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Analytic Simplification of a System of Ordinary Differential Equations at an Irregular-Type Singularity

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Abstract: Let $\mathbf{1}_n(\mu) = \mathrm{diag}\,(\mu_1, \ldots, \mu_n)$ for given complex μ_k. If $\mathrm{Re}\ \mu_k \geqslant 0$ $(1 \leqslant k \leqslant n)$, then the $(m+n)$-system $x^{\sigma+1}y' = F(x,z)y$, $xz' = \mathbf{1}_n(\mu)z$ is simplified to $x^{\sigma+1}Y' = G(x,Z)Y$, $xZ' = \mathbf{1}_n(\mu)Z$ by a transformation T defined as $y = Y + P(x,Z)Y$, $z = Z$ in a sector having property-$\mathcal{T}$ with respect to $\left\{(\lambda_i - \lambda_j)(\sigma x^\sigma)^{-1} \,|\, i,j=1,\ldots,s\ (i \neq j)\right\}$, where λ_i $(i=1,2,\ldots,s)$ are distinct eigenvalues of $F(0,0)$ and $G(x,Z)$ is in block-diagonal form agreeing with the Jordan canonical form of $F(0,0)$.

I. INTRODUCTION

1. Singularity of Nonlinear Equations

For a system of nonlinear differential equations given by

$$xw' = h(x,w) \tag{1.1}$$

where $w' = d/dx$, w and h are s-column vectors, x is a complex variable, the quantity $h(0,0) = 0$, and every component of $h(x,w)$ is holomorphic at $(0,0)$, the singular point $x = 0$ is said to be of the Briot-Bouquet type. Since the work of C. C. A. Briot and J. C. Bouquet [1] in 1856, many authors, including H. Dulac, E. Picard, H. Poincaré, P. Painlevé, J. Malmquist, and M. Hukuhara and T. Kimura (cf. [2]) have devoted study to this type of singularity. Recently, M. Iwano published a series of papers [3-8] devising a method to find general solutions of Eq. (1.1) when the matrix $h_w(0,0)$ is singular, and particularly in the form

$$h_w(0,0) = \begin{pmatrix} 0 & 0 \\ 0 & H \end{pmatrix},$$

where H is a nonsingular matrix whose size is smaller than s. In doing this, he encountered a system of equations of the form

$$x^{\sigma+1}y' = f(x,y,z), \qquad xz' = g(x,y,z) \tag{1.2}$$

with irregular-type singularity, where y and f are m-column vectors, z and g are n-column vectors, σ is a positive integer, f and g are holomorphic in a neighborhood of $(0,0,0)$, $f(0,0,0) = 0$ and $g(0,0,0) = 0$,

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$A = f_y(0,0,0)$ is nonsingular, $g_y(0,0,0) = 0$ and, furthermore, $g_z(0,0,0) = \operatorname{diag}(\mu_1,\mu_2,\ldots,\mu_n)$, with $\operatorname{Re}\mu_k > 0$ for $k = 1,2,\ldots,n$. It should be noted that $f_z(0,0,0) = 0$, $f_x(0,0,0) = 0$ and $g_y(0,0,0) = 0$ are nonrestrictive conditions under the assumption that A is nonsingular. However, one of the assumptions Iwano [6] imposed is that the eigenvalues of A are mutually distinct. In order to relax this assumption, it is necessary to reduce $f_y(x,0,z)$ to the simplest form. This is the purpose of the present paper; we want to reduce $f_y(x,0,z)$ to a block-diagonal form such that the simplified matrix coincides with the Jordan canonical form of A at $x = 0$, $z = 0$.

2. Notations and Definitions

In order to simplify expressions we introduce several notations and definitions.

The m by m unit-matrix is denoted by $\mathbf{1}_m$. For an m-column vector y with elements $\{y_j\}$, $\mathbf{1}_m(y)$ denotes an m by m diagonal matrix with diagonal elements $\{y_j\}$.

If u is an m-column vector with elements $\{u_j\}$, $[u]$ denotes an m-column vector with elements $\{|u_j|\}$. For another m-column vector $\tilde{u}$ with elements $\{\tilde{u}_j\}$, $[u] \leqslant [\tilde{u}]$ means that $|u_j| \leqslant |\tilde{u}_j|$ for each index j.

For an m-row vector $q = (q_1,q_2,\ldots,q_m)$, the components are all nonnegative integers and we define

$$|q| = q_1 + \ldots + q_m. \tag{2.1}$$

For an m-column vector y with elements $\{y_j\}$, the symbol y^q stands for the scalar quantity

$$y^q = y_1^{q_1} \ldots y_m^{q_m}. \tag{2.2}$$

The norm of an m-column vector y with elements $\{y_j\}$ is

$$\|y\| = \max_{j=1}^{m} |y_j|. \tag{2.3}$$

For a scalar t and an m-row vector y with elements $\{y_j\}$

$$t^y = (t^{y_1},\ldots,t^{y_m}), \tag{2.4}$$

$$\exp y = (\exp y_1,\ldots,\exp y_m) \tag{2.5}$$

and

$$\operatorname{Re} y = (\operatorname{Re} y_1,\ldots,\operatorname{Re} y_m), \qquad \operatorname{Im} y = (\operatorname{Im} y_1,\ldots,\operatorname{Im} y_m). \tag{2.6}$$

If y is a column vector, then t^y, $\exp y$, $\operatorname{Re} y$, and $\operatorname{Im} y$ are all column vectors.

For an m-column vector y with elements $\{y_j\}$ and an n-column vector function $f(x,y)$ with elements $\{f_j(x,y)\}$, the notation $f_y(x,y)$ denotes an n by m matrix given by

$$f_y(x,y) = \left(\frac{\partial}{\partial y_1} f(x,y),\ldots,\frac{\partial}{\partial y_m} f(x,y)\right). \tag{2.7}$$

A function $f(x)$ which is holomorphic and bounded in x for

$$0 < |x| < a, \qquad \underline{\Theta} < \arg x < \overline{\Theta} \tag{2.8}$$

where a, $\underline{\Theta}$, and $\overline{\Theta}$ are given constants and admits an asymptotic expansion in powers of x as x tends to 0 is said to *belong to class* $\mathcal{C}(\underline{\Theta},\overline{\Theta},a)$.

A vector $f(x,y,z)$ which is holomorphic in (x,y,z) for

$$0 < |x| < a, \qquad \underline{\Theta} < \arg x < \overline{\Theta}, \qquad \|y\| < b, \qquad \|z\| < c \tag{2.9}$$

is said to have Property-$\mathfrak{U}$ with respect to y and z in Eq. (2.9) if its components admit uniformly convergent expansions in powers of y and z for Eq. (2.9) and if the coefficients of these expansions belong to class $\mathcal{C}(\underline{\Theta},\overline{\Theta},a)$.

Given a finite number of monomials of x^{-1} of the same degree σ

$$\Omega_j(x) = -\frac{\gamma_j}{\sigma x^{\sigma}} \qquad (j = 1, 2, \ldots, M),$$

the sectors of the form

$$\frac{1}{\sigma}\left(\arg \gamma_j - \frac{\pi}{2} + 2\pi h_j\right) < \arg x < \frac{1}{\sigma}\left(\arg \gamma_j + \frac{\pi}{2} + 2\pi h_j\right) \tag{2.10}$$

and

$$\frac{1}{\sigma}\left(\arg \gamma_j + \frac{\pi}{2} + 2\pi h_j'\right) < \arg x < \frac{1}{\sigma}\left(\arg \gamma_j + \frac{3\pi}{2} + 2\pi h_j'\right) \tag{2.11}$$

where h_j and h_j' are integers, are said to be a *maximal negative region* of $\Omega_j(x)$ and a *maximal positive region* of $\Omega_j(x)$, respectively, indicating the sign of the quantity Re $\Omega(x)$ in these regions.

A sector $\underline{\Theta} < \arg x < \overline{\Theta}$ is said to have Property-$\mathfrak{T}$ with respect to monomials $\{\Omega_1(x), \ldots, \Omega_m(x)\}$ if this sector does not contain any maximal negative region of $\Omega_j(x)$ for each index j and if in this sector there is a direction for each index j such that, as x tends to the origin along this direction, exp (Re $\Omega_j(x)$) tends to infinity exponentially.

3. Main Theorem

Consider a system of differential equations

$$x^{\sigma+1}y' = F(x,z)y, \qquad xz' = \mathbf{1}_n(\mu)z \tag{E$_1$}$$

where y is an m-column vector, μ and z are n-column vectors, with constant elements $\{\mu_k\}$ for μ, σ is a positive integer, and $F(x,z)$ is an m by m matrix holomorphic in (x,z) with Property-$\mathfrak{U}$ with respect to z in

$$0 < |x| < a, \qquad \underline{\Theta} < \arg x < \overline{\Theta}, \qquad \|z\| < c \tag{3.1}$$

with a, $\underline{\Theta}$, $\overline{\Theta}$, and c as given constants.

Let F^0 denote the matrix

$$F^0 = \lim_{x\to 0} F(x,0), \text{ for } \underline{\Theta} < \arg x < \overline{\Theta}.$$

We further assume

(i) *the matrix* F^0 *has distinct eigenvalues* $\lambda_1, \lambda_2, \ldots, \lambda_s$ *with multiplicity* $m_1, m_2, \ldots, m_s$, *respectively* ($m_1 + m_2 + \ldots + m_s = m$);

(ii) *every component of* μ *satisfies*

$$\operatorname{Re} \mu_k \geqslant 0 \qquad (k = 1, 2, \ldots, n). \tag{3.2}$$

By assumption (i) we further assume, without loss of generality, that

$$F^0 = \operatorname{diag}(F_1^0, F_2^0, \ldots, F_s^0) \tag{3.3}$$

where F_j^0 is an m_j by m_j matrix of the form

$$F_j^0 = \lambda_j 1_{m_j} + D_j \tag{3.4}$$

with

$$D_j = \begin{pmatrix} 0 & & & & \bigcirc \\ \delta_{j2} & \ddots & & & \\ & \ddots & \ddots & & \\ & & \ddots & \ddots & \\ \bigcirc & & & \delta_{jm_j} & 0 \end{pmatrix} \qquad (\delta_{ji} = 0 \text{ or } 1).$$

Put

$$\Lambda_{ij}(x) = -\frac{\lambda_i - \lambda_j}{\sigma x^\sigma} \qquad (i, j = 1, 2, \ldots, s; i \neq j). \tag{3.5}$$

Then, the main theorem is

THEOREM M. *Assume that (i) and (ii) are satisfied and that the sector* $\underline{\Theta} < \arg x < \overline{\Theta}$ *has Property-*$\mathfrak{T}$ *with respect to* $\{\Lambda_{ij}(x)|, i,j=1,2,\ldots,s; i\neq j\}$. *Then there exists a transformation* T *defined as*

$$x^{\sigma+1}Y' = Y + P(x,Z)Y, \qquad z = Z \tag{T}$$

such that

(i) *Eq.* (E_1) *is reduced to*

$$x^{\sigma+1}Y' = G(x,Z)Y, \qquad xZ' = \mathbf{1}_n(\mu)Z \tag{E$_2$}$$

where $G(x,Z)$ *is an* m *by* m *block-diagonal matrix of the form*

$$G(x,Z) = \operatorname{diag}(G_1(x,Z), G_2(x,Z), \ldots, G_s(x,Z)) \tag{3.6}$$

with the $G_j(x,Z)$ *being* m_j *by* m_j *matrices; and*

(ii) *the entries of* $P(x,Z)$ *and* $G(x,Z)$ *are holomorphic in* (x,Z), *have Property-*$\mathfrak{U}$ *with respect to* Z *in*

$$0 < |x| < a_0, \qquad \underline{\Theta} < \arg x < \overline{\Theta}, \qquad ||Z|| < c_0, \tag{3.7}$$

where $0 < a_0 \leqslant a,\ 0 < c_0 \leqslant c$, *and satisfy*

$$P(x,0) = 0 \text{ and } \lim_{x \to 0} G(x,0) = F^0, \text{ for } \underline{\Theta} < \arg x < \overline{\Theta}. \tag{3.8}$$

Assumption (ii) includes the case where the vector $\mu = 0$, namely, z is a parameter independent of x. In this case, Theorem M includes the special cases of the simplification of equations containing parameters studied by W. J. Trijitzinski [9], M. Hukuhara [10], H. L. Turrittin [11], Y. Sibuya [12], and P. F. Hsieh [13].

Chapter II will be devoted to preliminary algorithm, reducing the proof of Theorem M to two types of nonlinear differential equations. In order to find the solutions of these equations, two fundamental existence theorems are needed. These theorems, Theorem A and Theorem B, will be stated and proved in Chapters III through V.

Theorem A and Theorem B resemble two theorems proved by M. Iwano [7] using a fixed-point theory devised by M. Hukuhara. Recently, they were proved by P. F. Hsieh [14] using the successive-approximations method. The first theorem actually is in simpler form than earlier results. However, because of the complication and also the resemblance in the proof of Theorems A and B, the sketch of the proof of Theorem A will be given in Chapter III. Theorem B is a refinement of earlier result, due to the fact that assumption (ii) is broader than earlier assumptions.

II. PROOF OF MAIN THEOREM AND NONLINEAR EQUATIONS

4. The Leading Term

Since the matrix $F(x,z)$ has Property-$\mathfrak{U}$ with respect to z, it can be expanded into a uniformly convergent series of the form

$$F(x,z) = F_0(x) + \sum_{|q|=1}^{\infty} z^q F_q(x) \tag{4.1}$$

for x, z in Eq. (3.1), where $F_0(x)$ and $F_q(x)$ are m by m matrices holomorphic in

$$0 < |x| < a, \qquad \underline{\Theta} < \arg c < \overline{\Theta} \tag{4.2}$$

and belong to $\mathcal{C}(\underline{\Theta}, \overline{\Theta}, a)$.

As the first step in proving Theorem M, it is necessary to block-diagonalize $F_0(x)$ according to the form of Eq. (3.3). This process itself resembles the proof of the Main Theorem.

LEMMA 1. *Given a system of m equations*

$$x^{\sigma+1} y' = F_0(x) y \tag{4.3}$$

where $F_0(x)$ *is an m by m matrix holomorphic in Eq. (4.2), belongs to* $\mathcal{C}(\underline{\Theta}, \overline{\Theta}, a)$, *and satisfies*

$$\lim_{x \to 0} F_0(x) = F^0, \text{ for } \underline{\Theta} < \arg x < \overline{\Theta}. \tag{4.4}$$

Then, there exists a transformation

$$y = \eta + Q(x)\eta \tag{4.5}$$

such that the system of Eq. (4.3) is reduced to

$$x^{\sigma+1}\eta' = \left(F^0 + \tilde{G}_0(x)\right)\eta \tag{4.6}$$

where $Q(x)$ and $\hat{G}_0(x)$ are m by m matrices holomorphic in

$$0 < |x| < a', \text{ and } \underline{\Theta} < \arg x < \overline{\Theta}, \text{ for } 0 < a' \leqslant a, \tag{4.7}$$

belong to $\mathcal{C}(\underline{\Theta},\overline{\Theta},a')$, and satisfy

$$\lim_{x\to 0} Q(x) = 0 \text{ and } \lim_{x\to 0} \tilde{G}_0(x) = 0, \text{ for } \underline{\Theta} < \arg x < \overline{\Theta}. \tag{4.8}$$

Further, $G_0(x)$ is in the same block-diagonal form as that in Eq. (3.3).

In order to prove this lemma, put

$$F_0(x) = F^0 + \tilde{F}_0(x). \tag{4.9}$$

Then $\tilde{F}_0(x)$ is holomorphic in Eq. (4.2), belongs to $\mathcal{C}(\underline{\Theta},\overline{\Theta},a)$, and

$$\lim_{x\to 0} \tilde{F}_0(x) = 0, \text{ for } \underline{\Theta} < \arg x < \overline{\Theta}. \tag{4.10}$$

From Eqs. (4.3), (4.5), and (4.6), we have

$$x^{\sigma+1}Q' = F_0(x)\left(1_m + Q(x)\right) - \left(1_m + Q(x)\right)\left(F^0 + \tilde{G}_0(x)\right).$$

Or, by Eq. (4.9),

$$x^{\sigma+1}Q' = (F^0Q - QF^0) + \left(\tilde{F}_0(x) - \tilde{G}_0(x)\right) + \left(\tilde{F}_0(x)Q - Q\tilde{G}_0(x)\right). \tag{4.11}$$

In order to find $Q(x)$ and $\tilde{G}_0(x)$ satisfying Eq. (4.11) and the properties described in Lemma 1, let us denote $\tilde{F}_0(x)$, $P(x)$, and $\tilde{G}_0(x)$ in block forms according to that of F^0 as shown in Eq. (3.3). Due to the first formula of Eq. (3.8), it is sufficient to find $Q(x)$ in off-block-diagonal form. Thus, put

$$\tilde{F}_0(x) = \begin{pmatrix} F^0_{11}(x) & F^0_{12}(x) & \dots & F^0_{1s}(x) \\ F^0_{21}(x) & F^0_{22}(x) & \dots & F^0_{2s}(x) \\ \vdots & \vdots & & \vdots \\ F^0_{s1}(x) & F^0_{s2}(x) & \dots & F^0_{ss}(x) \end{pmatrix},$$

$$\tilde{G}_0(x) = \begin{pmatrix} G_1{}^0(x) & & & \bigcirc \\ & G_2{}^0(x) & & \\ & & \ddots & \\ \bigcirc & & & G_s{}^0(x) \end{pmatrix}, \tag{4.12}$$

and

$$Q(x) = \begin{pmatrix} 0 & Q_{12}(x) & \dots & Q_{1s}(x) \\ Q_{21}(x) & 0 & \dots & Q_{2s}(x) \\ \vdots & \vdots & & \vdots \\ Q_{s1}(x) & Q_{s2}(x) & \dots & 0 \end{pmatrix}$$

where $F_{ij}^0(x)$ and $Q_{ij}(x)$ are m_i by m_j matrices, and $G_i^0(x)$ are m_i by m_i matrices. Then the $F_{ij}^0(x)$ belong to $\mathcal{C}(\underline{\Theta},\overline{\Theta},a)$ and satisfy

$$\lim_{x\to 0} F_{ij}^0(x) = 0, \text{ for } \underline{\Theta} < \arg x < \overline{\Theta}. \tag{4.13}$$

Substituting Eq. (4.12) into Eq. (4.11), we have

$$\left.\begin{aligned} G_i^0(x) &= F_{ii}^0(x) + \sum_{j\neq i} F_{ij}^0(x)Q_{ji} \\ x^{\sigma+1}Q_{ij}' &= (F_i^0Q_{ij} - Q_{ij}F_j^0) + \sum_{h\neq i} F_{ih}^0(x)Q_{hj} - Q_{ij}G_j^0(x) + F_{ij}^0(x) \\ &\qquad (i\neq j) \end{aligned}\right\}. \tag{4.14}$$

By finding the solutions $Q_{ij}(x)$ from the second equation, then $G_i^0(x)$ can be obtained from the first equation.

Substituting the first equation of Eq. (4.14) into the second, we have

$$x^{\sigma+1}Q_{ij}' = (F_i^0Q_{ij} - Q_{ij}F_j^0) + \sum_{h\neq i} F_{ih}^0Q_{hj} - Q_{ij}\left(F_{jj}^0 + \sum_{h\neq j} F_{jh}^0Q_{hj}\right) + F_{ij}^0, \quad (i\neq j). \tag{4.15}$$

By picking the entries in each of the Q_{ij} suitably, Eq. (4.15) is an $(m_2 - \Sigma m_i^2)$-column vectorial nonlinear equation of the form

$$x^{\sigma+1}y' = f(x,y) \tag{4.16}$$

where $f(x,y)$ is holomorphic in (x,y), has Property-$\mathfrak{U}$ for

$$0 < |x| < a, \qquad \underline{\Theta} < \arg x < \overline{\Theta}, \qquad \|y\| < b \tag{4.17}$$

and satisfies

$$\lim_{x\to 0} F(x,0) = 0, \text{ for } \underline{\Theta} < \arg x < \overline{\Theta}. \tag{4.18}$$

Moreover, the matrix $F_y(0,0)$ is nonsingular and in the lower triangular form with eigenvalues $\lambda_i - \lambda_j$ $(i,j=1,2,\ldots,s;\ i\neq j)$.

From these facts, we know that Eq. (4.16) has a formal solution of the form

$$y \sim \sum_{\ell=1}^{\infty} x^{\ell} g_{\ell} \tag{4.19}$$

where the g_ℓ are constant vectors. Further, due to the fact that the sector $\underline{\Theta} < \arg x < \overline{\Theta}$ has Property-$\mathfrak{J}$ with respect to $\{\Lambda_{ij}(x)|i,j=1,2,\ldots,s\ (i\neq j)\}$ by Theorem A to be proved in Section III, we know that Equation (4.16) has a solution $y(x)$ which is holomorphic and bounded in

$$0 < |x| < a' \text{ and } \underline{\Theta} < \arg x < \overline{\Theta}, \text{ for } 0 < a' \leqslant a \tag{4.20}$$

and belongs to $\mathcal{C}(\underline{\Theta},\overline{\Theta},a')$ with asymptotic expansion given by Eq. (4.19).

By the use of this result, we know that the Eqs. (4.14) have solutions $G_i^0(x)$ and $Q_{ij}(x)$ which are holomorphic, bounded in Eq. (4.20), and belong to $\mathcal{C}(\underline{\Theta},\overline{\Theta},a')$ with asymptotic expansions of the form

$$G_i^0(x) = \sum_{\ell=1}^{\infty} x^{\ell} G_{i\ell}^0 \qquad (i = 1, 2, \ldots, s)$$

$$Q_{ij}(x) = \sum_{\ell=1}^{\infty} x^{\ell} Q_{ij\ell} \qquad (i, j = 1, 2, \ldots, s; i \neq j).$$

5. Nonlinear Equations

In the light of Lemma 1, we can assume without loss of generality, that $F_0(x)$ is in the block-diagonal form of $F^0 + \tilde{G}_0(x)$. Similar to the process in Section 4, put

$$F(x,z) = F_0(x) + H(x,z), \qquad G(x,z) = F_0(x) + \hat{G}(x,z). \tag{5.1}$$

Then $H(x,z)$ and $\hat{G}(x,z)$ are holomorphic in

$$0 < |x| < a', \qquad \underline{\Theta} < \arg x < \overline{\Theta}, \qquad \|z\| < c \tag{5.2}$$

and

$$H(x,0) \equiv 0, \qquad \hat{G}(x,0) \equiv 0. \tag{5.3}$$

From Eqs. (E_1), (T), (E_2), and (5.1), $P(x,Z)$ and $\hat{G}(x,Z)$ satisfy

$$\left.\begin{aligned} x^{\sigma+1}P' &= \big(F_0(x)P - PF_0(x)\big) + \big(H(x,Z) - \hat{G}(x,Z)\big) + \big(H(x,Z)P - P\hat{G}(x,Z)\big), \\ xZ' &= \mathbf{1}_n(\mu)Z. \end{aligned}\right\} \tag{5.4}$$

Put

$$F_0(x) \begin{pmatrix} \hat{F}_1(x) & & & \bigcirc \\ & \hat{F}_2(x) & & \\ & & \ddots & \\ \bigcirc & & & \hat{F}_s(x) \end{pmatrix}$$

$$H(x,Z) = \begin{pmatrix} H_{11} & H_{12} & \cdots & H_{1s} \\ H_{21} & H_{22} & \cdots & H_{2s} \\ \vdots & \vdots & & \vdots \\ H_{s1} & H_{s2} & \cdots & H_{ss} \end{pmatrix}$$

$$\left.\begin{aligned}
\hat{G}(x,Z) &= \begin{pmatrix} \hat{G}_1 & & & \bigcirc \\ & \hat{G}_2 & & \\ & & \ddots & \\ \bigcirc & & & \hat{G}_2 \end{pmatrix} \\
P(x,Z) &= \begin{pmatrix} 0 & P_{12} & \dots & P_{1s} \\ P_{21} & 0 & \dots & P_{2s} \\ \vdots & \vdots & & \vdots \\ P_{s1} & P_{s2} & \dots & 0 \end{pmatrix}
\end{aligned}\right\}. \quad (5.5)$$

Then

$$\left.\begin{aligned}
\hat{G}_i(x,Z) &= H_{ii}(x,Z) + \sum_{j\neq i} H_{ij}(x,Z)P_{ji}(x,Z) \\
x^{\sigma+1}P'_{ij} &= \left(\hat{F}_i(x)P_{ij} - P_{ij}\hat{F}_j(x)\right) + \sum_{h\neq i} H_{ih}P_{hj} - P_{ij}\hat{G}_j + H_{ij} \\
&\qquad\qquad\qquad\qquad\qquad\qquad (i\neq j) \\
xZ' &= \mathbf{1}_n(\mu)Z
\end{aligned}\right\}. \quad (5.6)$$

Or, by substituting the first equation into the second,

$$\left.\begin{aligned}
x^{\sigma+1}P'_{ij} &= (\hat{F}_iP_{ij} - P_{ij}\hat{F}_j) + \sum_{h\neq i} H_{ih}P_{hj} - P_{ij}(H_{jj} + \sum_{h\neq j} H_{jh}P_{hj}) + H_{jj} \\
&\qquad\qquad\qquad\qquad\qquad\qquad (i\neq j) \\
xZ' &= \mathbf{1}_n(\mu)Z
\end{aligned}\right\}. \quad (5.7)$$

Similar to Eq. (4.15), Eq. (5.7) can be written as an $(m_2 - \Sigma m_i^2)$-column vectorial equation of the form

$$\left.\begin{aligned}
x^{\sigma+1}y' &= f(x,y,z) \\
xz' &= \mathbf{1}_n(\mu)z
\end{aligned}\right\} \quad (5.8)$$

where y and f are $(m^2 - \Sigma m_i^2)$-column vectors, and $F(x,y,z)$ is holomorphic in (x,y,z) and has Property-$\mathfrak{U}$ with respect to y and z in

$$0 < |x| < a', \qquad \underline{\Theta} < \arg x < \overline{\Theta}, \qquad \|y\| < b, \qquad \|z\| < c. \quad (5.9)$$

Furthermore,

$$f(x,0,0) \equiv 0 \quad (5.10)$$

and the matrix

$$A^0 = \lim_{x\to 0} f_y(x,0,0), \text{ for } \underline{\Theta} < \arg x < \overline{\Theta}, \tag{5.11}$$

is in the lower triangular form with nonzero eigenvalues $\lambda_i - \lambda_j$ $(i,j=1,2,\ldots,s; i\neq j)$.

Thus the problem is reduced to finding the solution $y = R(x, V(x))$ for (5.8), where $V(x)$ is a holomorphic solution of $xz' = 1_n(\mu)z$, such that $R(x,z)$ is holomorphic in (x,z) and has Property-$\mathfrak{U}$ with respect to z in Eq. (3.7).

6. Formal Solution

By the assumptions on $f(x,y,z)$, it can be expanded in the following uniformly convergent series

$$f(x,y,z) = f_0(x,z) + A(x,z)y + \sum_{|p|=2}^{\infty} y^p f_p(x,z) \tag{6.1}$$

where p is an $(m^2 - \Sigma m_i^2)$-row vector with nonnegative integer components, and the vectors $f_0(x,z)$, $f_p(x,z)$ and the matrix $A(x,z)$ are holomorphic in (x,z) and have Property-$\mathfrak{U}$ with respect to z for

$$0 < |x| < a', \qquad \underline{\Theta} < \arg x < \overline{\Theta}, \qquad \|z\| < c. \tag{6.2}$$

Moreover,

$$f_0(x,0) \equiv 0$$

and

$$\lim_{x\to 0} A(x,0) = A^0, \text{ for } \underline{\Theta} < \arg x < \overline{\Theta}.$$

Thus we have the following uniformly convergent series expansions

$$f_0(x,z) = \sum_{|q|=1}^{\infty} z^q f_{0q}(x)$$

$$f_p(x,z) = \sum_{|q|=0}^{\infty} z^q f_{pq}(x)$$

$$A(x,z) = \sum_{|q|=0}^{\infty} z^q A_q(x)$$

where q is an n-row vector with nonnegative components, and $f_q(x)$, $f_{pq}(x)$, and $A_q(x)$ are in the class $\mathcal{C}(\underline{\Theta},\overline{\Theta},a')$.

In order to find a formal solution of the form

$$\hat{y} \sim \sum_{|q|=1}^{\infty} z^q R_q(x) \tag{6.3}$$

for Eq. (5.8), differentiate Eq. (6.3) formally, and, by the fact that $xz' = \mathbf{1}_n(\mu)z$, we have

$$x^{\sigma+1}y' \sim x^{\sigma+1} \sum_{|q|=1}^{\infty} z^q R_q'(x) + x \sum_{|q|=1}^{\infty} q \cdot \mu z^q R_q(x). \tag{6.4}$$

On the other hand

$$\begin{aligned} x^{\sigma+1}y' \sim & \sum_{|q|=1}^{\infty} z^q f_{0q}(x) + \left(\sum_{|q|=0}^{\infty} z^q A_q(x) \right) \left(\sum_{|q|=1}^{\infty} z^q R_q(x) \right) \\ & + \sum_{|p|=2}^{\infty} \left(\sum_{|q|=1}^{\infty} z^q R_q(x) \right)^p \left(\sum_{|q|=0}^{\infty} z^q f_{pq}(x) \right). \end{aligned} \tag{6.5}$$

Equating the coefficients of z^q in the right-hand members of Eqs. (6.4) and (6.5), we have

$$x^{\sigma+1}R_q' = \left[A_0(x) - (x^{\sigma} q \cdot \mu) \mathbf{1}_{(m^2 - \Sigma m_1^2)} \right] R_q + H_q(x) \tag{6.6}$$

where the $H_q(x)$ are linear combinations of f_{0q}, f_{pq}, and A_q, with coefficients being polynomials of $R_q'(|q'| < |q|)$.

Since $\lim A_0(x) = A^0$ is nonsingular, we can find a unique formal solution, successively for q, in the form

$$\hat{R}_q(x) \sim \sum_{\ell=0}^{\infty} x^{\ell} R_{q\ell} \, . \tag{6.7}$$

However, by Theorem A to be given in Chapter III, there exists a unique solution $R_q(x)$ which is holomorphic and bounded in

$$0 < |x| < a', \qquad \underline{\Theta} < \arg x < \overline{\Theta} \tag{6.8}$$

and belongs to $\mathcal{C}(\underline{\Theta},\overline{\Theta},a')$ with asymptotic expansion given by Eq. (6.7). Let $z = V(x)$ be a holomorphic solution of $xz' = \mathbf{1}_n(\mu)z$. Then we get a formal solution

$$\hat{y} \sim \sum_{|q|=1}^{\infty} V(x)^q R_q(x) \tag{6.9}$$

for Eq. (5.8).

Remark. Since Eq. (6.6) is a linear differential equation with $A_0(x)$ independent of q, we thus have $R_q(x)$ holomorphic in (6.8) for all q.

7. Analytic Solution

Since the sector $\underline{\Theta} < \arg x < \overline{\Theta}$ has Property-$\mathcal{T}$ with respect to $\{\Lambda_{ij}(x)|i,j=1,2,\ldots,s;\ i\neq j\}$, Theorem B, to be given in Chapter IV, assures that Eq. (5.8) has a solution

$$y = R(x, V(x)) \tag{7.1}$$

such that $R(x,v)$ is holomorphic in x, v for

$$0 < |x| < a_0, \qquad \underline{\Theta} < \arg x < \overline{\Theta}, \qquad ||v|| < c_0 \tag{7.2}$$

where $0 < a_0 \leqslant a'$, $0 < c_0 \leqslant c$, and it possess a uniformly convergent power series expansion (6.9) whenever $(x, V(x))$ satisfies Eq. (7.2). Furthermore,

$$R(x,0) \equiv 0. \tag{7.3}$$

By the application of these results to Eqs. (5.7) and (5.6), Theorem M is proved.

III. FIRST EXISTENCE THEOREM

8. Theorem A and Its Equivalent Problem

The remainder of this report will be devoted to two existence theorems mentioned above for the proof of Theorem M.

For the first theorem, consider a system of nonlinear differential equations

$$x^{\sigma+1}y' = f(x,y) \tag{E$_3$}$$

where y and f are m-column vectors, and $f(x,y)$ is holomorphic, bounded in (x,y) for

$$0 < |x| < a, \qquad \underline{\Theta} < \arg x < \overline{\Theta}, \qquad ||y|| < b \tag{8.1}$$

and belongs to $\mathcal{C}\,(\underline{\Theta},\overline{\Theta},a)$. Further, we assume that

(i) the matrix

$$A^0 = \lim_{x\to 0} f_y(x,0), \text{ for } \underline{\Theta} < \arg x < \overline{\Theta}, \tag{8.2}$$

is in the form of

$$A^0 = \mathbf{1}_m(\gamma) + D, \qquad \det \mathbf{1}_m(\gamma) \neq 0 \tag{8.3}$$

where γ is an m-column vector with elements $\{\gamma_j\}$ and D is an m by m nilpotent matrix of lower triangular form; and

(ii) equation (E$_3$) possesses a formal solution

$$y \sim \sum_{\ell=0}^{\infty} x^{\ell} g_{\ell} \tag{8.4}$$

where the g_ℓ are constant m-column vectors, and in particular

$$\|g_0\| < b. \tag{8.5}$$

Let

$$\Omega_j(x) = \frac{-\gamma_j}{\sigma x^\sigma} \qquad (j = 1, 2, \ldots, m). \tag{8.6}$$

The first existence theorem is

THEOREM A. *Assume that the sector $\underline{\Theta} < \arg x < \overline{\Theta}$ has Property-$\mathfrak{J}$ with respect to $\{\Omega_1(x), \ldots, \Omega_m(x)\}$. Then Eq. (E_3) has a unique solution $\Phi(x)$ which is holomorphic and bounded for*

$$0 < |x| < a_0, \qquad \underline{\Theta} < \arg x < \overline{\Theta} \tag{8.7}$$

where $0 < a_0 \leqslant a$, and admits the asymptotic expansion (8.4) as x tends to 0 in the sector (8.7).

In order to prove Theorem A, let N be a positive integer and consider the following transformations to Eq. (E_3):

$$y = \sum_{\ell=0}^{N-1} x^\ell g_\ell + w_N \tag{8.8}$$

and

$$w_N = \mathbf{1}_m(e^{\Omega(x)})\eta_N \tag{8.9}$$

where $\Omega(x)$ is the m-column vector with elements $\{\Omega_j(x)\}$. Then $\eta_N(x)$ satisfies

$$x^{\sigma+1}\eta_N' = \mathbf{1}_m(e^{-\Omega(x)})\hat{f}(x, \mathbf{1}_m(e^{\Omega(x)})\eta_N) \tag{8.10}$$

where $f(x, w)$ is an m-column vector having Property-$\mathfrak{U}$ with respect to w in

$$0 < |x| < a_N, \qquad \underline{\Theta} < \arg x < \overline{\Theta}, \qquad \|w\| < b_N \tag{8.11}$$

and satisfies the inequality

$$\|\hat{f}(x, w)\| \leqslant H\|w\| + B_N|x|^N \tag{8.12}$$

where a_N, B_N, H, and b_N are constants, with H independent of N. Further, $\hat{f}(x,w)$ satisfies a Lipschitz condition

$$\|\hat{f}(x, w^1) - \hat{f}(x, w^2)\| \leqslant H\|w^1 - w^2\| \tag{8.13}$$

for (x, w^1) ar[illegible] (8.11).

Sinc[illegible] atrix [illegible] $= D$ and is nilpotent, we can assume, without loss of generality, that H sati[illegible]

$$4H < \|\gamma\|' \sin 2\sigma\epsilon, \text{ for } \|\gamma\|' = \min_{j=1}^{m} |\gamma_j|, \tag{8.14}$$

for a preassigned positive constant ϵ.

Then, the proof of Theorem A is reduced to solving

Problem A. If we have Eq. (8.14), then there exists a unique solution $\varphi_N(x)$ of Eq. (8.10) such that, for a suitable choice of a'_N and K_N,

(i) *$\varphi_N(x)$ is a holomorphic and bounded m-column vector for*

$$0 < |x| < a'_N, \qquad \underline{\Theta} < \arg x < \overline{\Theta} \tag{8.15}$$

and

(ii) *$\varphi_N(x)$ satisfies*

$$[\varphi_N(x)] \leqslant K_N |x|^N [e^{-\operatorname{Re}\Omega(x)}] \tag{8.16}$$

for x in Eq. (8.11).

Furthermore, a solution of (8.7) satisfying

$$[\varphi_N(x)] = O(|x|^N)[e^{-\operatorname{Re}\Omega(x)}] \tag{8.17}$$

is unique.

Theorem A can be derived from the solution of this problem by an argument similar to that to be given in Section 13 below.

9. A Stable Domain

In order to find the solution of Problem A, it is necessary to replace (8.15) by a domain of the form

$$0 < |x| < a'_N \omega(\arg x), \qquad \underline{\Theta} < \arg x < \overline{\Theta} \tag{9.1}$$

where $\omega(\tau)$ is a strictly positive-valued, bounded, continuous function of τ for $\underline{\Theta} \leqslant \tau \leqslant \overline{\Theta}$, to be defined soon. The domains (8.15) and (9.1) are equivalent in the sense that any point in (9.1) is contained in (8.15), if a'_N is suitably chosen, and vice versa. The domain given by Eq. (9.1) is called a *stable domain* of Problem A.

The directions $\arg x = \theta_j$ in the sector

$$\underline{\Theta} < \arg x < \overline{\Theta}, \tag{9.2}$$

such that $\operatorname{Re} \Omega_j(x) = 0$ for $\arg x = \theta_j$, are called singular directions of $\Omega_j(x)$ and are given by

$$\frac{1}{\sigma}\left(\arg \gamma_j + \frac{\pi}{2} + 2\pi h\right) \tag{9.3}$$

or

$$\frac{1}{\sigma}\left(\arg \gamma_j - \frac{\pi}{2} + 2\pi h'\right) \tag{9.3$'$}$$

where h and h' are some integers. Singular directions of the form (9.3) are called *ascending singular directions* of $\Omega_j(x)$, and those of the form (9.3)′ are called *descending singular directions.* When $\mathrm{Re}\,\Omega_j(x)$ is regarded as a function of $\arg x = \theta$, it is a monotonic increasing (or decreasing) function of $\arg x$ in a small neighborhood of each direction of the form (9.3) (or of the form (9.3)′).

For the indices j such that $\mathrm{Re}\,\Omega_j(x)$ change their signs in (9.2), we choose $\arg \gamma_j$ so that at least one of the two singular directions

$$\theta_{j+} = \frac{1}{\sigma}\left(\arg \gamma_j + \frac{\pi}{2}\right) \tag{9.4$^+$}$$

$$\theta_{j-} = \frac{1}{\sigma}\left(\arg \gamma_j + \frac{3\pi}{2}\right) \tag{9.4$^-$}$$

is contained in Eq. (9.2). By the assumption that Eq. (9.2) has Property-$\mathcal{J}$ with respect to $\{\Omega_1(x), \ldots, \Omega_m(x)\}$, we can classify the set $J = \{1,2,\ldots,m\}$ of indices j into four classes:

$$J_0 = \{j; \mathrm{Re}\,\Omega_j(x) > 0 \text{ for } \underline{\Theta} < \arg x < \overline{\Theta}\},$$

$$J_1 = \{j;\ \underline{\Theta} < \theta_{j+} < \theta_{j-} < \overline{\Theta}\},$$

$$J_2 = \{j;\ \underline{\Theta} < \theta_{j+} < \overline{\Theta} < \theta_{j-}\},$$

$$J_3 = \{j;\ \theta_{j+} < \underline{\Theta} < \theta_{j-} < \overline{\Theta}\}.$$

For $j \epsilon J_2$ we define θ_{j-} by Eq. (9.4)$^-$, and for $j \epsilon J_3$ we define θ_{j+} by Eq. (9.4)$^+$. Some of these four sets may be empty. Especially, either J_0 or J_1 is empty because (a) when J_0 is not empty, $\overline{\Theta} - \underline{\Theta} < \pi/\sigma$, and (b) when J_1 is not empty, $\overline{\Theta} - \underline{\Theta} > \pi/\sigma$. Therefore, $J = J_1 \cup J_2 \cup J_3$, or $J = J_0 \cup J_2 \cup J_3$.

Since the sector given by Eq. (9.2) has Property-$\mathcal{J}$ with respect to $\{\Omega_1(x), \ldots, \Omega_m(x)\}$, for a sufficiently small positive number ϵ, $\underline{\Theta}$ and $\overline{\Theta}$ must satisfy the inequality

$$\max_{j=1}^{m} \theta_{j+} - \left(\frac{\pi}{\sigma} + 6\epsilon\right) \leqslant \underline{\Theta} < \overline{\Theta} \leqslant \min_{j=1}^{m} \theta_{j-} + \left(\frac{\pi}{\sigma} - 6\epsilon\right) \tag{9.5}$$

for all $j \epsilon J_1 \cup J_2 \cup J_3$ or $J_0 \cup J_2 \cup J_3$. Put

$$\Theta_{k+} = \max_{j \epsilon J_k} \theta_{j+} \text{ and } \Theta_{k-} = \min_{j \epsilon J_k} \theta_{j-}. \tag{9.6}$$

A scalar function $A(\tau)$ in $\underline{\Theta} \leqslant \tau \leqslant \overline{\Theta}$ is defined as

$$A(\tau) = \begin{cases} \sigma(\tau - \Theta_{3-} + 4\epsilon), \text{ for } \Theta_{3-} + \dfrac{\pi}{2\sigma} - 4\epsilon \leqslant \tau \leqslant \overline{\Theta} \\[2ex] \dfrac{\pi}{2}, \text{ for } \Theta_{2+} - \dfrac{\pi}{2\sigma} + 4\epsilon \leqslant \tau \leqslant \Theta_{3-} + \dfrac{\pi}{2\sigma} - 4\epsilon \\[2ex] \sigma(\tau - \Theta_{2+} - 4\epsilon) + \pi, \text{ for } \underline{\Theta} \leqslant \tau \leqslant \Theta_{2+} - \dfrac{\pi}{2\sigma} + 4\epsilon . \end{cases} \tag{9.7}$$

Then $A(\tau)$ satisfies

$$2\sigma\epsilon \leqslant A(\tau) \leqslant \pi - 2\sigma\epsilon, \text{ for } \underline{\Theta} \leqslant \tau \leqslant \overline{\Theta}. \tag{9.8}$$

The function $\omega(\tau)$ is defined as

$$\omega(\tau) = \exp\left\{\int_{\theta_0}^{\tau} \cot A(t)dt\right\}, \tag{9.9}$$

where θ_0 is an arbitrary angle in (9.2).

10. A Fundamental Inequality

In order to find the solution to Problem A, we need an integral inequality.

LEMMA 2. *Let x_1 be an arbitrary point in the domain (9.1). Then there exists an m-vector path Γ_{x_1} with elements $\{\Gamma_{jx_1}\}$ such that*

(i) *each curve Γ_{jx_1} joins the point x_1 with the origin and is contained in (9.1), except for the origin, and*

(ii) *if a_N'' satisfies*

$$2N\left(a_N'' \max_{\underline{\Theta}\leqslant\tau\leqslant\overline{\Theta}} \omega(\tau)\right)^{\sigma} \leqslant \|\gamma\|' \sin 2\sigma\epsilon, \tag{10.1}$$

then

$$\int_{\Gamma_{jx_1}} |x|^{N-\sigma-1} e^{-\mathrm{Re}\,\Omega_j(x)} ds_j \leqslant \frac{2}{\|\gamma\|' \sin 2\sigma\epsilon} |x_1|^N e^{-\mathrm{Re}\,\Omega_j(x_1)}. \tag{10.2}$$

Here s_j is the arc length of Γ_{jx_1} measured from the origin to the variable point x on this curve.

In order to define Γ_{jx_1}, we define first an m-vector $a(\tau)$ with elements $a_j(\tau)$ in the closure of Eq. (9.2). If $j\epsilon J_0$,

$$a_j(\tau) = \frac{\pi}{2}, \text{ for } \underline{\Theta} \leqslant \tau \leqslant \overline{\Theta}. \tag{10.3}$$

If $j\epsilon J_1$,

$$a_j(\tau) = \begin{cases} \sigma(\tau - \theta_{j-} + 4\epsilon), \text{ for } \theta_{j-} - 2\epsilon \leqslant \tau \leqslant \overline{\Theta} \\ \frac{\pi}{2}, \text{ for } \theta_{j+} + 2\epsilon < \tau < \theta_{j-} - 2\epsilon \\ \sigma(\tau - \theta_{j-} - 4\epsilon) + \pi, \text{ for } \underline{\Theta} \leqslant \tau \leqslant \theta_{j+} + 2\epsilon \ . \end{cases} \tag{10.4}$$

If $j \epsilon J_2$,

$$a_j(\tau) = \begin{cases} \dfrac{\pi}{2}, \text{ for } \theta_{j+} + 2\epsilon \leqslant \tau \leqslant \overline{\Theta} \\ \sigma(\tau - \theta_{j+} - 4\epsilon) + \pi, \text{ for } \underline{\Theta} \leqslant \tau \leqslant \theta_{j+} + 2\epsilon \ . \end{cases} \tag{10.5}$$

If $j \epsilon J_3$,

$$a_j(\tau) = \begin{cases} \sigma(\tau - \theta_{j-} + 4\epsilon), \text{ for } \theta_{j-} - 2\epsilon \leqslant \tau \leqslant \overline{\Theta} \\ \dfrac{\pi}{2}, \text{ for } \underline{\Theta} \leqslant \tau \leqslant \theta_{j-} + 2\epsilon \ . \end{cases} \tag{10.6}$$

By virtue of Eq. (9.4), we have

$$2\sigma\epsilon \leqslant a_j(\tau) \leqslant \pi - 2\sigma\epsilon, \text{ for } \underline{\Theta} \leqslant \tau \leqslant \overline{\Theta}, \tag{10.7}$$

and by Eq. (9.5),

$$\begin{cases} a_j(\tau) \leqslant A(\tau), \text{ for } \theta_{j-} - 2\epsilon \leqslant \tau \leqslant \overline{\Theta} \quad (j\epsilon J_1, J_3) \\ a_j(\tau) \leqslant A(\tau), \text{ for } \underline{\Theta} \leqslant \tau \leqslant \theta_{j+} + 2\epsilon \quad (j\epsilon J_1, J_2) \ . \end{cases} \tag{10.8}$$

Hence, we have

$$\int_\theta^\tau \cot a_j(t)dt \leqslant \int_\theta^\tau \cot A(t)dt \tag{10.9}$$

for $\theta \leqslant \tau \leqslant \theta_{j+} + 2\epsilon$ $(j\epsilon J_1, J_2)$, and for $\theta_{j-} - 2\epsilon \leqslant \tau \leqslant \theta$ $(j\epsilon J_1, J_3)$.

Let (r,θ) and (ρ,τ) be the polar coordinates of the point x_1 and of the variable point x on the curve Γ_{jx_1} respectively. Then the curve Γ_{jx_1} is defined as follows:

(i) If $\theta < \theta_{j+} + 2\epsilon$ or $\theta_{j-} - 2\epsilon < \theta$, the curve Γ_{jx_1} consists of a curvilinear part Γ_j' given by

$$\left.\begin{array}{l} \rho = r \exp \displaystyle\int_\theta^\tau \cot a_j(t)dt \\ \\ \theta \leqslant \tau \leqslant \theta_{j+} + 2\epsilon \quad \text{or} \quad \theta_{j-} - 2\epsilon \leqslant \tau \leqslant \theta, \end{array}\right\} \tag{10.10}$$

and of a rectilinear part Γ_j'' given by

$$\left.\begin{array}{l} 0 \leqslant \rho \leqslant r \exp \displaystyle\int_\theta^\tau \cot a_j(t)dt \\ \\ \tau = \theta_{j+} + 2\epsilon \quad \text{or} \quad \theta_{j-} - 2\epsilon. \end{array}\right\} \tag{10.11}$$

(ii) If $\theta_{j+} + 2\epsilon \leqslant \theta \leqslant \theta_{j-} - 2\epsilon$, the curve Γ_{jx_1} consists only of a rectilinear part Γ_j'' given by

$$0 \leqslant \rho \leqslant r, \qquad \tau = \theta. \tag{10.12}$$

By virtue of Eq. (10.9), the curves Γ_{jx_1} defined by Eqs. (10.10) and (10.11), or (10.12), are contained entirely in the domain given by Eq. (10.1), except for the origin. This proves assertion (i) of Lemma 1.

In order to prove assertion (ii), we will prove

$$\frac{d}{ds_j}(|x|^N e^{-\mathrm{Re}\Omega_j(x)}) \geqslant \frac{||\gamma||' \sin 2\sigma\epsilon}{2} |x|^{N-\sigma-1} e^{-\mathrm{Re}\Omega_j(x)}, \tag{10.13}$$

instead, for x on Γ_{jx_1}, except for x_{ij}^*, the joint of Γ_j' and Γ_j''. By the fact that $|x|^N e^{-\mathrm{Re}\Omega_j(x)}$ is bounded in the neighborhood of x_{ij}^*, Eq. (10.2) follows immediately by a limit process and by Eq. (10.13).

For x on the curvilinear part Γ_j', ρ is a function of τ given by Eq. (10.10). By a simple computation, we have

$$\frac{dx}{ds_j} = \frac{dx}{d\tau}\frac{d\tau}{ds_j} = \pm e^{i\tau}(\cot a_j(\tau) + i) \sin a_j(\tau) = \pm e^{(a_j(\tau)+\tau)i} \tag{10.14}$$

where the negative sign is for $\theta \leqslant \tau \leqslant \theta_{j+} + 2\epsilon$, and the positive for $\theta_{j-} - 2\epsilon \leqslant \tau \leqslant \theta$.

Hence, we have

$$\frac{d}{ds_j}(-\mathrm{Re}\ \Omega_j(x)) = \pm\rho^{-\sigma-1}|\gamma_j| \cos(a_j(\tau) - \sigma\tau + \arg\gamma_j), \tag{10.15}$$

and consequently,

$$\frac{d}{ds_j} e^{-\mathrm{Re}\Omega_j(x)} = \pm\rho^{-\sigma-1}|\gamma_j| e^{-\mathrm{Re}\Omega_j(x)} \cos(a_j(\tau) - \sigma\tau + \arg\gamma_j) \tag{10.16}$$

with the positive sign for $\theta \leqslant \tau \leqslant \theta_{j+} + 2\epsilon$, and the negative for $\theta_{j-} - 2\epsilon \leqslant \tau \leqslant \theta$.

On the other hand, from the definition of $a_j(\tau)$ and of the angles θ_{j+} and θ_{j-}, we have

$$a_j(\tau) - \sigma\tau + \arg\gamma_j = \begin{cases} \dfrac{\pi}{2} - 4\sigma\epsilon, \text{ for } \theta \leqslant \tau \leqslant \theta_{j+} + 2\epsilon \\[2ex] \dfrac{\pi}{2} + 4\sigma\epsilon, \text{ for } \theta_{j-} - 2\epsilon \leqslant \tau \leqslant \theta \end{cases}.$$

Hence

$$\pm\cos(a_j(\tau) - \sigma\tau + \arg\gamma_j) = \sin 4\sigma\epsilon > \sin 2\sigma\epsilon,$$

and we have

$$\frac{d}{ds_j} e^{-\mathrm{Re}\Omega_j(x)} \geqslant |x|^{-\sigma-1} e^{-\mathrm{Re}\Omega_j(x)} ||\gamma||' \sin 2\sigma\epsilon \tag{10.17}$$

for x on Γ_j'.

Also, since s_j is real, we have

$$|x|^{-1}\frac{d|x|}{ds_j} = \frac{d}{ds_j}\log|x| = \mathrm{Re}\; x^{-1}\frac{dx}{ds_j}. \tag{10.18}$$

Thus, by Eq. (10.14),

$$|x|^{-1}\frac{d|x|}{ds_j} \geqslant -|x| \tag{10.19}$$

for x on Γ_j'.

On the rectilinear part Γ_j'', we have $s_j = \rho = |x|$. Thus

$$\frac{d}{ds_j}e^{-\mathrm{Re}\Omega_j(x)} = -e^{-\mathrm{Re}\Omega_j(x)}\frac{d}{d\rho}\mathrm{Re}\,\Omega_j(x)$$

$$= -e^{-\mathrm{Re}\Omega_j(x)}\rho^{-\sigma-1}|\gamma_j|\cos(\arg\gamma_j - \sigma\theta) \geqslant e^{-\mathrm{Re}\Omega_j(x)}\rho^{-\sigma-1}|\gamma_j|\sin 2\sigma\epsilon$$

because $\theta_{j+} + 2\epsilon \leqslant \theta \leqslant \theta_{j-} - 2\epsilon$. Therefore Eq. (10.17) holds also for Γ_j''. Equation (10.18) follows immediately for x on Γ_j'' since $|x| = s_j$.

By the fact that Eqs. (10.17) and (10.19) hold for Γ_{jx_1}, we have

$$\frac{d}{ds_j}(|x|^N e^{-\mathrm{Re}\Omega_j(x)}) \geqslant |x|^{N-\sigma-1}e^{-\mathrm{Re}\Omega_j(x)}(||\gamma||'\sin 2\sigma\epsilon - N|x|^\sigma),$$

and Eq. (10.13) follows immediately from Eq. (10.1). Thus, assertion (ii) is proved. This completes the proof of Lemma 2.

11. Solution of Problem A

For an arbitrary point x_1 in the domain given by Eq. (9.1), consider a system of integral equations given by

$$\varphi(x_1) = \int_{\Gamma_{x_1}} x^{-\sigma-1}1_m(e^{-\Omega(x)})\hat{f}(x, 1_m(e^{\Omega(x)})\varphi(x))dx, \tag{11.1}$$

which is equivalent to Eq. (8.11). By the use of Lemma 2 and a discussion analogous to that in Chapter V, below the solution of Problem A can be found.

Thus Theorem A is proved.

IV. SECOND EXISTENCE THEOREM

12. Statement of Theorem B

Consider a system of $m+n$ equations given by

$$x^{\sigma+1}y' = f(x,y,z), \qquad xz' = 1_n(\mu)z, \tag{E$_4$}$$

where y and f are m-column vectors and μ and z are n-column vectors. Here we assume that

(i) $f(x,y,z)$ is holomorphic in (x,y,z) and has Property-$\mathfrak{U}$ with respect to y and z in

$$0 < |x| < a, \qquad \underline{\Theta} < \arg x < \overline{\Theta}, \qquad \|y\| < b, \qquad \|z\| < c; \tag{12.1}$$

(ii) the matrix

$$\lim_{x\to 0} f(x,0,0) \equiv A^0 = \mathbf{1}_m(\gamma) + D, \text{ for } \underline{\Theta} < \arg x < \overline{\Theta} \tag{12.2}$$

where γ is an m-column vector with *nonzero* elements $\{\gamma_j\}$, and D is an m by m nilpotent matrix of lower triangular form;

(iii)′ every component of μ satisfies

$$\operatorname{Re} \mu_k \geqslant 0 \qquad (k = 1, 2, \ldots, n); \tag{12.3}$$

and

(iv) for a holomorphic solution $V(x)$ of the equation $xz' = \mathbf{1}_n(\mu)z$, Eq. (E_4) possesses a formal solution

$$\tilde{y} \sim \sum_{|q|=0}^{\infty} V(x)^q g_q(x) \tag{12.4}$$

where $g_q(x)$ are m-column vector functions belonging to $\mathcal{C}(\underline{\Theta},\overline{\Theta},a)$, and, in particular,

$$\|g_0(x)\| < b. \tag{12.5}$$

Now, the second existence theorem is

THEOREM B. *Assume that the sector $\underline{\Theta} < \arg x < \overline{\Theta}$ has Property-$\mathfrak{T}$ with respect to $\{\Omega_1(x), \ldots, \Omega_m(x)\}$, where the $\Omega_j(x)$ are given by Eq. (8.7). Then Eq. (E_4) has a solution of the form $\{\tilde{\Phi}(x, V(x)), V(x)\}$, where x and $V(x)$ are in the domain*

$$0 < |x| < a_0, \qquad \underline{\Theta} < \arg x < \overline{\Theta}, \qquad \|v\| < c_0, \tag{12.6}$$

and $0 < a_0 \leqslant a$, $0 < c_0 \leqslant c$. Furthermore, $y = \tilde{\Phi}(x, V(x))$ admits uniformly convergent expansion of the form (12.4) so that $\tilde{\Phi}(x,v)$ has Property-$\mathfrak{U}$ with respect to v in the domain given by Eq. (12.6).

This theorem is similar to a theorem proved in Refs. 6 and 14. However, the conditions given by Eq. (12.3) are milder than those assumed for earlier results. Also, Eq. (12.3) includes the case that $\mu_k = 0$ (k=1,2, . . .,n) which, in turn, reduce z to a parameter independent of x, and Theorem B becomes a special case of problems studied in Refs. 9-13.

13. Reduction of Theorem B

In order to prove Theorem B, we first consider, for a positive integer N, the following transformations to Eq. (E_4):

$$y = \sum_{|q|=0}^{N-1} V(x)^q g_q(x) + w_N \tag{13.1}$$

and

$$w_N = \mathbf{1}_m(e^{\Omega(x)})\eta_N. \tag{13.2}$$

Then, $\eta_N(x)$ satisfies

$$x^{\sigma+1}\eta_N' = \mathbf{1}_m(e^{-\Omega(x)})\hat{f}\big(x, V(x); \mathbf{1}_m(e^{\Omega(x)})\eta_N\big) \tag{13.3}$$

where $\hat{f}(x,v;w)$ is a holomorphic and bounded m-column vector function of $(x,v;w)$ for

$$0 < |x| < a_N, \qquad \underline{\Theta} < \arg x < \overline{\Theta}, \qquad \|v\| < c_N, \qquad \|w\| < b_N. \tag{13.4}$$

Here a_N, c_N, and b_N are positive constants depending on N, $a_N \leqslant a$, $c_N \leqslant c$, and b_N depends on b, a_N, and c_N. Further,

$$\hat{f}(0,0;0) = 0, \text{ and } \lim_{x\to 0} \hat{f}_w(x,0;0) = D, \text{ for } \underline{\Theta} < \arg x < \overline{\Theta}, \tag{13.5}$$

and Eq. (13.3) has a formal solution

$$\eta_N(x) = \mathbf{1}_m(e^{-\Omega(x)}) \sum_{|q|=N}^{\infty} V(x)^q g_q(x). \tag{13.6}$$

Since D is nilpotent, we have the inequalities

$$\|\hat{f}(x,v;w)\| \leqslant H\|w\| + B_N\|v\|^N \tag{13.7}$$

and

$$\|\hat{f}(x,v;w^1) - \hat{f}(x,v;w^2)\| \leqslant H\|w^1 - w^2\| \quad \text{(Lipschitz condition)} \tag{13.8}$$

for $(x,v;w)$, $(x,v;w^1)$, and $(x,v;w^2)$ in Eq. (13.4), where H can be taken, without loss of generality, to satisfy

$$4H < \|\gamma\|' \sin 2\sigma\epsilon. \tag{13.9}$$

Thus, the proof of Theorem B is reduced to solving

Problem B. If Eq. (13.9) is satisfied, then there exists a solution $\varphi_N(x,V(x))$ of Eq. (13.3) such that for suitably chosen a_N', c_N', and K_N

(i) *$\varphi_N(x,v)$ is a holomorphic and bounded m-column vector function for*

$$0 < |x| < a_N', \qquad \underline{\Theta} < \arg x < \overline{\Theta}, \qquad \|v\| < c_N'; \tag*{(13.10)$_N$}$$

(ii) *$\varphi_N(x,v)$ satisfies the inequality*

$$[\varphi_N(x,v)] \leqslant K_N\|v\|^N[e^{-\mathrm{Re}\,\Omega(x)}] \tag*{(13.11)$_N$}$$

for (x,v) in Eq. $(13.10)_N$.

Moreover, a solution of Eq. (13.3) satisfying

$$[\eta_N(x, V(x))] = O(\|V(x)\|^N)[e^{-\mathrm{Re}\,\Omega(x)}] \tag{13.12}$$

is unique.

In fact, we can prove Theorem B from the solution of Problem B in the following manner. Owing to Eqs. (13.1) and (13.2), the quantity

$$\sum_{|q|=0}^{N-1} V(x)^q g_q(x) + \mathbf{1}_m(e^{\Omega(x)})\varphi_N(x, V(x)) \tag{13.13}$$

is a solution of Eq. (E_4) provided that $(x, V(x))$ is in the domain defined by Eq. $(13.10)_N$. Let N' be an integer greater than N. Then

$$\mathbf{1}_m(e^{-\Omega(x)})\sum_{|q|=N}^{N'-1} V(x)^q g_q(x) + \varphi_{N'}(x, V(x)) \tag{13.14}$$

is a solution of Eq. (13.3) satisfying Eq. $(13.12)_N$ if $(x, V(x))$ belongs to the common part of Eqs. $(13.10)_N$ and $(13.10)_{N'}$. Hence, by the uniqueness of the solution, the solution (13.14) must coincide with $\varphi_N(x, V(x))$. Thus the solution of Eq. (E_4) expressed by Eq. (13.13) is independent of N. We denote this solution by $\tilde{\Phi}(x, V(x))$. By analytic continuation, the function $\tilde{\Phi}(x, V(x))$ is defined in the domain of the form shown in Eq. (12.6) with $a_0 = \sup a'_N$, $c_0 = \sup c'_N$.

On the other hand, $\nu = 0$ is an interior point in which $\tilde{\Phi}(x, \nu)$ is defined. Therefore, by Cauchy's theorem, $\tilde{\Phi}(x, V(x))$ can be expanded into a uniformly convergent power series of $V(x)$ whenever $(x, V(x))$ is in the domain defined by Eq. (12.6). Clearly, from Problem B, $\tilde{\Phi}(x, V(x))$ admits the asymptotic expansion of Eq. (12.4). By the uniqueness of asymptotic expansions, this asymptotic expansion must coincide with the uniformly convergent expansion. This proves the uniform convergence of the formal solution, Eq. (12.4).

Thus Theorem B is proved.

14. Stable Domain For Problem B

In order to find the solution of Problem B, it is necessary to replace Eq. $(13.10)_N$ by an equivalent stable domain defined by

$$0 < |x| < a''_N\omega(\arg x), \qquad \underline{\Theta} < \arg x < \overline{\Theta}, \qquad [\nu] < c''_N[\mathrm{X}(\arg x)] \tag{14.1}$$

where $\omega(\tau)$ is given by Eq. (9.9) and $\mathrm{X}(\tau)$ is an n-column vector function with elements $\{\mathrm{X}_k(\tau)\}$ defined by

$$\mathrm{X}_k(\tau) = \exp\left\{(\mathrm{Re}\,\mu_k)\int_{\theta_0}^{\tau} \cot A(t)dt + (\mathrm{Im}\,\mu_k)(\theta_0 - \tau)\right\}, \tag{14.2}$$

with θ_0 being a fixed angle satisfying $\underline{\Theta} \leqslant \theta_0 \leqslant \overline{\Theta}$.

Instead of finding the solution of Problem B, we shall prove

THEOREM B′. *There exist positive constants* a_N'', c_N'', *and* K_N *such that Eq. (13.3) has a unique solution* $\tilde{\varphi}_N(x, V(x))$ *satisfying*

(i) $\tilde{\varphi}_N(x,v)$ *is a holomorphic and bounded m-column vector function for* (x,v) *in Eq. (14.1); and*
(ii) $\tilde{\varphi}_N(x,v)$ *satisfies the inequality*

$$[\tilde{\varphi}_N(x,v)] \leqslant K_N \|v\|^N [e^{-\mathrm{Re}\,\Omega(x)}] \tag{14.3}$$

for (x,v) *in Eq. (14.1).*

This Theorem will be proved in Chapter V below.

15. Fundamental Inequalities for Problem B

In order to prove Theorem B′, we must prove fundamental inequalities stated in

LEMMA 3. *Let* (x_1, v^1) *be an arbitrary point in the domain of the form*

$$0 < |x| < a_N\omega(\arg x), \qquad \underline{\Theta} < \arg x < \overline{\Theta}, \qquad [v] < c_N[\mathrm{X}(\arg x)]. \tag{15.1}$$

Let $V(x) = 1_n(x^\mu)C$, *with* $C = 1_n(x_1{}^{-\mu})v^1$; *namely* $V(x_1) = v^1$. *Then there exists an m-vector path* Γ_{jx_1} *with elements* $\{\Gamma_{jx_1}\}$ *such that*

(i) *each curve* Γ_{jx_1} *joins the point* x_1 *with the origin and is contained in the domain*

$$0 < |x| < a_N\omega(\arg x), \qquad \underline{\Theta} < \arg x < \overline{\Theta}, \tag{15.2}$$

except for the origin;

(ii) *as x moves on* Γ_{jx_1}, *we have*

$$[V(x)] < c_N[\mathrm{X}(\arg x)], \qquad \underline{\Theta} < \arg x < \overline{\Theta}; \tag{15.3}$$

(iii) *if* a_N *satisfies*

$$2N\|\mu\|(a_N \max_{\underline{\Theta} \leqslant \tau \leqslant \overline{\Theta}} \omega(\tau))^\sigma \leqslant \|\gamma\|' \sin 2\sigma\epsilon, \tag{15.4}$$

then

$$\int_{\Gamma_{jx_1}} |x|^{-\sigma-1} \|V(x)\|^N e^{-\mathrm{Re}\,\Omega_j(x)} ds_j \leqslant \frac{2}{\|\gamma\|' \sin 2\sigma\epsilon} \|V(x_1)\|^N e^{-\mathrm{Re}\,\Omega_j(x_1)}. \tag{15.5}$$

Here s_j *is the arc length measured from the origin to the variable x on* Γ_{jx_1}.

The curves Γ_{jx_1} are defined exactly the same way as in Section 10 above. Then assertion (i) is evidently satisfied.

For assertion (ii), we introduce the polar coordinates $x_1 = r\,e^{i\theta}$ and $x = \rho\,e^{i\tau}$. Let the components of $V(x)$, v^1, and μ be $\{V_k(x)\}$, $\{v_k{}^1\}$, and $\{\mu_k\}$ respectively. Then Eq. (15.3) is equivalent to n inequalities

$$|V_k(x)| < c_N \exp\left\{(\mathrm{Re}\,\mu_k)\int_{\theta_0}^{\tau} \cot A(t)dt + (\mathrm{Im}\,\mu_k)(\theta_0 - \tau)\right\} \tag{15.6.k}$$

as x moves on the curve Γ_{jx_1}. Observe that the curve consists of the two parts Γ_j' and Γ_j'', in general, and we have $V_k(x) = v_k^1(x/x_1)^{\mu_k}$. Thus

$$|V_k(x)| = |v_k^1|\left(\frac{\rho}{r}\right)^{\mathrm{Re}\mu_k}\exp\{(\mathrm{Im}\,\mu_k)(\theta-\tau)\}. \tag{15.7.k}$$

For x on Γ_j', ρ is a function of τ given by Eq. (10.10), and we have

$$|V_k(x)| = |v_k^1|\exp\left\{(\mathrm{Re}\,\mu_k)\int_\theta^\tau \cot a_j(t)\,dt + (\mathrm{Im}\,\mu_k)(\theta-\tau)\right\};$$

consequently, by Eq. (10.9),

$$|V_k(x)| \leqslant |v_k^1|\exp\left\{(\mathrm{Re}\,\mu_k)\int_\theta^\tau \cot A(t)dt + (\mathrm{Im}\,\mu_k)(\theta-\tau)\right\}. \tag{15.8.k}$$

On the other hand, since $v^1 = V(x_1)$, v_k^1 must satisfy the inequality given by Eq. (15.6.k) with $\tau = \theta$. Namely,

$$|v_k^1| < c_N\exp\left\{(\mathrm{Re}\,\mu_k)\int_{\theta_0}^\tau \cot A(t)dt + (\mathrm{Im}\,\mu_k)(\theta_0-\theta)\right\}. \tag{15.9.k}$$

Hence, by Eqs. (15.8.k) and (15.9.k), Eq. (15.3) holds for x on Γ_j'.

For x on Γ_j'', $\rho \leqslant r$ and τ is constant. Hence, by virtue of Eq. (15.7.k), $|V_k(x)|$ is a nonincreasing function of ρ, since Re $\mu_k > 0$. Therefore, Eq. (15.3) continues to hold if it holds at the starting point of Γ_j''. Thus assertion (ii) is proved.

For assertion (iii), Eq. (15.5) is reduced to Eq. (10.2) if $\mu = 0$. If $\mu \neq 0$, notice that, analogous to Eq. (10.8),

$$|V_k(x)|^{-1}\frac{d}{ds_j}|V_k(x)| = \mathrm{Re}\left(V_k(x)^{-1}\frac{d}{ds_j}V_k(x)\right) = \mathrm{Re}\left(\mu_k x^{-1}\frac{dx}{ds_j}\right). \tag{15.10}$$

Since $|dx/ds_j| = 1$, except for x_{ij}^* (the joint of Γ_j' and Γ_j''), we have

$$\frac{d}{ds_j}\|V(x)\| \geqslant -|x|^{-1}\|\mu\|\,\|V(x)\| \tag{15.11}$$

for x on Γ_{jx_1}, except for x_{ij}^*. Thus

$$\frac{d}{ds_j}(\|V(x)\|^N e^{-\mathrm{Re}\Omega_j(x)}) \geqslant x^{-\sigma-1}\|V(x)\|^N e^{-\mathrm{Re}\Omega_j(x)}(\|\gamma\|'\sin 2\sigma\epsilon - N\|\mu\|\,|x|^\sigma) \tag{15.12}$$

for x on Γ_{jx_1}, except for x_{ij}^*. Thus, if a_N satisfies Eq. (15.4), then

$$\frac{d}{ds_j}(\|V(x)\|^N e^{-\mathrm{Re}\Omega_j(x)}) \geqslant \frac{\|\gamma\|'\sin 2\sigma\epsilon}{2}|x|^{-\sigma-1}\|V(x)\|^N e^{-\mathrm{Re}\Omega_j(x)} \tag{15.13}$$

for x on Γ_{jx_1}, except for x_{ij}^*.

By the fact that $|x|^{-\sigma-1}\,||V(x)||^N e^{-\mathrm{Re}\,\Omega_j(x)}$ is bounded in the neighborhood of x_{ij}^*, Eq. (15.5) follows immediately by a limit process at x_{ij}^* and by Eq. (15.13).

Thus Lemma 3 is proved.

V. PROOF OF THEOREM B′

16. Successive Approximations

We shall prove Theorem B′ in this chapter by means of successive approximations which involve improper contour integrals and analyticity with respect to several complex variables.

Let (x_1, ν^1) be an arbitrary point in the domain given by Eq. (4.1), where a_N'' and c_N'' are to be specified shortly. Let $V(x)$ be an n-column vector function defined by

$$\left.\begin{aligned} V(x) &= \nu^1, && \text{for } \mu = 0 \\ V(x) &= \mathbf{1}_n\left(\left(\frac{x}{x_1}\right)^{\mu}\right)\nu^1, && \text{for } \mu \neq 0 \end{aligned}\right\}.$$

Namely, $V(x)$ is a holomorphic solution of $xz' = \mathbf{1}_n(\mu)z$ such that $V(x_1) = \nu^1$. It is evident that the system of Eq. (13.3) is equivalent to

$$\Phi(x_1, \nu^1) = \int_{\Gamma_{x_1}} x^{-\sigma-1}\mathbf{1}_m(e^{-\Omega(x)})\hat{f}(x, V(x); \mathbf{1}_m(e^{\Omega(x)})\Phi(x, V(x))dx. \tag{16.1}$$

The successive approximations for Eq. (16.1) are defined to be the sequence $\{\Phi^{(\ell)}(x_1, \nu^1) | \ell = 0,1,2\ldots\}$ where

$$\Phi^{(0)}(x_1, \nu^1) = 0, \tag{16.2}$$

$$\Phi^{(\ell+1)}(x_1, \nu^1) = \int_{\Gamma_{x_1}} x^{-\sigma-1}\mathbf{1}_m(e^{-\Omega(x)})\hat{f}(x, V(x); \mathbf{1}_m(e^{\Omega(x)})\Phi^{(\ell)}(x, V(x))dx. \tag{16.3.ℓ+1}$$

We shall prove that such a defined sequence as $\{\Phi^{(\ell)}(x_1, \nu^1)\}$ converges to the desired solution of Eq. (16.1), or equivalently, that of Eq. (13.3), in the following steps:

(I) Each term of the sequence $\{\Phi^{(\ell)}(x_1, \nu^1)\}$ is well defined and holomorphic in (x_1, ν^1) for Eq. (14.1).

(II) The sequence $\{\Phi^{(\ell)}(x_1, \nu^1)\}$ converges uniformly to $\varphi(x_1, \nu^1)$ in any compact subset of Eq. (14.1).

(III) The limit function $\varphi(x_1, \nu^1)$ satisfies the integral equation given by Eq. (16.1), namely, the contour integral and the limiting process are interchangeable.

(IV) The function $\varphi(x_1, \nu^1)$ satisfies the integral equation given by Eq. (13.3).

(V) A solution of Eq. (13.3) satisfying Eq. (14.3) is unique.

Due to the relationship between ν^1 and x_1 through $V(x)$, Step IV is not an immediate consequence of Step III.

If Steps I to V are proven, the unique solution $\varphi(x, \nu)$ will be denoted by $\tilde{\varphi}_N(x, \nu)$. Thus Theorem B′ is proved.

17. The Function $\Phi^{(1)}(x,\nu)$

We shall prove Step I by means of mathematical induction. In the meantime, the constants a_N'', c_N'', and K_N will be specified.

Let us define

$$\Phi^{(1)}(x_1,\nu^1) = \int_{\Gamma_{jx_1}} f^{(1)}\big(x,V(x)\big)dx \tag{17.1}$$

where

$$f^{(1)}(x,\nu) = x^{-\sigma-1}1_m(e^{-\Omega(x)})\hat{f}(x,\nu;0).$$

Existence of Integrals–From the definition of Γ_{x_1} in Section 10, we know that each element Γ_{jx_1} or Γ_{x_1} has rectilinear portion Γ_j'' of positive length. Furthermore, the jth component $f_j^{(1)}$ of $f^{(1)}$ tends to zero exponentially as x approaches the origin along Γ_j''. Thus each component $\Phi_j^{(1)}$ of the integral given by Eq. (17.1) exists at $x = 0$.

At the joint x_{ij}^* of Γ_j' and Γ_j'', the integrand $\Phi_j^{(1)}$ is bounded. Also, the arc Γ_{jx_1} is rectifiable. Hence, $\Phi_j^{(1)}$ exists at x_{ij}^*.

Thus, $\Phi_j^{(1)}(x_1,\nu^1)$ exists for (x_1,ν^1) in Eq. (14.1).

Upper Bound–By Eq. (13.7), we have

$$[f^{(1)}(x,V(x))] \leqslant |x|^{-\sigma-1}B_N\|V(x)\|^N[e^{-\mathrm{Re}\,\Omega(x)}] \tag{17.2}$$

for

$$0 < |x| < a_N''\omega(\arg x), \qquad \underline{\Theta} < \arg x < \overline{\Theta}. \tag{17.3}$$

Choose a_N'' so small that

$$2N\|\mu\|\big(a_N'' \max_{\underline{\Theta}\leqslant\tau\leqslant\overline{\Theta}} \omega(\tau)\big)^\sigma \leqslant \|\gamma\|' \sin 2\sigma\epsilon. \tag{17.4}$$

Then, by Lemma 3, we have

$$[\Phi^{(1)}(x_1,\nu^1)] \leqslant \frac{2B_N}{\|\gamma\|' \sin 2\sigma\epsilon}\|\nu^1\|^N[e^{-\mathrm{Re}\,\Omega(x_1)}] \tag{17.5}$$

for (x_1,ν^1) in Eq. (14.1).

Now, we can choose K_N and c_N'' such that

$$K_N = \frac{2B_N}{\|\gamma\|' \sin 2\sigma\epsilon - 2H} \tag{17.6}$$

and c_N'' satisfies

$$K_N\{c_N'' \max_{\underline{\Theta}\leqslant\tau\leqslant\overline{\Theta}} \|X(\tau)\|\}^N < c_N. \tag{17.7}$$

These inequalities are needed in defining $\Phi^{(\ell)}(x,\nu)$ by Eq. (16.3).

Analyticity—First of all, when x_1 is fixed, Eq. (17.5) implies that the integral given by Eq. (17.1) converges uniformly with respect to ν^1. Thus $\Phi^{(1)}(x_1,\nu^1)$ is holomorphic in ν^1 for $[\nu^1] \leqslant c_N''[X(\arg x)]$ when x_1 is fixed.

Next we shall prove that $\Phi^{(1)}(x_1,\nu^1)$ is holomorphic in x_1 for Eq. (17.3) when ν^1 is fixed. This is trivial when $\mu = 0$. Thus, it is sufficient to prove for $\mu \neq 0$.

Let x_0 be a point in Eq. (17.3) and sufficiently near x_1, and we want to show that

$$\int_{\Gamma_{x_1}} f^{(1)}\big(x, V(x)\big)dx = \int_{\Gamma_{x_0}} f^{(1)}\big(x, V(x)\big)dx + \int_{\overline{x_0x_1}} f^{(1)}\big(x, V(x)\big)dx. \tag{17.8}$$

In fact, let t_0 and t_1 be, respectively, the intersections of the paths Γ_{jx_0} and Γ_{jx_1} with a small circle $|x| = \delta$. Since ν^1 is fixed and $f_j^{(1)}(x, V(x))$ is holomorphic in Eq. (17.3), the jth component of Eq. (17.8) is an immediate consequence of Cauchy's integral theorem and

$$\left|\int_{\widehat{t_0t_1}} f_j^{(1)}\big(x, V(x)\big)\,dx\right| \to 0 \qquad \text{as } \delta \to 0. \tag{17.9.j}$$

Here $\widehat{t_0t_1}$ denotes the circular arc of $|x| = \delta$ in Eq. (17.3) joining t_0 with t_1. However, from the construction of Γ_{jx_0} and Γ_{jx_1}, we know that Re $\Omega_j(x) > 0$ for x on $\widehat{t_0t_1}$. Thus the left-hand side of Eq. (17.9.j) tends to zero exponentially as δ tends to zero, and Eq. (17.8) is proved component-wise.

Now let $V(x)$ be specifically denoted by $W(x,x_1,\nu^1)$, namely, $W(x_1,x_1,\nu^1) = \nu^1$. Let $\hat{\nu}^1 = W(\hat{x}_1,x_1,\nu^1)$. Then, by Eq. (17.8)

$$\begin{aligned}
\Phi^{(1)}(x_1,\nu^1) - \Phi^{(1)}(\hat{x}_1,\nu^1) &= \left\{\Phi^{(1)}(x_1,\nu^1) - \Phi^{(1)}(\hat{x}_1,\hat{\nu}^1)\right\} \\
&\quad + \left\{\Phi^{(1)}(\hat{x}_1,\hat{\nu}^1) - \Phi^{(1)}(\hat{x}_1,\nu^1)\right\} \\
&= \int_{\Gamma_{x_1}} f^{(1)}\big(x,W(x,x_1,\nu^1)\big)dx - \int_{\Gamma_{\hat{x}_1}} f^{(1)}\big(x,W(x,\hat{x}_1,\hat{\nu}^1)\big)dx \\
&\quad + \left\{\Phi^{(1)}(\hat{x}_1,\hat{\nu}^1) - \Phi^{(1)}(\hat{x}_1,\nu^1)\right\} \\
&= -\int_{\overline{x_1\hat{x}_1}} f^{(1)}\big(x,W(x,x_1,\nu^1)\big)dx + \left\{\Phi^{(1)}(\hat{x}_1,\hat{\nu}^1) - \Phi^{(1)}(\hat{x}_1,\nu^1)\right\}.
\end{aligned}$$

Thus, we have that

$$\lim_{\hat{x}_1 \to x_1} \frac{\Phi^{(1)}(x_1,\nu^1) - \Phi^{(1)}(\hat{x}_1,\nu^1)}{\hat{x}_1 - x_1} = -f^{(1)}(x_1,\nu^1) + \Phi_\nu^{(1)}(x,\nu)\frac{d\nu}{dx}\bigg|_{\substack{\nu = \nu^1 \\ x = x_1}}$$

exists, since we have just proved that the matrix $\Phi_\nu^{(1)}(x_1,\nu^1)$ exists. Therefore, $\Phi^{(1)}(x_1,\nu^1)$ is holomorphic with respect to x_1 for Eq. (17.3) when ν^1 is fixed.

Hence by Hartog's theorem, $\Phi^{(1)}(x_1,\nu^1)$ is holomorphic in (x_1,ν^1) for Eq. (14.1).

18. The Functions $\{\Phi^{(\ell)}(x,\nu)\}$

Let $\Pi(\ell)$ denote the proposition

$\Pi(\ell)$. (i) *The function* $\Phi^{(\ell)}(x_1,\nu^1)$ *is well defined and holomorphic for* (x_1,ν^1) *in Eq. (14.1);*
(ii) $\Phi^{(\ell)}(x_1,\nu^1)$ *satisfies*

$$[\Phi^{(\ell)}(x_1,\nu^1) - \Phi^{(\ell-1)}(x_1,\nu^1)] \leqslant \frac{B_N}{H}\left(\frac{2H}{\|\gamma\|' \sin 2\sigma\epsilon}\right)^{\ell} \|\nu^1\|^N [e^{-\mathrm{Re}\,\Omega(x_1)}] \qquad (18.1.\ell)$$

and

$$[\Phi^{(\ell)}(x_1,\nu^1)] \leqslant \frac{2B_N}{\|\gamma\|' \sin 2\sigma\epsilon}\left\{1+\ldots+\left(\frac{2H}{\|\gamma\|' \sin 2\sigma\epsilon}\right)^{\ell-1}\right\}\|\nu^1\|^N [e^{-\mathrm{Re}\,\Omega(x_1)}] \qquad (18.2.\ell)$$

for (x_1,ν^1) *in Eq. (14.1).*

We have seen that $\Pi(\ell)$ is valid in Section 16 above.

Suppose that $\Pi(\ell)$ is true for $\ell = 1,2,\ldots,a$. We want to show that $\Pi(a+1)$ is true.

First of all, by Eqs. (17.6), (17.7), and (18.2) with $\ell = a$, the function $\hat{f}(x, V(x))$; $1_m(e^{\Omega(x)})\Phi^{(a)}(x, V(x))$ is well defined and holomorphic in Eq. (17.3). Thus, $\Phi^{(a+1)}(x_1,\nu^1)$ is given by Eq. (16.3) for $\ell + 1 = a + 1$, which does exist, by the same reasoning as that in the first subsection of Section 17.

By Eqs. (13.8), (18.1) for $\ell = a$, and (15.5), we have

$$[\Phi^{(a+1)}(x_1,\nu^1) - \Phi^{(a)}(x_1,\nu^1)] \leqslant \frac{B_N}{H}\left(\frac{2H}{\|\gamma\|' \sin 2\sigma\epsilon}\right)^{a+1} \|\nu^1\|^N [e^{-\mathrm{Re}\,\Omega(x_1)}] \qquad (18.1.\,a+1)$$

and, consequently,

$$[\Phi^{(a+1)}(x_1,\nu^1)] \leqslant \frac{2B_N}{\|\gamma\|' \sin 2\sigma\epsilon}\left\{1+\ldots+\left(\frac{2H}{\|\gamma\|' \sin 2\sigma\epsilon}\right)^{a}\right\}\|\nu^1\|^N [e^{-\mathrm{Re}\,\Omega(x_1)}]. \qquad (18.2.a+1)$$

Thus, $\Pi(a + 1)$ is true. Hence, by mathematical induction, $\Pi(\ell)$ is true for all positive integers ℓ. Therefore Step I is proved and we have the estimates given by Eqs. (18.1.ℓ) and (18.2.ℓ).

19. Convergence of $\{\Phi^{(\ell)}(x,\nu)\}$

By Eqs. (18.1.ℓ), (13.8), (15.5), (17.6), and (13.9), we have

$$[\Phi^{(\ell+1)}(x_1,\nu^1) - \Phi^{(\ell)}(x_1,\nu^1)] \leqslant \frac{K_N}{2^{\ell}} \|\nu^1\|^N [e^{-\mathrm{Re}\,\Omega(x_1)}] \qquad (19.1)$$

for x_1,ν^1 in Eq. (14.1). Thus, the sum

$$\Phi^{(\ell)}(x_1,\nu^1) = \sum_{a=0}^{\ell-1}\{\Phi^{(a+1)}(x_1,\nu^1) - \Phi^{(a)}(x_1,\nu^1)\} \qquad (19.2)$$

converges absolutely and uniformly in any compact subset of Eq. (14.1) as ℓ tends to ∞. Denote the limit function by $\varphi(x_1,\nu^1)$. Since each term $\Phi^{(\ell)}(x_1,\nu^1)$ is holomorphic in Eq. (14.1), $\varphi(x_1,\nu^1)$ is also holomorphic in Eq. (14.1). Moreover, we have

$$[\varphi(x_1,\nu^1)] \leqslant K_N \|\nu^1\|^N [e^{-\mathrm{Re}\,\Omega(x_1)}] \tag{19.3}$$

for (x_1,ν^1) in Eq. (14.1). Thus Step II is proved.

20. Integral Expression of $\varphi(x,\nu)$

We want to show that

$$\varphi(x_1,\nu^1) = \int_{\Gamma_{x_1}} x^{-\sigma-1} \mathbf{1}_m(e^{-\Omega(x)}) \hat{f}\left(x, V(x);\ \lim_{\ell\to\infty} \mathbf{1}_m(e^{\Omega(x)})\Phi^{(\ell)}\big(x, V(x)\big)\right) dx. \tag{20.1}$$

Let the jth component of φ be φ_j; it is sufficient to prove that for a given $\delta > 0$, there exists a positive integer $L(\delta,x_1)$ such that

$$\left| \int_{\Gamma_{jx_1}} x^{-\sigma-1} e^{-\Omega_j(x)} \left\{ \hat{f}_j\Big(x, V(x);\ \varphi\big(x, V(x)\big)\Big) - \hat{f}_j\Big(x, V(x);\ \Phi^{(\ell)}\big(x, V(x)\big)\Big) \right\} dx \right| < \delta \tag{20.2}$$

for $\ell \geqslant L(\delta,x_1)$ and all indices j.

In fact, since the integral in the left member of Eq. (20.2) exists at the origin, we can choose a point x_{ij}^0, independent of ℓ, on Γ_j'' such that the portion of the integral in Eq. (20.2) from the origin to x_{ji}^0 is less than $\delta/2$. On the other hand, by the uniform convergence of $\Phi^{(\ell)}$ to φ, we can choose $L(\delta,x_1)$ such that, when $\ell \geqslant L(\delta,x_1)$, the portion of the integral in Eq. (20.2) along Γ_{jx_1} from x_{j1}^0 to x_1 is less than $\delta/2$. Hence, Eq. (20.2) is proved, and consequently, Step III is shown.

21. $\varphi(x, V(x))$ as a Solution of Eq. (13.3)

For the sake of simplicity, rewrite the integral equation satisfied by $\varphi(x,\nu)$ as

$$\varphi(x_1,\nu^1) = \int_{\Gamma_{x_1}} \Psi\big(x, V(x)\big)\, dx \tag{21.1}$$

where

$$\Psi(x,\nu) = x^{-\sigma-1} \mathbf{1}_m(e^{-\Omega(x)}) \hat{f}\big(x, \nu;\ \mathbf{1}_m(e^{\Omega(x)})\varphi(x,\nu)\big)$$

and $V(x) = W(x,x_1,\nu^1)$. In order to show that $\varphi(x, V(x))$ satisfies Eq. (13.3) whenever $(x, V(x))$ is in Eq. (14.1), it is sufficient to prove that

$$\frac{d}{dx_0}\varphi(x_0,\nu^0) = \Psi(x_0,\nu^0) \tag{21.2}$$

where $\nu^0 = W(x_0,x_1,\nu^0)$.

Since $W(x,x_0,\nu^0) = W(x,x_1,\nu^1)$, Eq. (21.1) can be written as

$$\varphi(x_0,\nu^0) = \int_{\Gamma_{x_0}} \Psi(x,x_0,\nu^0))dx. \tag{21.3}$$

Hence,

$$\frac{d}{dx_0}\varphi(x_0,\nu^0) = \Psi(x_0,\nu^0) + \int_{\Gamma_{x_0}} \frac{\partial\Psi(x,W)}{\partial W}\left\{\frac{\partial W(x,x_0,\nu^0)}{\partial x_0} + \frac{\partial W(x,x_0,\nu^0)}{\partial \nu_0}\cdot\frac{\partial W(x_0,x_1,\nu^1)}{\partial x_0}\right\}dx. \tag{21.4}$$

However, for any constant ξ, the quantity $\eta = W(\xi,x,\nu)$ is an integral of the equation $x\nu' = \mathbf{1}_n(\mu)\nu$. Thus, $\eta = W(\xi,x_0,\nu^0) = W(\xi,x_1,\nu^1)$, and

$$\frac{d}{dx_0}W(\xi,x_0,\nu^0) = 0.$$

Hence the expression in the braces of the integrand of Eq. (21.4) vanishes identically. Thus, Eq. (21.2) holds and Step IV is proved.

22. Uniqueness

Suppose that there are two solutions of Eq. (13.3) satisfying Eq. (14.3). Let $\psi(x,V(x))$ be the difference of these two solutions. Then there exists a positive constant K such that

$$[\psi(x_1,\nu^1)] \leqslant K\|\nu^1\|^N[e^{-\mathrm{Re}\Omega(x_1)}] \tag{22.1}$$

for (x_1,ν^1) in Eq. (14.1). By the Lipschitz condition given in Eq. (13.8), and by Eqs. (15.5) and (13.9), we have

$$[\psi(x_1,\nu^1)] \leqslant \frac{K}{2}\|\nu^1\|^N[e^{-\mathrm{Re}\Omega(x_1)}]$$

for (x_1,ν^1) in Eq. (14.1). Repeating this process, we have, for any positive integer p,

$$[\psi(x_1,\nu^1)] \leqslant \frac{K}{2^p}\|\nu^1\|^N[e^{-\mathrm{Re}\Omega(x_1)}]$$

for (x_1,ν^1) in Eq. (14.1). Hence,

$$\psi(x_1,\nu^1) \equiv 0$$

for (x_1,ν^1) in Eq. (14.1), and Step V is proved.

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13. ABSTRACT

Let $1_n(\mu) = \text{diag}\,(\mu_1,\ldots,\mu_n)$ for given complex μ_k. If $\text{Re}\,\mu_k \geqslant 0$ $(1 \leqslant k \leqslant n)$, then the $(m+n)$-system $x^{\sigma+1}y' = F(x,z)y$, $xz' = 1_n(\mu)z$ is simplified to $x^{\sigma+1}Y' = G(x,Z)Y$, $xZ' = 1_n(\mu)Z$ by a transformation T defined as $y = Y + P(x,Z)Y$, $z = Z$ in a sector having property $\mathfrak{J}$ with respect to $\left\{(\lambda_i - \lambda_j)(\sigma x^\sigma)^{-1} \,|\, i,j = 1,\ldots,s,\ (i \neq j)\right\}$, where λ_i $(i=1,2,\ldots,s)$ are distinct eigenvalues of $F(0,0)$ and $G(x,Z)$ is in block-diagonal form agreeing with the Jordan canonical form of $F(0,0)$.

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