) = h^{-1}(z_g^i(l)),$$

where  $z_c^r(l)$ ,  $z_c^i(l)$ ,  $z_g^r(l)$ , and  $z_g^i(l)$  are the inphase and the quadrature components for the Cauchy and Gaussian random variables respectively,  $h^{-1}(\cdot)$  is the nonlinear function, and  $l$  indicates the  $l$ th sample. The inverse transformation  $h(\cdot)$  is defined by

$$z_g^r(l) = h(z_c^r(l))$$

and

$$z_g^i(l) = h(z_c^i(l)).$$

The nonlinear functions are specified by the requirement that the distribution functions over each variable must be equal and are

$$\int_{-\infty}^{h(z_c^i(l))} P_{z_g^i}(z_g^i(l)) dz_g^i(l) = \int_{-\infty}^{z_c^i(l)} P_{z_c^i}(z_c^i(l)) dz_c^i(l)$$

and

$$\int_{-\infty}^{z_g^r(l)} P_{z_g^r}(z_g^r(l)) dz_g^r(l) = \int_{-\infty}^{h^{-1}(z_g^r(l))} P_{z_c^r}(z_c^r(l)) dz_c^r(l).$$

The Gaussian density is

$$P_{z_g^r}(z_g^r(l)) = \frac{1}{\sqrt{2\pi}\sigma_{z_g^r}} \exp \left\{ -\frac{z_g^r(l)^2}{2\sigma_{z_g^r}^2} \right\},$$

and the Cauchy density is

$$P_{z_c^r}(z_c^r(l)) = \frac{1}{\pi b \left\{ 1 + \left[ \frac{z_c^r(l)}{b} \right]^2 \right\}},$$

where  $\sigma_{z_g^r}$  is the standard deviation of the Gaussian noise and  $b$  is the Cauchy parameter. After integration the nonlinear function is

$$h^{-1}(z_g^{\bullet}(l)) = b \tan \left\{ \frac{\pi}{2} - \frac{\pi}{2} \operatorname{erfc} \left[ \frac{z_g^{\bullet}(l)}{\sqrt{2}} \right] \right\}$$

$$h(z_c^{\bullet}(l)) = \sqrt{2} \operatorname{erfc}^{-1} \left\{ 1 - \frac{2}{\pi} \tan^{-1} \left[ \frac{z_c^{\bullet}(l)}{b} \right] \right\},$$

where the complementary error functions are defined as

$$\operatorname{erfc}(x) = \frac{2}{\sqrt{\pi}} \int_x^{\infty} e^{-t^2} dt$$

and the dot represents the in phase  $r$  and quadrature  $i$  components. The integrations are classic ones in that the integration over the Gaussian density yields an error function and over the Cauchy yields an inverse tangent function.

The next step is to correlate the data. Since the sum of Cauchy processes is still Cauchy (shown subsequently), the data are correlated by using some of the same samples in sums for each of the new random variables. Furthermore, the results are to be obtained for both the independent inphase and quadrature channels. The following sum operations over  $K$  variables accomplish these goals:

$$x_1^r = \frac{1}{K} \sum_{l=1}^K z_c^r(l),$$

$$x_2^r = \frac{1}{K} \sum_{l=2}^{K+1} z_c^r(l),$$

$$x_1^i = \frac{1}{K} \sum_{l=1}^K z_c^i(l), \text{ and}$$

$$x_2^i = \frac{1}{K} \sum_{l=2}^{K+1} z_c^i(l).$$

The marginal probability densities of each of these new random variables  $x_1^r$ ,  $x_2^r$ ,  $x_1^i$ , and  $x_2^i$  are all identical and are Cauchy, given by the density function

$$P_{x_i^*}(x_i^*) = \frac{1}{\pi b \left\{ 1 + \left( \frac{x_i^*}{b} \right)^2 \right\}}$$

The bivariate density

$$P_{x_1^r, x_2^r, x_1^i, x_2^i}(x_1^r, x_2^r, x_1^i, x_2^i) = P_{x_1^r, x_2^r}(x_1^r, x_2^r) P_{x_1^i, x_2^i}(x_1^i, x_2^i)$$

is derived in the next section. The first samples  $x_1^r$  and  $x_1^i$  can be made highly correlated to the second set of samples  $x_2^r$  and  $x_2^i$  if  $K$  is large. Note that the moments are not defined for the Cauchy process.

## BIVARIATE CAUCHY PROBABILITY DENSITY

The bivariate Cauchy probability density is found by using characteristic functions. Because  $P_{x_1^r, x_2^r}(x_1^r, x_2^r)$  and  $P_{x_1^i, x_2^i}(x_1^i, x_2^i)$  are identical in form, only the one for the inphase component is developed. The characteristic function of  $P_{x_1^r, x_2^r}(x_1^r, x_2^r)$  is first found, then  $P_{x_1^r, x_2^r}(x_1^r, x_2^r)$  is found by using the inverse Fourier transform of it.

The characteristic function  $C_{x_1^r, x_2^r}(jw_1, jw_2)$  of  $P_{x_1^r, x_2^r}(x_1^r, x_2^r)$  is defined by

$$C_{x_1^r, x_2^r}(jw_1, jw_2) = E \left\{ e^{jw_1 x_1^r + jw_2 x_2^r} \right\}$$

where  $E$  is the expected value with respect to  $x_1^r$  and  $x_2^r$ . Substituting for  $x_1^r$  and  $x_2^r$ , the characteristic function becomes

$$C_{x_1^r, x_2^r}(jw_1, jw_2) = E \left\{ e^{j \frac{w_1}{K} \sum_{l=1}^K z_c^r(l)} e^{j \frac{w_2}{K} \sum_{l=2}^{K+1} z_c^r(l)} \right\},$$

where now the expected value is over  $z_c^r(l)$  for  $l = 1, \dots, K+1$ . This expression can then be written as

$$C_{x_1^r, x_2^r}(jw_1, jw_2) = C_{z_c^r(1)} \left[ \frac{jw_1}{K} \right] C_{z_c^r(K+1)} \left[ \frac{jw_2}{K} \right] \prod_{l=2}^K C_{z_c^r(l)} \left[ \frac{j(w_1 + w_2)}{K} \right]$$

by noting that all the  $z_c^r(l)$ 's are independent and that each of the  $(K+1)$  terms are by definition a characteristic function. The characteristic functions  $C_{z_c^r(l)}(\cdot)$  are characteristic functions of Cauchy processes and are

$$C_{z_c^r(1)} \left[ j \frac{w_1}{K} \right] = \exp \left\{ -\frac{|w_1|b}{K} \right\}$$

$$C_{z_c^r(K+1)} \left[ j \frac{w_2}{K} \right] = \exp \left\{ -\frac{|w_2|b}{K} \right\}$$

$$C_{z_c^r(l)} \left[ \frac{j(w_1 + w_2)}{K} \right] = \exp \left\{ -\frac{|w_1 + w_2|b}{K} \right\}.$$

The desired characteristic function then is

$$C_{x_1^r, x_2^r}(jw_1, jw_2) = \exp \left\{ -\frac{b}{K} (|w_1| + |w_2| + (K-1)|w_1 + w_2|) \right\}.$$

The bivariate probability density is obtained by taking the inverse Fourier transform of the characteristic function and is defined by

$$P_{x_1^r, x_2^r}(x_1^r, x_2^r) = \frac{1}{(2\pi)^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} C_{x_1^r, x_2^r}(jw_1, jw_2) \exp \left\{ -j(w_1 x_1^r + w_2 x_2^r) \right\} dw_1 dw_2.$$

The integration is performed in parts over different regions because of the absolute value signs in the characteristic function. The regions are defined by the boundaries

$$\begin{array}{ll}
 I_0 : w_1 > 0, & w_2 > 0 & I'_1 : w_1 < 0, & w_2 > 0, & |w_1| > |w_2| \\
 I'_0 : w_1 < 0, & w_1 < 0 & I_2 : w_1 > 0, & w_2 < 0, & |w_1| < |w_2| \\
 I_1 : w_1 > 0, & w_2 < 0, & |w_1| > |w_2| & I'_2 : w_1 < 0, & w_2 > 0, & |w_1| < |w_2|
 \end{array}$$

and  $I_0, I'_0, I_1, I'_1, I_2$  and  $I'_2$  are the values of the integrals over those regions. Consequently

$$P_{x'_1, x'_2}(x'_1, x'_2) = I_0 + I'_0 + I_1 + I'_1 + I_2 + I'_2.$$

The integrations are sketched out in Appendix A. The result is

$$\begin{aligned}
 P_{x'_1, x'_2}(x'_1, x'_2) = & \frac{2}{(2\pi)^2 b^2} \left\{ \frac{\left[ 1 - \left( \frac{x'_1}{b} \right) \left( \frac{x'_2}{b} \right) \right]}{\left[ 1 + \left( \frac{x'_1}{b} \right)^2 \right] \left[ 1 + \left( \frac{x'_2}{b} \right)^2 \right]} \right. \\
 & - \frac{\left[ (1 - \rho_x) - \left( \frac{x'_1}{b} \right) \left( \frac{x'_2}{b} \right) \right]}{\left[ (1 - \rho_x)^2 + \left( \frac{x'_2}{b} \right)^2 \right] \left[ 1 + \left( \frac{x'_1}{b} \right)^2 \right]} \\
 & - \frac{\left[ (1 - \rho_x) - \left( \frac{x'_1}{b} \right) \left( \frac{x'_2}{b} \right) \right]}{\left[ (1 - \rho_x)^2 + \left( \frac{x'_1}{b} \right)^2 \right] \left[ 1 + \left( \frac{x'_2}{b} \right)^2 \right]} \\
 & + \frac{\left[ \rho_x(1 - \rho_x) - \left( \frac{x'_2}{b} \right) \left( \frac{x'_1}{b} - \frac{x'_2}{b} \right) \right]}{\left[ (1 - \rho_x)^2 + \left( \frac{x'_2}{b} \right)^2 \right] \left[ \rho_x^2 + \left( \frac{x'_1}{b} - \frac{x'_2}{b} \right)^2 \right]} \\
 & \left. + \frac{\left[ \rho_x(1 - \rho_x) - \left( \frac{x'_1}{b} \right) \left( \frac{x'_2}{b} - \frac{x'_1}{b} \right) \right]}{\left[ (1 - \rho_x)^2 + \left( \frac{x'_1}{b} \right)^2 \right] \left[ \rho_x^2 + \left( \frac{x'_1}{b} - \frac{x'_2}{b} \right)^2 \right]} \right\}. \tag{1}
 \end{aligned}$$

where  $\rho_x = 2/K$ . A small  $\delta x$  indicates a large correlation.

The density  $P_{x_1^i, x_2^i}(x_1^i, x_2^i)$  is identical to  $P_{x_1^r, x_2^r}(x_1^r, x_2^r)$  except superscript  $r$  is replaced with  $i$ . The density was checked to see if a simple Cauchy was obtained after numerical integration over one of the variables. This was obtained for a number of cases. This fact is also sufficient to guarantee that the area under the density is one. These numerical integrations served as checks to the mathematics.

### BIVARIATE WEIBULL AND LOG-NORMAL PROBABILITY DENSITIES

Two noise distributions of interest are specified by the Weibull or log-normal marginal distributions on the magnitude of the signal. To map the bivariate Cauchy variables just derived to these distributions on the marginals, the distribution of the magnitudes of the random variables given by

$$|x_k| = \sqrt{(x_k^r)^2 + (x_k^i)^2}$$

must first be found. Then the nonlinear mapping must be specified.

The distribution of the magnitude and phase of the Cauchy random variables is given by

$$P_{|x_k|, \theta_k}(|x_k|, \theta_k) = J_{x_k} P_{x_k^r}(x_k^r) P_{x_k^i}(x_k^i),$$

where  $J_{x_k}$  is the Jacobian transformations. Given

$$x_k^r = |x_k| \cos \theta_k$$

$$x_k^i = |x_k| \sin \theta_k,$$

$$J_{x_k} = \begin{vmatrix} \frac{\partial x_k^r}{\partial |x_k|} & \frac{\partial x_k^i}{\partial |x_k|} \\ \frac{\partial x_k^r}{\partial \theta_k} & \frac{\partial x_k^i}{\partial \theta_k} \end{vmatrix} = |x_k|,$$

and

$$P_{x_k^r}(x_k^r) = \frac{1}{\pi b \left[ 1 + \left( \frac{x_k^r}{b} \right)^2 \right]}$$

$$P_{x_k^i}(x_k^i) = \frac{1}{\pi b \left[ 1 + \left( \frac{x_k^i}{b} \right)^2 \right]},$$

then

$$P_{|x_k|, \theta_k}(|x_k|, \theta_k) = \frac{|x_k|}{\pi^2 b^2} \left\{ \frac{1}{\left[ 1 + \left( \frac{|x_k|}{b} \right)^2 \right]^2 + \left( \frac{|x_k|}{\sqrt{2}b} \right)^4 \sin^2 2\theta_k} \right\}.$$

Integrating over  $\theta$ , from  $-\pi$  to  $\pi$  yields

$$P_{|x_k|}(|x_k|) = \frac{4|x_k|}{\pi} \left\{ \frac{1}{(|x_k|^2 + 2b^2) \sqrt{1 + \left(\frac{|x_k|}{b}\right)^2}} \right\}$$

where the integral

$$\int \frac{dx}{a + b \sin^2 x} = \frac{a}{\sqrt{a(a+b)}} \tan^{-1} \left[ \sqrt{\frac{a+b}{a}} \tan x \right]$$

found on page 152 in Ref. 3 is used.

The mapping used to map the bivariate Cauchy into a bivariate Weibull is

$$y_k^r = \frac{g^{-1}(|x_k|)}{|x_k|} x_k^r$$

$$y_k^i = \frac{g^{-1}(|x_k|)}{|x_k|} x_k^i,$$

and the inverses are

$$x_k^r = \frac{g(|y_k|)}{|y_k|} y_k^r \tag{2}$$

$$x_k^i = \frac{g(|y_k|)}{|y_k|} y_k^i \tag{3}$$

for  $k = 1$  and  $k = 2$ . The nonlinear functions are specified from the requirement that the distribution function on  $\{y_k\}$  and  $\{x_k\}$  must be equal. This requires

$$\int_0^{g(|y_k|)} P_{|x_k|}(|x_k|) d|x_k| = \int_0^{|y_k|} P_{|y_k|}(|y_k|) d|y_k|$$

and

$$\int_0^{|x_k|} P_{|x_k|}(|x_k|) d|x_k| = \int_0^{g^{-1}(|x_k|)} P_{|y_k|}(|y_k|) d|y_k|.$$

The Weibull distribution is given by

$$P_{|y_k|}(|y_k|) = \alpha \ln 2 \left[ \frac{|y_k|}{M_y} \right]^{\alpha-1} \exp \left[ \ln 2 \left[ \frac{|y_k|}{M_y} \right]^\alpha \right],$$

where  $M_y$  is the median value and  $\alpha$  is the Weibull parameter that ranges from  $\sim .5$  to 2 for radar clutter. For  $\alpha = 2$ , the Weibull reduces to the Rayleigh distribution. The lognormal is given by

$$P_{|y_k|}(|y_k|) = \frac{2}{\sqrt{2\pi}\sigma_l |y_k|} \exp - \left\{ \frac{1}{2\sigma_l^2} \left[ 2 \ln \left( \frac{|y_k|}{M_y} \right) \right]^2 \right\},$$

where  $M_y$  is the median value and  $\sigma_l$  is the standard deviation of  $(\ln |y_k|)^2$ . By performing the integrals over the defined marginal densities, the nonlinear functions are found to be

$$g(|y_k|) = b \sqrt{\left[ \tan \left\{ \frac{\pi}{2} - \frac{\pi}{4} \exp \left\{ -\ln 2 \left( \frac{|y_k|}{M_y} \right)^\alpha \right\} \right\} \right]^2 - 1}$$

$$g^{-1}(|x_k|) = M_y \left[ \frac{\ln \left\{ \frac{1}{2 \left[ 1 - \frac{2}{\pi} \tan^{-1} \left( \frac{\sqrt{|x_k|^2 + b^2}}{b} \right) \right]} \right\}}{\ln 2} \right]^{1/\alpha}$$

for the Weibull distribution and

$$g(|y_k|) = b \sqrt{\left\{ \tan \frac{\pi}{8} \operatorname{erfc} \left[ \frac{\sqrt{2}}{\sigma_l} \ln \left( \frac{|x_k|}{M_y} \right) \right] \right\}^2 - 1}$$

$$g^{-1}(|x_k|) = M_y \exp \left\{ \frac{\sigma_l}{\sqrt{2}} \operatorname{erfc}^{-1} 4 \left[ 1 - \frac{1}{\pi} \tan^{-1} \left( \frac{\sqrt{|x_k|^2 + b^2}}{b} \right) \right] \right\}$$

for the lognormal. The function  $\operatorname{erfc}$  is the complimentary error function,  $\operatorname{erfc}^{-1}$  is its inverse, and  $\ln$  is the natural logarithm. Appendix B provides the details for finding these functions.

Finally the new bivariate Weibull or bivariate lognormal density can be specified in terms of the bivariate Cauchy by

$$P_{y_1^r, y_2^r, y_1^i, y_2^i} (y_1^r, y_1^i, y_2^r, y_2^i) = P_{x_1^r, x_2^r} (x_1^r, x_2^r) P_{x_1^i, x_2^i} (x_1^i, x_2^i) J_y (y_1^r, y_1^i, y_2^r, y_2^i) \begin{cases} x_1^r = \frac{g(|y_1^r|)}{|y_1^r|} y_1^r \\ \cdot \\ \cdot \\ \cdot \\ x_1^i = \frac{g(|y_1^i|)}{|y_1^i|} y_1^i \end{cases} \quad (4)$$

where

$$J_y(y_1^r, y_1^i, y_2^r, y_2^i) = \begin{vmatrix} J_1 & \underline{0} \\ \underline{0} & J_2 \end{vmatrix} = J_1 J_2 \quad (5)$$

and

$$J_k = \begin{vmatrix} \frac{\partial x_k^r}{\partial y_k^r} & \frac{\partial x_k^i}{\partial y_k^r} \\ \frac{\partial x_k^r}{\partial y_k^i} & \frac{\partial x_k^i}{\partial y_k^i} \end{vmatrix} = \frac{g(|y_k|)g'(|y_k|)}{|y_k|} \quad (6)$$

where  $g'(\cdot)$  is the derivature of  $g(\cdot)$  with respect to  $|y_k|$ . The Jacobian for the Weibull is

$$J_k = \left[ \frac{\sigma b^2 \ln 2}{M_y} \right] \left[ \frac{|y_k|}{M_y} \right]^{\alpha-2} \psi \tan \left[ \frac{\pi}{2} - \psi \right] \sec^2 \left[ \frac{\pi}{2} - \psi \right]$$

where

$$\psi = \frac{\pi}{4} \exp \left\{ -\ln 2 \left[ |y_k| / M_y \right] \right\},$$

and for lognormal

$$J_k = \left[ \frac{-b^2 \sqrt{2\pi}}{4\sigma |y_k|^2} \right] \exp - \left\{ \frac{\sqrt{2}}{\sigma_l} \ln \left[ \frac{|y_k|}{M_y} \right] \right\}^2 \tan \psi \sec^2 \psi$$

where  $\psi$  for this case is

$$\psi = \frac{\pi}{8} \operatorname{erfc} \left[ \frac{\sqrt{2}}{\sigma_l} \ln \left[ \frac{|y_k|}{M_y} \right] \right].$$

The bivariate Weibull distribution was checked by numerical integration. The signals were converted to magnitude and phase through another transform and then integrated over the two phases and one magnitude. The remaining marginal density was Weibull as desired; consequently the integration over all variables is one.

### NEYMAN-PEARSON TEST

The classical radar solution for optimally detecting a signal in additive noise is the Neyman-Pearson procedure for a binary hypothesis test. This processor maximizes the probability of detection given a fixed false alarm rate. The processor is specified by the likelihood ratio and in this case is

$$\lambda_{np} = \frac{P_{y_1^r, y_2^r, y_1^i, y_2^i}((y_1^r - s_1^r), (y_1^i - s_1^i), (y_2^r - s_2^r), (y_2^i - s_2^i))}{P_{y_1^r, y_2^r, y_1^i, y_2^i}(y_1^r, y_1^i, y_2^r, y_2^i)}$$

where

$$s_1^r = S \cos(\phi_s),$$

$$s_1^i = S \sin(\phi_s),$$

$$s_2^r = S \cos(\phi_s + \Delta), \text{ and}$$

$$s_2^i = S \sin(\phi_s + \Delta).$$

Here,  $S$  and  $\theta_s$  are the signal's unknown amplitude and phase, and  $\Delta$  is the phase rotation of the signal between samples 1 and 2. The signal-to-noise ratio is defined to be

$$(S/N) = S^2/2\sigma_y^2,$$

where

$$\sigma_y^2 = \text{VAR}(y_k^r) = \text{VAR}(y_k^i)$$

and  $\text{VAR}(\cdot)$  is the variance. A very lengthy closed form expression for the likelihood ratio can be written by combining Eqs. (2) through (6). However, the test is very difficult to implement and contains unknown parameters. Consequently an approximation to this processor is next considered.

### APPROXIMATE NEYMAN-PEARSON TEST

The noise is assumed to be heavily correlated ( $\rho_x$  very small), and the signal is assumed to be small with respect to the noise. Equations (2) and (3) under signal condition can be approximated from Ref. 2 as

$$x_k^r \approx \frac{g(|y_k|)}{|y_k|} y_k^r - cs_k^r$$

$$x_k^i \approx \frac{g(|y_k|)}{|y_k|} y_k^i - cs_k^i.$$

By using this approximation, the likelihood ratio becomes

$$\lambda_{NP} = \frac{P_{x_1^r, x_2^r}(x_1^r, x_2^r) P_{x_1^i, x_2^i}(x_1^i, x_2^i) \mathcal{J}_y(y_1^r, y_1^i, \dots)}{P_{x_1^r, x_2^r}(x_1^r, x_2^r) P_{x_1^i, x_2^i}(x_1^i, x_2^i) \mathcal{J}_y(y_1^r, y_1^i, \dots)}$$

$x_1^r = \frac{g(|y_1|)}{|y_1|} y_1^r - cs_1^r$

$x_1^i = \frac{g(|y_1|)}{|y_1|} y_1^i - cs_1^i$

Assuming that the Jacobians cancel and that  $x_1^r, x_2^r, x_1^i$ , and  $x_2^i$  are the transformed samples, the likelihood function is

$$\lambda_{np} = \frac{P_{x_1^r, x_2^r} [(x_1^r - cs_1^r), (x_2^r - cs_2^r)] P_{x_1^i, x_2^i} [(x_1^i - cs_1^i), (x_2^i - cs_2^i)]}{P_{x_1^r, x_2^r} (x_1^r, x_2^r) P_{x_1^i, x_2^i} (x_1^i, x_2^i)}$$

The expressions for  $P_{x_1^r, x_2^r} (x_1^r, x_2^r)$  and  $P_{x_1^i, x_2^i} (x_1^i, x_2^i)$  are simplified by noting that the factor

$$\rho_x^2 + \left( \frac{x_1^r}{b} - \frac{x_2^r}{b} \right)^2$$

in the denominator of Eq. (1) that defines  $P_{x_1^r, x_2^r} (x_1^r, x_2^r)$  basically controls the central peak of the probability density, since  $\rho_x$  is small and  $x_1^r \approx x_2^r$  under high correlation conditions. Consequently,

$$P_{x_1^r, x_2^r} (x_1^r, x_2^r) \approx \frac{\gamma}{\rho_x^2 + \left( \frac{x_1^r}{b} - \frac{x_2^r}{b} \right)^2},$$

and

$$P_{x_1^i, x_2^i} (x_1^i, x_2^i) = \frac{\gamma}{\rho_x^2 + \left( \frac{x_1^i}{b} - \frac{x_2^i}{b} \right)^2},$$

Next,  $\lambda_a$  is defined by

$$\lambda_{np} \approx \lambda_a = \lambda_a^r \lambda_a^i,$$

where

$$\lambda_a^r = \frac{b^2 \rho_x^2 + (x_1^r - x_2^r)^2}{b^2 \rho_x^2 + [(x_1^r - cs_1^r) - (x_2^r - cs_2^r)]^2}$$

and

$$\lambda_a^i = \frac{b^2 \rho_x^2 + (x_1^i - x_2^i)^2}{b^2 \rho_x^2 + [(x_1^i - cs_1^i) - (x_2^i - cs_2^i)]^2}.$$

These can be approximated by

$$\lambda_a^r = 1 + 2c(s_1^r - s_2^r)(x_1^r - x_2^r),$$

$$\lambda_a^i = 1 + 2c(s_1^i - s_2^i)(x_1^i - x_2^i).$$

Then

$$\lambda_a \approx (s_1^i - s_2^i)(x_1^i - x_2^i) + (s_1^r - s_2^r)(x_1^r - x_2^r)$$

where the higher order term has been ignored and the other coefficients have been incorporated into  $\lambda_a$ . This expression states that the data are transformed back to the bivariate Cauchy random variable and then the correlation between samples is removed by subtraction. This seems reasonable since it was in the Cauchy domain that the data were correlated. Consequently, the correlation probably should be removed in this domain. However the Cauchy domain is not an easy one in which to set the thresholds because of the large tails on the distribution. Since the Cauchy random variables were generated from Gaussian ones, the subtracted random variables are transformed to Gaussian by  $h$ . The detector finally used is then given by

$$\lambda_n = \sqrt{[h(x_1^r - x_2^r)]^2 + [(h(x_1^i - x_2^i))]^2}.$$

The detector transforms the measurements from bivariate Weibull or lognormal to bivariate Cauchy, removes the correlation by subtraction in both the  $r$  and the  $i$  components, transforms the remaining Cauchy to Gaussian, and computes the energy in both  $r$  and  $i$  signal components. This is intuitively satisfying in the sense that the processor inverts the way the data are generated.

### SIMULATION PERFORMANCE OF AN EXAMPLE

The performance of the detector is evaluated by using data samples obtained from the constructed multivariate density of the clutter. The performance curves are defined as the probability-of-detection vs signal-to-noise ratio given a fixed probability of false alarm for various problem conditions. The performance of the approximate Neyman-Pearson test is compared to that of the matched filter defined by

$$\lambda_{mf} = |\bar{\mathbf{S}}^t \mathbf{R}_y^{-1} \mathbf{Y}|,$$

where the traditional complex notation has been used to represent the data vector

$$\mathbf{Y} = \begin{bmatrix} y_1^r + jy_1^i \\ y_2^r + jy_2^i \end{bmatrix}$$

where  $j = \sqrt{-1}$ . The covariance matrix is

$$\mathbf{R}_y = \sigma_y^2 \begin{bmatrix} 1 & \rho_y \\ \rho_y & 1 \end{bmatrix},$$

where  $\rho_y$  is the correlation coefficient of the Weibull noise that has an even spectrum, and the signal vector is

$$\mathbf{S} = \begin{bmatrix} s_1^r + js_1^i \\ s_2^r + js_2^i \end{bmatrix}.$$

The bar represents the conjugate,  $t$  represents the transpose,  $-1$  represents the inverse, and  $|\cdot|$  represents the absolute value.

For this example  $N = 2$ ,  $\bar{N} = 40$ ,  $b = 1$ ,  $M_y = 1$ ,  $\sigma_{z_s} = 1$ ,  $\phi_s$  is arbitrary, and the steering vector is  $s_1^r = 1/\sqrt{2}$ ,  $s_1^i = 0$ ,  $s_2^r = 0$ , and  $s_2^i = 1/\sqrt{2}$ . For Weibull clutter of parameter  $\gamma = 1$ , 1.5, and 2, it was found by simulation that  $\sigma_y = 1.42$ , .97, and .84, respectively, and that  $\rho_y = .972$  for all three values.

The data used in the simulation are generated by converting independent Gaussian distributed samples to Cauchy distributed samples, correlating them through a summer, and then converting them to Weibull distributed samples on the marginals. This data generation is used to set the thresholds through the probability of false alarm. Signals are then added with random phase to the bivariate Weibull distributed samples to obtain the data required for generating the operating characteristics of the detectors.

The probability of false alarm  $p_{fa}$  vs threshold is computed by using Monte Carlo simulation techniques. Random samples of noise  $y_1^r, y_2^r, y_1^i, y_2^i$  are generated. A normalized histogram of the number of samples having values in a small interval for each detector is found. After many trials, a curve of the probability density of the output of each detector vs  $\lambda_n$ , or  $\lambda_{mf}$  is obtained. The probability of false alarm is defined as the sum of all values of the density from  $\lambda_n^*$  or  $\lambda_{mf}^*$  to infinity. For small values of the probability of false alarm, an importance sampling technique is used. This procedure distorts the generation of random samples so that more false alarms occur than should and then compensates for this in the weightings used to generate the histogram. This technique is outlined in Appendix C. The threshold settings vs probability of false alarm for the new approximate detector and the matched filter are shown in Table 1 for  $\alpha = 1, 1.5,$  and  $2.0$  and for probability of false alarms as low as  $10^{-7}$ . The reason the thresholds for the approximate detector are independent of  $\alpha$  is that the detector maps the Weibull distributed measurements to an identical Cauchy distribution regardless of what  $\alpha$  is. Although the thresholds are independent of  $\alpha$ , this does not mean that the detector performance is independent of  $\alpha$ .

The performances of the detectors are compared by observing the probability of detection vs signal-to-noise ratio for a fixed probability of false alarm. The probability of detection is computed by using a Monte Carlo simulation. The fraction of time the detector output exceeds the threshold for a set of randomly generated samples is computed. This number is the probability of detection.

Performance results are shown in Figs. 1 to 3 for Weibull parameters of  $\alpha = 1.0, 1.5,$  and  $2.0,$  respectively. In all cases, the probability of false alarm is  $10^{-7}$ . Figures 1 to 3 show that the approximate detector performs better than the matched filter. In all cases, the new approximate detector performs better than the matched filter; that is optimum for stationary Gaussian bivariate distributions. This is true even when  $\alpha = 2$  and the marginal densities become Gaussianly distributed. Even though the marginals are Gaussian, the data are not bivariate Gaussian and consequently the new detector is better. A functional flow of the detector is shown in Fig. 4.

Table 1 — Probability of False Alarm vs Thresholds for Weibull Clutter

	$\alpha$	$P_{fa}$						
		$10^{-1}$	$10^{-2}$	$10^{-3}$	$10^{-4}$	$10^{-5}$	$10^{-6}$	$10^{-7}$
Approximate filter $\lambda_a^*$	1.0	.51	1.65	2.52	3.30	3.85	4.15	4.35
	1.5	.51	1.65	2.52	3.30	3.85	4.15	4.35
	2.0	.51	1.65	2.52	3.30	3.85	4.15	4.35
Matched filter $\lambda_{mf}^*$	1.0	5.0	30.	75.	115.	155.	195.	230.
	1.5	9.0	45.	95.	135.	170.	200.	230.
	2.0	9.0	54.	104.	137.	165.	189.	210.

Fig. 1 — Detector operating characteristics for Weibull clutter,  $\alpha = 1.0$  and  $p_{fa} = 10^{-7}$

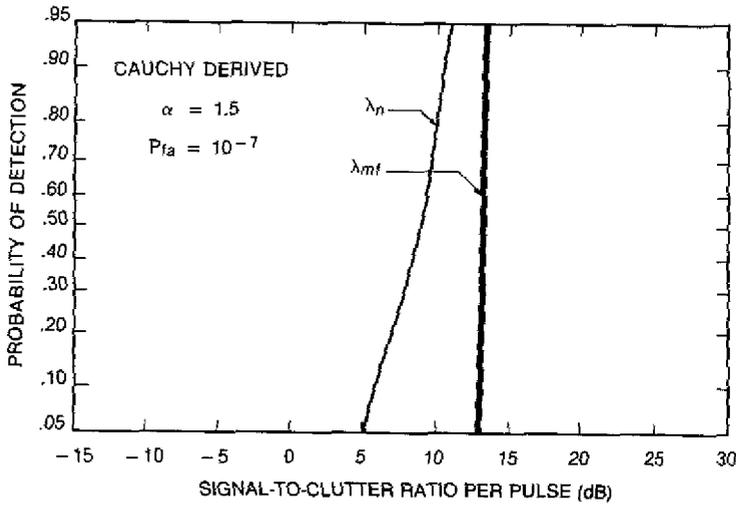
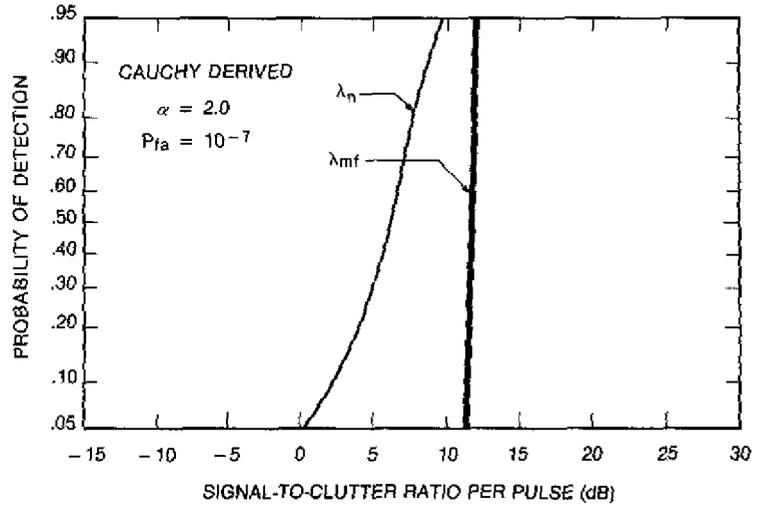
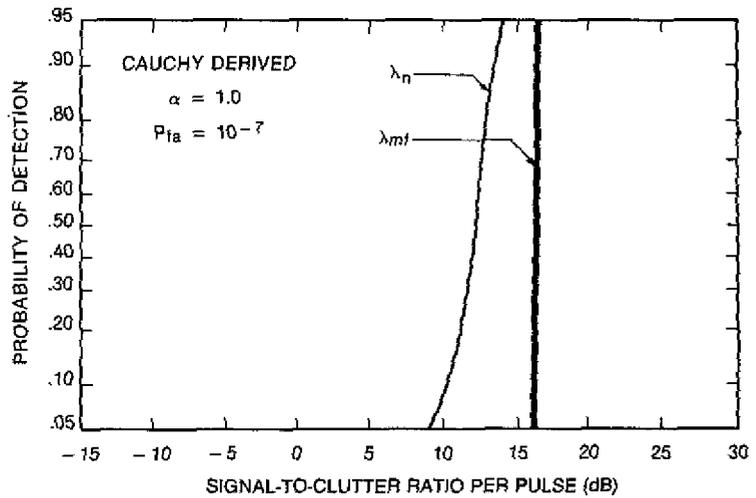


Fig. 2 — Detector operating characteristics for Weibull clutter,  $\alpha = 1.5$  and  $p_{fa} = 10^{-7}$

Fig. 3 — Detector operating characteristics for Weibull clutter,  $\alpha = 2.0$  and  $p_{fa} = 10^{-7}$



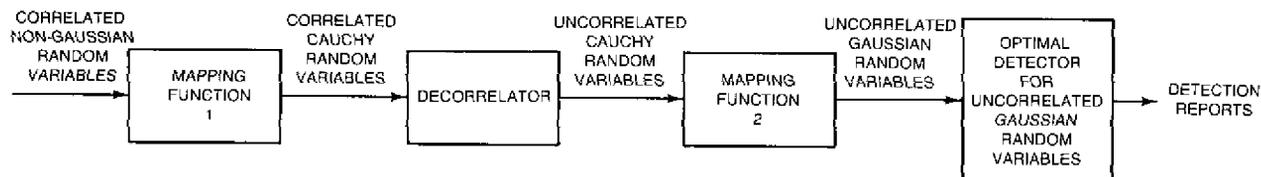


Fig. 4—Signal processor functional diagram

## SUMMARY

A procedure to detect a target in non-Gaussian correlated noise is obtained. Since the bivariate probability density is unknown, an approximate one is constructed. The constructed density matches the true density in the marginals and first two moments. The mappings required are found for both the Weibull and the lognormal clutter distributions. The bivariate density is formed from first mapping an independent Gaussian random variable into Cauchy distributed random variables, correlating with summers, and then mapping so that the new bivariate density has the desired marginals and first two moments.

A Neyman-Pearson test was obtained for detecting an additive signal in this bivariate distributed noise. Since the test was complicated and contained signals with unknown amplitude and phase, it was simplified. The simpler test transforms the correlated Weibull or log-normal measurements back to Cauchy distributed, removes the correlation, and finally transforms back to "Gaussian-like distributed" variables where the result is compared to a threshold. The performance of this detector was compared to that of the matched filter operating on the same data set. It was found that the new detector performed better than the matched filter. One could speculate that the best detector always transforms the noisy data to that domain where the correlation is best removed and then transforms it so that stationary Gaussian random variables are obtained on all examples.

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**Appendix A**  
**INTEGRATIONS FOR BIVARIATE CAUCHY**

The integrations  $I_0$  and  $I'_0$  are

$$I_0 = \frac{1}{(2\pi)^2} \int_0^\infty \int_0^\infty \exp \{ -b[w_1 + w_2 + (\bar{N} - 1)(w_1 + w_2)] - jw_1x_1 - jw_2x_2 \} dw_1 dw_2$$

$$I'_0 = \frac{1}{(2\pi)^2} \int_{-\infty}^0 \int_{-\infty}^0 \exp \{ -b[-w_1 - w_2 - (\bar{N} - 1)(w_1 + w_2)] - jw_1x_1 - jw_2x_2 \} dw_1 dw_2$$

where the superscript  $r$  and  $i$  have been dropped for convenience. By changing variables  $w'_1 = -w_1$  and  $w'_2 = -w_2$ , the two integrations can be combined

$$I_0 + I'_0 = \frac{1}{(2\pi)^2} \int_0^\infty \int_0^\infty \exp \{ -\bar{N}b(w_1 + w_2) \} [ \exp \{ -j(x_1w_1 + x_2w_2) \} + \exp \{ +j(x_1w_1 + x_2w_2) \} ] dw_1 dw_2.$$

This can be simplified to

$$\begin{aligned} I_0 + I'_0 &= \frac{2}{(2\pi)^2} \int_0^\infty e^{-\bar{N}bw_1} \cos x_1w_1 dw_1 \int_0^\infty e^{-\bar{N}bw_2} \cos x_2w_2 dw_2 \\ &\quad - \frac{2}{(2\pi)^2} \int_0^\infty e^{-\bar{N}bw_1} \sin x_1w_1 dw_1 \int_0^\infty e^{-\bar{N}bw_2} \sin x_2w_2 dw_2. \end{aligned}$$

These integrals are standard table integrals and consequently

$$I_0 + I'_0 = \frac{2}{(2\pi)^2} \left[ \frac{(\bar{N}b)^2 - x_1x_2}{[(\bar{N}b)^2 + x_1^2] [(\bar{N}b)^2 + x_2^2]} \right].$$

The integrals  $I_1$  and  $I'_1$  are

$$I_1 = \frac{1}{(2\pi)^2} \int_0^\infty \int_{-w_1}^0 \exp \{ -b[w_1 - w_2 + (\bar{N} - 1)(w_1 + w_2)] - jw_1x_1 - jw_2x_2 \} dw_2 dw_1$$

$$I'_1 = \frac{1}{(2\pi)^2} \int_{-\infty}^0 \int_0^{-w_1} \exp \{ -b[-w_1 + w_2 - (\bar{N} - 1)(w_1 + w_2)] - jw_1x_1 - jw_2x_2 \} dw_2 dw_1.$$

These can be combined after changing variables to yield

$$\begin{aligned} I_1 + I'_1 &= \frac{1}{(2\pi)^2} \int_0^\infty \int_{-w_1}^0 \exp \{ -\bar{N}bw_1 - (\bar{N} - 2)bw_2 \} [ \exp \{ -j(x_1w_1 + x_2w_2) \} \\ &\quad + \exp \{ +j(x_1w_1 + x_2w_2) \} ] dw_2 dw_1 \end{aligned}$$

This can be simplified to

$$I_1 + I'_1 = \frac{2}{(2\pi)^2} \int_0^\infty e^{-\bar{N}bw_1} \cos w_1 y_1 \left[ \int_0^{w_1} e^{+(\bar{N}-2)bw_2} \cos w_2 y_2 dw_2 \right] dw_1$$

$$+ \frac{2}{(2\pi)^2} \int_0^\infty e^{-\bar{N}bw_1} \sin w_1 y_1 \left[ \int_0^{w_1} e^{+(\bar{N}-2)bw_2} \sin w_2 y_2 dw_2 \right] dw_1.$$

After integrating over  $w_2$ ,  $I_1 + I'_1$  become

$$I_1 + I'_1 = \frac{2}{(2\pi)^2 \{(\bar{N}-2)b)^2 + x_2^2\}} \left\{ (\bar{N}-2)b \int_0^\infty e^{-2bw_1} \cos w_1(x_1 - x_2) dw_1 \right.$$

$$- x_2 \int_0^\infty e^{-2bw_1} \sin w_1(x_1 - x_2) dw_1 - (\bar{N}-2)b \int_0^\infty e^{-\bar{N}bw_1} \cos w_1 x_1 dw_1$$

$$\left. + x_2 \int_0^\infty e^{-\bar{N}bw_1} \sin w_1 x_1 dw_1 \right\}.$$

After integration over  $w_1$ , then

$$I_1 + I'_1 = \frac{2 [2(\bar{N}-2)b^2 - x_2(x_1 - x_2)]}{(2\pi)^2 \{[(\bar{N}-2)b]^2 + x_2^2\} [(2b)^2 + (x_1 - x_2)^2]}$$

$$- \frac{2[\bar{N}(\bar{N}-2)b^2 - x_1 x_2]}{(2\pi)^2 \{[(\bar{N}-2)b]^2 + x_2^2\} [(\bar{N}b)^2 + x_1^2]}$$

The final integrations for  $I_2$  and  $I'_2$  are similar and are

$$I_2 = \frac{1}{(2\pi)^2} \int_{-\infty}^0 \int_0^{-w_2} \exp \{ -b[w_1 - w_2 - (\bar{N}-1)(w_1 + w_2)] - jw_1 x_1 - jw_2 x_2 \} dw_1 dw_2$$

$$I'_2 = \frac{1}{(2\pi)^2} \int_0^\infty \int_{-w_2}^0 \exp \{ -b[-w_1 + w_2 + (\bar{N}-1)(w_1 + w_2)] - jw_1 x_1 - jw_2 x_2 \} dw_1 dw_2.$$

Combining and changing variables yield

$$I_2 + I'_2 = \frac{2}{(2\pi)^2} \int_0^\infty e^{-\bar{N}bw_2} \cos w_1 y_2 \left[ \int_0^{w_2} e^{+(\bar{N}-2)bw_1} \cos w_1 y_1 dw_1 \right] dw_2$$

$$+ \frac{2}{(2\pi)^2} \int_0^\infty e^{-\bar{N}bw_2} \sin w_2 y_2 \left[ \int_0^{w_2} e^{+(\bar{N}-2)bw_1} \sin w_1 y_1 dw_1 \right] dw_2.$$

In a similar manner to  $I_1 + I'_1$ , first one integrates over  $w_1$  then  $w_2$  by using standard integration tables; the result is

$$I_1 + I'_2 = \frac{2[2(\bar{N} - 2)b^2 - x_1(x_2 - x_1)]}{(2\pi)^2\{[(\bar{N} - 2)b]^2 + x_1^2\} [(2b)^2 + (x_2 - x_1)^2]} \\ - \frac{2[\bar{N}(\bar{N} - 2)b^2 - x_1x_2]}{(2\pi)^2\{[(\bar{N} - 2)b]^2 + x_1^2\} [(\bar{N}b)^2 + x_2^2]}.$$

## Appendix B DISTRIBUTION FUNCTIONS

The distribution function for the magnitude of the Cauchy variables over the in-phase and quadrature components is

$$\int_0^{|x_k|} \frac{4|x_k|}{\pi \left\{ (|x_k|^2 + 2b^2) \sqrt{1 + \left( \frac{|x_k|}{b} \right)^2} \right\}} d|x_k|.$$

Changing variables to

$$\xi^2 = |x_k|^2 + b^2$$

yields

$$\int_{+b}^{\sqrt{|x_k|^2 + b^2}} \frac{1}{\xi^2 + b^2} d\xi = \frac{4}{\pi} \tan^{-1} \left\{ \frac{\sqrt{|x_k|^2 + b^2}}{b} \right\} - 1.$$

The distribution function for Weibull is

$$\int_0^{|y_k|} \alpha \ln 2 \left\{ \frac{|y_k|}{M_y} \right\}^{\alpha-1} \exp \left[ \ln 2 \left\{ \frac{|y_k|}{M_y} \right\}^\alpha \right] d|y_k|$$

and is obtained by first noting

$$\int_0^{|y_k|} \exp \left[ \ln 2 \left\{ \frac{|y_k|}{M_y} \right\}^\alpha \right] d \left[ \ln 2 \left\{ \frac{|y_k|}{M_y} \right\}^\alpha \right].$$

Then we obtain

$$1 - \exp \left\{ - \ln 2 \left\{ \frac{|y_k|}{M_y} \right\}^\alpha \right\}.$$

The distribution function for the lognormal is

$$\int_0^{|y_k|} \frac{2}{\sqrt{2\pi}\sigma_l |y_k|} \exp - \left\{ \frac{1}{2\sigma_l^2} \left[ 2 \ln \left\{ \frac{|x_k|}{M_y} \right\} \right]^2 \right\} d|y_k|,$$

which is

$$\int_0^{|y_k|} \frac{1}{\sqrt{2\pi}\sigma_l} \exp - \left\{ \frac{1}{2\sigma_l^2} \left[ 2 \ln \left( \frac{|y_k|}{M_y} \right) \right]^2 \right\} d \left[ 2 \ln \left( \frac{|y_k|}{M_y} \right) \right].$$

This is the form of an integral over a Gaussian, which is

$$\frac{1}{2} + \frac{1}{2} \operatorname{erf} \left[ \frac{\sqrt{2}}{\sigma_l} \ln \left( \frac{|y_k|}{M_y} \right) \right].$$

These distribution functions are used to define the mappings  $g(\cdot)$  and  $g^{-1}(\cdot)$ .

## Appendix C IMPORTANCE SAMPLING

The importance-sampling procedure [C1,C2] is a Monte Carlo technique that distorts the generation of the random number so that the events of interest occur more frequently than, but in the same manner as, the events occur in nature. The probability of the event occurring is then compensated for with a weighting factor so that the true probability of the event is obtained.

To compute the probability density of the detector, the filter values are quantized by

$$\lambda = m \Delta\lambda ,$$

where

$$m = 0, \dots, M - 1.$$

The probability density is computed by

$$p(\lambda \text{ is between } (m - 1) \Delta\lambda \text{ and } m \Delta\lambda) = \frac{1}{N} \sum_{k=1}^N \delta_k$$

where  $N$  is the number of Monte Carlo samples and  $\delta_k = 1$  if no importance sampling is used. The equation is simply counting the percentage of time the samples fall in the  $m$ th interval. For importance sampling and this case, the first and last sample of the independent Gaussian variables are distorted in their standard deviation. The weight factor is computed by ratioing the true density to the distorted density evaluated with the distorted data samples. In this case

$$\delta_k = \left[ \frac{\sigma_d}{\sigma_g} \exp - \frac{1}{2} \left\{ \frac{[z_g^r(1)]^2 + [z_g^i(1)]^2}{\sigma_g^2} - \frac{[z_g^r(1)]^2 + [z_g^i(1)]^2}{\sigma_d^2} \right\} \right] \left[ \frac{\sigma_d}{\sigma_g} \exp - \frac{1}{2} \left\{ \frac{[z_g^r(\bar{N} + 1)]^2 + [z_g^i(\bar{N} + 1)]^2}{\sigma_g^2} - \frac{[z_g^r(\bar{N} + 1)]^2 + [z_g^i(\bar{N} + 1)]^2}{\sigma_d^2} \right\} \right]$$

where  $\sigma_d$  is the distortion of the standard deviation on the first and last samples. The values  $z_g^r(1)$ ,  $z_g^i(1)$ ,  $z_g^r(\bar{N} + 1)$ , and  $z_g^i(\bar{N} + 1)$  are the sample values under distortion. This distortion decreases the correlation in the final data and raises its variance slightly. Consequently more false alarms for the same threshold are produced after distortion but the probability is reduced because of  $\delta_k$ .

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