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Scattering from a Periodic Corrugated Surface - Part 1 - Semi-Infinite Inhomogeneously Filled Plates with Soft Boundaries

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CONTENTS

ABSTRACT

An incident plane wave is scattered from a surface which is corrugated in one dimension and which is described by an infinite number of periodically spaced semi-infinite parallel plates (comb) having soft boundary conditions. An additional plate is placed between each set of adjacent plates, thus dividing the periodicity interval into two regions, one of which doesn't differ in its properties from the region above the plates, while the second region is inhomogeneously filled. The latter means that this region differs in wavenumber and density from the surrounding media. Both the wavenumber and the density are here assumed to be constant. The solutions of the Helmholtz equation are assumed to be upgoing plane waves above the plates and, between the plates, standing waves along the periodicity direction and downgoing waves along the plate slots. The solutions have unknown amplitude coefficients. Continuity of pressure and velocity across the common boundary yield linear equations relating the amplitudes in the various regions. The latter are shown to be similar to the residue series of integrals of certain meromorphic functions. The amplitudes are expressed as values or residues of these functions, which are explicitly constructed. The two examples treated in detail are (a) zero-thickness plates with arbitrary incident angle, and (b) arbitrarily thick (inhomogeneous) plates at normal incidence.

PROBLEM STATUS

This is a final report on one phase of the problem; work on other phases is continuing.

AUTHORIZATION

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SCATTERING FROM A PERIODIC CORRUGATED SURFACE

Part 1—Semi-Infinite Inhomogeneously Filled Plates with Soft Boundaries

1. INTRODUCTION

The problem considered in this report is the calculation of the scattered and diffracted fields when a plane wave is incident on a one-dimensional periodically corrugated surface. The surface consists of periodically spaced, infinitesimally thin, parallel plates extending to infinity in the remaining two dimensions of the problem. Each period is further divided by a parallel plate into two regions which are, respectively, free and inhomogeneously filled. The inhomogeneity is expressed by having different wavenumbers and densities in the various regions. Soft boundary conditions are considered on the plates. In a sense this report is a generalization of the classic diffraction problem of plane waves incident on thin, periodically spaced, parallel plates, first solved by Carlson and Heins (1) using Wiener-Hopf methods, and by Berz (2) and Whitehead (3) using a residue calculus method from complex function theory.

The basic formalism of the problem is presented in Sec. 2. The velocity potential solutions ψ of the scalar Helmholtz equation are written in the various geometric regions of the problem. Above the plates, ψ is expressed as an incident plane wave plus a scattered field consisting of a superposition of upgoing plane waves, and evanescent waves whose propagation directions are given by the grating equation. The solutions in both free and inhomogeneously filled plate wells are expressed as standing waves in the direction of periodicity, and propagating or evanescent waves in the other dimension. Continuity of pressure and velocity across the common boundary yield linear equations relating the amplitude coefficients. Green's theorem is used to derive a flux conservation relationship between the reflection and transmission coefficients. The general mathematical procedure is to relate these sets of linear equations to the residue series of integrals of certain constructed meromorphic functions. The amplitude coefficients are thus related to residues or values of the functions. The most general sets of linear equations (for arbitrary incidence and arbitrary inhomogeneity) are derived, but this most general set cannot be solved by the present methods. Instead, two special cases are considered.

In Sec. 3 the case of an arbitrary angle of incidence and no inhomogeneity (zero-thickness plates) is considered. This is just the Carlson-Heins problem and is solved using a slightly different residue calculus method from that of Berz (2) in order to illustrate the general procedure.

Also in Sec. 3 the case of normal angle of incidence and arbitrary inhomogeneity (arbitrarily thick* plates) is treated. It is necessary to modify the residue calculus technique by an iterative procedure due to Mittra et al. (4) in order to satisfy a necessary symmetry property of the meromorphic function. In both cases the amplitudes of the various waves are related to the values and residues of the constructed function, and the behavior of the fields near a plate edge (edge condition) is demonstrated.

The summary and conclusions are contained in Sec. 4. This report is restricted to the analytic problems involved; the numerical evaluation of the reflection and transmission coefficients will be presented elsewhere.

There are also four appendices. Appendix A covers the properties of the infinite products used to construct the functions. The edge condition is derived in App. B and is related to techniques used in the iterative procedure in App. C. Finally, in App. D the asymptotic algebraic behavior of the residues of the functions is discussed.

2. BASIC FORMALISM

Scalar Wave Function

We wish to solve the two-dimensional Helmholtz equation for a plane wave incident at an angle θ_i on an infinite number of periodically spaced (period 2ℓ) half planes which are alternately filled with an inhomogeneous material. These planes are illustrated in Fig. 1. The x direction is the direction of periodicity, and the planes extend to $z = -\infty$ and $y = \pm\infty$ (perpendicular to the plane of the figure). The separation between adjacent inhomogeneous planes is $2a$. The Helmholtz equation is**

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial z^2} + k^2 \right) \psi(x,z) = 0 \tag{2.1}$$

where the wavenumber $k = 2\pi/\lambda$ (λ is the wavelength), and ψ is the scalar wave function or velocity potential given by (see Fig. 1)

*The word "thick" as used in this report is not the conventional usage, i.e., it does not mean that I am discussing a *single* plate having parallel sides and one end *closed* (the other open to infinity). "Thick" refers to the *region* between two infinitesimally thin and *adjacent* plates (region C in Fig. 1) which is filled with a material having density and wavenumber parameter values which differ from those in the surrounding media (regions A and B). A limiting case of these parameter values yields a thick plate in the conventional sense (see case for $t \neq 1$ in Sec. 3 and App. C).

**$e^{-i\omega t}$ is suppressed throughout.

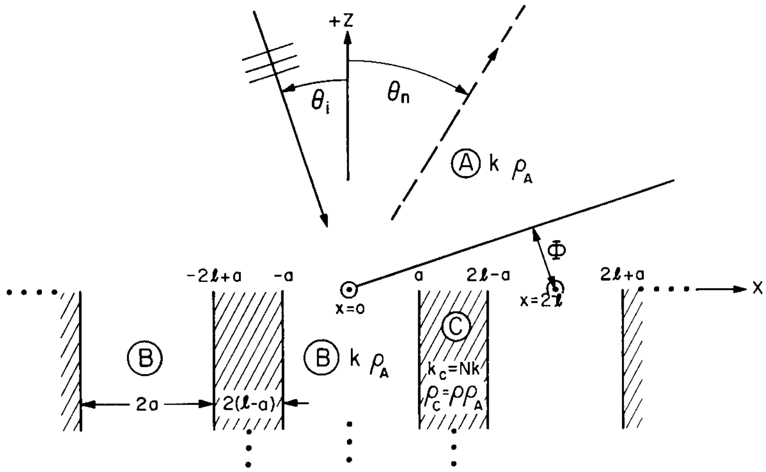

Fig. 1—Plane wave incident at an angle θ_i on an infinite grating of parallel plates which extends to $z = -\infty$ and $y = \pm\infty$ (perpendicular to the plane of the paper). Region C, bounded by two infinitesimally thin plates, is filled with a material having different density and wavenumber values from those of the surrounding media (regions A and B). The wavenumber k_C is defined by $k_C = Nk$, and the density ρ_C by $\rho_C = \rho\rho_A$. The discrete scattering angles are θ_n, the periodicity is 2ℓ, $2a$ is the distance between empty plates, and the parameter t is defined by $t \equiv \ell/a$. Region A is $z \geqslant 0$, and regions B and C are $z \leqslant 0$; ϕ is the phase lag for the wavefront striking $x = 2\ell$ as opposed to that striking $x = 0$.

$$\psi(x,z) = \left\{ \begin{array}{ll} \psi_A(x,z),\ z \geqslant 0 \text{ (region A)} & \\ \psi_B(x,z), & \begin{cases} z \leqslant 0 \text{ (region B where} \\ -a+2m\ell \leqslant x \leqslant a+2m\ell;\ m = 0,\pm1,\pm2,\ldots) \end{cases} \end{array} \right\}. \tag{2.2}$$

In addition in region C where $z \leqslant 0$ and $a+2m\ell \leqslant x \leqslant 2(m+1)\ell-a$ $(m = 0,\pm1,\pm2,\ldots)$, ψ is given by ψ_C which satisfies

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial z^2} + k_C^2\right) \psi_C(x,z) = 0. \tag{2.3}$$

Thus region C is filled with a constant inhomogeneity represented by the fact that $k_C \neq k$. It is convenient to define the dimensionless quantity N by $k_C = Nk$. (The case of no inhomogeneity will of course follow by setting $N = 1$ in our results.)

In addition to the soft boundary condition (satisfied by ψ_B or ψ_C, as appropriate), we have

$$\psi(x_0,z) = 0 \text{ for } z \leqslant 0 \text{ and } x_0 = \pm a+2m\ell \ (m = 0,\pm1,\pm2,\ldots). \tag{2.4}$$

The function ψ satisfies the following restrictions:

a. ψ and $\nabla\psi$ are finite in each subregion, except at the sharp edges of the plates where $\psi = 0\,(r^{(1/2)+\epsilon})$ and $|\nabla\psi| = 0(r^{-(1/2)+\epsilon})$ as the edge of a plate is approached along a radial direction ($r \to 0$).* The number ϵ depends on N and ρ, the ratio of the density in region C to that in region A (Fig. 1). Its exact form is derived in App. C and there we also show that $|\epsilon| \leqslant 1/6$. This full remark is the edge condition.

b. ψ and $\nabla\psi$ are continuous in each subregion, and the pressure and velocity are continuous across the $z = 0$ interface.

c. Apart from the incident wave, ψ represents outgoing waves as $z \to \pm\infty$.

(In the analogous electromagnetic problem, ψ is the single component of the electric field which is in the y direction. The magnetic vector is in the xz plane and is calculated from ψ via Maxwell's equations.)

We have the following wave functions in the various regions. For $z \geqslant 0$ (region A), ψ is written as

$$\psi_A(x,z) = \psi_i(x,z) + \psi_{sc}(x,z) \tag{2.5}$$

where ψ_i is the plane wave incident at angle θ_i and is given by

$$\psi_i(x,z) = \exp[ik(\alpha_0 x - \beta_0 z)] \tag{2.6}$$

with $\alpha_0 = \sin\theta_i$ and $\beta_0 = \cos\theta_i$. The scattered wave ψ_{sc} is written as a superposition of plane waves propagating (or decaying) in the positive z direction as

$$\psi_{sc}(x,z) = \sum_{n=-\infty}^{\infty} A_n \exp[ik(\alpha_n x + \beta_n z)] \tag{2.7}$$

where $\alpha_n = \sin\theta_n$ and

$$\beta_n = \cos\theta_n = \left\{ \begin{array}{l} (1-\alpha_n^2)^{1/2},\ \alpha_n^2 < 1 \\ +i(\alpha_n^2-1)^{1/2},\ \alpha_n^2 > 1 \end{array} \right\}.$$

The scattering angle θ_n is given by the grating equation below. Note that the restriction on β_n insures that ψ_{sc} satisfies restriction (c) above. The coefficients A_n are to be determined. The scattering coefficient $R(x,z)$ is defined by

$$R(x,z) = \psi_{sc}(x,z)/\psi_i(x,z). \tag{2.8}$$

*The symbol "0" is the order symbol. The remark $\psi = 0(r^{1/2})$ as $r \to 0$ means that $\lim_{r\to 0} r^{-1/2}\psi =$ constant $\neq 0$.

Since the surface is periodic, so is R. Thus

$$R(x+2\ell,z) = R(x,z)$$

which, using Eqs. (2.6) and (2.7), implies the grating equation

$$\alpha_n = \alpha_0 + n\Lambda \tag{2.9}$$

where $\Lambda = \lambda/2\ell$ and n is an integer. For $z \leqslant 0$ and $-a \leqslant x \leqslant a$ (region B), ψ is written as

$$\psi_B(x,z) = \sum_{j=1}^{\infty} B_j \sin\left(\frac{j\pi(x+a)}{2a}\right) e^{-ikq_jz} \tag{2.10}$$

with

$$q_j = \left\{ \begin{array}{ll} \left[1-\left(\dfrac{j\pi}{2ka}\right)^2\right]^{1/2} = \left[1-\left(\dfrac{j\Lambda t}{2}\right)^2\right]^{1/2}, & \left(\dfrac{j\Lambda t}{2}\right)^2 < 1 \\ +i\left[\left(\dfrac{j\Lambda t}{2}\right)^2 - 1\right]^{1/2}, & \left(\dfrac{j\Lambda t}{2}\right)^2 > 1 \end{array} \right\} \tag{2.11}$$

where ψ_B satisfies both Eq. (2.4), at $x=\pm a$, and restriction (c) via Eq. (2.11). The parameter t is equal to ℓ/a. Also it is convenient to define

$$p_j = j\pi/2ka = j\Lambda t/2.$$

For the remaining values of x, ψ_B is given below. For $z \leqslant 0$ and $a \leqslant x \leqslant 2\ell-a$ (region C), ψ is written as

$$\psi_C(x,z) = \sum_{j=1}^{\infty} C_j \sin\left(\frac{j\pi(x-a)}{2\ell-2a}\right) e^{-ikr_jz} \tag{2.12}$$

with

$$r_j = \left\{ \begin{array}{ll} \left[N^2 - \left(\dfrac{j\pi}{k(2\ell-2a)}\right)^2\right]^{1/2} \equiv \{N^2 - (ju)^2\}^{1/2} & (ju)^2 < N^2 \\ +i[(ju)^2 - N^2]^{1/2}, & (ju)^2 > N^2 \end{array} \right\} \tag{2.13}$$

where

$$u \equiv \frac{\pi}{k(2\ell-2a)} = \frac{\Lambda t}{2(t-1)} . \tag{2.14}$$

It is easily seen that ψ_C satisfies Eq. (2.4) at $x = a$ and $x = 2\ell-a$. Equation (2.13) insures that restriction (c) is satisfied. It is also convenient to define $u_j \equiv r_j|_{N=1}$.

Both ψ_B and ψ_C are periodic functions (up to a phase lag term) so that, to determine these functions for values of x other than those above, we write for both ψ_B and ψ_C

$$[\psi_{B,C}(x,z)\exp(-ik\alpha_0 x)]_{x=x_1+2m\ell} = [\psi_{B,C}(x,z)\exp(-ik\alpha_0 x)]_{x=x_1}$$

where $m = 0,\pm1,\pm2,\ldots$, $-a \leqslant x_1 \leqslant a$ for $\psi_B{}'$ and $a \leqslant x_1 \leqslant 2\ell - a$ for ψ_C. The term $\exp(-ik\alpha_0 x)$ yields the phase lag term and expresses the phase delay (or advance) of the plane wave as it strikes two different surface points.

A_n and B_m Equations

The various unknown amplitude coefficients will be shown to be related via sets of linear equations. These follow from the continuity of pressure and normal velocity across the $z = 0$ interface. In terms of the velocity potential ψ, the pressure p and the normal velocity v_n are given by

$$p = -i\omega\rho_0\psi \text{ and } v_n = -\partial\psi/\partial n \tag{2.15}$$

where ρ_0 is the density in the particular region and n is the normal. We are assuming that $\rho_A = \rho_B$ and $\rho_C = \rho\rho_A$ where ρ is a number. To relate A_n and B_m, use these continuity conditions which, for $-a \leqslant x \leqslant a$, are given by

$$\psi_A(x,0) = \psi_B(x,0)$$

and

$$\frac{\partial\psi_A}{\partial z}(x,0) = \frac{\partial\psi_B}{\partial z}(x,0). \tag{2.16}$$

Substituting ψ_A and ψ_B in Eq. (2.16), multiplying the resulting equations by $\sin(m\pi(x+a)/2a)$, and integrating on x from $-a$ to a yields, after manipulation, the set of equations

$$\sum_{n=-\infty}^{\infty} \frac{A_n I_{nm}}{\beta_n \pm q_m} - \frac{I_{0m}}{\beta_0 \mp q_m} + \frac{2\pi q_m}{\Lambda t p_m}\begin{Bmatrix} B_m \\ 0 \end{Bmatrix} = 0 \tag{2.17}$$

where the upper sign is to be read with B_m of the last term, and the lower sign with 0. I_{nm} is given by

$$I_{nm} = e^{-\pi i\alpha_n/\Lambda t} - (-)^m e^{\pi i\alpha_n/\Lambda t}. \tag{2.18}$$

Equations (2.17) relate the A_n and B_m amplitudes as desired.

A_n and C_m Equations

To relate A_n and C_m, use continuity of pressure and velocity in region C where $a \leqslant x \leqslant 2\ell - a$, which gives

$$\psi_A(x,0) = \rho\ \psi_C(x,0)$$

and (2.19)

$$\frac{\partial \psi_A}{\partial z}(x,0) = \frac{\partial \psi_C}{\partial z}(x,0).$$

Substituting ψ_A and ψ_C in Eq. (2.19), multiplying the resulting equations by $\sin[m\pi(x-a)/(2\ell-2a)]$, and integrating on x from a to $2\ell-a$ yields the sets of equations

$$\sum_{n=-\infty}^{\infty} \frac{(A_n+\delta_{n0})J_{nm}}{\beta_n^2-u_m^2} - \frac{\rho\pi(t-1)C_m}{\Lambda tmu} = 0 \tag{2.20}$$

and

$$\sum_{n=-\infty}^{\infty} \frac{(A_n-\delta_{n0})\beta_n J_{nm}}{\beta_n^2-u_m^2} + \frac{\pi(t-1)r_m C_m}{\Lambda tmu} = 0 \tag{2.21}$$

where

$$J_{nm} = e^{\pi i\alpha_n/\Lambda t}\,(1-(-)^m e^{\pi i\alpha_n/u}) \tag{2.22}$$

and δ_{n0} is the Kronecker delta function (equal to 1 for $n = 0$, and equal to 0 otherwise). Next, multiply Eq. (2.20) by u_m. The resulting equation is successively added to and subtracted from Eq. (2.21). These operations yield the sets of equations

$$\sum_{n=-\infty}^{\infty} \frac{A_n J_{nm}}{\beta_n-u_m} - \frac{J_{0m}}{\beta_0+u_m} + \frac{(r_m-\rho u_m)\pi(t-1)C_m}{\Lambda tmu} = 0 \tag{2.23}$$

and

$$\sum_{n=-\infty}^{\infty} \frac{A_n J_{nm}}{\beta_n+u_m} - \frac{J_{0m}}{\beta_0-u_m} + \frac{(r_m+\rho u_m)\pi(t-1)C_m}{\Lambda tmu} = 0. \tag{2.24}$$

Define

$$\sigma_m = \frac{r_m-\rho u_m}{r_m+\rho u_m}. \tag{2.25}$$

Multiply Eq. (2.24) by σ_m and subtract Eq. (2.23) from the resulting equation, which yields

$$\sum_{n=-\infty}^{\infty} A_n J_{nm}\left(\frac{\sigma_m}{\beta_n+u_m} - \frac{1}{\beta_n-u_m}\right) - J_{0m}\left(\frac{\sigma_m}{\beta_0-u_m} - \frac{1}{\beta_0+u_m}\right) = 0. \tag{2.26}$$

We shall use Eqs. (2.26) and (2.24) as the two sets of linear equations for the A and C regions.

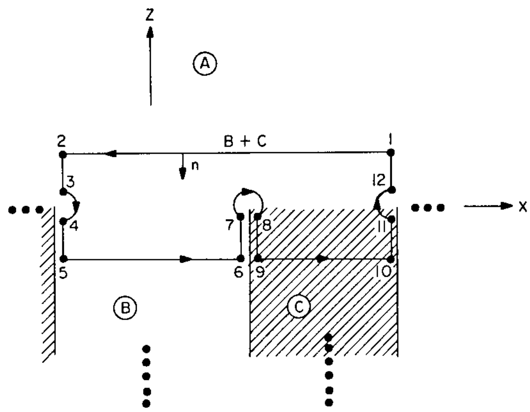

Fig. 2—Contour B+C used with Green's theorem to derive the flux relation. The inward normal is n.

Flux Conservation

In order to calculate a relation which expresses the conservation of flux across the $z = 0$ interface (i.e., to calculate the reflection and transmission coefficients for the surface), consider integrals of the form

$$\oint_{B+C} d\mu\rho_0 \psi^*(x,z)\, \overleftrightarrow{\partial}_n \psi(x,z) = 0 \tag{2.27}$$

where ψ^* is the complex conjugate of ψ, n is the inward normal to the closed contour B+C = 1,2,...,12,1 illustrated in Fig. 2, and $f\overleftrightarrow{\partial}g \equiv f(\partial g)-(\partial f)g$. Since both ψ and ψ^* satisfy the same Helmholtz equation (k in each region is real) in the region of integration, the right-hand side of Eq. (2.27) vanishes. This follows from Green's theorem. It is easily seen that the only contributions to Eq. (2.27) arise from the paths $\widehat{12}$, $\widehat{56}$, and $\widehat{9\ 10}$ in the B+C contour. The sum of integrals along $\widehat{23}$ and $\widehat{12\ 1}$ vanish due to the periodicity in ψ. Integrals along $\widehat{45}$ and $\widehat{67}$ vanish since $\psi_B = 0$ along these paths, and $\widehat{89}$ and $\widehat{10\ 11}$ vanish since $\psi_C = 0$. Integrals along the three semicircles $\widehat{34}$, $\widehat{78}$ and $\widehat{11\ 12}$ can be shown to vanish as their radii shrink to zero. This follows from the edge condition. Substituting appropriate values of ψ in Eq. (2.27) and carrying out the integrals and evaluating the contributing contours at $z = 0$ yields the form

$$R + T = 1 \tag{2.28}$$

for Eq. (2.27), where the reflection coefficient R is given by

$$R = \sum_n |A_n|^2(\beta_n/\beta_0) \equiv \sum_n R_n \tag{2.29}$$

with the sum running over all $n = 0,\pm1,\ldots,$ such that β_n is real (i.e., over real scattering orders). The R_n are obviously defined and are the individual spectral reflection coefficients. The transmission coefficient T is given by the sum of the two transmission coefficients T_B and T_C corresponding to transmission into regions B and C, respectively,

$$T = T_B + T_C \tag{2.30}$$

where

$$T_B = \sum_m |B_m|^2 (q_m/2t\beta_0) \tag{2.31}$$

and

$$T_C = \rho(t-1) \sum_m |C_m|^2 \, (r_m/2t\beta_0) \tag{2.32}$$

with the sums running over $m = 1,2,\ldots,$ such that q_m and r_m are real (i.e., real propagating orders in the plate regions). Similarly, individual spectral transmission coefficients can be defined as

$$T_{Bm} = |B_m|^2 (q_m/2t\beta_0) \tag{2.33}$$

and

$$T_{Cm} = \rho(t-1)|C_m|^2 (r_m/2t\beta_0). \tag{2.34}$$

Note that for $t = 1$, $T_C = 0$. This is because the region C is not present for $t = 1$.

3. DISCUSSION OF THE EQUATIONS

We wish to solve Eqs. (2.17), (2.24), and (2.26) for the amplitude coefficients A_n, B_m, and C_m by matching the equations to the residue series of certain meromorphic functions. It is not possible to do this for the most general of the parameters α_0 and t. Instead, in this section we present two special cases for the soft boundary condition we have been using. First, the case of arbitrary α_0 and $t = 1$ is solved; then the case of normal incidence ($\alpha_0 = 0$) and arbitrary t (and arbitrary N and ρ) is solved.

Arbitrary α_0 and $t = 1$

The first case is for a plane wave incident at an arbitrary angle on a surface. The surface consists of periodically spaced, semi-infinite, infinitesimally thin parallel plates. It is a classic diffraction problem solved by Carlson and Heins (1) using the Wiener-Hopf method, and by Berz (2) and Whitehead (3) using the residue calculus method (RCM). The two methods are related (5). The method we present below is slightly different from that of Berz, but the amplitudes we calculate can be shown to be equivalent. It should also be noted that our geometry differs from that of Berz.

For $t = 1$, Eqs. (2.17), (2.24), and (2.26) simplify considerably. Recall that u_m is just r_m at $N = 1$. Equations (2.13) and (2.14) indicate that for $t = 1$, u_m is infinite. Hence, Eqs. (2.24) and (2.26) are identically satisfied. Using the grating equation, Eq. (2.9), and $t = 1$ in Eq. (2.18) yields

$$I_{nm}|_{t=1} = (-)^n I_{0m}.$$

Thus, I_{0m} can be factored out of the sums in Eqs. (2.17). The latter in turn can thus be written as

$$\sum_{n=-\infty}^{\infty} \frac{(-)^n A_n}{\beta_n \pm q_m} - \frac{1}{\beta_0 \mp q_m} + \frac{2\pi q_m}{\Lambda p_m} I_{0m}^{-1} \begin{Bmatrix} B_m \\ 0 \end{Bmatrix} = 0. \tag{3.1}$$

Now construct a meromorphic function $f(\omega)$ which is assumed to have the following properties:

a. $f(\omega)$ has simple poles at $\omega = \beta_n$ $(n = 0,\pm1,\pm2,...)$ and a simple pole at $\omega = -\beta_0$.

b. $f(\omega)$ has simple zeroes at $\omega = q_m$ $(m = 1,2,3,...)$.

c. $f(\omega) = 0(\omega^{-3/2})$ as $|\omega| \to \infty$.

d. The wave function $\psi = 0(r^{1/2})$ as a plate edge is approached (r is the radial distance from an edge), or, equivalently, $\partial\psi/\partial r = 0(r^{-1/2})$ as $r \to 0$. The connection of this edge behavior and the asymptotic behavior of $f(\omega)$ will be demonstrated. The derivation of the edge condition is given in App. B and C ($\epsilon = 0$).

Consider integrals of the form

$$\frac{1}{2\pi i} \int_{C_s} \frac{f(\omega)d\omega}{\omega \pm q_m} \tag{3.2}$$

where the closed contour C_s is chosen to enclose the points $\omega = \pm q_m$ (for $m \leqslant s$), β_n (for $n = 0,\pm1,...,\pm s$), and $-\beta_0$. The contour is illustrated in Fig. 3. Application of the residue theorem yields

$$\frac{1}{2\pi i} \int_{C_s} \frac{f(\omega)\, d\omega}{\omega \pm q_m} = \sum_{n=-s}^{s} \frac{r(\beta_n)}{\beta_n \pm q_m} - \frac{r(-\beta_0)}{\beta_0 \mp q_m} + \begin{Bmatrix} f(-q_m) \\ 0 \end{Bmatrix} \tag{3.3}$$

where $r(\beta_n)$ is the residue of $f(\omega)$ at $\omega = \beta_n$. As s approaches infinity, C_s approaches an infinite contour, and the integrals on the left-hand side of Eq. (3.3) approach zero because of property (c) of $f(\omega)$. The resulting residue series are thus given by

$$\sum_{n=-\infty}^{\infty} \frac{r(\beta_n)}{\beta_n \pm q_m} - \frac{r(-\beta_0)}{\beta_0 \mp q_m} + \begin{Bmatrix} f(-q_m) \\ 0 \end{Bmatrix} = 0. \tag{3.4}$$

Equations (3.4) are similar to Eqs. (3.1) if we make the identifications

$$(-)^n A_n = r(\beta_n) \tag{3.5}$$

$$1 = r(-\beta_0) \tag{3.6}$$

$$B_m = \frac{\Lambda p_m}{2\pi q_m} I_{0m} f(-q_m). \tag{3.7}$$

Thus the amplitudes are known via Eqs. (3.5) and (3.7) once the function $f(\omega)$ is known. Equation (3.6) will be used in the construction of $f(\omega)$.

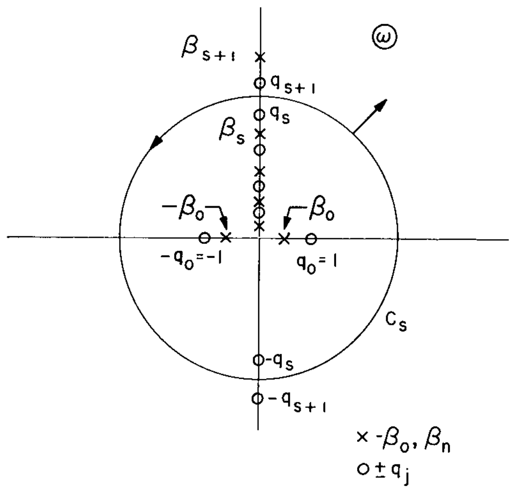

Fig. 3—The contour of integration C_s for the residue calculus technique discussed in Sec. 3 of this report. A possible configuration of the poles β_n and $-\beta_0$ and points $\pm q_m$ is shown.

In order to construct $f(\omega)$, consider the following infinite products, which are defined and discussed in App. A [Eqs. (A.6)-(A.8)]:

$$\Pi_1(\omega,\beta) = \prod_{n=1}^{\infty} (1-\omega/\beta_n)(\beta_n/in\Lambda)e^{\omega/in\Lambda} \tag{A.6}$$

$$\Pi_2(\omega,\beta) = \prod_{n=1}^{\infty} (1-\omega/\beta_{-n})(\beta_{-n}/in\Lambda)e^{\omega/in\Lambda} \tag{A.7}$$

$$\Pi(\omega,q) = \prod_{m=1}^{\infty} (1-\omega/q_m)(2q_m/im\Lambda)e^{2\omega/im\Lambda}. \tag{A.8}$$

(The latter product is actually Eq. (A.8) with $t = 1$.) Using these products, and the definition $\Pi_{12}(\omega,\beta) \equiv \Pi_1(\omega,\beta)\Pi_2(\omega,\beta)$, an $f(\omega)$ satisfying properties (a) and (b) can be written as

$$f(\omega) = \frac{g(\omega)}{\omega^2-\beta_0^2} \frac{\Pi(\omega,q)}{\Pi_{12}(\omega,\beta)} \tag{3.8}$$

with $g(\omega)$ being an entire function which will be determined. Using the asymptotic properties of the infinite products found in App. A, it can be shown, via Eq. (3.8), that as $\omega \to \infty$ for $-3\pi/2 < \arg(\omega) < \pi/2$,

$$f(\omega) \approx g(\omega)\, \omega^{-3/2}\, e^{-i\omega H} \tag{3.9}$$

where $H = (2 \ln 2)/\Lambda$ and some constant terms have been absorbed into $g(\omega)$. The domain containing $\arg(\omega) = \pi/2$ can be included (see App. A) and doesn't alter the choice of the entire function $g(\omega)$ given by

$$g(\omega) = (g_0 + g_1 \omega) e^{i\omega H}. \tag{3.10}$$

Using Eqs. (3.9) and (3.10) we find that property (c) of $f(\omega)$ is thus satisfied if $g_1 = 0$. In fact, $f(\omega)$ vanishes algebraically as $|\omega| \to \infty$. It will be shown that if $g_1 \neq 0$, $\partial\psi/\partial r$ will behave like $r^{-3/2}$ near a plate edge. This is too singular (6). Hence $g_1 = 0$ by the edge condition, and, using Eq. (3.6) to calculate g_0, $f(\omega)$ can be written as

$$f(\omega) = \frac{2\beta_0}{\beta_0^2-\omega^2}\, \frac{\Pi(\omega,q)}{\Pi(-\beta_0,q)}\, \frac{\Pi_{12}(-\beta_0,\beta)}{\Pi_{12}(\omega,\beta)}\, e^{iH(\omega+\beta_0)}. \tag{3.11}$$

In order to calculate A_n via Eqs. (3.5) and (3.11), it is necessary to know

$$L_n \equiv \lim_{\omega\to\beta_n} \left(\frac{\omega-\beta_n}{\Pi_{12}(\omega,\beta)} \right) \qquad (n \neq 0).$$

Using Eq. (A.31) we can write

$$[\Pi_{12}(\omega,\beta)]^{-1} = \frac{\Pi_{12}(-\omega,\beta)\,(\omega^2-\beta_0^2)\,(\pi/\Lambda)^2}{\sin(\pi(\alpha_0+\sqrt{1-\omega^2})/\Lambda)\,\sin(\pi(\alpha_0-\sqrt{1-\omega^2})/\Lambda)}$$

which exposes the poles via the sine factor. The residue calculation is now straightforward and yields

$$L_n = \frac{\alpha\pi_n}{\Lambda\beta_n}\, \frac{(\alpha_0^2-\alpha_n^2)\Pi_{12}(-\beta_n,\beta)}{\sin(2\pi\alpha_0/\Lambda)}. \tag{3.12}$$

From Eq. (A.31) we can also write

$$\Pi_{12}(-\beta_0,\beta) = \frac{\sin(2\pi\alpha_0/\Lambda)}{(2\pi\alpha_0/\Lambda)}\, [\Pi_{12}(\beta_0,\beta)]^{-1}. \tag{3.13}$$

Using Eqs. (3.5), (3.11), (3.12), and (3.13), A_n can thus be written as

$$A_n = (-)^{n+1} \frac{\beta_0\alpha_n}{\alpha_0\beta_n}\, \frac{\Pi_{12}(-\beta_n,\beta)}{\Pi_{12}(\beta_0,\beta)}\, \frac{\Pi(\beta_n,q)}{\Pi(-\beta_0,q)}\, e^{iH(\beta_n+\beta_0)}. \tag{3.14}$$

Note that the amplitude is not singular when either α_0 or β_n vanishes. For $\alpha_0 = 0$ there is a zero in the numerator which arises from the fact that there is an integer $m = 2n$ such that $q_m = \beta_n$. A zero also arises from Π_{12} when $\beta_n = 0$. (An explicit form for A_n when $\alpha_0 = 0$ is given in the following subsection on the normal incidence case.)

In order to check property (d), write ψ_A from Eq. (2.5) in polar coordinates ($x - a = r\sin\theta \equiv r\alpha$, and $z = r\cos\theta \equiv r\beta$) about the edge $x = a$, $z = 0$. The radial derivative of ψ_i is bounded as $r \to 0$. The radial derivative of ψ_{sc} is, from Eq. (2.7),

$$\frac{\partial \psi_{sc}}{\partial r} = ik \sum_{n=-\infty}^{\infty} A_n(\alpha_n\alpha+\beta_n\beta)e^{ikr(\alpha_n\alpha+\beta_n\beta)+ik\alpha_n a}. \tag{3.15}$$

The sum in Eq. (3.15) can be bounded above and below as

$$-\sum \leqslant \frac{1}{ik}\frac{\partial \psi_{sc}}{\partial r} \leqslant \sum \tag{3.16}$$

where Σ is defined by

$$\sum \equiv \sum_{n=-\infty}^{\infty} |A_n||\alpha_n\alpha+\beta_n\beta|e^{-kr\beta\ \mathrm{Im}(\beta_n)}, \tag{3.17}$$

and Im means "imaginary part of." To show that $\Sigma = 0(r^{-1/2})$ as $r \to 0$, break Σ up into a sum for $n \geqslant 1$ (Σ_+), $n \leqslant -1$ (Σ_-), and $n = 0$ (which is bounded as $r \to 0$). For n large, $A_n = 0(n^{-3/2})$ from Eq. (3.14) and $\alpha_n \approx n\Lambda$, $\beta_n \approx i|n|\Lambda$ so that, *up to a bounded function*, Σ_+, e.g., behaves like the sum

$$\sum\nolimits_+' \equiv \sum_{n=1}^{\infty} \frac{e^{-k\beta\Lambda nr}}{n^{1/2}}. \tag{3.18}$$

Next, Σ_+' is bounded above and below by

$$\int_1^{\infty} \frac{e^{-(k\beta\Lambda r)n}}{n^{1/2}}\,dn \leqslant \Sigma_+' \leqslant \int_0^{\infty} \frac{e^{-(k\beta\Lambda r)n}}{n^{1/2}}\,dn. \tag{3.19}$$

The integral on the right-hand side of Eq. (3.19) is known (7) in terms of gamma functions as

$$\int_0^{\infty} \frac{e^{-(k\beta\Lambda r)n}}{n^{1/2}}\,dn = \frac{\Gamma(1/2)}{(k\beta\Lambda r)^{1/2}}. \tag{3.20}$$

The integral on the left-hand side of Eq. (3.19) can be written as the integral from 0 to ∞ [given by Eq. (3.20)] minus the integral from 0 to 1. The exponential in the latter integral can be expanded for small r and yields a bounded result as $r \to 0$. Hence as $r \to 0$, $\Sigma_+' = 0(r^{-1/2})$ by Eq. (3.20), and thus $\Sigma_+ = 0(r^{-1/2})$ also. Similar results can be found for Σ_-.

Hence $\Sigma = 0(r^{-1/2})$ as $r \to 0$ and, by Eq. (3.16), $\partial\psi_{sc}/\partial r = 0(r^{-1/2})$ as $r \to 0$, thus completing the proof. By a similar analysis it can easily be seen that if $g_1 \neq 0$, $\partial\psi_{sc}/\partial r = 0(r^{-3/2})$ as $r \to 0$, which is too singular.

For completeness, we also write the value of B_m from Eqs. (3.7) and (3.11) as

$$B_m = \frac{\Lambda p_m \beta_0 I_{0m}}{\pi q_m (\beta_0^2 - q_m^2)} \frac{\Pi(-q_m, q)}{\Pi(-\beta_0, q)} \frac{\Pi_{12}(-\beta_0, \beta)}{\Pi_{12}(-q_m, \beta)} e^{iH(\beta_0 - q_m)} \tag{3.21}$$

with I_{0m} given by Eq. (2.21) with $t = 1$.

Finally, the flux conservation relation of Sec. 2.3 becomes, for $t = 1$,

$$\sum_n |A_n|^2 (\beta_n/\beta_0) + \sum_m |B_m|^2 (q_m/2\beta_0) = 1, \tag{3.22}$$

the first term being the reflection coefficient and the second term the transmission coefficient. The sums are over those n and m such that β_n and q_m are real.

Normal Incidence ($\alpha_0 = 0$) and $t \neq 1$

The second case is for normal incidence on arbitrarily thick plates ($t \neq 1$) filled with an arbitrary inhomogeneity of wavenumber N and density ρ. For $\alpha_0 = 0$, Eqs. (2.17), (2.24), and (2.26) again simplify considerably. Since the incident field and the geometry of the problem (see Fig. 1) are symmetric about the $x = 0$ plane, we have $A_n = A_{-n}$. Also $\beta_n = \beta_{-n}$ since $\alpha_0 = 0$. The sums in Eqs. (2.17) can thus be written as

$$\sum_{n=-\infty}^{\infty} \frac{A_n I_{nm}}{\beta_n \pm q_m} = \frac{A_0 I_{0m}}{\beta_0 \pm q_m} + \sum_{n=1}^{\infty} \frac{A_n (I_{nm} + I_{-nm})}{\beta_n \pm q_m}. \tag{3.23}$$

From Eq. (2.18) it follows that, for $\alpha_0 = 0$,

$$I_{nm} + I_{-nm} = 2\cos(\pi n/t)\,(1-(-)^m) = 2\cos(\pi n/t)\, I_{0m}. \tag{3.24}$$

Using Eqs. (3.23) and (3.24), Eqs. (2.17) become

$$(1-(-)^m) \left(\sum_{n=0}^{\infty} \frac{A_n \epsilon_n \cos(\pi n/t)}{\beta_n \pm q_m} - \frac{1}{1 \mp q_m} \right) + \frac{2\pi q_m}{\Lambda t p_m} \begin{Bmatrix} B_m \\ 0 \end{Bmatrix} = 0 \tag{3.25}$$

where $\epsilon_0 = 1$ and $\epsilon_n = 2\,(n > 0)$. Thus for m even

$$B_m = 0 \ (m \text{ even}), \tag{3.26}$$

and for m odd, Eqs. (3.25) become

$$\sum_{n=0}^{\infty} \frac{A_n \epsilon_n \cos(\pi n/t)}{\beta_n \pm q_m} - \frac{1}{1 \mp q_m} + \frac{\pi q_m}{\Lambda t p_m} \begin{Bmatrix} B_m \\ 0 \end{Bmatrix} = 0. \tag{3.27}$$

Similarly, from Eq. (2.22), for $\alpha_0 = 0$,

$$J_{nm} + J_{-nm} = 2\cos(\pi n/t)\,(1-(-)^m) = 2\cos(\pi n/t)\, J_{0m}. \tag{3.28}$$

Using Eq. (3.28), Eqs. (2.26) and (2.24) can be rewritten, for m odd, as

$$\sum_{n=0}^{\infty} A_n \epsilon_n \cos(\pi n/t) \left(\frac{\sigma_m}{\beta_n + u_m} - \frac{1}{\beta_n - u_m} \right) - \left(\frac{\sigma_m}{1-u_m} - \frac{1}{1+u_m} \right) = 0 \tag{3.29}$$

and

$$\sum_{n=0}^{\infty} \frac{A_n \epsilon_n \cos(\pi n/t)}{\beta_n + u_m} - \frac{1}{1-u_m} + \frac{\pi}{4mu^2} (r_m + \rho u_m) C_m = 0. \tag{3.30}$$

For m even, Eq. (2.26) is identically satisfied and Eq. (2.24) implies that

$$C_m = 0 \ (m \text{ even}). \tag{3.31}$$

These equations have been reduced to a form similar to the linear equations arising in the problem of an inhomogeneously filled bifurcated waveguide as discussed by Mittra et al. (4), who also gave the modification of the residue calculus method necessary to solve the equations.

Consider the meromorphic function $F(\omega)$ with the following five properties:

a. simple poles at $\omega = \beta_n$ (n = 1,2,...) and $\omega = \pm\beta_0 = \pm 1$,

b. simple zeroes at $\omega = q_m$ (m = 1,3,5,...),

c. simple zeroes at $\omega = u'_m$ (m = 1,3,5,...), where the u'_m are shifted an amount δ_m from u_m

$$u'_m = u_m + \delta_m.$$

The δ_m shifts are calculated via an iteration procedure from the functional relation

$$\sigma_m F(-u_m) = F(u_m) \qquad (m = 1,3,5,\ldots). \tag{3.32}$$

In particular, the asymptotic value of these shifts

$$\delta = \lim_{m \to \infty} \delta_m$$

is important in properties (d) and (e) below. It is shown in App. C that

$$\epsilon = \delta/2iu = \pi^{-1}\sin^{-1}(\sigma/2)$$

where $\sigma = \lim_{m\to\infty} \sigma_m$. Thus a specific value of δ is known in terms of the parameters Λ, t, N, and ρ of the problem.

d. $F(\omega) = 0(\omega^{-(3/2)-\epsilon})$ as $|\omega| \to \infty$, and

e. $\psi = 0(r^{(1/2)+\epsilon})$ and $\partial\psi/\partial r = 0(r^{(1/2)+\epsilon})$ as an edge is approached. This is the edge behavior and depends on N and ρ through ϵ (see App. C).

Integrals of the form

$$\frac{1}{2\pi i} \int_C \frac{F(\omega)}{\omega \pm q_m} d\omega,$$

where C is a closed contour at infinity, yield residue series

$$\sum_{n=0}^{\infty} \frac{R(\beta_n)}{\beta_n \pm q_m} - \frac{R(-1)}{1 \mp q_m} + \left\{ \begin{matrix} F(-q_m) \\ 0 \end{matrix} \right\} = 0 \tag{3.33}$$

where $R(\beta)$ is the residue of $F(\omega)$ at $\omega = \beta$. These equations are similar to Eqs. (3.27) if we identify

$$A_n \epsilon_n \cos(\pi n/t) = R(\beta_n), \tag{3.34}$$

$$1 = R(-1), \tag{3.35}$$

and

$$B_m = \frac{\Lambda t p_m}{\pi q_m} F(-q_m) \qquad (m \text{ odd}). \tag{3.36}$$

Integrals of the form

$$\frac{1}{2\pi i} \int_C F(\omega) \left(\frac{\sigma_m}{\omega + u_m} - \frac{1}{\omega - u_m} \right) d\omega$$

yield residue series similar to Eq. (3.29) if we use Eqs. (3.32), (3.34), and (3.35). Finally, integrals of the form

$$\frac{1}{2\pi i} \int_C \frac{F(\omega)}{\omega + u_m} d\omega$$

yield residue series similar to Eq. (3.30) if we identify

$$C_m = \frac{4mu^2}{\pi(r_m+\rho u_m)} F(-u_m) \qquad (m \text{ odd}). \tag{3.37}$$

The function $F(\omega)$ satisfying properties (a), (b), and (c) can be written as

$$F(\omega) = \frac{G(\omega)}{\omega^2-1} \frac{\Pi_0(\omega,q)\Pi_0(\omega,u')}{\Pi_1(\omega,\beta)} \tag{3.38}$$

where $G(\omega)$ is an entire function to be determined, $\Pi_1(\omega,\beta)$ is given by Eq. (A.6) and is used to satisfy property (a), $\Pi_0(\omega,q)$ is given by Eq. (A.10) and is used to satisfy (b), and $\Pi_0(\omega,u')$ is given by Eq. (A.25) and is used to satisfy property (c). All the products are with $\alpha_0 = 0$. The asymptotic properties of these products (see App. A) yield, as $|\omega| \to \infty$ for $-3\pi/2 < \arg(\omega) < \pi/2$ (where $\arg(\omega) = \pi/2$ can be included as in the previous section,

$$F(\omega) \approx G(\omega)\, \omega^{-(3/2)-\epsilon}\, e^{-i\omega L} \tag{3.39}$$

where

$$\epsilon = \delta/2iu$$

and

$$L = [2t \ln(2)+(t-1)\ln(t-1)-t\ln(t)]/\Lambda t \tag{3.40}$$

and δ is defined in property (c). $F(\omega) \to 0$ as $|\omega| \to \infty$ if we choose the entire function as

$$G(\omega) = (G_0+G_1\omega)e^{i\omega L} \tag{3.41}$$

where G_0 and G_1 are constants. It can be shown by methods similar to those used in the previous section that if $G_1 \neq 0$, $\partial\psi/\partial r = 0(r^{-(3/2)+\epsilon})$ as $r \to 0$ (r is the distance to an edge). This result is too singular. Similarly, it can be shown that if $G_1 = 0$, $\partial\psi/\partial r = 0(r^{-(1/2)+\epsilon})$ as $r \to 0$. This is the usual edge condition. Thus, properties (d) and (e) are shown to hold, as well as the relationship between properties (c), (d), and (e).

Setting $G_1 = 0$ in Eq. (3.41) and using Eq. (3.35) to calculate G_0 yields

$$F(\omega) = \frac{2e^{iL(\omega+1)}}{1-\omega^2} \frac{\Pi_0(\omega,q)}{\Pi_0(-1,q)} \frac{\Pi_0(\omega,u')}{\Pi_0(-1,u')} \frac{\Pi_1(-1,\beta)}{\Pi_1(\omega,\beta)}. \tag{3.42}$$

Using the symmetry property

$$[\Pi_1(\omega,\beta)]^{-1} = \frac{\pi\sqrt{1-\omega^2}/\Lambda}{\sin(\pi\sqrt{1-\omega^2}/\Lambda)} \Pi_1(-\omega,\beta) \tag{3.43}$$

(which follows from Eq. (A.31) for $\alpha_0 = 0$), the residue $R(\beta_n)$ can easily be calculated using Eq. (3.42). Using Eq. (3.43) for $\omega = -1$ in the result yields, with Eq. (3.34),

$$A_n = \frac{(-)^{n+1} e^{iL(1+\beta_n)}}{\beta_n \cos(\pi n/t)} \frac{\Pi_0(\beta_n,q)}{\Pi_0(-1,q)} \frac{\Pi_0(\beta_n,u')}{\Pi_0(-1,u')} \frac{\Pi_1(-\beta_n,\beta)}{\Pi_1(1,\beta)}.$$

Finally, use of the symmetry condition Eq. (A.30) yields

$$A_n = \frac{(-)^{n+1} e^{iL(1+\beta_n)}}{\beta_n} \frac{\Pi_0(1,q)}{\Pi_0(-\beta_n,q)} \frac{\Pi_0(\beta_n,u')}{\Pi_0(-1,u')} \frac{\Pi_1(-\beta_n,\beta)}{\Pi_1(1,\beta)}. \tag{3.44}$$

Since $\alpha_0 = 0$, it is easily seen from Eq. (3.44) that $A_n = A_{-n}$. Also, using Eqs. (3.36), (3.37), and (3.42) we can write for (m odd)

$$B_m = \frac{2\Lambda t}{\pi p_m q_m} \frac{\Pi_0(-q_m,q)}{\Pi_0(-1,q)} \frac{\Pi_0(-q_m,u')}{\Pi_0(-1,u')} \frac{\Pi_1(-1,\beta)}{\Pi_1(-q_m,\beta)} e^{iL(1-q_m)} \tag{3.45}$$

and

$$C_m = \frac{8}{\pi m(r_m+\rho u_m)} \frac{\Pi_0(-u_m,q)}{\Pi_0(-1,q)} \frac{\Pi_0(-u_m,u')}{\Pi_0(-1,u')} \frac{\Pi_1(-1,\beta)}{\Pi_1(-u_m,\beta)} e^{iL(1-u_m)}. \tag{3.46}$$

Recall also that $B_m = C_m = 0$ for m even.

In the limit where $t = 1$, Eq. (3.43) yields

$$A_n|_{\substack{\alpha_0=0\\ t=1}} = \frac{-\Pi_0(\beta_n,q)\Pi_1(-\beta_n,\beta)}{\beta_n \Pi_0(-1,q)\Pi_1(1,\beta)} e^{2i \ln 2(1+\beta_n)/\Lambda}. \tag{3.47}$$

This is the same result as letting $\alpha_0 = 0$ in Eq. (3.14). (The latter derivation is accomplished by noting that, for $\alpha_0 = 0$ and $t = 1$, $\Pi_1(\omega,\beta) = \Pi_2(\omega,\beta) = \Pi_e(\omega,q)$ where $\Pi_e(\omega,q)$ is defined by Eq. (A.9). Finally, the use of symmetry properties in App. A completes the derivation.) A similar equality of the B_m amplitudes can be shown to hold by setting $t = 1$ in Eq. (3.45) or $\alpha_0 = 0$ in Eq. (3.21). For $t = 1$, C_m of course is zero. This follows directly from Eq. (3.46).

Finally we must calculate the δ_m shifts from property (c). The procedure is due to Mittra et al (4). Substituting Eq. (3.42) into Eq. (3.32) and manipulating the result yields

$$\frac{\Pi_0(u_m,u')}{\Pi_0(-u_m,u')} = \sigma_m \frac{\Pi_0(-u_m,q)\Pi_1(u_m,\beta)}{\Pi_0(u_m,q)\Pi_1(-u_m,\beta)} e^{-2iLu_m}. \tag{3.48}$$

Using Eq. (A.25) and the definition $u_m' = u_m + \delta_m$ from property (c) (recall that m is odd), the left-hand side (LHS) of Eq. (3.48) can be written as

$$\text{LHS} = \prod_{n=1}^{\infty} \frac{\delta_{2n-1} + u_{2n-1} - u_m}{\delta_{2n-1} + u_{2n-1} + u_m} e^{-2iu_m/[(2n-1)u]}. \tag{3.49}$$

Factoring the product in Eq. (3.49), substituting the result into Eq. (3.48), and rearranging terms yields

$$\frac{\delta_m{}^{(j+1)}}{\delta_m{}^{(j+1)}+2u_m}\, e^{-2iu_m/mu} \prod_{n=1}^{n_1-1} \frac{\delta_{2n-1}^{(j+1)}+u_{2n-1}-u_m}{\delta_{2n-1}^{(j+1)}+u_{2n-1}+u_m}\, e^{-2iu_m/(2n-1)u}$$

$$= \text{RHS} \times \prod_{n=n_1+1}^{\infty} \frac{\delta_{2n-1}^{(j)}+u_{2n-1}+u_m}{\delta_{2n-1}^{(j)}+u_{2n-1}-u_m}\, e^{2iu_m/(2n-1)u} \qquad (3.50)$$

where RHS stands for the right-hand side of Eq. (3.48) and $2n_1-1 = m$. Equation (3.50) is to be used as an iterative equation, and thus superscripts have been added to the δ_m terms. The procedure is as follows:

1. For large m (m odd), $\delta_m \approx \delta$. This is assumed to be the zeroth iteration and is substituted into the right-hand side of Eq. (3.50).

2. First iterations $\delta_1^{(1)}, \delta_3^{(1)}, \ldots, \delta_M^{(1)}$ are calculated sequentially up to an $m = M$ such that $|\delta-\delta_{M+2}^{(1)}|, |\delta-\delta_{M+4}^{(1)}|, \ldots,$ are zero to any desired accuracy. These latter terms, $\delta_{M+2}, \ldots,$ are set equal to δ throughout successive iterations. Further iterations are calculated only on the iteration set $\delta_1^{(1)}, \delta_3^{(1)}, \ldots, \delta_M^{(1)}$.

3. Higher iterations are calculated until Eq. (3.32) is satisfied to any desired accuracy.

This iterative procedure was used by both Mittra et al. (4) and the present author (8) and found to be rapidly converging. The chief advantage of the method is that it is not necessay to use matrix inversion to solve our original sets of linear equations. Specific numerical results for this problem will be presented elsewhere.

4. SUMMARY

It has been shown how to analytically solve for the scattered field when a plane wave is normally incident on a one-dimensional periodic corrugated surface consisting of alternate free and inhomogeneously filled parallel plates having soft boundary conditions. The solution resulted from the use of complex function techniques and an iterative procedure. Matrix inversion was avoided and, save for the normal incidence, there are no parameter restrictions. Plane and standing wave amplitude coefficients in the various geometric regions were expressed as residues or values of a constructed meromorphic function. Edge conditions on the field were derived, explicitly demonstrated, and shown to be related to the iterative procedure used. Equations (3.44-(3.46) are used to calculate the amplitude coefficients, and Eqs. (2.29) through (2.34) define the reflection and transmission coefficients.

There remain the numerical evaluation of the reflection and transmission coefficients and, using this, a discussion of the anomalies involved in scattering from these grating surfaces. We also wish to consider this problem with different boundary conditions. All these will be presented in future publications.

REFERENCES

1. Carlson, J.F., and Heins, A.E., *Quart. Appl. Math.*, 4:313 (1947), and Heins, A.E., and Carlson, J.F., 5:82 (1947).

2. Berz, F., *Proc. IEE (London)* 98(Part III):47 (1951).

3. Whitehead, E.A.N., *Proc. IEE (London)* 98(Part III):133 (1951).

4. Mittra, R., Lee, S.W., and Vanblaricum, G.F., *Int. J. Eng. Sci.* 6:395 (1968); also, R. Mittra and S.W. Lee, "*Analytical Techniques in the Theory of Guided Waves,*" New York: Macmillan, 1971.

5. Noble, B., "*Methods Based on the Wiener-Hopf Technique for the Solution of Partial Differential Equations,*" p. 174, New York: Pergamon, 1958.

6. Bouwkamp, C.J., *Rep. Progr. Phys.*, 17:35 (1954); also, M. Born and E. Wolf, "*Principles of Optics,*" 4th ed., New York: Pergamon Press, 1970.

7. Bateman Manuscript Project, "*Higher Transcendental Functions,*" Vol. 1, Ch. 1, ed. by A. Erdélyi, New York: McGraw-Hill, 1953).

8. DeSanto, J.A., "Scattering from a Periodic Corrugated Structure: Thin Comb with Soft Boundaries," *J. Math. Phys.*, 12:1913 (1971) and "Scattering from a Periodic Corrugated Structure II: Thin Comb with Hard Boundaries," *J. Math. Phys.*, (to be published March or April 1972).

APPENDIX A
PROPERTIES OF INFINITE PRODUCTS

Define the infinite product $\Pi(\omega)$ which vanishes at the positive integers as

$$\Pi(\omega) = \prod_{n=1}^{\infty} (1-\omega/n)e^{\omega/n}. \tag{A.1}$$

The exponential term is necessary to insure the absolute and uniform convergence of the product.* The product $\Pi(\omega)$ is related to the gamma function $\Gamma(\omega)$ via

$$\Pi(\omega) = -e^{\gamma\omega}/\omega\Gamma(-\omega), \tag{A.2}$$

which follows from Weierstrass's definition of $\Gamma(\omega)$, where γ is the Euler-Mascheroni constant.

The general form for products which vanish at points $\omega = D_m$ $(m = 1,2,3,...)$, where, for m large, $D_m \approx im\Delta$ with Δ a real positive quantity, is given by

$$\Pi(\omega,D) = \prod_{m=1}^{\infty} (1-\omega/D_m)(D_m/im\Delta)e^{\omega/im\Delta} \tag{A.3}$$

which is absolutely and uniformly convergent provided that the product $\prod_{m=1}^{\infty} (D_m/im\Delta)$ is. Dividing Eq. (A.3) term by term by a modified form of Eq. (A.1) given by $\Pi(\omega/i\Delta)$, we easily see that for large ω the result is unity, viz.,

$$\Pi(\omega,D) \approx \Pi(\omega/i\Delta). \tag{A.4}$$

Also, relating $\Pi(\omega/i\Delta)$ to the gamma function via Eq. (A.2) and using Stirling's approximation to the gamma function*, it is easily seen that, as $|\omega| \to \infty$ for $-3\pi/2 < \arg(\omega) < \pi/2$,

$$\Pi(\omega/i\Delta) \approx e^{-\pi i/4}(\Delta/2\pi\omega)^{1/2}\, e^{(\gamma+\ln\omega-\ln\Delta-1+\pi i/2)\omega/i\Delta}. \tag{A.5}$$

The domain $\arg(\omega) = \pi/2$ is discussed below.

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*Whittaker, E.T., and Watson, G.N., "*A Course of Modern Analysis,*" Cambridge:Cambridge Univ. Press, 4th ed., reprinted, 1962).

Some useful infinite products include the product which vanishes at $\omega = \beta_n$ ($n = 1,2,3,\ldots$)

$$\Pi_1(\omega,\beta) = \prod_{n=1}^{\infty} (1-\omega/\beta_n)(\beta_n/in\Lambda)e^{\omega/in\Lambda}, \tag{A.6}$$

the product which vanishes at $\omega = \beta_n$ ($n = -1,-2,-3,\ldots$)

$$\Pi_2(\omega,\beta) = \prod_{n=-1}^{-\infty} (1-\omega/\beta_n)(\beta_n/i|n|\Lambda)e^{\omega/i|n|\Lambda}$$

$$= \prod_{n=1}^{\infty} (1-\omega/\beta_{-n})(\beta_{-n}/in\Lambda)e^{\omega/in\Lambda}, \tag{A.7}$$

the product which vanishes at $\omega = q_m$ ($m = 1,2,3,\ldots$)

$$\Pi(\omega,q) = \prod_{m=1}^{\infty} (1-\omega/q_m)(2q_m/im\Lambda t)e^{2\omega/im\Lambda t}, \tag{A.8}$$

the product which vanishes at $\omega = q_m$ ($m = 2,4,6,\ldots,m$ even)

$$\Pi_e(\omega,q) = \prod_{m=1}^{\infty} (1-\omega/q_{2m})(q_{2m}/im\Lambda t)e^{\omega/im\Lambda t}, \tag{A.9}$$

and the product which vanishes at $\omega = q_m$ ($m = 1,3,5,\ldots,$ m odd)

$$\Pi_0(\omega,q) = \frac{\Pi(\omega,q)}{\Pi_e(\omega,q)} = \prod_{m=1}^{\infty} (1-\omega/q_{2m-1})(2q_{2m-1}/i(2m-1)\Lambda t) \times e^{2\omega/i(2m-1)\Lambda t}. \tag{A.10}$$

Note that for $\alpha_0 \neq 0$ the products $\Pi_1(\omega,\beta)$ and $\Pi_2(\omega,\beta)$ don't separately converge since the products $\prod_{n=1}^{\infty} (\beta_{\pm n}/in\Lambda)$ don't converge. The latter is true because of the α_0 factor in $\beta_{\pm n}$. However, $\Pi_{12}(\omega,\beta) = \Pi_1(\omega,\beta)\Pi_2(\omega,\beta)$ does converge since $\prod_{n=1}^{\infty} (-\beta_n\beta_{-n}/n^2\Lambda^2)$ does.

Following the derivation of Eq. (A.4) we write, for large ω,

$$\Pi_1(\omega,\beta) \approx \Pi_2(\omega,\beta) \approx \Pi(\omega/i\Lambda) \tag{A.11}$$

$$\Pi(\omega,q) \approx \Pi(2\omega/i\Lambda t) \tag{A.12}$$

$$\Pi_e(\omega,q) \approx \Pi(\omega/i\Lambda t) \tag{A.13}$$

$$\Pi_0(\omega,q) = \frac{\Pi(\omega,q)}{\Pi_e(\omega,q)} \approx \frac{\Pi(2\omega/i\Lambda t)}{\Pi(\omega/i\Lambda t)}. \tag{A.14}$$

Applying Eq. (A.5) to the right-hand sides of Eqs. (A.11), (A.12), and (A.14) yields, as $|\omega| \to \infty$, $-3\pi/2 < \arg(\omega) < \pi/2$

$$\Pi_1(\omega,\beta) \approx \Pi_2(\omega,\beta) \approx e^{-\pi i/4}(\Lambda/2\pi\omega)^{1/2} e^{(\gamma+\ln(\omega/\Lambda)-1+\pi i/2)(\omega/i\Lambda)} \tag{A.15}$$

$$\Pi(\omega,q) \approx e^{-\pi i/4}(\Lambda t/4\pi\omega)^{1/2} e^{(\gamma+\ln(2\omega/\Lambda t)-1+\pi i/2)(2\omega/i\Lambda t)} \tag{A.16}$$

$$\Pi_0(\omega,q) \approx 2^{-1/2} e^{(\gamma+\ln(4\omega/\Lambda t)-1+\pi i/2)(\omega/i\Lambda t)}. \tag{A.17}$$

A further useful product is the one which vanishes at $\omega = u'_m = u_m + \delta_m$ for $m = 1,3,5,\ldots$. In order to construct this product and find its asymptotic properties, first write the product and find its asymptotic properties, first write the product which vanishes at $\omega = u'_m = u_m + \delta_m$ for $m = 1,2,3,\ldots$, which is given by

$$\Pi(\omega,u') = \prod_{m=1}^{\infty} (1-\omega/u'_m)[u'_m/(imu+\delta)]e^{\omega/imu} \tag{A.18}$$

where, as $m \to \infty$, $u_m \approx imu$ and $\delta_m \to \delta$. Following Eq. (A.4), $\Pi(\omega,u')$ is, for large ω, asymptotic to the product

$$\Pi(\omega,\delta) = \prod_{m=1}^{\infty} \left(1 - \frac{\omega}{imu+\delta}\right) e^{\omega/imu} \tag{A.19}$$

which can be related to the gamma function* by

$$\Pi(\omega,\delta) = \frac{\Gamma(1-i\delta/u)e^{-i\gamma\omega/u}}{\Gamma[(1-i\delta/u)+i\omega/u]}. \tag{A.20}$$

Using the relation (Ref. 7)

$$\lim_{|z|\to\infty} \frac{\Gamma(z+\tau)}{z^{\tau}\Gamma(z)} = 1 \tag{A.21}$$

in Eq. (A.20) and combining the above results, we see that, for large ω and $-3\pi/2 < \arg(\omega) < \pi/2$,

$$\Pi(\omega,u') \approx \Pi(\omega,\delta) \approx (2\pi u)^{-1/2}(i/u)^{-1+i\delta/u}\Gamma(1-i\delta/u)e^{\pi i/4}$$
$$\times\, (\omega)^{-(1/2)+i\delta/u} e^{(\gamma+\ln(\omega/u)-1+\pi i/2)(\omega/iu)} \tag{A.22}$$

*See previous footnote.

where we have again used Stirling's approximation on the remaining gamma function. The product which vanishes at $\omega = u'_m = u_m + \delta_m$ for m even ($m = 2,4,6,\ldots$) is given by

$$\Pi_e(\omega,u') = \prod_{m=1}^{\infty} (1-\omega/u'_{2m})[u'_{2m}/(2imu+\delta)]e^{\omega/2imu}. \tag{A.23}$$

Its asymptotic value can be found directly from Eq. (A.22) by replacing u by $2u$, and for large ω is

$$\Pi_e(\omega,u') \approx (4\pi u)^{-1/2}(i/2u)^{-1+(i\delta/2u)}\Gamma(1-i\delta/2u)e^{\pi i/4}$$
$$\times\ (\omega)^{-(1/2)+i\delta/2u}e^{(\gamma+\ln(\omega/2u)-1+\pi i/2)(\omega/2iu)}. \tag{A.24}$$

Using the above results we can write the product which vanishes at $\omega = u'_m = u_m + \delta_m$ for m odd ($m = 1,3,5,\ldots$) as

$$\Pi_0(\omega,u') = \frac{\Pi(\omega,u')}{\Pi_e(\omega,u')} = \prod_{m=1}^{\infty} (1-\omega/u'_{2m-1})\ \{u'_{2m-1}/[(2m-1)iu+\delta]\}$$
$$\times\ e^{\omega/[(2m-1)iu]} \tag{A.25}$$

whose asymptotic value follows from Eqs. (A.22) and (A.24) and is, for $|\omega| \to \infty$ and $-3\pi/2 < \arg(\omega) < \pi/2$,

$$\Pi_0(\omega,u') \approx 2^{-(1/2)+i\delta/2u}(i/u)^{i\delta/2u}\Gamma(1-i\delta/u)\Gamma^{-1}(1-i\delta/2u)$$
$$\times\ (\omega)^{i\delta/2u}e^{(\gamma+\ln(2\omega/u)-1+\pi i/2)(\omega/2ui)}. \tag{A.26}$$

Note that the value of δ is the sole factor in the algebraic part of the asymptotic behavior of $\Pi_0(\omega,u')$. Thus shifting zeroes in an infinite product has the effect of changing the asymptotic algebraic behavior of the product. The connection of this result and the behavior of fields near an edge is discussed in App. C.

When evaluating residues, it is useful to have certain symmetry properties of the products. There can be derived from the relation

$$\Pi(\omega)\Pi(-\omega) = \prod_{n=1}^{\infty} [1-(\omega/n)^2] = \sin(\pi\omega)/(\pi\omega) \tag{A.27}$$

which follows from the duplication formula of the gamma function and from Eq. (A.2). The following symmetry properties can be derived in a straightforward way by multiplying the definitions of the products involved and using Eq. (A.27):

$$\Pi(\omega,q)\Pi(-\omega,q) = \sin(2\pi\sqrt{1-\omega^2}/\Lambda t)/(2\pi\sqrt{1-\omega^2}/\Lambda t) \tag{A.28}$$

$$\Pi_e(\omega,q)\Pi_e(-\omega,q) = \sin(\pi\sqrt{1-\omega^2}/\Lambda t)/(\pi\sqrt{1-\omega^2}/\Lambda t) \tag{A.29}$$

$$\Pi_0(\omega,q)\Pi_0(-\omega,q) = \cos(\pi\sqrt{1-\omega^2}/\Lambda t) \tag{A.30}$$

$$\Pi_{12}(\omega,\beta)\Pi_{12}(-\omega,\beta) = \frac{\sin(\pi(\alpha_0+\sqrt{1-\omega^2})/\Lambda)\sin(\pi(\alpha_0-\sqrt{1-\omega^2})/\Lambda)}{(\pi(\alpha_0+\sqrt{1-\omega^2})/\Lambda)(\pi(\alpha_0-\sqrt{1-\omega^2})/\Lambda)}. \tag{A.31}$$

The asymptotic properties of the infinite products in the region $\arg(\omega) = \pi/2$ can be found from these symmetry properties. For example, to find the asymptotic expansion of $\Pi(\omega,q)$ for $\arg(\omega) = \pi/2$, write Eq. (A.29) as

$$\Pi(\omega,q) = \frac{\sin(2\pi\sqrt{1-\omega^2}/\Lambda t)}{(2\pi\sqrt{1-\omega^2}/\Lambda t)\Pi(-\omega,q)} \tag{A.32}$$

and find the asymptotic value of $\Pi(-\omega,q)$. For large ω

$$\Pi(-\omega,q) \approx \Pi(-2\omega/i\Lambda t). \tag{A.33}$$

The latter product is related to the gamma function by using Eq. (A.2), and then applying Stirling's approximation to the gamma function. Stirling's approximation has its domain of validity rotated an angle π due to the minus sign in the product. Using Eq. (A.32) the result is, for $|\omega| \to \infty$ and $-\pi/2 < \arg(\omega) < 3\pi/2$,

$$\Pi(\omega,q) \approx \sin(2\pi\sqrt{1-\omega^2}/\Lambda t)\,(\Lambda t/\pi\omega)^{1/2}e^{-\pi i/4}$$

$$\times\, e^{(\gamma+\ln(2\omega/\Lambda t)-1-\pi i/2)\,(2\omega/i\Lambda t)}. \tag{A.34}$$

In particular, the expansion is valid for $\arg(\omega) = \pi/2$. Expansions of the other products in this region can be easily written down using the above procedure. When they are substituted into the meromorphic functions it can be easily seen that they don't affect the choices of the various entire functions.

APPENDIX B
DERIVATION OF THE EDGE CONDITION

We wish to derive the behavior of the velocity potential or field ψ in the neighborhood of a wedge of angle ϕ_1 for the case of a wedge with a soft boundary. We consider no density or wavenumber variations. The geometry is illustrated in Fig. B1. We must solve the Helmholtz equation

$$(\nabla^2 + k^2)\psi = 0 \tag{B.1}$$

within a boundary in the neighborhood of the wedge. The solution of Eq. (B.1) can be separated in polar coordinates (r,ϕ) as

$$\psi(r,\phi) = R(r)\,\Phi(\phi) \tag{B.2}$$

and when substituted back into Eq. (B.1) yields the equations

$$\frac{d^2R}{dr^2} + \frac{1}{r}\frac{dR}{dr} + \left(k^2 - \frac{s^2}{r^2}\right)R = 0 \tag{B.3}$$

and

$$\frac{d^2\Phi}{d\Phi^2} + s^2\Phi = 0 \tag{B.4}$$

where s^2 is a separation constant. The equations have solutions

$$R(r) = A\,J_s(kr) + B\,N_s(kr) \tag{B.5}$$

and

$$\Phi(\phi) = C\,\sin(s\phi) + D\,\cos(s\phi) \tag{B.6}$$

where J_s and N_s are respectively, the Bessel and Neumann functions of order s, and A, B, C, and D are constants.

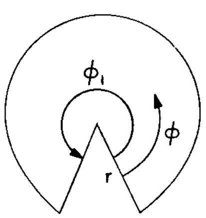

Fig. B1—The neighborhood of a wedge of angle ϕ_1 with polar coordinates r and ϕ.

The soft boundary condition is given by

$$\psi(r,0) = \psi(r,\phi_1) = 0 \tag{B.7}$$

which implies that $D = 0$ and $s = n\pi/\phi_1$, where n is any integer. The full solution of Eq. (B.1) can thus be written

$$\psi(r,\phi) = \sum_{n=-\infty}^{\infty} \sin(n\pi\phi/\phi_1)\ [C_n J_{n\pi/\phi_1}(kr) + D_n N_{n\pi/\phi_1}(kr)] \tag{B.8}$$

where C_n and D_n are now constants. An additional result independent of ϕ must be satisfied and is given by

$$\lim_{r\to 0} \psi(r,\phi) = 0 \tag{B.9}$$

which implies that $D_n = 0$ (all n) and $C_n = 0$ ($n = -1,-2,\ldots$). Thus Eq. (B.8) becomes

$$\psi(r,\phi) = \sum_{n=1}^{\infty} C_n \sin(n\pi\phi/\phi_1)\, J_{n\pi/\phi_1}(kr). \tag{B.10}$$

Using properties of Bessel functions it is easily seen that, as $r \to 0$,

$$\psi(r,\phi) = 0(r^{\pi/\phi_1})$$

which follows from the leading term ($n = 1$) in the series. For an infinitely sharp edge ($\phi_1 = 2\pi$), $\psi = 0(r^{1/2})$ and $\partial\psi/\partial r = 0(r^{-1/2})$. This was the case for arbitrary α_0 in Sec. 3, and also for $\alpha_0 = 0$, N finite, and $\rho = 1$ in Sec. 3. For $N = \infty$ (impenetrably thick plate, $\phi_1 = 3\pi/2$), $\psi = 0(r^{2/3})$ and $\partial\psi/\partial r = 0(r^{-1/3})$. This was also a special case in Sec. 3 ($\alpha_0 = 0$) for $\epsilon = 1/6$.

Note in particular that as long as N is finite and $\rho = 1$, the geometric edge is infinitely sharp ($\phi_1 = 2\pi$). Finally note that we have shown that the edge conditions follow from the boundary conditions.

APPENDIX C
EDGE BEHAVIOR AND ZERO SHIFTS

In App. A we pointed out the asymptotic behavior of infinite products which depend on shifted zeros. The corresponding behavior of the function $F(\omega)$ was pointed out in Eq. (3.39). The edge conditions were derived in App. B for N finite (infinitely sharp edge) and N infinite (impenetrably thick plate and right-angle wedge), and it was shown in Sec. 3 (arbitrary α_0) how the radial derivative of a field ψ explicitly satisfies the edge condition. There remains the question of tying all this together and showing the how and why of the choice of δ—the asymptotic value of the δ_m zero shifts.

Using Eqs. (3.39) and (3.41) and the fact that $G_1 = 0$, we have, for large ω,

$$F(\omega) \approx G_0 \omega^{-(3/2)-\epsilon} \tag{C.1}$$

with $\epsilon = \delta/2iu$.

From Eq. (3.34) and the remark that the residue function has the same asymptotic algebraic behavior as the function (the proof of this remark is in App. D), we see that for large n

$$A_n = 0(n^{-(3/2)-\epsilon}) \tag{C.2}$$

which follows from the fact that, for n large, $\beta_n \approx i|n|\Lambda$.

We wish to show how ϵ is related to the edge behavior by using the ψ_A field from Eq. (2.5). Since the radial derivative of ψ_i is bounded near an edge, the edge behavior arises from ψ_{sc}. As in Sec. 3, write ψ_{sc} in polar coordinates and take its radial derivative. The latter is bounded

$$-\Sigma \leqslant (ik)^{-1}\partial\psi_{sc}/\partial r \leqslant \Sigma \tag{3.16}$$

where

$$\Sigma = \sum_{n=-\infty}^{\infty} |A_n||\alpha_n\alpha+\beta_n\beta|e^{-kr\beta \mathrm{Im}(\beta_n)}. \tag{3.17}$$

Again break Σ up into three parts: the $n = 0$ term, and sums from 1 to ∞ (Σ_+) and -1 to $-\infty$ (Σ_-). Using Eq. (C.2) and large n values of α_n and α_n, Σ_+ behaves, up to a bounded function, like the sum defined by

$$\Sigma'_+ \equiv \sum_{n=1}^{\infty} n^{-(1/2)-\epsilon} e^{-k\beta\Lambda nr} \tag{C.3}$$

$$= \Gamma(1/2)-\epsilon)(kr\beta\Lambda)^{-(1/2)+\epsilon} \tag{C.4}$$

so that, as $r \to 0$, $\Sigma_+ = 0(r^{-(1/2)+\epsilon})$. Σ_- behaves in a similar way, and the $n = 0$ term is bounded. Thus, as $r \to 0$, by Eq. (3.16) and the fact that $\partial\psi_i/\partial r$ is bounded we have

$$\frac{\partial\psi_A}{\partial r} = 0(r^{-(1/2)+\epsilon}). \tag{C.5}$$

Thus it is seen how ϵ is connected to the edge behavior of the fields.

We must next derive a specific value for ϵ in terms of the other parameters of the problem. This follows from Eq. (3.32) which we write, using Eq. (3.42) and recalling that m is odd, as

$$\sigma_m e^{-2iLu_m} \frac{\Pi_0(-u_m,q)}{\Pi_0(u_m,q)} \frac{\Pi_1(u_m,\beta)}{\Pi_1(-u_m,\beta)} = \frac{\Pi_0(u_m,u')}{\Pi_0(-u_m,u')}. \tag{C.6}$$

Using Eqs. (A.30) and (A.31), the latter with $\alpha_0 = 0$ and $\Pi_1 = \Pi_2$, the left-hand side (LHS) of Eq. (C.6) can be written

$$\text{LHS} = \sigma_m e^{-2iLu_m} \left(\frac{\Pi_0(-u_m,q)}{\Pi_1(-u_m,\beta)}\right)^2 \frac{\sin(\pi mu/\Lambda)}{(\pi mu/\Lambda)\cos(\pi mu/\Lambda t)}. \tag{C.7}$$

From App. A we can write, for m large,

$$\Pi_0(-u_m,q) \approx \frac{\Pi(-2mu/\Lambda t)}{\Pi(-mu/\Lambda t)} \approx 2^{-1/2}\exp\left\{\frac{-m}{2(t-1)}\,(\gamma-1+\ln[2m/(t-1)]\right\}$$

and

$$\Pi_1(-u_m,\beta) \approx \Pi(-mu/\Lambda) \approx (\Lambda/2\pi mu)^{1/2}\exp\left\{\frac{-mt}{2(t-1)}\,(\gamma-1+\ln[mt/2(t-1)]\right\}.$$

Thus Eq. (C.7) is, for m large,

$$\text{LHS} \approx \sigma\,\frac{\sin(\pi mu/\Lambda)}{\cos(\pi mu/\Lambda t)}\,e^{m[\gamma-1+\ln(2m)]}$$

$$= \sigma\,\sin(m\pi/2)e^{[\gamma-1+\ln(2m)]} \tag{C.8}$$

where

$$\sigma = \lim_{m\to\infty} \sigma_m = \left\{ \begin{array}{ll} 0, & \rho = 1, N \text{ finite} \\ -1, & \rho = \infty, N \text{ finite} \\ 1, & \rho = \text{finite}, N = \infty \\ \dfrac{1-\rho}{1+p} & \rho \text{ and } N \text{ are finite} \end{array} \right\}. \tag{C.9}$$

To evaluate the right-hand side (RHS) of Eq. (C.6) first write

$$\text{RHS} = \frac{\Pi(u_m,u')}{\Pi(-u_m,u')}\,\frac{\Pi_e(-u_m,u')}{\Pi_e(u_m,u')}. \tag{C.10}$$

Substituting explicit forms for these products, we can for example write, for m large,

$$\frac{\Pi(u_m,u')}{\Pi(-u_m,u')} \approx \frac{\prod_{n=1}^{\infty}\,[1-m/(n+2\epsilon)]\,e^{m/n}}{\prod_{n=1}^{\infty}\,[1+m/(n+2\epsilon)]\,e^{-m/n}} = \frac{\Pi(m-2\epsilon)}{\Pi[-(m+2\epsilon)]} \tag{C.11}$$

where $2\epsilon = \delta/iu$. Equation (C.11) can be rewritten as

$$\frac{\Pi(m-2\epsilon)}{\Pi[-(m+2\epsilon)]} = \frac{\Pi(m-2\epsilon)\,\Pi[-(m-2\epsilon)]}{\Pi[-(m+2\epsilon)]\,\Pi[-(m-2\epsilon)]} = \frac{\sin[\pi(m-2\epsilon)]}{\pi(m-2\epsilon)}\,\frac{1}{\Pi[-(m+2\epsilon)]\,\Pi[-(m-2\epsilon)]}$$

$$\approx 2\sin[\pi(m-2\epsilon)]\,e^{2m[\gamma-1+\ln(m)]} \tag{C.12}$$

where the final form follows for m large. The ratio of Π_e products follows by replacing m by $m/2$ and ϵ by $\epsilon/2$ in Eq. (C.12). Thus we can write

$$\frac{\Pi_e(u_m,u')}{\Pi_e(-u_m,u')} \approx 2\sin[\pi(m/2-\epsilon)]\,e^{m[\gamma-1+\ln(m/2)]}. \tag{C.13}$$

Using these results, Eq. (C.10) can be written, for large m, as

$$\text{RHS} \approx 2\sin(\pi m/2)\,\sin(\pi\epsilon)e^{m[\gamma-1+\ln(2m)]} \tag{C.14}$$

Combining Eqs. (C.8) and (C.14) yields

$$\epsilon = \pi^{-1}\sin^{-1}(\sigma/2) = \left\{ \begin{array}{ll} 0, & \rho = 1, N \text{ finite} \\ -1/6, & \rho = \infty, N \text{ finite} \\ +1/6, & \rho = \text{finite}, N = \infty \\ \pi^{-1}\sin^{-1}\left(\dfrac{1}{2}\dfrac{1-\rho}{1+\rho}\right), & \rho \text{ and } N \text{ finite} \end{array} \right\}. \tag{C.15}$$

This is the explicit form for ϵ which we presented in the second part of Sec. 3 and which was associated with the edge condition.

APPENDIX D
ALGEBRAIC BEHAVIOR OF THE RESIDUE FUNCTION

In Appendix C it was stated that the residue function has the same asymptotic algebraic behavior as the function. This is proved here. For simplicity we set $\epsilon = 0$. From Eq. (A.5) for large ω (neglecting the exponential behavior) we have

$$\Pi(\omega)^{-1} = 0(\omega^{1/2}). \tag{D.1}$$

The algebraic (large m) behavior of the residue function defined by

$$R(m) = \lim_{\omega \to m} [(\omega - m)/\Pi(\omega)] \tag{D.2}$$

is found by substituting Eq. (A.27) into Eq. (D.2) to get

$$R(m) = m\pi\Pi(-m) \lim_{\omega \to m} [(\omega - m)/\sin(\pi\omega)]$$

$$= (-)^m m\Pi(-m) \tag{D.3}$$

and then using Eq. (D.1) on $\Pi(-m)$ for large m to yield

$$R(m) = 0(m^{1/2}). \tag{D.4}$$

This procedure is easily generalized to the full functions $f(\omega)$ and $F(\omega)$ and their residues and thus, e.g., we have from Eqs. (3.5) and (3.9) that, for n large,

$$A_n \approx r(\beta_n) \approx f(\beta_n) \approx n^{-3/2}. \tag{D.5}$$

This completes the proof of the remark.

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13. ABSTRACT

An incident plane wave is scattered from a surface which is corrugated in one dimension and which is described by an infinite number of periodically spaced semi-infinite parallel plates (comb) having soft boundary conditions. An additional plate is placed between each set of adjacent plates, thus dividing the periodicity interval into two regions, one of which doesn't differ in its properties from the region above the plates, while the second region is inhomogeneously filled. The latter means that this region differs in wavenumber and density from the surrounding media. Both the wavenumber and the density are here assumed to be constant. The solutions of the Helmholtz equation are assumed to be upgoing plane waves above the plates and, between the plates, standing waves along the periodicity direction and downgoing waves along the plate slots. The solutions have unknown amplitude coefficients. Continuity of pressure and velocity across the common boundary yield linear equations relating the amplitudes in the various regions. The latter are shown to be similar to the residue series of integrals of certain meromorphic functions. The amplitudes are expressed as values or residues of these functions, which are explicitly constructed. The two examples treated in detail are (a) zero-thickness plates with arbitrary incident angle, and (b) arbitrarily thick (inhomogeneous) plates at normal incidence.

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