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Aerodynamic Coefficient Derivatives of Sonic Missiles via Slender Body Theory

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This investigation was initiated to develop analytic procedures for estimating aerodynamic coefficient derivatives for missiles. The analytic estimates will depend primarily on the geometrical configurations of the missiles.

The problem of determining the coefficient derivatives becomes reasonably tractable for thin airfoil-body combinations with moderate finite aspect ratios and flying at sonic speeds. Starting with the equations for a perfect gas, a linearization of them is achieved by assuming flow over a thin

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profile. A further assumption of a speed of Mach 1 gives rise to slender body theory. The problem is thus reduced to a potential boundary value problem in a cross-flow plane. Upon consideration of the total momentum in a cross-flow slab, it is found that the resultant lateral force may be expressed as a contour integral of the velocity potential. The effects of missile angle of attack and control-surface angle are incorporated by way of Neumann-type conditions on the boundary contour.

For the special case where the missile cross section is a circle with midwing, there is an analytic solution for the potential-flow problem. For the case where there is an arbitrary missile cross section, a computer program has been developed which addresses the problem using a source distribution approach. Results are given for several sample cross sections. It is shown how the cross-flow results may be applied to a typical missile configuration to obtain the aerodynamic coefficient derivatives.

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AERODYNAMIC COEFFICIENT DERIVATIVES OF SONIC MISSILES VIA SLENDER BODY THEORY

INTRODUCTION

The Effectiveness of Navy Electronic Warfare Systems (ENEWS) Program is involved in continuing development of simulation facilities at NRL to allow U.S. Navy investigation of the interaction between anti-ship missiles and shipboard defensive electronic warfare (EW) systems. Over the past four years, the ENEWS Program has developed techniques for assessing evolving EW systems using mathematical models in simulation programs on digital computers.

The advent of this new generation of models is supplemented by the arrival of new methods of missile analysis for accomplishing the intent of EW simulations. To achieve realistic simulation, it is imperative that the aerodynamic properties of the threat missiles to be modeled must be closely approximated using valid analytic results. This report forms a link in the chain of continuing investigations intended to assess missile threats, current and projected. This continuing assessment is an essential input to the evaluation and enhancement of electronic warfare systems intended to counter missile threats.

At its current stage, this assessment program is developing expeditious and versatile procedures for estimating transonic aerodynamic coefficient derivatives. These values are required in computer simulations that incorporate missile flight behavior. This information is also useful in determining outer limits for the maneuvering capability of a missile, thus providing inputs to bounding the potential capabilities of threats. An analytic ability to determine aerodynamic coefficient derivatives is essential for the comprehensive study of missile-EW engagements. These coefficients depend upon the geometric configuration of the missile, including control surface positions, the velocity of the air stream relative to the missile, and the orientation of the missile relative to the air stream. The values of the various aerodynamic coefficient derivatives may be estimated using wind-tunnel tests on missile models, or alternatively, by means of analytic calculation. Although wind-tunnel tests are preferred from the point of view of accuracy and validity, the great expense involved in performing them must be taken into account. Wind-tunnel tests will continue to be used to determine the aerodynamic coefficient derivatives for a few of the more important missiles. For the multitude of others, it would be preferable to have and utilize a capability for making analytic estimates of these coefficients.

This problem becomes tractable for missiles with moderate aspect ratios at sonic speeds. In such cases assumptions may be introduced which greatly simplify the mathematical description of the problem. By treating a missile of moderate aspect ratio as a thin profile and regarding the flow about the missile as a perturbation of a uniform flow, it follows that the basic equation of a perfect gas can be rendered linear. Requiring, in addition, that the flow be sonic reduces the formulaton to that of slender body theory.

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In addition to satisfying a number of requirements for sonic missile analysis, these evaluations would be useful in missile analysis outside the sonic range. Slender body theory may be used to find wing-body interference factors that can be used to determine valid estimates of the subsonic and supersonic values.

Caution should be exercised however in applying the results of slender body theory. The direct results of this theory should be tempered with some understanding of transonic flow and its many complexities, otherwise the uninformed user could potentially be led into attributing properties to transonic vehicles which are erroneous.

The theoretical development on which the estimation of these coefficient derivatives is based will be outlined herein. This theory reduces the problem to one of solving a two-dimensional potential cross-flow problem. In the special case where a cross section of the missile is a circle with mid-wing, there are analytic solutions for the potential-flow problem. For other cross sections, a computer program has been developed in the course of this investigation that can be used to solve the two-dimensional potential cross-flow problem.

SUMMARY OF PROCEDURE FOR COEFFICIENT DERIVATIVE DETERMINATION

This report embodies a procedure for determining the sonic aerodynamic coefficient derivatives of a missile, given its geometric configuration. This procedure is summarized and certain equations are referred to that will be discussed or derived later in the report. According to this procedure, certain cross section profiles are chosen at various locations along the length of the missile. Each individual cross-section profile is subjected to analysis using a computer program designed to determine cross flows. To obtain input for this computer program, it is necessary to express the cross-section profile quantitatively. To accomplish this, the profile may be broken down into line segments and arcs, representing each with a few numerical parameters. Another input to the computer program is a set of boundary conditions that is determined by body angle and control surface deflections. From these inputs, the cross-flow program computes the apparent area, that is a quantity representing the overall aerodynamic effect of a cross section. The cross-flow program represents the major portion of the computational effort in determining the aerodynamic coefficients. By use of Eqs. (38)-(41), the apparent areas are translated into the aerodynamic coefficient derivatives characteristic of the individual cross sections.

The aerodynamic coefficient derivatives characteristic of the overall missile are obtained from the coefficient derivatives characteristic of cross sections using Eqs. (42)-(51). Among the other inputs to these equations are the positions of chosen cross sections and the moment reference center.

AERODYNAMIC COEFFICIENT DERIVATIVES DEFINED

A number of the more important aerodynamic coefficient derivatives will be defined after the introduction of notations and figures that describe and illustrate the mathematical expressions and sign conventions used. All angles are expressed in degrees.

The notations used in defining the aerodynamic coefficient derivatives are presented next and in Figs. 1 and 2.

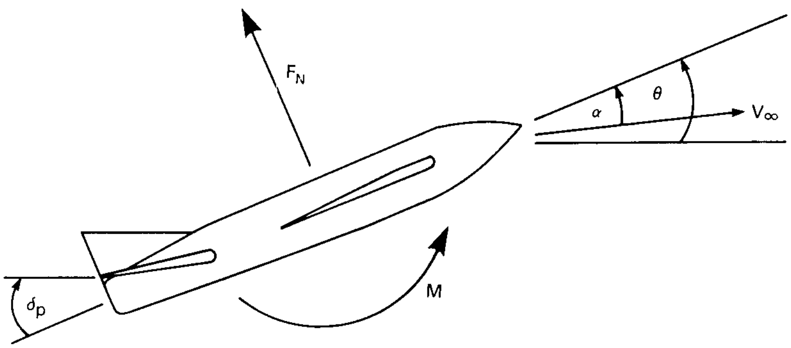


Fig. 1 — Pitch notation and conventions

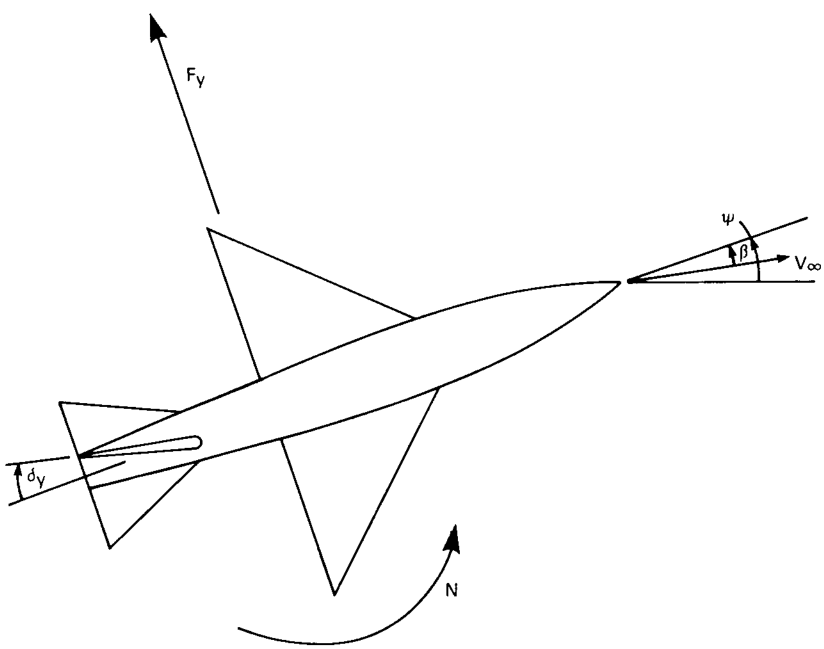


Fig. 2 — Yaw notation and conventions

b	(reference diameter)
$F_N(\alpha,\ \delta_P)$	(normal force)
$F_Y(\beta,\ \delta_Y)$	(side force)
$M(\alpha,\ \delta_P, \dot{\theta})$	(pitching torque)
$N(\beta,\ \delta_Y,\ \dot{\psi})$	(yawing torque)
$q = \frac{1}{2}\rho V_\infty^2$	(dynamic pressure)
$S = \pi b^2/4$	(reference area)
V_∞	(free stream velocity)
α	(angle of attack)
β	(sideslip angle)
δ_P	(elevator angle)
δ_Y	(rudder angle)
$\dot{\theta}$	(angular velocity of pitch)
ρ	(density of air)
$\dot{\psi}$	(yawing angular velocity)

The pitch aerodynamic coefficient derivatives considered are defined as follows:

$$C_{N\alpha} = \frac{1}{qS}\ \frac{\partial F_N}{\partial \alpha}, \tag{1}$$

$$C_{N\delta} = \frac{1}{qS}\ \frac{\partial F_N}{\partial \delta_P}, \tag{2}$$

$$C_{m\alpha} = \frac{1}{qSb}\ \frac{\partial M}{\partial \alpha}, \tag{3}$$

$$C_{mq} = \frac{2V_\infty}{qSb^2}\ \frac{\partial M}{\partial \dot{\theta}}, \tag{4}$$

and

$$C_{m\delta} = \frac{1}{qSb}\ \frac{\partial M}{\partial \delta_P}. \tag{5}$$

These sets of coefficient derivatives may be used to express the normal force due to angle of attack and elevator angle. Thus,

$$F_N = qS(C_{N\alpha}\alpha + C_{N\delta}\delta). \tag{6}$$

The equation for pitching torque due to angle of attack, elevator angle, and angular velocity of pitch is

$$M = qSb(C_{m\alpha}\alpha + C_{m\delta}\delta_P) + \frac{qSb^2}{2V_\infty}\ C_{mq}\ \dot{\theta}. \tag{7}$$

The yaw aerodynamic coefficient derivatives used herein are defined as follows:

$$C_{Y\beta} = \frac{1}{qS}\ \frac{\partial F_Y}{\partial \beta}, \tag{8}$$

$$C_{Y\delta} = \frac{1}{qS} \frac{\partial F_Y}{\partial \delta_Y}, \tag{9}$$

$$C_{n\beta} = \frac{1}{qSb} \frac{\partial N}{\partial \beta}, \tag{10}$$

$$C_{nr} = \frac{2V_\infty}{qSb^2} \frac{\partial N}{\partial \dot{\psi}}, \tag{11}$$

and

$$C_{n\delta} = \frac{1}{qSb} \frac{\partial N}{\partial \delta_Y}. \tag{12}$$

The yaw coefficient derivatives may be used to express the side force due to sideslip angle and rudder angle. Thus,

$$F_Y = qS(C_{Y\beta}\beta + C_{Y\delta}\delta_Y). \tag{13}$$

Yawing torque due to sideslip angle, rudder angle, and angular velocity of yaw is expressed as

$$N = qSb(C_{n\beta}\beta + C_{n\delta}\delta_Y) + \frac{qSb^2}{2V_\infty} C_{nr}\dot{\psi}. \tag{14}$$

SLENDER BODY THEORY

In setting forth the rationale underlying the use of slender body theory, it is assumed that we are dealing with the flow of a perfect gas about a solid body. The velocity potential Φ for the flow of this compressible fluid is governed by the following partial differential equation [1]:

$$\begin{aligned} &\left[1 - \frac{1}{c^2}\left(\frac{\partial \Phi}{\partial x}\right)^2\right]\frac{\partial^2 \Phi}{\partial x^2} + \left[1 - \frac{1}{c^2}\left(\frac{\partial \Phi}{\partial y}\right)^2\right]\frac{\partial^2 \Phi}{\partial y^2} \\ &+ \left[1 - \frac{1}{c^2}\left(\frac{\partial \Phi}{\partial z}\right)^2\right]\frac{\partial^2 \Phi}{\partial z^2} - \frac{2}{c^2}\frac{\partial \Phi}{\partial x}\frac{\partial \Phi}{\partial y}\frac{\partial^2 \Phi}{\partial x \partial y} \\ &- \frac{2}{c^2}\frac{\partial \Phi}{\partial y}\frac{\partial \Phi}{\partial z}\frac{\partial^2 \Phi}{\partial y \partial z} - \frac{2}{c^2}\frac{\partial \Phi}{\partial x}\frac{\partial \Phi}{\partial z}\frac{\partial^2 \Phi}{\partial x \partial z} \\ &= 0, \end{aligned} \tag{15}$$

where the spatial coordinates are denoted by x, y, and z. The local velocity of sound c is given by

$$c^2 = c_0^2 - \frac{k-1}{2}\left[\left(\frac{\partial \Phi}{\partial x}\right)^2 + \left(\frac{\partial \Phi}{\partial y}\right)^2 + \left(\frac{\partial \Phi}{\partial z}\right)^2\right], \tag{16}$$

where the following notation is used:

c_0	(stagnation value for the speed of sound)
$k = C_p/C_v$	(ratio of the specific heats)
C_p	(specific heat at constant pressure)
C_v	(specific heat at constant volume)

Next, the previously given partial differential equation is linearized. To do this, consider a flow about a thin profile which is parallel to the free stream. This flow may be represented as a uniform flow plus a small perturbation. In the following equation the velocity potential Φ is represented as a uniform flow potential $V_\infty x$ plus a velocity perturbation potential ϕ.

$$\Phi = V_\infty x + \phi. \tag{17}$$

Substituting Eq. (17) into Eqs. (15) and (16) results in a differential equation for the velocity perturbation potential. Upon linearization, the equation becomes

$$\gamma^2 \frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} + \frac{\partial^2\phi}{\partial z^2} = 0, \tag{18}$$

where

$$\gamma^2 = 1 - M_\infty^2, \tag{19}$$

$$M_\infty = V_\infty / c_\infty, \tag{20}$$

and

$$c_\infty^2 = c_0^2 - \frac{k-1}{2} V_\infty^2. \tag{21}$$

The development that results when the first term in Eq. (18) is neglected is referred to as "slender body theory". This theory yields the following equation:

$$\frac{\partial^2\phi}{\partial y^2} + \frac{\partial^2\phi}{\partial z^2} = 0. \tag{22}$$

According to this approximation, only flow perturbations confined to planes perpendicular to the body are considered significant.

The primary application of slender body theory is, as the name suggests, to slender bodies. These are configurations whose lateral dimensions such as span and thickness are small compared to their length. And, for this type of missile, the expression $\partial^2\phi/\partial x^2$ in the first term of Eq. (18) may be neglected, giving rise to Eq. (22).

Slender body theory may also apply to sonic flows about non-slender bodies. The small disturbance assumption is still valid for a thin wing-body combination with a moderate finite aspect ratio, and for sonic flows γ in Eq. 18 may be neglected, again resulting in Eq. (22). Under these conditions, slender body theory will provide a first-order approximation to sonic lifting flows [2].

Next, it is shown just how estimates of the aerodynamic coefficient derivatives may be obtained using slender body theory. Consider an infinite slab of air of thickness dx perpendicular to and moving with the free stream. According to Eq. (22), a two-dimensional potential flow within the slab is assumed. The problem of the missile cutting through the slab, as illustrated in Fig. 3, may be treated as a plane flow with an immersed two-dimensional region. The moving boundary contour of this region provides boundary conditions for the potential flow. The effect of missile angle of attack is to impart a uniform velocity to the contour within the plane, as illustrated in Fig. 4.

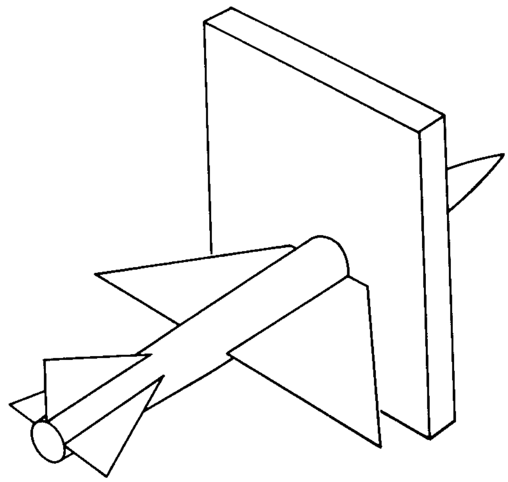


Fig. 3 — Missile cutting through a slab of air

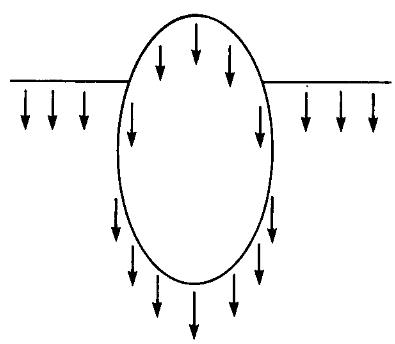


Fig. 4 — Effect of body angle
on boundary conditions

The effect of control surface deflection is to impart a velocity to that part of the contour corresponding to the control surface as illustrated in Fig. 5. An increase in size and extent of the missile cross section as it cuts through the slab gives a point on the contour an outward velocity as shown in Fig. 6.

An additional boundary condition is provided by a zero fluid velocity at infinity in the cross-flow plane. These conditions determine (except for an additive constant) the two-dimensional velocity potential ϕ within the slab.

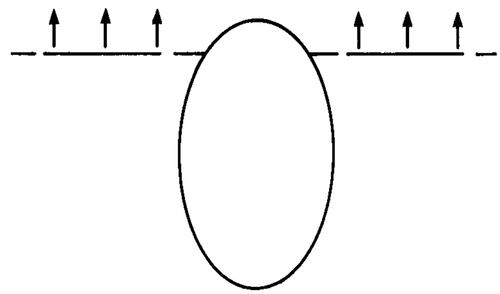


Fig. 5 — Effect of control surface angle on boundary conditions

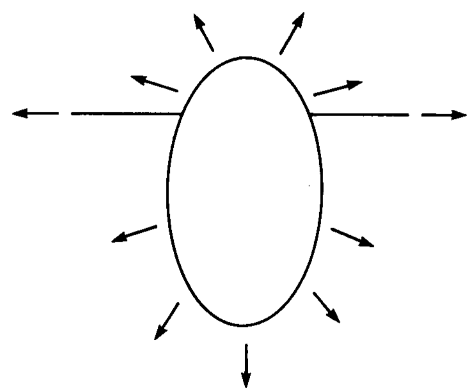


Fig. 6 — Effect of expanding cross section on boundary conditions

The total momentum vector of the flow $d\mathbf{P}$ is expressed in the following equation as a surface integral of the momentum density, over the region exterior to the boundary contour:

$$d\mathbf{P} = dx \int\int \rho \, \mathbf{V} \, dA, \tag{23}$$

where the following notation is used:

$d\mathbf{P}$	(total momentum vector in slab)
dx	(thickness of slab)
$\mathbf{V}$	(local velocity vector of fluid within plane slab)
dA	(element of surface area)

The surface integral in Eq. (23) may be expressed as a line integral around the boundary contour as follows:

$$d\mathbf{P} = dx \, \rho \oint \phi \, d\mathbf{z}, \tag{24}$$

where $d\mathbf{z}$ is an outward normal vector representing an element of arc length along the boundary contour.

The force vector $d\mathbf{F}$ on a missile cross-sectional element will be equal and opposite to the rate of increase of momentum in the slab.

This is expressed as

$$d\mathbf{F} = -\frac{d}{dt}(d\mathbf{P}). \tag{25}$$

Taking into consideration that the missile is cutting through the slab with a velocity V_∞, Eq. (25) becomes

$$d\mathbf{F} = -V_\infty \frac{d}{dx}(d\mathbf{P}). \tag{26}$$

Slender body theory has greatest validity for wings with monotonically increasing span [3]. In calculating particular aerodynamic coefficient derivatives, the analyst will not, in most cases, be dealing with the simple case of a gradually expanding contour. The body may be tapered so that it gives a contracting contour, or, as at the trailing edge of a wing, there may be a discontinuous reduction in the extent of the contour.

In order to deal with these situations, it will be taken as a hypothesis that Eq. (26) gives the $d\mathbf{F}$ only when the magnitude of $d\mathbf{P}$ is increasing along the axis of the missile. Otherwise, $d\mathbf{F}$ will be taken to be zero.

To obtain the total lateral force vector $\mathbf{F}$ on a missile, the force vector $\mathbf{dF}$ on a cross sectional element is integrated along the length of the missile, the coordinate x being used to denote positions along that length down stream from the nose. Consider an interval along the length of the missile over which the magnitude of $\mathbf{dP}$ is increasing. The total lateral force on this interval is expressed as

$$\mathbf{F} = \int_{x=x_A}^{x=x_B} d\mathbf{F}. \tag{27}$$

Here, x_A and x_B are the end points of the interval. The integration is carried out after substituting Eq. (26) into Eq. (27), thus obtaining

$$\mathbf{F} = -V_\infty \left. \frac{d\mathbf{P}}{dx} \right|_{x=x_A}^{x=x_B}. \tag{28}$$

Substituting Eq. (24) into Eq. (28) results in the following equation for the total lateral force on the interval:

$$\mathbf{F} = -V_\infty \rho \left. \oint \phi \, d\mathbf{z} \right|_{x=x_A}^{x=x_B}. \tag{29}$$

A particular missile cross section with a specified direction of motion within a cross-flow slab could be characterized using the conventional quantity *apparent mass.* However, we shall define a related quantity, the *apparent area* of the cross section, a more convenient unit to express this concept. We define this vector quantity according to the following equation, where the surface integral is over the region exterior to the boundary contour:

$$\mathbf{A} = \frac{1}{V_c} \int \int \mathbf{V} \, dA. \tag{30}$$

The expression V_c is a velocity naturally associated with the motion of the contour in the cross-flow plane. This will generally be either the uniform velocity of the contour due to body angle or the velocity of that portion of the boundary representing a control-surface deflection. It is given as follows:

$$V_c = \left[\frac{\pi}{180}\right] V_\infty \, \eta. \tag{31}$$

Here, η is the body angle or control-surface deflection. The velocity V_c may also be due to the angular velocity of pitch or yaw denoted by ω. In this case it is given by

$$V_c = \left[\frac{\pi}{180}\right] |x - x_c| \, \omega. \tag{32}$$

Here, x_c denotes the position of the moment reference center measured downstream from the nose and chosen to coincide with the center of gravity.

The significance of the apparent area is that the apparent mass of the cross section is given by $\rho|\mathbf{A}|dx$, and the momentum of the cross flow is given by $\rho V_c \mathbf{A} dx$. The apparent area may be expressed as the following contour integral:

$$\mathbf{A} = \frac{1}{V_c} \oint \phi \, d\mathbf{z}. \tag{33}$$

Note that the forgoing expression does not depend on the magnitude of V_c since the boundary condition values and, thus, the contour integral are proportional to this velocity.

The transverse force and the moment of force on a missile may be expressed in terms of the apparent areas of the missile cross sections. The transverse force due to body angle or control-surface deflection on an interval taken along the length of a missile where the apparent area is increasing is:

$$\mathbf{F} = -\left(\frac{\pi}{180}\right) \rho \, V_\infty^2 \, \eta \, \mathbf{A} \Big|_{x=x_A}^{x=x_B}. \tag{34}$$

The moment of force on this interval is given by

$$\mathbf{M} = \int_{x=x_A}^{x=x_B} (x - x_c) \, \mathbf{i} \times d\mathbf{F}. \tag{35}$$

Here, $\mathbf{i}$ is a unit vector pointing downstream.

Substituting Eq. (34) in Eq. (35) yields the following for the moment of force on this interval:

$$\mathbf{M} = -\left(\frac{\pi}{180}\right) \rho\, V_\infty^2\, \eta \int_{x=x_A}^{x=x_B} (x - x_c)\, \mathbf{i} \times d\mathbf{A}. \tag{36}$$

In a similar manner, Eq. (37) is obtained for the moment of force due to angular velocity of pitch or yaw:

$$\mathbf{M} = -\left(\frac{\pi}{180}\right) \rho\, V_\infty\, \omega \int_{x=x_A}^{x=x_B} (x - x_c)^2\, \mathbf{i} \times d\mathbf{A}. \tag{37}$$

APPLICATION TO A TYPICAL MISSILE CONFIGURATION

A method of estimating the aerodynamic coefficient derivatives of a typical missile configuration is described here. Top and side views of the missile configuration under consideration are shown in Figs. 7 and 8. Various points along the length of the missile have been numbered as follows:

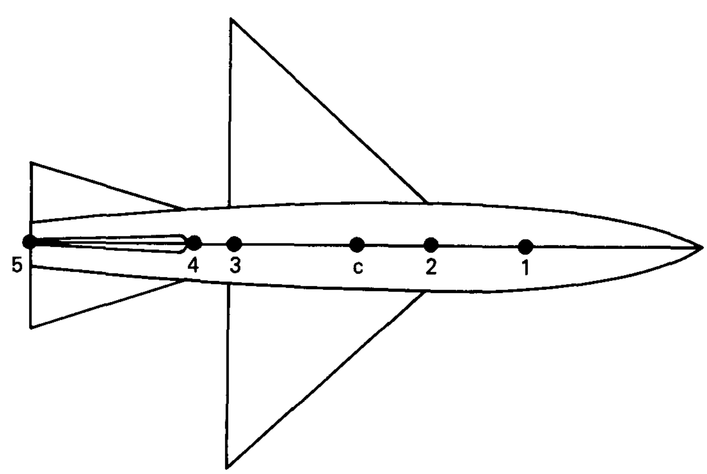


Fig. 7 — Top view of typical missile configuration

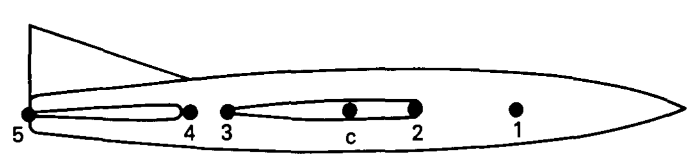


Fig. 8 — Side view of typical missile configuration

1 (a point where the cross section of the nose attains a maximum),
2 (beginning of wing),
C (moment reference center),
3 (trailing edge of wing),
4 (beginning of tail),
5 (trailing edge of elevator and rudder).

The value of x at the ith point along the length of the missile is denoted by x_i.

One aspect of carrying out the method for determining the aerodynamic coefficient derivatives is to determine apparent areas for the cross sections at a number of points along the length of the missile. Notation is given below for apparent areas as given by Eq. (33) for the cross section located at the station denoted by x. For angle of attack the apparent area is denoted by $\mathbf{A}_{N\alpha}(x)$, where in this case η in Eq. (31) represents the angle of attack of the missile, α. The effect of this α for small angles is to give the boundary contour a rigid motion with velocity V_c, as illustrated in Fig. 4, thus determining the boundary conditions for ϕ.

For elevator deflection the apparent area is denoted by $\mathbf{A}_{N\delta}(x)$, where, in this case, η in Eq. 31 represents the elevator deflection δ_P. The effect of this for small deflections is to contribute to that part of the boundary representing the control surfaces a velocity V_c, while the remainder of the contour assumes a zero velocity. This is illustrated in Fig. 5 and provides the boundary conditions for ϕ.

For side slip angles the apparent area is denoted by $\mathbf{A}_{Y\beta}(x)$, where, in this case, η in Eq. 31 represents the angle of sideslip of the missile β. The effect of this β for small angles is to give the boundary contour a rigid motion with velocity V_c, thus determining the boundary conditions for ϕ.

For rudder deflection the apparent area is denoted by $\mathbf{A}_{Y\delta}(x)$, where, in this case, η in Eq. (31) represents the rudder deflection δ_Y. The effect of this for small deflections is to contribute to that part of the boundary representing the control surface a velocity V_c, while the remainder of the contour assumes a zero velocity. This provides the boundary conditions for ϕ.

The following equations introduce cross section coefficient derivatives. These are related in a simple way to the apparent areas and are useful in expressing the aerodynamic coefficient derivatives of the missile. Thus:

$$C'_{N\alpha}(x) = -\frac{\pi}{90\,S}\,\mathbf{A}_{N\alpha}(x)\cdot\mathbf{j}, \tag{38}$$

$$C'_{N\delta}(x) = -\frac{\pi}{90\,S}\,\mathbf{A}_{N\delta}(x)\cdot\mathbf{j}, \tag{39}$$

$$C'_{Y\beta}(x) = -\frac{\pi}{90\,S}\,\mathbf{A}_{Y\beta}(x)\cdot\mathbf{k}, \tag{40}$$

and

$$C'_{Y\delta}(x) = -\frac{\pi}{90\,S}\,\mathbf{A}_{Y\delta}(x)\cdot\mathbf{k}. \tag{41}$$

Here, **j** and **k** are unit vectors in the cross-flow plane and in the directions of F_N in Fig. 1 and F_Y in Fig. 2, respectively.

Formulas for estimating the aerodynamic coefficient derivatives of the missile configuration are listed next.

$$C_{N\alpha} = C'_{N\alpha}(x_3) - C'_{N\alpha}(x_4) + C'_{N\alpha}(x_5) \tag{42}$$

$$C_{N\delta} = C'_{N\delta}(x_5) \tag{43}$$

$$C_{m\alpha} = -\frac{1}{b}\left[\int_0^{x_1} + \int_{x_2}^{x_3} + \int_{x_4}^{x_5}\right](x - x_c)\, dC'_{N\alpha}(x) \tag{44}$$

$$C_{mq} = -\frac{2}{b^2}\left[\int_0^{x_1} + \int_{x_2}^{x_3} + \int_{x_4}^{x_5}\right](x - x_c)^2\, dC'_{N\alpha}(x) \tag{45}$$

$$C_{m\delta} = -\frac{1}{b}\int_{x_4}^{x_5}(x - x_c)\, dC'_{N\delta}(x) \tag{46}$$

$$C_{Y\beta} = C'_{Y\beta}(x_3) - C'_{Y\beta}(x_4) + C'_{Y\beta}(x_5) \tag{47}$$

$$C_{Y\delta} = C'_{Y\delta}(x_5) \tag{48}$$

$$C_{n\beta} = -\frac{1}{b}\left[\int_0^{x_1} + \int_{x_2}^{x_3} + \int_{x_4}^{x_5}\right](x - x_c)\, dC'_{Y\beta}(x) \tag{49}$$

$$C_{nr} = -\frac{2}{b_2}\left[\int_0^{x_1} + \int_{x_2}^{x_3} + \int_{x_4}^{x_5}\right](x - x_c)^2\, dC'_{Y\beta}(x) \tag{50}$$

$$C_{n\delta} = -\frac{1}{b}\int_{x_4}^{x_5}(x - x_c)\, dC'_{Y\delta} \tag{51}$$

In obtaining the normal force due to body angle, two intervals are considered. One of these extends from the nose tip to the trailing edge of the wing and the other extends from the beginning of the tail to the trailing edge of the elevator. Using Eqs. (1), (34), and (38), the coefficient derivatives for the normal force due to body angle are expressed by Eq. (42). In a similar manner the coefficient for side force due to side-slip angle is expressed by Eq. (47).

The normal force due to elevator deflection is obtained as the force on the interval extending over this control surface. Using Eqs. (2), (34), and (39), the coefficient derivative for this force is expressed by Eq. (43). Similarly, the coefficient derivative for side force due to rudder deflection is expressed by Eq. (48).

In obtaining the pitching moments due to body angle and angular velocity, three intervals are considered. The first extends over the nose, the second over the wings, and the third over the horizontal stabilizer. Using Eqs. (3), (36), and (38), the coefficient derivative for the pitching moment due to body angle is expressed by Eq. (44). Using Eqs. (4), (37), and (38), the coefficient derivative for the pitching moment due to angular velocity of pitch is expressed by Eq. (45).

In a similar manner the coefficient for yawing moment due to side-slip angle is expressed by Eq. (49), and the coefficient for yawing moment due to angular velocity of yaw is expressed by Eq. (50).

The pitching moment due to elevator deflection is obtained as the moment on the interval extending over this control surface. Using Eqs. (5), (36), and (39), the coefficient derivative for this force is expressed by Eq. (46). Similarly, the coefficient for yawing moment due to rudder deflection is expressed by Eq. (51).

It is apparent from Eqs. (42), (43), (47), and (48) that the coefficient derivatives for the normal and side forces, namely, $C_{N\alpha}$, $C_{N\delta}$, $C_{Y\beta}$ and $C_{Y\delta}$, are dependent on the cross flows at the three stations along the length of the missile numbered 3, 4, and 5. The other coefficients involve integrations and are evaluated as approximating sums. This evaluation requires analyzing a number of cross flows taken along the intervals of integration.

THE CIRCLE-WITH-MIDWING CROSS SECTION

In the special case, where the cross-section contour of a missile of angle of attack α is a circle-with-midwing, the potential cross-flow problem has an analytic solution. The boundary condition for a problem of this type is illustrated in Fig. 9. The following notation is also indicated:

$R(x_i)$ (radius of circular cross section of missile)
$s(x_i)$ (semi-span of wing)

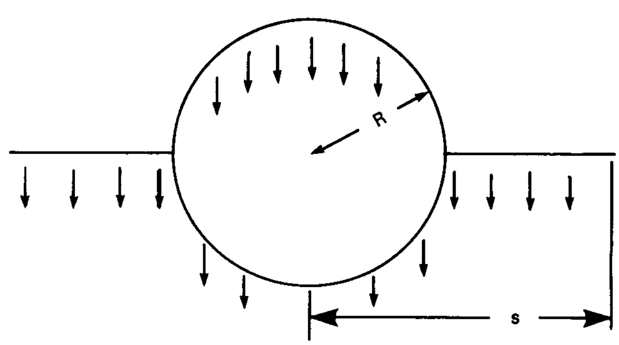


Fig. 9 — Circle-with-midwing cross section

For this case, the cross-section coefficients [4] are given by

$$C'_{N\alpha}(x_i) = \frac{\pi^2}{90\,S}\left\{s^2(x_i) - R^2(x_i) + \frac{R^4(x_i)}{s^2(x_i)}\right\}. \tag{52}$$

Applying this formula, the pitch coefficient $C_{N\alpha}$ may be estimated for a missile with the configuration shown in Figs. 7 and 8, and which has circle-with-midwing cross sections. This coefficient derivative is given by the formula:

$$C_{N\alpha} = \frac{\pi^2}{90\,S}\left[s^2(x_3) - R^2(x_3) + \frac{R^4(x_3)}{s^2(x_3)} - R^2(x_4) + s^2(x_5) - R^2(x_5) + \frac{R^4(x_5)}{s^2(x_5)}\right]. \tag{53}$$

The formula given by Eq. (52) may also be applied in determining the coefficient derivatives $C_{m\alpha}$ and C_{mq}. This is accomplished by substituting into Eqs. (44) and (45).

A COMPUTER PROGRAM FOR ARBITRARY CROSS SECTIONS

A computer program has been developed which solves the potential flow problem for shapes of the type encountered with missile cross sections. This program also determines the apparent area of the cross section through evaluation of the integral expressed in Eq. (33).

A missile cross section can generally be represented by a region representing the body along with line segments representing wings and various other airfoils. The boundary of this configuration will be the internal boundary for the potential-flow problem. Body angle or control-surface deflection is represented by specifying the normal gradient along this boundary. It is further assumed that the gradient of the potential goes to zero at infinity.

The calculation is based upon source distribution methods similar to those which have been used by J. L. Hess and others [5—7] in several computer programs. With these methods, a general potential function is represented as a linear sum of a fundamental system of potential functions, each of which is derived from a source of some type. The source may be a point monopole or dipole or a continuous distribution of one of these types.

Thus we may represent a general potential function as

$$\phi(x_1, x_2) = \sum_{i=1}^{N} C_i\,\phi_i(x_1, x_2). \tag{54}$$

Here, x_1 and x_2 are coordinates in the cross-flow plane. The expression

$$\phi_i(x_1, x_2) \qquad i = 1,\ldots, N$$

represents the fundamental system of source-derived functions. Each of these functions takes on a weight denoted by

$$C_i \qquad i = 1,\ldots,N.$$

We may derive the velocity field from the potential by taking the gradient of the latter, as shown by the following equations:

$$\mathbf{V}(x_1, x_2) = \left[\frac{\partial \phi}{\partial x_1}, \frac{\partial \phi}{\partial x_2} \right] \tag{55}$$

and

$$\mathbf{V}_i(x_1, x_2) = \left[\frac{\partial \phi_i}{\partial x_1}, \frac{\partial \phi_i}{\partial x_2} \right]. \tag{56}$$

Applying these equations to Eq. (54) results in the following general expression for the velocity field:

$$\mathbf{V}(x_1, x_2) = \sum_{i=1}^{N} C_i \, \mathbf{V}_i(x_1, x_2). \tag{57}$$

A set of N points are chosen on the boundary of the contour as locations for imposing the boundary condition. The coordinates of these points are denoted by $x_1{}^j$ and $x_2{}^j$ where $(j = 1,...,N)$. Also, the unit normal to the boundary at each of these locations is denoted by $\mathbf{n}_j$ where $(j = 1,...,N)$. As a boundary condition, the normal velocity W_j is specified at each of these points. The imposition of this condition is expressed by

$$W_j = \sum_{i=1}^{N} C_i \, \mathbf{V}_i(x_1{}^j, x_2{}^j) \cdot \mathbf{n}_j. \tag{58}$$

This relation represents N simultaneous linear equations for the weight factors C_i. Solving this system of equations determines the value of each C_i, whereupon the velocity potential ϕ and the velocity $\mathbf{V}$ may be calculated from Eqs. (54) and (57). The apparent area vector may be determined by evaluating Eq. (33) with a numerical integration process. The general representation of the velocity potential is based on its expression in terms of sources. This may be done in a variety of ways. Many investigators (See Ref. 6 for citation of additional refs.) have, for instance, utilized continuous density distributions on the body surface.

For this investigation, the body has been treated differently from the airfoils. For the body, point sources located within and at some distance from the boundary contour have been utilized. For a point source, the velocity potential and the velocity at the point (x_1, x_2) are given by the formulas

$$\phi = \frac{1}{2} \, ln \left[(x_1 - y_1)^2 + (x_2 - y_2)^2 \right] \tag{59}$$

and

$$\mathbf{V} = \frac{(x_1 - y_1, \, x_2 - y_2)}{(x_1 - y_1)^2 + (x_2 - y_2)^2}. \tag{60}$$

Here, y_1 and y_2 are the coordinates of the point source. For airfoils which are represented as line segments, continuous dipole distributions are utilized. The line segment representing a wing is divided into small elemental segments, and, on each of these, the dipole distribution is represented by a quadratic function. For each elemental segment, we denote its length by Δs and introduce coordinates x_1 and x_2 such that the segment lies on the x_1 axis centered about the origin. To accomplish the general quadratic representation, three fundamental potential functions are introduced corresponding to each elemental segment. These three functions are derived from dipole distributions, the first of which is constant while the second and third are proportional to x_1 and x_1^2, respectively.

For a constant dipole distribution on a line segment, the velocity potential and the velocity at the point (x_1, x_2) are given by the following formulas:

$$\phi_C = Incl \left\{ x_1^2 + x_2^2 - \left(\frac{\Delta s}{2} \right)^2, \ \Delta s \ x_2 \right\} \tag{61*}$$

and

$$\mathbf{V}_C = \frac{\left\{ -2\Delta s \ x_1 x_2, \ \Delta s \left(x_1^2 - x_2^2 - \left(\frac{\Delta s}{2} \right)^2 \right) \right\}}{\left[\left(x_1 + \frac{\Delta s}{2} \right)^2 + x_2^2 \right] \left[\left(x_1 - \frac{\Delta s}{2} \right)^2 + x_2^2 \right]}. \tag{62}$$

For a dipole distribution which is proportional to x_1, the velocity potential and the velocity are given by

$$\phi_L = \frac{x_1 \phi_c - x_2 \psi}{\Delta s} \tag{63}$$

and

$$\mathbf{V}_L = \frac{(\phi_{c}, - \psi) + x_1 \mathbf{V}_c - x_2 \mathbf{U}}{\Delta s}, \tag{64}$$

where we have introduced the following auxiliary quantities:

$$\psi = \frac{1}{2} \ ln \left[\frac{\left(x_1 + \frac{\Delta s}{2} \right)^2 + x_2^2}{\left(x_1 - \frac{\Delta s}{2} \right)^2 + x_2^2} \right] \tag{65}$$

and

$$\mathbf{U} = \left[\frac{\left\{ x_1 + \frac{\Delta s}{2}, \ x_2 \right\}}{\left(x_1 + \frac{\Delta s}{2} \right)^2 + x_2^2} - \frac{\left\{ x_1 - \frac{\Delta s}{2}, \ x_2 \right\}}{\left(x_1 - \frac{\Delta s}{2} \right)^2 + x_2^2} \right]. \tag{66}$$

For a dipole distribution which is proportional to x_1^2, the velocity potential and the velocity are given by

*The function *Incl* **w** is defined as the angle of inclination of **w** in radians where $-\pi < Incl \ \mathbf{w} \leqslant \pi$.

$$\phi_Q = \frac{(x_1^2 - x_2^2)\,\phi_c - 2\,x_1 x_2 \psi + \Delta s\, x_2}{\Delta s^2} \tag{67}$$

$$\mathbf{V}_Q = \frac{2\begin{bmatrix} x_1 \\ -x_2 \end{bmatrix} \phi_c + (x_1^2 - x_2^2)\,\mathbf{V}_c - 2\,\psi \begin{bmatrix} x_2 \\ x_1 \end{bmatrix} - 2\,x_1 x_2\,\mathbf{U} + \begin{bmatrix} 0 \\ \Delta s \end{bmatrix}}{\Delta s^2} \tag{68}$$

In order to determine the N weight factors C_i, an equal number of linear equations are established. Each of these equations are generated by specifying the normal velocity W_j at a point on the boundary. Those points where the normal velocities W_j are specified shall be referred to as control points. For each monopole source chosen internal to the body, a control point is selected on the body boundary. For each elemental segment of a wing, three fundamental functions are derived. This requires the determination of three corresponding weights C_i. Therefore, on each such elemental segment, three control points are selected.

In formulating a particular calculation, choices are made of the control points and of the sources from which the fundamental system of basic functions are derived. No definite procedure has been established for making these choices in a way which approaches the optimal. Rather, making satisfactory choices requires the use of good intuitive judgment in combination with knowledge learned from a moderate amount of computational experimentation.

Using a circle-with-midwing profile, a certain amount of computational experimentation was performed. For this profile, analytic answers are available for comparison. Figure 10 illustrates how the sources were chosen for this typical cross-flow calculation. The conclusions drawn from this undertaking are assumed to have general validity for bodies with wings.

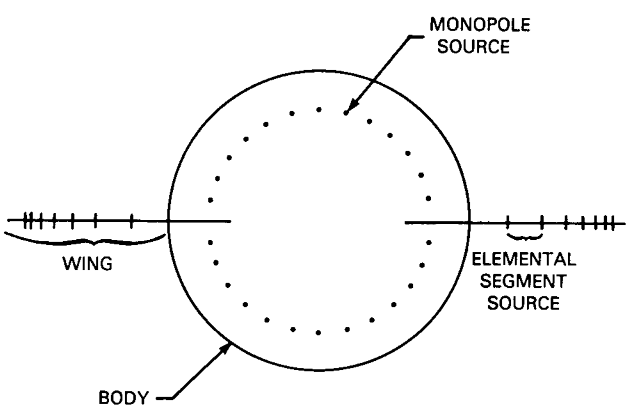


Fig. 10 — Choice of sources for missile cross-section profile

One conclusion deduced is that in order to obtain accurate results it is necessary to let the line segment representing an airfoil penetrate a distance into the body, as shown in Fig. 10. When this line segment was allowed to stop exactly at the body boundary, a good approximation to the true solution was unobtainable.

Another observation is that a modest increase in accuracy is obtained by letting the size of the elemental segments become smaller as the tip of the wing is approached.

The information defining a particular cross-flow problem is summarized next. Associated with the body is the position of each interior monopole along with information regarding elements on the body boundary. The latter consists of the location and orientation of each element along with control-point positions. Associated with an airfoil is the positon and orientation of the line-segment representation along with information regarding its partitioning into elemental segments; each elemental segment has three control points designated on it. Also included among the given information is the normal velocity W_j which is specified at each control point on a body boundary or airfoil.

Rather than inputting the given information with enumerative lists, various portions of it are generated by subroutines using analytic formulas. For example, the program contains a subroutine which generates all the necessary defining information for an airfoil. Another subroutine does this for a circular arc. A number of subroutines such as these can be used to piece together an entire cross-section contour. Upon execution, the computer program calculates the value of the velocity potential ϕ and the velocity $\mathbf{V}$ at each control point. It also calculates the weights C_i of the various fundamental functions. In addition, it calculates the apparent area vector $\mathbf{A}$ as given by Eq. (33).

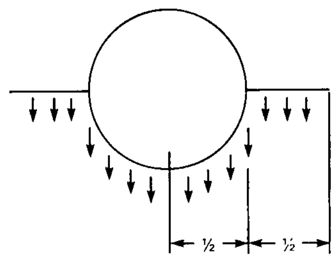
TEST CALCULATIONS

In order to test the computer program for accuracy and computation time, the program was executed on a CDC 6600 computer. Profiles were run for which answers could be obtained independently using analytic formulas. The profiles used were the circle, circle with midwing, and wing only. These are all special cases of the general circle-with-midwing problem for which an analytic result (Ref. 4) is available for comparison. The general expression for the apparent area magnitude A due to a vertical cross flow is given by

$$A = \pi(s^2 - R^2 + R^4/s^2) \qquad (69)$$

Here, R is the radius of the circle and s is the semi-span of the wing, as indicated in Fig. 9. Analytic comparison values of the apparent area for the profiles used in the test computations were obtained by using the following three sets of values in Eq. 69. The values used for R and s were: $R = s = 1$ for the circle; $R = 1/2$, $s = 1$ for the circle with midwing; and $R = 0$, $s = 1$ for the wing only. For the circle-with-midwing profile, test calculations were performed using various numbers of fundamental function generating elemental sources. These will be referred to as elements. The results of the test calculations using 24, 42, 84, and 168 elements are summarized in Table 1. In each case, the number of elements is broken down into the number of monopole types representing point monopoles within the body, and the number of dipole types representing continuous dipole distributions on an elemental segment of a wing. This table indicates the improvement in accuracy as the number of elements is increased. The 0.27% error for 42 elements, however, should be regarded as unusually low due to chance and not really fitting the trend. Table 1 also indicates the rapid increase in computation time as the number of elements is increased. It is apparent that a reasonable combination of accuracy and computation time is obtained when the number of source elements is on the order of 84. In order to test the program for the effect of varying the contour configuration, calculations were performed for the circle, circle-with-midwing, and wing only using 84 elements in each case. The results are shown in Table 2, indicating in each case the individual number of monopole and dipole elements used.

Table 1 – Circle With Midwing



APPARENT AREA (ANALYTIC) = 2.5525

No. of Elements	No. of Monopole Elements	No. of Dipole Elements	Apparent Area (Computed)	Error (%)	Execution Time (sec)
24	6	18	2.3257	8.89	0.972
42	12	30	2.5594	0.27	2.56
84	24	60	2.5638	0.44	12.5
168	48	120	2.5533	0.031	78.1

Table 2 – Various Profiles Using 84 Source Elements

Missile Configuration	Number of Monopole Elements	Number of Dipole Elements	Apparent Area [computed]	Apparent Area [analytic]	Error (%)
Circle	84	0	3.1416	3.1416	0
Circle-With-Midwing	24	60	2.5638	2.5525	0.44
Wing Only	0	84	3.1380	3.1416	0.11

It is observed in Table 2 that near-perfect accuracy is obtained with the circular profile (assumed typical of a body-only type of contour). The least accuracy or highest error (0.44%), is obtained using the circle-with-midwing. This profile (assumed typical of a wing-body combination) is the case most useful for revealing any possible error. It is found that an intermediate degree of accuracy, or next highest error (0.11%) is obtained with the wing-only profile.

SAMPLE CALCULATIONS

This computer program is of general utility for performing cross-flow calculations for almost any shape that would be encountered with a missile cross section. Sample calculations have been performed for a few cases where analytic results are not available. Apparent areas are calculated for boundary conditions representing pitch, elevator deflection, yaw, and rudder deflection. The results of these calculations are given in Tables 3, 4, and 5. In Table 3, a rudder has been added to the circle-with-midwing configuration. Table 4 shows results for a configuration consisting of simply a circular body and a rudder. Results for a tri-tail configuration are given in Table 5.

Table 3 – Circle With Midwing and Rudder (114 Elements)

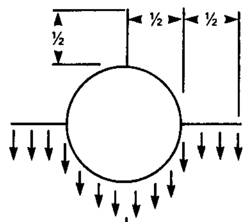
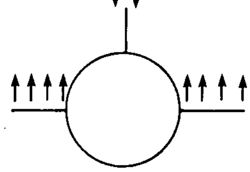
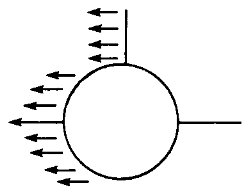
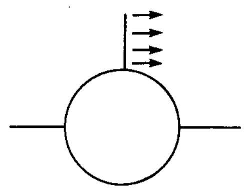
Type of Control	Configuration	Apparent Area Magnitude
Pitch		2.55
Elevator Deflection		1.14
Yaw		1.64
Rudder Deflection		0.548

Table 4 – Circle With Rudder (114 Elements)

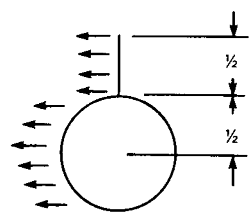
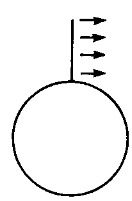
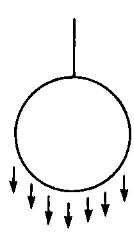
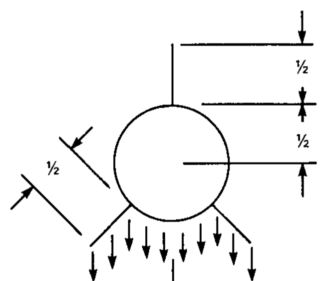
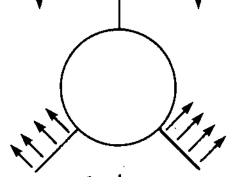
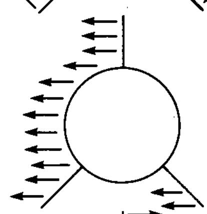
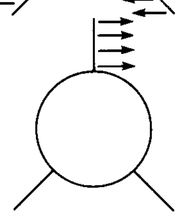
Type of Control	Configuration	Apparent Area Magnitude
Yaw		1.62
Rudder Deflection		0.538
Pitch		0.785

Table 5 — Tri-Tail Configuration (114 Elements)

Type of Control	Configuration	Apparent Area Magnitude
Pitch		2.14
Elevator Deflection		1.00
Yaw		2.14
Rudder Deflection		0.579

CONCLUSIONS AND IMPLICATIONS FOR FURTHER EFFORT

Expeditious and versatile procedures for estimating sonic aerodynamic coefficient derivatives have been developed. This endeavor was based upon the generation of a computer program to solve the potential cross-flow problem. This program is similar to certain other existing ones but has some necessary differences. The development is based on slender body theory and gives estimates which can be applied to missiles flying at sonic speeds. These estimates are also helpful in determining approximations for the coefficient derivatives in the subsonic and supersonic ranges.

There is a considerable need for these aerodynamic coefficient estimates as inputs to computer and hardware simulations. These simulations deal with a wide variety of analyses of the expected flight behavior of actual missiles under a variety of conditions. An extensive effort is needed to investigate the implications of certain postulated characteristics for future missiles. This effort would involve working out a missile design within the constraints of the postulated external characteristics. The resulting design would be extremely useful in *bounding the threat* by indicating the outer limits of a missile's maneuvering capability. Aerodynamic coefficients based on this design would provide valuable inputs for future simulations which could be used to study the expected outcome of missile-ship engagements involving newer missile types.

A capability for analytic determination of aerodynamic coefficient derivatives is essential for any comprehensive investigation of missile-ship engagements. It is needed not only for the analysis of existing missile threats but also for the examination of possible future missile threats.

REFERENCES

1. A. H. Shapiro, *The Dynamics and Thermodynamics of Compressible Fluid Flow,* Ronald Press Co., New York, Vol. I, p. 287, 1953.
2. H. Ashley and M. Landahl, *Aerodynamics of Wings and Bodies,* Addison-Wesley, Reading, Mass., p. 235, 1965.
3. Ibid., p. 123.
4. Ibid., p. 122.
5. J. L. Hess and A.M.O. Smith, "Calculation of Potential Flow About Arbitrary Bodies," *Progr. Aeronaut. Sci.,* **8**, 1-138 (1966).
6. J. L. Hess, "Higher Order Numerical Solution of the Integral Equation for the Two-Dimensional Neumann Problem," *Computer Methods in Applied Mechanics and Engineering,* **2** (No. 1) 1-15 (1973).
7. J. L. Hess, "Review of Integral-Equation Techniques for Solving Potential-Flow Problems with Emphasis on the Surface-Source Method," *Comput. Meth. Appl. Mech. Engrg.,* **5**, 145 - 196 (1975).