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The Steering Problem for the Mirror-Scan Tracking System

Part 1—The Stop-Go-Stop Procedure

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Tracking radar systems are being designed which use a lightweight mirror to steer the radar beam. Such systems can track several targets simultaneously, and the steering problem is to find the beam motion which covers a given collection of targets with minimum update time. In this report it is shown how the steering problem is a control-theoretic problem on the sphere. The suboptimal stop-go-stop solutions are discussed, and it is determined that the proposed Mirror Scan system can simultaneously track at least seven targets in a half-hemisphere with an update time of less than 1 second.

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THE STEERING PROBLEM FOR THE MIRROR-SCAN TRACKING SYSTEM PART 1 — THE STOP-GO-STOP PROCEDURE

1. INTRODUCTION

The Mirror Scan system employs a mirror to steer the beam of a tracking radar whose light weight (about 11 kg (25 lb)) permits it to be slewed much more rapidly than the antenna of a conventional system, so that the system can be used to track several targets simultaneously. Since tracking accuracy varies inversely with update time, the number of targets which can be simultaneously tracked depends on the efficiency of the steering algorithm. The *steering problem* is to determine the radar-beam motion which covers a given collection of targets with minimum update time, subject to two constraints:

- A physical constraint on the magnitudes of the torques that can be applied to the mirror;
- A constraint on the velocity of the beam motion as the beam passes through a target, so as to insure a required number of radar-pulse hits on the target.

1.1 General Synopsis

Sections 2 through 4 are devoted to a careful mathematical description of the steering problem, which is a difficult problem in control theory. As a first approach to this problem we will in this report examine the stop-go-stop (SGS) paths, that is, beam motions in which the beam is made to stop at each target. The problem of finding the optimal SGS path can be reduced to a standard problem in control theory with a known solution. This reduction of the problem is described in section 5. In section 7 we will show how calculations of update times of optimal SGS paths for various configurations of target positions lead to the conclusion that the mirror-scan system can track at least seven targets in a half-hemisphere with an update time of less than 1 second.

In section 6 we will discuss the *aiming problem*, which is the problem of determining the best system orientation for a given area of coverage. One might expect that the solution of this problem would be to point the system at the centroid of the area of coverage, and computer calculations suggest that this is in fact the case. However, the same calculations also reveal that the system orientation has only a slight effect on efficiency, provided that the departure of the pointing direction from the centroid of the area of coverage is not too "extreme."

Finally, in some situations the SGS procedure is nearly optimal, and in others SGS is clearly inefficient. In subsequent reports in this series we hope to obtain lower bounds on update time (as a function of number of targets and area of coverage), identify the precise

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conditions under which SGS is grossly inefficient, and construct better tracking algorithms for this case.

1.2 Technical Summary

A summary of the technical results of this report is as follows:

- The target position ξ and the corresponding mirror normal position $\mathbf{p}$ are related by

$$\xi = \mathbf{a} - 2(\mathbf{a} \cdot \mathbf{p})\mathbf{p}, \tag{2.1}$$

where $\mathbf{a}$ is a constant vector defining the orientation of the system. (The *aiming vector* is $-\mathbf{a}$.)

- The *system control* is defined by specifying the applied torque vector $\mathbf{u}$ as a function of time.

- The system control $\mathbf{u} = \mathbf{u}(t)$ and mirror motion $\mathbf{p} = \mathbf{p}(t)$ are related by

$$\mathbf{u} = I\mathbf{p} \times \ddot{\mathbf{p}}, \tag{3.1}$$

where I is a certain moment of inertia.

- The *steering problem* is to find the closed curve $\mathbf{p} = \mathbf{p}(t)$ which joins given points with minimum update time, subject to two constraints:

$$|\mathbf{p} \times \ddot{\mathbf{p}}| \leqslant \alpha \tag{4.C1}$$

and

$$|\dot{\xi}|_{\text{target}} \leqslant \beta. \tag{4.C2}$$

- The constraint (4.C1) can be expressed as a bound on the second covariant derivative $\delta^2\mathbf{p}/\delta t^2$ of $\mathbf{p}$, which is the tangential component of $\ddot{\mathbf{p}}$.

- The constraint (4.C2) is closely approximated in a more convenient form by

$$|\dot{p}| \leqslant \beta/2. \tag{4.CA2}$$

- In the stop-go-stop (SGS) procedure the constraint (4.C2) is automatically satisfied, and $\mathbf{p}$ moves in great-circle arcs, stopping at each "image" of a target position.

- The determination of the optimal point-to-point motion of $\mathbf{p}$ along an SGS path reduces to a solved problem in control theory, and the update time T of an SGS path is then given by

$$T = 2\sum \sqrt{Z_j/\alpha}, \tag{5.10}$$

where $\{Z_j\}$ are the arc lengths of the various links of the path.

- There are $(N-1)!/2$ SGS paths, where N is the number of targets. The optimal SGS path is the one which minimizes $\Sigma\sqrt{Z_j}$ (and therefore T) and can be found by straightforward enumeration of all cases, provided that N is not too large.

- For the current design value of α it can be predicted from the calculated results that the Mirror Scan system can simultaneously track at least seven targets in a half-hemisphere with an update time of less than 1 second.

- The orientation of the system (defined by **a**) has only a slight effect on system efficiency, provided that the departure of the aiming vector (− **a**) from the centroid of the area of coverage is not too extreme.

2. SYSTEM GEOMETRY

Radiation from a feed (which is fixed with respect to the ship) passes through a hole in a mirror M and falls upon a fixed reflector S, which reflects it back toward M. The radiation is linearly polarized, and S is composed of wires aligned parallel to the direction of polarization. The mirror acts as a half-wave plate and rotates the angle of polarization by 90°, so that the radiation passes out through S after reflection by M. Let **a** be the unit vector defining the direction of the radiation incident on the mirror. Then **a** is fixed to the ship and points into the ship, and its negative (− **a**) will be called the *aiming vector*.

Let

p = unit vector normal to the mirror, pointing outward;

ξ = "reflected" unit vector, pointing to the target.

Then, since the angle of incidence on the mirror equals the angle of reflection, the vectors **a**, **p**, and ξ are related by (Fig. 1)

$$\xi = \mathbf{a} - 2(\mathbf{a}\cdot\mathbf{p})\mathbf{p} \tag{2.1}$$

and

$$p = \frac{\xi - \mathbf{a}}{|\xi - \mathbf{a}|} \quad . \tag{2.2}$$

Since target ranges have no relevance to our problem, we can identify each target position with a point on a unit sphere, whose position is ξ. The corresponding value of **p** (given by (2.2)) will be called the *p-image* of the target.

Let $\xi^{(1)}, \ldots, \xi^{(N)}$ be an arbitrary collection of N target positions, and let $\mathbf{p}^{(1)}, \ldots, \mathbf{p}^{(N)}$ be the corresponding p-images. The steering problem is to find the closed curve $\mathbf{p} = \mathbf{p}(t)$ $(0 \leqslant t \leqslant T)$, subject to certain constraints, which

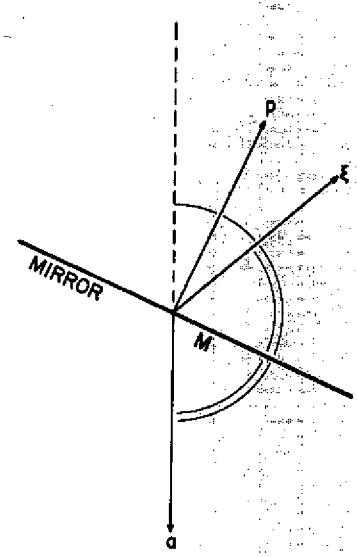

Fig. 1 — Relation of the vectors **a**, **p**, and ξ.

joins all the p-images with a minimum update time T. In the steering problem it is assumed that the value of the aiming vector $(-\mathbf{a})$ is fixed and given. The *aiming problem* is to make an intelligent choice of $\mathbf{a}$ for each given area of coverage.

We shall fix a right-handed rectangular $X_1X_2X_3$ coordinate system to the ship, with corresponding spherical coordinates r, θ, and ϕ. As shown in Fig. 2, the mirror-scan system is attached to the port side of the ship, the positive X_1 axis points toward the bow, the positive X_3 axis is vertical, and the coordinates of a point on the unit sphere $(r = 1)$ are

$$\left.\begin{aligned} x_1 &= \sin\theta \cos\phi \\ x_2 &= \sin\theta \sin\phi \\ x_3 &= \cos\theta \end{aligned}\right\}. \tag{2.3}$$

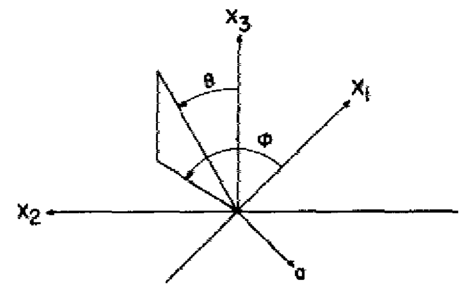

Fig. 2 — Relation of coordinate systems

Since $\mathbf{a}$ points into the ship, its X_2 coordinate is negative. Similarly, the X_2 coordinates of $\mathbf{p}$ and ξ are always positive.

3. THE CONTROL

Mathematically, the mirror motion is defined by a curve $\mathbf{p} = \mathbf{p}(t)$ lying on the unit sphere, that is, by a description of the motion of the unit normal $\mathbf{p}$. Physically, the mirror is moved by applying torques to the mirror around certain instantaneous axes of rotation, and the system *control* is defined by specifying the total torque vector $\mathbf{u}$ as a function of time: $\mathbf{u} = \mathbf{u}(t)$. The relationship between the control $\mathbf{u}$ and motion of $\mathbf{p}$ is

$$\mathbf{u} = I\mathbf{p} \times \ddot{\mathbf{p}}, \tag{3.1}$$

where I is a certain moment of inertia.

Equation (3.1) is a special case of Euler's equation for the motion of a rigid body, and in its derivation two assumptions are made:

A1. The mirror is a disk, a square, or some other body having certain symmetry properties. For a disk, I is the moment of inertia about a diameter.

A2. The torques are applied in such a manner that the mirror motion has no rotational component around the axis $\mathbf{p}$, because the existence of such a component would imply the expenditure of energy to produce motions of the mirror which have no effect on its aim.

It turns out to be more convenient to write (3.1) in the form

$$\mathbf{u} = I\mathbf{p} \times \delta^2\mathbf{p}/\delta t^2, \tag{3.1*}$$

where, by definition,

$$\delta^2\mathbf{p}/\delta t^2 = \ddot{\mathbf{p}} + |\dot{\mathbf{p}}|^2\mathbf{p}. \tag{3.2}$$

The derivation of (3.1*) and thus of (3.1) is given in Appendix A. The mathematical equivalence of (3.1) and (3.1*) is a trivial consequence of the identity $\mathbf{p} \times \mathbf{p} = 0$, which allows *any* scaler multiple of $\mathbf{p}$ to be added to the $\ddot{\mathbf{p}}$ term in the right-hand side of (3.1) without changing its value. The replacement of $\ddot{\mathbf{p}}$ in (3.1) by $\ddot{\mathbf{p}} + |\dot{\mathbf{p}}|^2\mathbf{p}$ and thus by $\delta^2\mathbf{p}/\delta t^2$ may therefore appear to be highly arbitrary, but it is justified by the identity

$$|\mathbf{p} \times \ddot{\mathbf{p}}| = |\delta^2\mathbf{p}/\delta t^2|, \tag{3.3}$$

so that the constraint on the torque magnitude $|\mathbf{u}|$ reduces to a constraint on $|\delta^2\mathbf{p}/\delta t^2|$. To show that (3.3) is an identity, one differentiates the identity $\mathbf{p} \cdot \mathbf{p} = 1$ twice and obtains

$$\mathbf{p} \cdot \dot{\mathbf{p}} = 0 \text{ and } \mathbf{p} \cdot \ddot{\mathbf{p}} + |\dot{\mathbf{p}}|^2 = 0. \tag{3.4}$$

Equation (3.3) is a consequence of (3.4) and of the vector identities

$$\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = -\mathbf{c} \cdot (\mathbf{b} \times \mathbf{a})$$

and

$$\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = \mathbf{b}(\mathbf{a} \cdot \mathbf{c}) - \mathbf{c}(\mathbf{a} \cdot \mathbf{b}).$$

From (3.2) and (3.4) we also have

$$|\delta^2\mathbf{p}/\delta t^2|^2 = |\ddot{\mathbf{p}}|^2 - |\dot{\mathbf{p}}|^4. \tag{3.5}$$

The quantity $\delta^2\mathbf{p}/\delta t^2$ is called the *second covariant derivative* of $\mathbf{p} = \mathbf{p}(t)$, and in Appendix B it is shown that $\delta^2\mathbf{p}/\delta t^2$ is the tangential component of $\ddot{\mathbf{p}}$, that is, the projection of the *vector* $\ddot{\mathbf{p}}(t)$ onto the plane tangent to the unit sphere at the *point* $\mathbf{p}(t)$. If the curve $\mathbf{p} = \mathbf{p}(t)$ is taken to be the description of the motion of a particle of unit mass, then $\delta^2\mathbf{p}/\delta t^2$ is equal to the tangential component of the force acting on the particle. In particular, the condition $\delta^2\mathbf{p}/\delta t^2 = 0$ is precisely the condition that the motion be inertial, that is, on a great-circle arc with constant speed. Hence (3.1*) has the following consequence: if the applied torque $\mathbf{u}$ is cut off at a certain time, then $\mathbf{p} = \mathbf{p}(t)$ will thereafter move along a great-circle route with constant speed.

4. THE STEERING PROBLEM

In this section we will define the steering problem in its most general form. The constraints of this general problem are easy to express and hard to apply, but in our subsequent discussion of the stop-go-stop procedure (section 5) we will show how these constraints reduce to a single constraint which is simply and directly related to the motion of $\mathbf{p}$.

The first constraint can be stated simply as

$$|\mathbf{u}| \leqslant \alpha', \tag{4.1}$$

where α' is the largest torque that can be applied to the mirror. This constraint is related to the motion of $\mathbf{p}$ via (3.1) or (3.1*) and can be written in one of the following equivalent forms, each of which will hereafter be referred to as the constraint (4.C1):

$$\left.\begin{aligned} &|\mathbf{p} \times \ddot{\mathbf{p}}| \leqslant \alpha, \\ \text{or} \\ &|\delta^2\mathbf{p}/\delta t^2| \leqslant \alpha, \\ \text{or} \\ &|\ddot{\mathbf{p}}|^2 - |\dot{\mathbf{p}}|^4 \leqslant \alpha, \end{aligned}\right\} \tag{4.C1}$$

where $\alpha = \alpha'/I$, I being the moment of inertia which appears in (3.1) and (3.1*). The mathematical equivalence of these forms is a consequence of (3.3) and (3.5).

The second constraint, referring to the beam velocity $\dot{\xi}(t)$ as the beam passes through a target, will be expressed as

$$|\dot{\xi}|_{\text{target}} \leqslant \beta, \tag{4.2}$$

where β is the largest possible beam velocity which allows the required number of radar pulse hits on the target. Differentiating (2.1) to express (4.2) in terms of the motion of $\mathbf{p}$, we get the constraint

$$|(\mathbf{a} \cdot \dot{\mathbf{p}})\mathbf{p} + (\mathbf{a} \cdot \mathbf{p})\dot{\mathbf{p}}|_{\text{target}} \leqslant \beta/2. \tag{4.C2}$$

Using the same notation introduced in section 2, we let $\xi^{(1)}, ..., \xi^{(N)}$ be the position vectors of N targets and let $\mathbf{p}^{(1)}, ..., \mathbf{p}^{(N)}$ be the corresponding p-images. We let $\xi = \xi(t)$ be a curve which joins the points $\xi^{(1)}, ..., \xi^{(N)}$, and we let $\mathbf{p} = \mathbf{p}(t)$ be the p-image of this curve, so that for each value of t, $\mathbf{p}(t)$ and $\xi(t)$ are related by (2.1) and (2.2). We can now state the steering problem in its most general form:

The Steering Problem. Find the closed curve $\mathbf{p} = \mathbf{p}(t)$ which satisfies the constraints (4.C1) and (4.C2) and joins all the points $\mathbf{p}^{(1)}, ..., \mathbf{p}^{(N)}$ with minimum update time.

Remark 4.1. Later, when the design of the system has been made more definite, it may be necessary to replace the constraint (4.1) (and hence constraint (4.C1)) by a constraint imposing separate bounds on the components of $\mathbf{u}$ about certain instantaneous axes of rotation. A further complication might also be required if the moments of inertia about these axes should have too strong a functional dependence on position. In this case the constraint (4.1) would be only an approximation but a safe one, since the values of α' and α would always be adjusted so as to guarantee that no component of $\mathbf{u}$ exceeds a certain amount, and the estimation of the update time T thus obtained would be *greater* than its true value.

Remark 4.2. An attractive alternative to constraint (4.C2) would be the "approximation"

$$2|\dot{\mathbf{p}}| \leqslant \beta. \tag{4.C2A}$$

In fact, by setting $\mathbf{a} \cdot \mathbf{p} = -\cos \psi$, it is easy to show that $|\mathbf{a} \cdot \dot{\mathbf{p}}| \leqslant \sin \psi |\dot{\mathbf{p}}|$, so that from (2.1) one obtains

$$
\begin{aligned}
|\dot{\xi}|^2 &= 4\, |(\mathbf{a} \cdot \dot{\mathbf{p}})\mathbf{p} + (\mathbf{a} \cdot \mathbf{p})\dot{\mathbf{p}}|^2 \\
&= 4[(\mathbf{a} \cdot \dot{\mathbf{p}})^2 + (\mathbf{a} \cdot \mathbf{p})^2 |\dot{\mathbf{p}}|^2] \\
&\leqslant 4[\sin^2 \psi + \cos^2 \psi] |\dot{\mathbf{p}}|^2.
\end{aligned}
$$

In other words

$$|\dot{\xi}| \leqslant 2|\dot{\mathbf{p}}|, \tag{4.3}$$

so that approximation (4.C2A) is safe in the same sense as in the previous remark. Moreover the accuracy of the approximation $|\dot{\xi}| = 2|\dot{\mathbf{p}}|$ is usually quite good; it depends on the maximum permissible excursion of $\mathbf{p}$ from $-\mathbf{a}$ and therefore depends on the area of coverage. For example, if the area of coverage is a quarter-hemisphere, then $|\dot{\xi}|$ and $2|\dot{\mathbf{p}}|$ differ by less than 8 percent.

5. THE STOP-GO-STOP PROCEDURE

As before, we let $\xi^{(1)}, \ldots, \xi^{(N)}$ be N target positions and let $\mathbf{p}^{(1)}, \ldots, \mathbf{p}^{(N)}$ be the corresponding p-images. In the stop-go-stop (SGS) procedure $\mathbf{p} = \mathbf{p}(t)$ moves along great-circle arcs in a closed path joining all the $\mathbf{p}^{(j)}$ and comes to a complete stop at each of the $\mathbf{p}^{(j)}$. Hence constraint (4.C2) is automatically satisfied, and we will now show how constraint (4.C1) reduces to a mathematically simpler form, which will be denoted by (5.CSGS).

Our first task is to determine the curve $p = p(t)$ which joins two points, say $\mathbf{p}^{(1)}$ and $\mathbf{p}^{(2)}$, in the shortest time, subject to constraint (4.C1). We let $Z(1, 2)$ be the great-circle distance between $\mathbf{p}^{(1)}$ and $\mathbf{p}^{(2)}$ (the arc length of the short great-circle arc joining these points). Then

$$Z(1, 2) = \cos^{-1} [\mathbf{p}^{(1)} \cdot \mathbf{p}^{(2)}]. \tag{5.1}$$

Let $\mathbf{x} = \mathbf{x}(s)$ $(0 \leqslant s \leqslant Z(1,2))$ be the short great-circle arc joining $\mathbf{p}^{(1)}$ to $\mathbf{p}^{(2)}$ parameterized by s = arc length. Then $\mathbf{x} = \mathbf{x}(s)$ is (geodesic) motion along a great-circle arc with constant unit speed, and the acceleration vector is a unit vector which is always directed toward the center. Hence, letting primes denote differentiation with respect to s, $(') = (d/ds)$, we have

$$|\mathbf{x}'| = 1, \tag{5.1}$$

$$\mathbf{x}' \cdot \mathbf{x}'' = 0, \tag{5.2}$$

and

$$\mathbf{x}'' + \mathbf{x} = 0. \tag{5.3}$$

(We recall from (3.2) that by definition $\delta^2\mathbf{x}/\delta s^2 = \mathbf{x}'' + |\mathbf{x}'|^2\mathbf{x}$, so that $\delta^2\mathbf{x}/\delta s^2 = 0$ by virtue of (5.1) and (5.3); that is, the motion $\mathbf{x} = \mathbf{x}(s)$ is *geodesic.*)

Now let

$$\mathbf{p}(t) = \mathbf{x}[s(t)] \tag{5.4}$$

be a general motion along the great-circle arc whose velocity $|\dot{\mathbf{p}}| = |d\mathbf{p}/dt|$ is not necessarily constant. Differentiating (5.4), we get

$$\dot{\mathbf{p}} = \dot{s}\,\mathbf{x}' \tag{5.5}$$

and

$$\ddot{\mathbf{p}} = \ddot{s}\,\mathbf{x}' + \dot{s}^2\mathbf{x}''. \tag{5.6}$$

But $\mathbf{p}(t)$ is merely a reparameterization of $\mathbf{x}(s)$, so that from (5.3) we have $\mathbf{x}'' = -\,\mathbf{p}$. Also, from (5.2), x' is a unit vector perpendicular to $\mathbf{p}$. Referring to (5.6), we therefore see that

$$\ddot{s}\mathbf{x}' = \text{tangential component of } \ddot{\mathbf{p}} = \delta^2\mathbf{p}/\delta t^2,$$

so that

$$|\mathbf{p} \times \ddot{\mathbf{p}}| = |\ddot{s}\mathbf{p} \times \mathbf{x}'| = |\ddot{s}|.$$

Hence constraint (4.C1) reduces to

$$|\ddot{s}| \leqslant \alpha, \tag{5.CSGS}$$

where α is the same constant as appears in (4.C1).

The problem, of finding the curve $\mathbf{p} = \mathbf{p}(t)$ joining $\mathbf{p}^{(1)}$ to $\mathbf{p}^{(2)}$ in minimum time T subject to constraint (4.C1) as well as the conditions $\dot{\mathbf{p}}(0) = \dot{\mathbf{p}}(T) = 0$, has now been reduced to the problem of finding the piecewise smooth function $s = s(t)$ for which

$$\text{(5.CSGS) is satisfied on each smooth piece,}$$

$$s(0) = 0 \;\text{ and }\; s(T) = Z(1,2),$$

$$\dot{s}(0) = \dot{s}(T) = 0,$$

and

$$T \text{ is minimum.}$$

This is a standard problem in control theory and has a well-known solution: one applies a maximum acceleration α for half the time and a maximum deceleration $-\,\alpha$ for the other half.* Hence we have

*E.B. Lee and L. Markus, *Foundations of Optimal Control Theory*, Wiley, New York, 1967, Ch. 1.

$$\frac{1}{2}\ Z(1,2) = \frac{1}{2}\ \alpha \left(\frac{T_{\min}}{2}\right)^2$$

or, solving for $T_{\min}$,

$$T_{\min} = 2\sqrt{Z(1,2)/\alpha}. \tag{5.7}$$

There are $(N-1)!/2$ SGS paths joining the N points $\mathbf{p}^{(1)}, \ldots, \mathbf{p}^{(N)}$, and we now consider the problem of finding the one which is optimal.

Consider for example the configuration of four points shown schematically in Fig. 3. The points are labeled 1, 2, 3, and 4, and there are three SGS paths labeled 1234, 1324, and 1342. For example, the path 1234 is the path which takes point 1 to 2, 2 to 3, 3 to 4, and 4 to 1. These three paths are the only three essentially different SGS paths in this case, since all the other paths differ from these only in their starting point or orientation. Thus, for our purposes, the path labeled 1234 is the same as paths labeled 2341, 1432, etc.

More generally, for a configuration of N points we can always label the starting point as 1, so that the N-tuple $(1, i_2, i_3, \ldots, i_N)$ runs through all the different path levels as the $(N-1)$-tuple $(i_2, i_3, \ldots, i_N)$ runs through all the permutations of the $(N-1)$ integers $\{2, 3, \ldots, N\}$. In fact, each path is represented *twice*, once for each orientation, so that there are only $(N-1)!/2$ essentially different SGS paths joining N points.

We let $Z(i,j)$ be the arc length of the great-circle arc joining $\mathbf{p}^{(i)}$ to $\mathbf{p}^{(j)}$ and let $T(i,j)$ be the transit time for the motion of $\mathbf{p}(t)$ from $\mathbf{p}^{(i)}$ to $\mathbf{p}^{(j)}$, the motion being "optimal" as previously described in this section. Then as in (5.1) and (5.7) we have

$$Z(i,j) = \cos^{-1}\ [\mathbf{p}^{(i)} \cdot \mathbf{p}^{(j)}] \tag{5.8}$$

and

$$T(i,j) = 2\sqrt{Z(i,j)/\alpha}, \tag{5.9}$$

and the update time T corresponding to the path $(1, i_2, i_3, \ldots, i_N)$ is

$$T = \frac{2}{\sqrt{\alpha}}\ [\sqrt{Z(1,i_2)} + \sqrt{Z(i_2,i_3)} + \ldots + \sqrt{Z(i_{N-1},i_N)} + \sqrt{Z(i_N,1)}]. \tag{5.10}$$

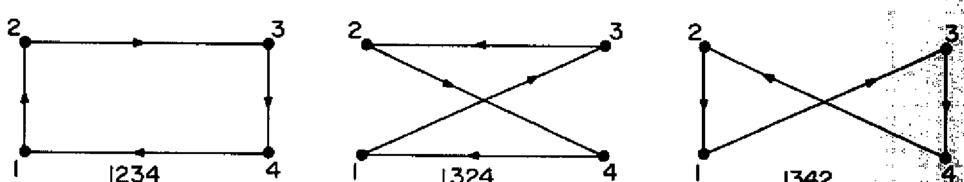

Fig. 3 — Paths joining four points

Hence the optimal SGS path is the one which minimizes the right-hand side of (5.10). In brief, we shall say that the optimal SGS path minimizes $\Sigma\sqrt{Z}$.

For the moderately small number of targets considered in this report ($N \leqslant 7$) the optimal SGS path can be found by the straightforward enumeration of all $(N-1)!/2$ cases. The general problem of minimizing a "cost function" such as (5.10) over a class of paths is known as the traveling-salesman problem, and algorithms have been constructed which make the search for the optimal path more efficient. However these algorithms are all inefficient in the sense that the computational time varies exponentially with N.* Much attention has been given to the search for optimal and near-optimal solutions to the problem of minimizing ΣZ. However, simple examples can be constructed which show that paths which minimize ΣZ do not necessarily minimize $\Sigma\sqrt{Z}$, and conversely.

6. THE AIMING PROBLEM

Figure 4 shows a half-hemisphere with four "extreme" points on the boundary and their coordinates. The coordinate system shown here is the same as that shown in Fig. 2. As ξ varies over the entire half-hemisphere, its p-image $\mathbf{p}$ will vary over a smaller region, and the range of $\mathbf{p}$ depends on the value of the aiming vector $\mathbf{a}$. This is shown in Fig. 5 for three values of $\mathbf{a}$. The outer curve is the boundary of the half-hemisphere, as seen by a distant eye having spherical coordinates (Fig. 2) $\theta = 60°$ and $\phi = 90°$. The inner curve is the boundary of the range of $\mathbf{p}$, and it is marked off into two subregions I and II which are the p-images of the quarter-hemispheres whose extreme points are (1, 2, 4) and (2, 3, 4) respectively.

Table 1 shows how the great-circle distances between the p-images of the extreme points varies with three values of the aiming vector $\mathbf{a}$. The arc lengths $Z(i, j)$, in degrees, are computed according to (5.8), in which the values of $\mathbf{p}^{(i)}$ are computed according to (2.2). $Z(1, 3)$ is always $90°$, because the position vectors ξ of points 1 and 3 are the negatives of each other, so that, from (2.2), $\mathbf{p}^{(1)} \cdot \mathbf{p}^{(3)} = 0$ for every value of $\mathbf{a}$.) The results shown in this table and other calculations suggest that the second value of $\mathbf{a}$ listed is optimal when the area of coverage is the entire half-hemisphere and that the third value is optimal when the area of coverage is restricted to the quarter-hemisphere (1, 2, 4).

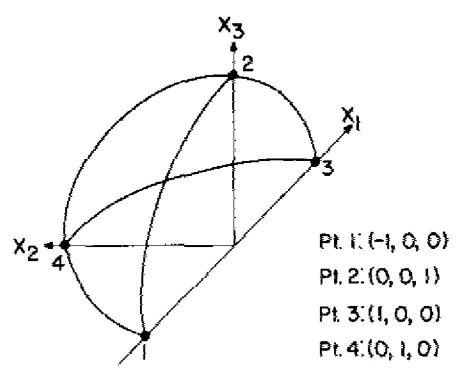

Fig. 4 — Coordinates of four "extreme" points on a half-hemisphere

*H.R. Lewis and C.H. Papadimitriou, "The Efficiency of Algorithms," *Scientific American* **238** (No. 1), 96-109 (Jan. 1978).

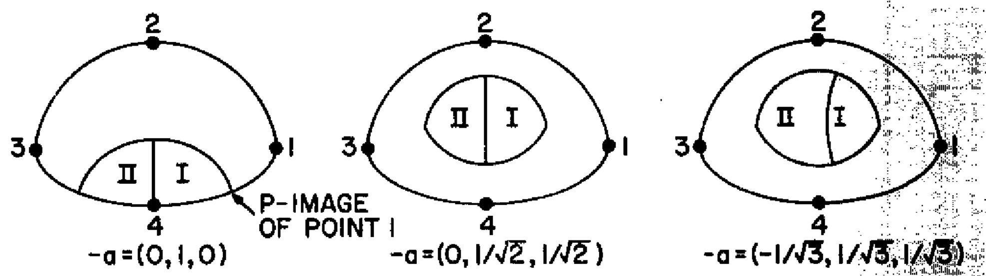

Fig. 5 — Ranges of p-images $\mathbf{p}$ for various aiming vectors $\mathbf{a}$, with the ranges of $\mathbf{p}$ subdivided into regions I and II (the ends of the subdividing arcs being p-images of points 2 and 4)

7. CONCLUSIONS

7.1 Calculations of Optimal SGS Paths and Update Times

Optimal SGS paths and update times were computed in the manner described in section 5, with the current design value of the constant α (defining the constraint) given by

$$\alpha = (100\sqrt{2}) \text{ radians/s}^2 \approx 8103 \text{ deg/s}^2.$$

Optimal paths and update times T were computed for various configurations of up to seven points and various values of $\mathbf{a}$. Some examples of our calculations are given in Fig. 6 for the three values of $\mathbf{a}$ listed in Table 1. The half-hemisphere shown in Fig. 4 is shown in Fig. 6 as seen by an observer who is in the X_2X_3 plane at large and positive x_3 and slightly negative x_2. The target positions ξ are indicated by heavy dots, and N is the number of targets.

7.2 Summary of Observations Concerning Calculated Results

The results of these calculations can be summarized in the following observations:

Observation 1. The Mirror Scan system can simultaneously track at least seven targets in a half-hemisphere with an update time of less than 1 second.

Observation 2. The value of the aiming vector $-\mathbf{a}$ has only a slight effect on the efficiency of the system; for the three values of $\mathbf{a}$ listed in Table 1, the variation in update time was about 5% or less.

Table 1 — Variation of $Z(i, j)$ with $\mathbf{a}$

$-\mathbf{a}$	$Z(1,2)$	$Z(1,3)$	$Z(1,4)$	$Z(2,3)$	$Z(2,4)$	$Z(3,4)$
$[0, 1, 0]$	60.0°	90.0°	45.0°	60.0°	45.0°	45.0°
$[0, 1/\sqrt{2}, 1/\sqrt{2}]$	49.2°	90.0°	49.2°	49.2°	45.0°	49.2°
$[-1/\sqrt{3}, 1/\sqrt{3}, 1/\sqrt{3}]$	46.9°	90.0°	46.9°	52.2°	46.9°	52.2°

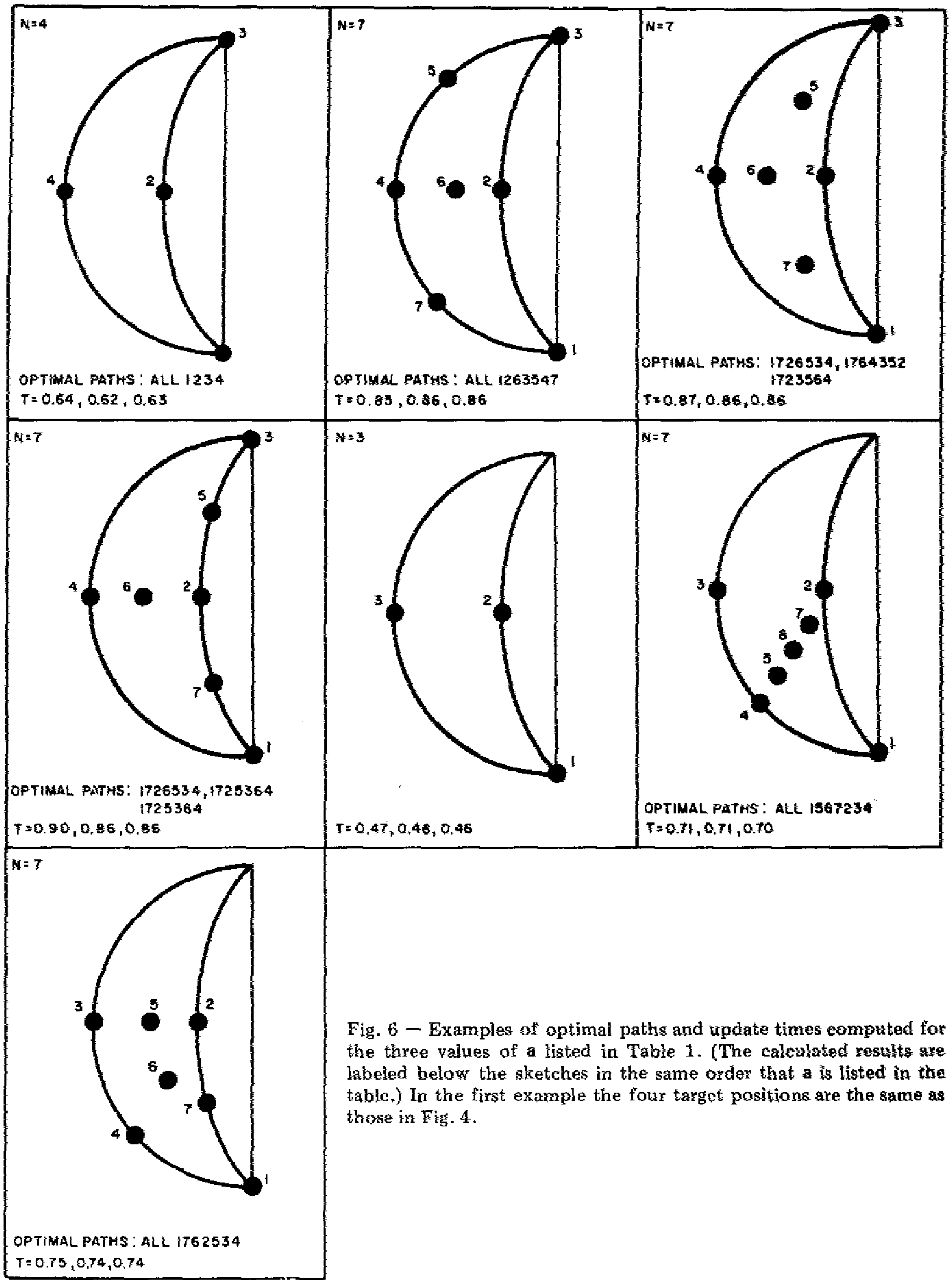

Fig. 6 — Examples of optimal paths and update times computed for the three values of a listed in Table 1. (The calculated results are labeled below the sketches in the same order that a is listed in the table.) In the first example the four target positions are the same as those in Fig. 4.

Appendix A

DERIVATION OF EQUATION (3.1)

As explained in section 3, equations (3.1) and (3.1*) are mathematically equivalent, and, to prove (3.1*), we shall first express $\ddot{\mathbf{p}}$ in terms of certain angular accelerations and velocities.

As shown in Fig. A, let $\mathbf{e}_1$ and $\mathbf{e}_2$ be two unit vectors which are fixed to the mirror M in such a way that $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{p}\}$ form a right-handed triple:

$$\mathbf{e}_1 \times \mathbf{e}_2 = \mathbf{p},\ \mathbf{e}_2 \times \mathbf{p} = \mathbf{e}_1,\ \mathbf{p} \times \mathbf{e}_1 = \mathbf{e}_2. \tag{A1}$$

Since $\mathbf{e}_1 \cdot \mathbf{e}_1 = 1$, we have $\mathbf{e}_1 \cdot \dot{\mathbf{e}}_1 = 0$, so that $\dot{\mathbf{e}}_1$ has no component in the $\mathbf{e}_1$ direction. Also, from assumption A2 in section 3, it follows that $\dot{\mathbf{e}}_1$ has no component in the $\mathbf{e}_2$ direction. Similar remarks hold for $\dot{\mathbf{e}}_2$, so that one can define angular velocities $\dot{\omega}_1$ and $\dot{\omega}_2$ satisfying

$$\dot{\mathbf{e}}_1 = -\dot{\omega}_1 \mathbf{p} \text{ and } \dot{\mathbf{e}}_2 = -\dot{\omega}_2 \mathbf{p}. \tag{A2}$$

From equations (A1) and (A2) we get

$$\dot{\mathbf{p}} = \dot{\omega}_1 \mathbf{e}_1 + \dot{\omega}_2 \mathbf{e}_2 \tag{A3}$$

and

$$\ddot{\mathbf{p}} = \ddot{\omega}_1 \mathbf{e}_1 + \ddot{\omega}_2 \mathbf{e}_2 - (\dot{\omega}_1^2 + \dot{\omega}_2^2)\mathbf{p}. \tag{A4}$$

Since $\delta^2\mathbf{p}/\delta t^2$ is by definition the tangential component of $\ddot{\mathbf{p}}$, it follows that

$$\delta^2\mathbf{p}/\delta t^2 = \ddot{\omega}_1 \mathbf{e}_1 + \ddot{\omega}_2 e_2. \tag{A5}$$

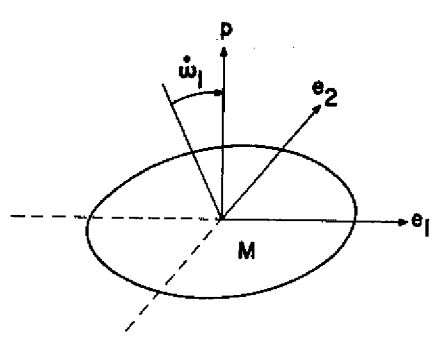

Fig. A — Vectors relative to a disk-shaped mirror

We will now compute the torque $\mathbf{u}$. It will be convenient to think of the mirror as being composed of particles all having the same unit mass. Let $\mathbf{r}_i$ be the position vector of the ith particle, and set

$$\mathsf{M} = \sum \mathbf{r}_i \times \dot{\mathbf{r}}_i.$$

Then, by definition,

$$\mathbf{u} = \dot{M} = \sum r_i \times \ddot{r}_i. \tag{A6}$$

Each particle has "local" coordinates x_i^1 and x_i^2 defined by

$$\mathbf{r}_i = x_i^1 \mathbf{e}_1 + x_i^2 \mathbf{e}_2,$$

and x_i^1 and x_i^2 are constant, since the moving axes $\mathbf{e}_1$ and $\mathbf{e}_2$ are fixed to the mirror. From (A6) one ultimately gets

$$\mathbf{u} = -[I_{12}\ddot{\omega}_1 + I_{22}\ddot{\omega}_2]\mathbf{e}_1 + [I_{11}\ddot{\omega}_1 + I_{12}\ddot{\omega}_2]\mathbf{e}_2$$
$$+ [I_{12}(\dot{\omega}_1^2 - \dot{\omega}_2^2) - (I_{11} - I_{22})\dot{\omega}_1\dot{\omega}_2]\mathbf{p},$$

where

$$I_{12} = \sum_i x_i^1 x_i^2, I_{11} = \sum_i (x_i^1)^2, \text{ and } I_{22} = \sum_i (x_i^2)^2.$$

From the symmetry properties of the disk we have

$$I_{12} = 0 \text{ and } I_{11} = I_{22} = I,$$

where I is the moment of inertia about a diameter. Hence for the disk we have

$$\mathbf{u} = I(-\ddot{\omega}_2\mathbf{e}_1 + \ddot{\omega}_1\mathbf{e}_2)$$
$$= I(\ddot{\omega}_2\mathbf{p} \times \mathbf{e}_2 + \ddot{\omega}_1\mathbf{p} \times \mathbf{e}_1),$$

and from (A5) we get (3.1*):

$$u = I\mathbf{p} \times \delta^2\mathbf{p}/\delta t^2.$$

Appendix B

COVARIANT DERIVATIVES

Let $\mathbf{p} = \mathbf{p}(t)$ be a curve on the unit sphere S. We want to show that the right-hand side of (3.2) is the tangential component of $\ddot{\mathbf{p}}$. Since the vector $\mathbf{p}$ is normal to S at the point $\mathbf{p}$, we have

$$\text{normal component of } \ddot{\mathbf{p}} = (\ddot{\mathbf{p}} \cdot \mathbf{p})\mathbf{p}.$$

Differentiating the identity $\mathbf{p} \cdot \mathbf{p} = 1$, we get

$$\dot{\mathbf{p}} \cdot \mathbf{p} = 0 \quad\text{and}\quad \ddot{\mathbf{p}} \cdot \mathbf{p} + |\dot{\mathbf{p}}|^2 = 0.$$

Hence

$$\text{normal component of } \ddot{\mathbf{p}} = -|\dot{\mathbf{p}}|^2\mathbf{p},$$

and it follows that

$$\begin{aligned}\text{tangential component} &= \ddot{\mathbf{p}} - \text{normal component}\\ &= \ddot{\mathbf{p}} + |\dot{\mathbf{p}}|^2\mathbf{p}.\end{aligned}$$

The preceding formulas apply only to spheres. More generally let S be any surface in Euclidean three-space defined by the parametric equations

$$X^i = X^i(y^1, y^2), \qquad i = 1, 2, 3,$$

where y^1 and y^2 are local surface coordinates. Let $\mathbf{p}(t) = \mathbf{X}(\mathbf{y}(t))$ be a smooth curve on S. Then the second covariant derivative of $\mathbf{p}(t)$, $\delta^2\mathbf{p}/\delta t^2$, is again defined as the tangential component of $\ddot{\mathbf{p}}(t)$ and is given by

$$\delta^2\mathbf{p}/\delta t^2 = \frac{\delta^2 y^1}{\delta t^2} \cdot \frac{\partial \mathbf{X}}{\partial y^1} + \frac{\delta^2 y^2}{\delta t^2} \cdot \frac{\partial \mathbf{X}}{\partial y^2} \;,$$

where

$$\frac{\delta^2 y^\alpha}{\delta t^2} = \ddot{y}^\alpha + \sum_{\beta,\gamma} \Gamma^\alpha_{\beta\gamma}\dot{y}^\beta\dot{y}^\gamma, \qquad \alpha = 1, 2,$$

and where $\Gamma^\alpha_{\beta\gamma}$ are the so-called Christoffel symbols. Details can be found in almost any elementary text in differential geometry (for example, A. J. McConnel, *Applications of Tensor Analysis*, Dover, New York, 1957).