

**FR-7007**

**A Review of Narrowband Ambiguity  
Functions**

**D.A. Swick  
Signal Processing Branch  
Acoustics Division**

**February 27, 1970**

## DOCUMENT CONTROL DATA - R &amp; D

(Security classification of title, body of abstract and indexing annotation must be entered when the overall report is classified)

1. ORIGINATING ACTIVITY (Corporate author) Naval Research Laboratory Washington, D.C. 20390		2a. REPORT SECURITY CLASSIFICATION Unclassified	
		2b. GROUP	
3. REPORT TITLE A REVIEW OF NARROWBAND AMBIGUITY FUNCTIONS			
4. DESCRIPTIVE NOTES (Type of report and inclusive dates) An interim report; work on the problem continues.			
5. AUTHOR(S) (First name, middle initial, last name) D. A. Swick			
6. REPORT DATE February 27, 1970		7a. TOTAL NO. OF PAGES 36	7b. NO. OF REFS 41
8a. CONTRACT OR GRANT NO. NRL Problem S02-15		9a. ORIGINATOR'S REPORT NUMBER(S) NRL Report 7007	
b. PROJECT NO. RF 05-552-401-6075		9b. OTHER REPORT NO(S) (Any other numbers that may be assigned this report)	
c.			
d.			
10. DISTRIBUTION STATEMENT This document has been approved for public release and sale; its distribution is unlimited.			
11. SUPPLEMENTARY NOTES		12. SPONSORING MILITARY ACTIVITY Department of the Navy (Office of Naval Research), Washington, D. C. 20360	
13. ABSTRACT The narrowband ambiguity function has been used to study the space-time resolution of radar and sonar signals. Its derivation, properties, and generalizations are reviewed in a unifying notation. A "most general ambiguity function" is defined, from which both narrowband and wideband ambiguity functions can be derived.			

DD FORM 1473 (PAGE 1)

33

S/N 0101-807-6801

Security Classification

14. KEY WORDS	LINK A		LINK B		LINK C	
	ROLE	WT	ROLE	WT	ROLE	WT
Narrowband ambiguity function Radar targets Ambiguity function (woodward) Wideband ambiguity function Doppler effect Sonar targets Echo resolution						

## CONTENTS

Abstract	ii
Problem Status	ii
Authorization	ii
INTRODUCTION	1
RESOLUTION AND AMBIGUITY	1
Range Ambiguity	1
Ambiguity in Range and Velocity	2
SOME PROPERTIES OF THE GENERALIZED AUTO-CORRELATION FUNCTION AND THE WOODWARD AMBIGUITY FUNCTION	4
Relationship to Matched Filter	5
Convolution Theorems	5
Uniqueness Theorem	6
Complex Energy Density Function	6
EXAMPLES OF NARROWBAND AMBIGUITY FUNCTIONS	7
Single Gaussian Pulse	7
Single Rectangular Pulse	8
Modulated Uniform Pulse Train	9
Gaussian Pulse Train	11
Single Gaussian Pulse with Linear Frequency Modulation	12
The Ideal Ambiguity Function and its Approximation	13
Pseudo-Random Sequences	13
GENERALIZATIONS OF NARROWBAND AMBIGUITY FUNCTIONS	15
Cross-Ambiguity Function	15
Convolution Theorems	17
Invariance Relations for the Real and Imaginary Parts of Ambiguity Functions of Analytic Waveforms	17
The Most General Ambiguity Function	18
Angular Ambiguity Function	19
ACKNOWLEDGMENT	21
REFERENCES	22
APPENDIX A – Proofs of Theorems 1 Through 4	25
APPENDIX B – GACF of a Single Gaussian Pulse	27
APPENDIX C – GACF of a Train of Delta Functions	28
APPENDIX D – Rotation of Axes of the Ambiguity Diagram of a Single Gaussian Pulse with Linear Frequency Modulation	29
APPENDIX E – Multiplicative Property of Pseudo-Random Signals	30

## ABSTRACT

The narrowband ambiguity function has been used to study the space-time resolution of radar and sonar signals. Its derivation, properties, and generalizations are reviewed in a unifying notation. A "most general ambiguity function" is defined, from which both narrowband and wideband ambiguity functions can be derived.

## PROBLEM STATUS

This is an interim report; work on the problem continues.

## AUTHORIZATION

NRL Problem S02-15  
Project RF 05-552-401-6075

Manuscript submitted October 16, 1969.

# A REVIEW OF NARROWBAND AMBIGUITY FUNCTIONS

## INTRODUCTION

The narrowband ambiguity function was introduced by Woodward (1) in 1953. Its properties have been extensively explored and used to study range and velocity resolution of radar and sonar targets by Woodward (1) and others (2-41). The wideband ambiguity function is of more recent origin and has not yet been as fully developed. References to both functions are scattered through the literature, in widely varying notations. This report reviews the derivations and properties of the narrowband function and some of its generalizations, using a consistent notation. A companion report ("A Review of Wideband Ambiguity Functions,") is concerned with the wideband function and the relationship between the two functions.

These reports originated as a set of notes for a Branch Seminar and have been revised for more widespread distribution. The intent was to elaborate on and explain Woodward's (1) Chapter 7 and then to discuss some of the subsequent work. To facilitate reference to Woodward, the initial sections follow his work closely. Notational changes were necessary for consistency in the sequel. Details of some of the developments have been relegated to the appendices.

## RESOLUTION AND AMBIGUITY

### Range Ambiguity

Let a signal transmitted at time  $t$  be represented by the real part of the analytic† signal  $\Psi(t)$ . We consider first the problem of range resolution of point targets, where range is determined by the known velocity of propagation and the measured delay in the signal echo. We assume no attenuation. To achieve maximum resolution, we would like the echo  $\Psi(t-\tau)$  to differ as much as possible from  $\Psi(t)$ . Using a mean-square criterion, we would like‡

$$\int |\Psi(t) - \Psi(t-\tau)|^2 dt \quad (1)$$

to be as large as possible, except, of course, near  $\tau = 0$ . That is, we wish to maximize

$$\begin{aligned} & \int [\Psi(t) - \Psi(t-\tau)] [\Psi^*(t) - \Psi^*(t-\tau)] dt \\ &= \int |\Psi(t)|^2 dt + \int |\Psi(t-\tau)|^2 dt - \int \Psi(t) \Psi^*(t-\tau) dt - \int \Psi(t-\tau) \Psi^*(t) dt \\ &= 2E - 2R \int \Psi(t) \Psi^*(t-\tau) dt \end{aligned}$$

where  $\int |\Psi(t)|^2 dt = \int |\Psi(t-\tau)|^2 dt = E$ , the total energy of the analytic signal or twice the total energy of the real signal. Equivalently, we wish to minimize

†A complex signal is said to be analytic if its imaginary part is the Hilbert transform of its real part:  $Im[\Psi(t)] = H\{Re[\Psi(t)]\}$ .

‡Here and in the sequel integration is over the entire space of the appropriate dimension,  $\pm\infty$  for all single integrals, unless otherwise stated.

$$R^{\mathcal{L}} [\int \Psi(t) \Psi^*(t-\tau) dt] , \quad (2)$$

except near  $\tau = 0$ . Let

$$\Psi(t) = u(t) \exp(i\omega t) . \quad (3)$$

Then

$$\Psi^*(t-\tau) = u^*(t-\tau) \exp[-i\omega(t-\tau)] ,$$

and Eq. (2) becomes

$$R^{\mathcal{L}} [\exp(i\omega\tau) \int u(t) u^*(t-\tau) dt] , \quad (4)$$

which oscillates with  $\tau$ .

The requirement is that  $|R(\tau)|$  be as small as possible, where

$$R(\tau) \triangleq \int u(t) u^*(t-\tau) dt \quad (5)$$

is the complex autocorrelation function, and the symbol  $\triangleq$  means "is equal by definition to." Since†

$$\mathcal{F}_{t \rightarrow f} [u(t-\tau)] = \int u(t-\tau) \exp(-2\pi i f t) dt = \exp(-2\pi i f \tau) U(f) ,$$

Parseval's theorem gives us from Eq. (5)

$$R(\tau) = \int |U(f)|^2 \exp(2\pi i f \tau) df = \mathcal{F}_{f \rightarrow \tau}^{-1} [|U(f)|^2] . \quad (6)$$

Furthermore,  $R(0) = \int |u(t)|^2 dt = \int |U(f)|^2 df = E$ .

As a measure of total signal ambiguity, Woodward (1) defines the time-resolution constant

$$T \triangleq \frac{1}{E^2} \int |R(\tau)|^2 d\tau = \frac{1}{E^2} \int |U(f)|^4 df , \quad (7)$$

where the last equality again uses Parseval's theorem.

#### Ambiguity in Range and Velocity

If the effect of moving targets is assumed to be adequately approximated by a simple shift in frequency, we can define in an analogous fashion in terms of the frequency shift a "frequency autocorrelation function"

$$\kappa(\Phi) \triangleq \int U(f) U^*(f-\Phi) df \quad (8)$$

$$= \int |u(t)|^2 \exp(-2\pi i \Phi t) dt = \mathcal{F}_{t \rightarrow \Phi} [ |u(t)|^2 ] , \quad (9)$$

†In general,  $\mathcal{F}_{x \rightarrow y} [f(x)] = g(y)$  symbolizes the Fourier transform  $g(y) = \int f(x) \exp(-2\pi i xy) dx$ . The inverse transform will be symbolized by  $\mathcal{F}_{y \rightarrow x}^{-1} [g(y)] = f(x) = \int g(y) \exp(2\pi i xy) dy$ .

since  $\mathcal{F}_{ft}^{-1}[U(f-\Phi)] = \exp(2\pi i\Phi t) u(t)$ .

Without loss of generality we can let  $E = 1$ , and by analogy with Eq. (7) define the frequency resolution constant

$$F \triangleq \int |\kappa(\Phi)|^2 d\Phi = \int |u(t)|^4 dt, \quad (10)$$

where the second form comes from Eq. (9), using Parseval's theorem.

If the targets are at different ranges and are moving with different radial velocities, we need a combined time and frequency correlation function (two-dimensional correlation function). For the time being we consider only the narrowband approximation:

The Doppler effect is approximated by a frequency shift  $\Phi$ , constant across the signal bandwidth.

The echo of  $\Psi(t)$  is thus given by  $\Psi(t-\tau) \exp[2\pi i\Phi(t-\tau)]$ . Again, we let  $\Psi(t) = u(t) \exp(i\omega t) = u(t) \exp(2\pi i f t)$ . The function to be minimized is now, by analogy with Eq. (2),

$$\begin{aligned} & R^2 \{ \exp(2\pi i\Phi\tau) \int \Psi(t) \Psi^*(t-\tau) \exp(-2\pi i\Phi t) dt \} \\ & = R^2 \{ \exp[2\pi i(f+\Phi)\tau] \int u(t) u^*(t-\tau) \exp(-2\pi i\Phi t) dt \}. \end{aligned}$$

We require the modulus of the combined time and frequency correlation function

$$\chi(\tau, \Phi) \triangleq \int u(t) u^*(t-\tau) \exp(-2\pi i\Phi t) dt \quad (11)$$

to be as small as possible, except near  $|\chi(0,0)| = E = 1$ .

The ambiguity function (Woodward ambiguity function, narrowband ambiguity function,  $n$ - $b$  autoambiguity function) is defined as

$$A(\tau, \Phi) \triangleq |\chi(\tau, \Phi)|^2. \quad (12)$$

Other definitions of the generalized autocorrelation function (GACF) which lead to the same ambiguity function appear in the literature. For example, let

$$\begin{aligned} \tilde{\chi}(\tau, \Phi) & \triangleq \int u(t+\tau/2) u^*(t-\tau/2) \exp(-2\pi i\Phi t) dt \\ & = \int u(\eta) u^*(\eta-\tau) \exp[-2\pi i\Phi(\eta-\tau/2)] d\eta \\ & = \exp(\pi i\Phi\tau) \chi(\tau, \Phi), \end{aligned}$$

so that

$$|\tilde{\chi}(\tau, \Phi)|^2 = |\chi(\tau, \Phi)|^2.$$

The present definition, Eq. (11), has some useful transformation properties which will be discussed later.

The name of the ambiguity function stems from the fact that it does not uniquely determine a waveform. For example, let the GACF of Eq. (11) be labeled  $\chi_u(\tau, \Phi)$  to distinguish it from  $\chi_v(\tau, \Phi) = \int v(t) v^*(t-\tau) \exp(-2\pi i\Phi t) dt$ , where  $v(t) \neq u(t)$ . Ambiguity is apparent if we consider the frequency-shifted and time-delayed waveform

$$v(t) = u(t - \epsilon) \exp(2\pi i \theta t) ,$$

where  $\epsilon \neq 0$  and  $\theta \neq 0$ . Then

$$|X_v(\tau, \Phi)|^2 = |X_u(\tau, \Phi)|^2 .$$

#### SOME PROPERTIES OF THE GENERALIZED AUTOCORRELATION FUNCTION AND THE WOODWARD AMBIGUITY FUNCTION

$$\chi(\tau, 0) = \int u(t) u^*(t - \tau) dt = R(\tau) \quad (13)$$

$$\chi(0, \Phi) = \int |u(t)|^2 \exp(-2\pi i \Phi t) dt = \kappa(\Phi) \quad (14)$$

$$\chi(0, 0) = \int |u(t)|^2 dt = E = 1 . \quad (15)$$

##### Theorem 1

The GACF as an integral in the frequency domain is given as

$$\chi(\tau, \Phi) = \int U^*(f) U(f + \Phi) \exp(2\pi i f \tau) df . \quad (16)$$

The proof of theorems 1 through 4 will be found in Appendix A.

##### Theorem 2

The combined time-frequency resolution constant is given as

$$\iint |\chi(\tau, \Phi)|^2 d\tau d\Phi = 1 . \quad (17)$$

"The effective 'area of ambiguity' in the time-frequency domain is independent of the transmitted waveform and is equal to unity." This theorem is called the "Radar Uncertainty Principle" by Siebert (3). It is one of the most important properties of the narrow-band ambiguity function and will be discussed further after some examples have been considered.

##### Theorem 3

$$|\chi(\tau, \Phi)|^2 \leq \chi^2(0, 0) = 1 . \quad (18)$$

##### Theorem 4 (Siebert's theorem (4))

The ambiguity function is its own two-dimensional Fourier transform:

$$\iint |\chi(\tau, \Phi)|^2 \exp[-2\pi i (f\tau - \Phi t)] d\tau d\Phi = |\chi(t, f)|^2 . \quad (19)$$

Theorem 2 can be obtained from Eq. (19) with  $t = f = 0$ .

Symmetry

$$\begin{aligned}
X(-\tau, -\Phi) &= \int u(t) u^*(t+\tau) \exp(2\pi i \Phi t) dt = \int u(\eta-\tau) u^*(\eta) \exp[2\pi i \Phi(\eta-\tau)] d\eta \\
&= \exp(-2\pi i \Phi \tau) X^*(\tau, \Phi) \\
|X(-\tau, -\Phi)|^2 &= |X(\tau, \Phi)|^2 .
\end{aligned} \tag{20}$$

Relationship to Matched Filter

Consider a filter matched to a real transmitted signal  $u(t)$ ; the impulse response is  $h(t) = u(T-t)$ . Let the input to this filter be the time-delayed and frequency-shifted waveform

$$x(t) = u(t+\tau) \exp[-2\pi i \Phi(t+\tau)] .$$

Then the output is  $y(t) = x(t) * h(t) = \int x(\alpha) h(t-\alpha) d\alpha$

$$\begin{aligned}
y(t) &= \int u(\alpha+\tau) \exp[-2\pi i \Phi(\alpha+\tau)] u(T-t+\alpha) d\alpha \\
&= \int u(\beta) u(\beta-\tau+T-t) \exp(-2\pi i \Phi \beta) d\beta
\end{aligned}$$

$$y(T) = X(\tau, \Phi) .$$

Thus, the GACF is the output of a filter matched to the transmitted signal in response to an echo with constant time delay and constant frequency shift (narrowband approximation to Doppler effect).

Convolution TheoremsTheorem 5

If two functions are convolved in the time (frequency) domain, their generalized autocorrelation functions are convolved in the time (frequency) coordinate.

Proof of frequency convolution case

Let  $u(t) = v(t) w(t)$ . Then  $U(f) = V(f) * W(f)$

$$\begin{aligned}
X_u(\tau, \Phi) &= \int U^*(f) U(f+\Phi) \exp(2\pi i f \tau) df \\
&= \iiint V^*(\mu) W^*(f-\mu) V(\nu) W(f+\Phi-\nu) \exp(2\pi i f \tau) d\mu d\nu df \\
&= \iiint V^*(\mu) V(\nu) W^*(\eta) W(\eta+\mu-\nu+\Phi) \exp[2\pi i (\eta+\mu)\tau] d\eta d\mu d\nu \\
&= \iint V^*(\mu) V(\nu) X_w(\tau, \mu-\nu+\Phi) \exp(2\pi i \mu \tau) d\mu d\nu \\
&= \iint V^*(\mu) V(\mu+\zeta) \exp(2\pi i \mu \tau) X_w(\tau, \Phi-\zeta) d\mu d\zeta \\
&= \int X_v(\tau, \zeta) X_w(\tau, \Phi-\zeta) d\zeta \\
&= X_v(\tau, \Phi) *_{\Phi} X_w(\tau, \Phi) ,
\end{aligned}$$

where  $\ast_{\Phi}$  indicates convolution with respect to  $\Phi$ . The proof of the time convolution case is similar. The operation of these two theorems is noncommutative (31).

### Uniqueness Theorem

#### Theorem 6

The function  $u(t)$  is uniquely determined to within a multiplicative constant of unit magnitude almost everywhere (a.e.) by its generalized autocorrelation function.

#### Proof

$$\begin{aligned} \int X_u(\tau, \Phi) d\tau &= \iint u(t) u^*(t-\tau) \exp(-2\pi i \Phi t) dt d\tau \\ &= \int u(t) \exp(-2\pi i \Phi t) \int u^*(\eta) d\eta dt = U(\Phi) U^*(0) . \end{aligned}$$

Therefore, if  $X_u(\tau, \Phi) = X_v(\tau, \Phi)$ ,  $V(\Phi) V^*(0) = U(\Phi) U^*(0)$  and hence if  $V^*(0) \neq 0$ ,  $V(\Phi) = c U(\Phi)$ , where  $c = U^*(0)/V^*(0)$ . Since two functions having the same Fourier transform are equal a.e., we have  $v(t) = cu(t)$ , a.e., and

$$X_u(\tau, \Phi) = X_v(\tau, \Phi) = \int cu(t) c^* u^*(t-\tau) \exp(-2\pi i \Phi t) dt = |c|^2 X_u(\tau, \Phi) .$$

so that  $|c| = 1$ .

### Complex Energy Density Function

Rihaczek (41) has recently defined a complex energy density function which may be obtained as the Fourier transform of the GACF.

Consider a real signal represented by  $R\ell[u(t)]$ , where  $u(t)$  is an analytic signal, and let

$$\int |u(t)|^2 dt = \int |U(f)|^2 df = E ,$$

where  $E$  is twice the total energy of the real signal.

Then  $|u(t)|^2$  represents power, or "energy density waveform," and  $|U(f)|^2$  is the energy density spectrum. If the autocorrelation function of  $u(t)$  is given by Eq. (5), then from Eq. (6), or by the Wiener-Khinchine theorem, the spectral density is

$$W(f) = \mathcal{F}_{\tau t} [R(\tau)] = |U(f)|^2 . \quad (21)$$

Similarly, the autocorrelation function in the frequency domain, Eq. (8), transforms into the "waveform density." From Eq. (9) we have

$$\mathcal{F}_{\Phi t}^{-1} [\kappa(\Phi)] = |u(t)|^2 . \quad (22)$$

By analogy, the two-dimensional Fourier transform of the GACF is

$$\begin{aligned}
\mathcal{F}_{\Phi t}^{-1} \mathcal{F}_{\tau f} [X(\tau, \Phi)] &\stackrel{\Delta}{=} \iint X(\tau, \Phi) \exp[-2\pi i(f\tau - \Phi t)] d\tau d\Phi \\
&= \iiint u(\eta) u^*(\eta - \tau) \exp(-2\pi i\Phi\eta) \exp[-2\pi i(f\tau - \Phi t)] d\eta d\tau d\Phi \\
&= u(t) \int u^*(t - \tau) \exp(-2\pi i f \tau) d\tau \\
&= u(t) \exp(-2\pi i f t) \int u^*(\zeta) \exp(2\pi i f \zeta) d\zeta \\
&= u(t) U^*(f) \exp(-2\pi i f t) .
\end{aligned} \tag{23}$$

Thus, just as the autocorrelation functions in time and in frequency each are the Fourier transforms of an energy density function, as shown in Eq. (21) and Eq. (22), we may define the complex energy density function in time and frequency from Eq. (23) as

$$\epsilon(t, f) \stackrel{\Delta}{=} u(t) U^*(f) \exp(-2\pi i f t) . \tag{24}$$

Then the energy density spectrum is

$$\int \epsilon(t, f) dt = |U(f)|^2 ;$$

the energy density waveform is

$$\int \epsilon(t, f) df = |u(t)|^2$$

and

$$\iint \epsilon(t, f) dt df = E ,$$

the total energy of the analytic signal.

The energy of the analytic signal within a "cell" of area  $TB$  centered at  $(t_0, f_0)$  in the time-frequency plane is given by

$$E_{TB} = \int_{t_0 - T/2}^{t_0 + T/2} \int_{f_0 - B/2}^{f_0 + B/2} \epsilon(t, f) df dt .$$

## EXAMPLES OF NARROWBAND AMBIGUITY FUNCTIONS

### Single Gaussian Pulse

We consider first an example discussed by Woodward (1). Only slightly more general than Woodward's simplest pulse is the single Gaussian pulse

$$u(t) = k \exp(-at^2) , \quad a > 0 . \tag{25}$$

where the parameter  $a$  determines the width of the pulse, and  $k = (2a/\pi)^{1/4}$  so that

$$\int u^2(t) dt = 1 .$$

It is shown in Appendix B that

$$\chi_u(\tau, \Phi) = \exp \left\{ -a\tau^2/2 - \pi^2\Phi^2/2a - \pi i\Phi\tau \right\} . \quad (26)$$

Thus,

$$|\chi_u(\tau, \Phi)|^2 = \exp \left[ -(a\tau^2 + \pi^2\Phi^2/a) \right] , \quad (27)$$

and contours of constant ambiguity are given by the ellipse  $a\tau^2 + \pi^2\Phi^2/a = \text{constant}$ , as shown in Fig. 1.

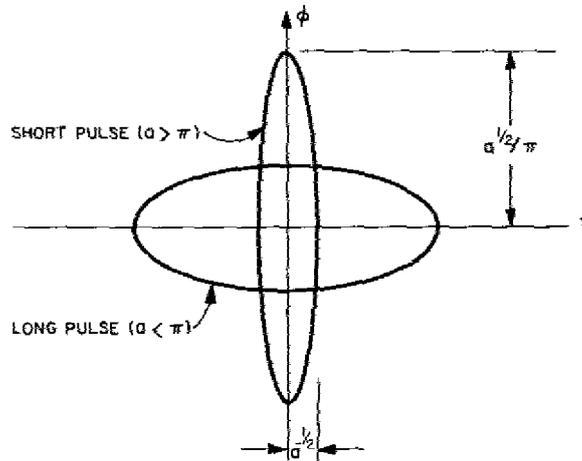


Fig. 1 - Curves of constant ambiguity:

$$|\chi|^2 = e^{-1}$$

Note that

$$|R_u(\tau)|^2 = |\chi_u(\tau, 0)|^2 = \exp(-a\tau^2)$$

and

$$|K_u(\Phi)|^2 = |\chi_u(0, \Phi)|^2 = \exp(-\pi^2\Phi^2/a)$$

so that in this case the ambiguity function is factorable:  $|\chi(\tau, \Phi)|^2 = |R(\tau)|^2 |K(\Phi)|^2$ . This result is not general. When it holds we have from Eqs. (7), (10), and (17) that  $TF = 1$ .

### Single Rectangular Pulse

Let  $u(t) = (2a)^{-1/2}$ ,  $|t| < a$ , zero elsewhere. In Woodward's (1) notation,  $u(t) = (2a)^{-1/2} \text{rect}(t/2a)$ . Then

$$\begin{aligned} \chi_u(\tau, \Phi) &= \mathcal{F}_{t\Phi}[u(t) u^*(t - \tau)] = \mathcal{F}_{t\Phi}\{(2a)^{-1} \text{rect} [(2t - \tau)/2(2a - |\tau|)]\} \\ &= \begin{cases} |(1 - |\tau|/2a)| \text{sinc} [(2a - |\tau|)\Phi] \exp(-\pi i\Phi\tau), & |\tau| \leq 2a \\ 0, & |\tau| > 2a \end{cases} \end{aligned} \quad (28)$$

and

$$|\chi_u(\tau, \Phi)|^2 = (1 - |\tau|/2a)^2 \text{sinc}^2 [(2a - |\tau|)\Phi], \quad |\tau| \leq 2a, \quad (29)$$

zero elsewhere, where  $\text{sinc } x \triangleq \sin \pi x / \pi x$ .

### Modulated Uniform Pulse Train

To facilitate Woodward's next example, the Gaussian pulse train, we consider a train of (Dirac) delta functions. Eschewing questions of rigor and using Woodward's notation, we let

$$w(t) = \sum_{-\infty}^{\infty} \delta(t - nT) = \text{rep}_T \delta(t).$$

Then

$$W(f) = \frac{1}{T} \text{comb}_{1/T} 1 = \frac{1}{T} \sum_{-\infty}^{\infty} \delta\left(f - \frac{n}{T}\right).$$

If the pulse repetition period  $T$  is unity, the train of delta functions Fourier transforms into itself.

The GACF of a train of delta functions is shown in Appendix C to be

$$\chi_w(\tau, \Phi) = \frac{1}{T} \sum_{n,m} \delta(\tau - nT) \delta\left(\Phi - \frac{m}{T}\right). \quad (30)$$

This "bed of nails" (5, 22, 33) is shown in Fig. 2, drawn as if the delta functions had finite amplitude.

Let a general pulse train be represented by

$$v(t) = \text{rep}_T u(t) = u(t) * \sum \delta(t - nT) = u(t) * w(t). \quad (31)$$

By Theorem 5 we have

$$\begin{aligned} \chi_v(\tau, \Phi) &= \chi_u(\tau, \Phi) *_{\tau} \chi_w(\tau, \Phi) \\ &= \frac{1}{T} \sum_{n,m} \delta\left(\Phi - \frac{m}{T}\right) \int \chi_u(\alpha, \Phi) \delta(\tau - nT - \alpha) d\alpha \\ &= \frac{1}{T} \sum_{n,m} \chi_u(\tau - nT, \Phi) \delta\left(\Phi - \frac{m}{T}\right). \end{aligned} \quad (32)$$

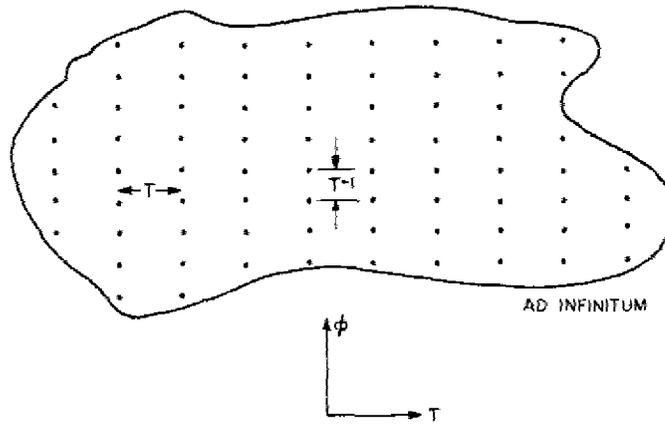


Fig. 2 - "Bed of Nails" generalized autocorrelation function

If the pulse train of Eq. (31) is, in turn, modulated by an envelope function  $x(t)$ , the resulting waveform and its spectrum may be represented by

$$y(t) = x(t)v(t) \quad \text{and} \quad Y(f) = X(f) * V(f).$$

Using Theorem 5 and Eq. (32) it follows that

$$\begin{aligned} X_y(\tau, \Phi) &= X_x(\tau, \Phi) *_{\Phi} X_v(\tau, \Phi) \\ &= \frac{1}{T} \sum_{n,m} \int X_x(\tau, \nu) X_u(\tau - nT, \Phi - \nu) \delta\left(\Phi - \frac{m}{T} - \nu\right) d\nu \\ &= \frac{1}{T} \sum_{n,m} X_x\left(\tau, \Phi - \frac{m}{T}\right) X_u\left(\tau - nT, \frac{m}{T}\right). \end{aligned}$$

Thus, the ambiguity function is

$$|X_y(\tau, \Phi)|^2 = \frac{1}{T^2} \sum_{k, \ell, m, n} X_x\left(\tau, \Phi - \frac{k}{T}\right) X_u\left(\tau - \ell T, \frac{k}{T}\right) X_x^*\left(\tau, \Phi - \frac{m}{T}\right) X_u^*\left(\tau - nT, \frac{m}{T}\right).$$

We now impose the additional conditions that the GACF of a single pulse vanishes for  $|\tau| > T$  and that the GACF of the envelope function vanishes for  $|\Phi| > 1/T$ . With these conditions we can write

$$X_u(\tau - \ell T, \Phi) X_u^*(\tau - nT, \Phi) = |X_u(\tau - \ell T, \Phi)|^2 \delta_{\ell n} \quad (33)$$

and

$$X_x\left(\tau, \Phi - \frac{k}{T}\right) X_x^*\left(\tau, \Phi - \frac{m}{T}\right) = \left|X_x\left(\tau, \Phi - \frac{k}{T}\right)\right|^2 \delta_{km},$$

where  $\delta_{mn}$  is the Kronecker delta. Hence,

$$|X_y(\tau, \Phi)|^2 = \frac{1}{T^2} \sum_{m,n} \left| X_x \left( \tau, \Phi - \frac{m}{T} \right) \right|^2 \left| X_u \left( \tau - nT, \frac{m}{T} \right) \right|^2. \quad (34)$$

For a rectangular pulse,  $u(t) = (2a)^{-1/2} \text{rect}(t/2a)$ , with a rectangular envelope,  $x(t) = (2b)^{-1/2} \text{rect}(t/2b)$ , Eq. (28) shows that the conditions Eq. (33) are approximately satisfied if  $2a < T \ll 2b$ . We then have

$$|X_y(\tau, \Phi)|^2 = \frac{(1 - |\tau|/2b)^2}{T^2} \sum_{n,m} \{(1 - |\tau - nT|/2a)^2 \times \text{sinc}^2[(2a - |\tau - nT|)m/T] \text{sinc}^2[(2b - |\tau|)(\Phi - m/T)]\}.$$

### Gaussian Pulse Train

If  $u(t)$  is a single short Gaussian pulse, we have from Eq. (26)

$$\begin{aligned} X_u(\tau - nT, \Phi) X_u^*(\tau - mT, \Phi) &= \exp \left\{ -\frac{a}{2} \left[ (\tau - nT)^2 + (\tau - mT)^2 \right] - \frac{\pi^2}{a} \Phi^2 - \pi i \Phi (m - n) T \right\} \\ &= (-1)^{(n-m)\Phi T} \exp \left\{ -\frac{a}{2} \left[ (\tau - nT)^2 + (\tau - mT)^2 \right] - \frac{\pi^2}{a} \Phi^2 \right\}. \end{aligned}$$

If  $n = m$ , this equation reduces to  $|X_u(\tau - nT, \Phi)|^2$  in agreement with the conditions in Eq. (33). However, for  $n \neq m$ , the product does not exactly vanish; there is some overlap of the Gaussian tails. We will have a close approximation if

$$\exp \left\{ -\frac{a}{2} \left[ (\tau - nT)^2 + (\tau - mT)^2 \right] \right\} \ll 1, \quad \forall m \neq n, \quad \forall \tau.$$

This requirement is satisfied if  $aT^2 \gg 0$ , as must be the case for a short Gaussian pulse — moderate  $T$  requires large  $a$ .

Similarly, if  $x(t)$  is a broad Gaussian envelope,  $x(t) = k' \exp(-bt^2)$ , we have again from Eq. (26) with  $b$  replacing  $a$ :

$$\begin{aligned} X_x \left( \tau, \Phi - \frac{n}{T} \right) X_x^* \left( \tau, \Phi - \frac{m}{T} \right) &= \exp \left\{ -b\tau^2 - \frac{\pi}{2b} \left[ \left( \Phi - \frac{n}{T} \right)^2 + \left( \Phi - \frac{m}{T} \right)^2 \right] - \frac{\pi i (m - n)}{T} \tau \right\} \\ &= (-1)^{(n-m)\tau/T} \exp \left\{ -b\tau^2 - \frac{\pi}{2b} \left[ \left( \Phi - \frac{n}{T} \right)^2 + \left( \Phi - \frac{m}{T} \right)^2 \right] \right\}. \end{aligned}$$

If  $m = n$ , this reduces to

$$\left| X_x \left( \tau, \Phi - \frac{n}{T} \right) \right|^2.$$

If  $m = n$ , we require  $1/bT^2 \gg 0$ . Thus, for  $T$  moderate, we must have  $b$  very small, consistent with the broad Gaussian envelope.

Thus, if  $b \ll T \ll a$ , the ambiguity function for a train of narrow Gaussian pulses with a broad envelope, obtained from Eqs. (34) and (27), is

$$|X_v(\tau, \Phi)|^2 = \frac{1}{T^2} \sum_{n,m} \exp \left[ -a(\tau - nT)^2 - \frac{\pi^2}{a} \left( \frac{m}{T} \right)^2 - b\tau^2 - \frac{\pi^2}{b} \left( \Phi - \frac{m}{T} \right)^2 \right] . \quad (35)$$

For each  $n$  and  $m$ , the contours of constant ambiguity are ellipses. Centers of ellipses are spaced at intervals of  $T$  in  $\tau$  and  $1/T$  in  $\Phi$ , and the overall ambiguity decreases exponentially in  $\tau$  and  $\Phi$ .

A sketch of this ambiguity function is given in Fig. 19 on page 122 of Ref. 1.

### Single Gaussian Pulse with Linear Frequency Modulation

If  $v(t) = u(t) \exp(ib t^2)$  for any  $u(t)$ , the "instantaneous frequency" is

$$f_i \triangleq \frac{1}{2\pi} \frac{d}{dt} (bt^2) = bt/\pi .$$

Thus,  $f_i$  varies linearly with time. The GACF is

$$\begin{aligned} X_v(\tau, \Phi) &= \int v(t) v^*(t - \tau) \exp(-2\pi i \Phi t) dt \\ &= \exp(-ib\tau^2) \int u(t) u^*(t - \tau) \exp[-2\pi i (\Phi - b\tau/\pi) t] dt \\ &= \exp(ib\tau^2) X_u(\tau, \Phi - b\tau/\pi) . \end{aligned}$$

Hence,

$$|X_v(\tau, \Phi)|^2 = |X_u(\tau, \Phi - b\tau/\pi)|^2 . \quad (36)$$

From Eq. (36) we see that

$$|X_v(\tau, b\tau/\pi)|^2 = |X_u(\tau, 0)|^2 .$$

That is, the ambiguity along a line  $\Phi = b\tau/\pi$ , whose slope is equal to the rate of change of the instantaneous frequency, is equal to the ambiguity along the  $\tau$  axis in the absence of frequency modulation.

For the single Gaussian pulse with linear FM, we have from Eqs. (27) and (36)

$$|X_v(\tau, \Phi)|^2 = \exp \left[ -a\tau^2 - \frac{\pi^2}{a} \left( \Phi - \frac{b\tau}{\pi} \right)^2 \right] .$$

Curves of constant ambiguity are again ellipses,

$$\left( a + \frac{b^2}{a} \right) \tau^2 - \frac{2\pi b}{a} \tau \Phi + \frac{\pi^2}{a} \Phi^2 = \text{constant} . \quad (37)$$

The eccentricity of these constant ambiguity ellipses depends on both  $a$  and  $b$ , and their axes are rotated with respect to the  $\tau, \Phi$  axes by an angle  $\theta$ , where

$$\tan 2\theta = 2\pi b / [\pi^2 - (a^2 + b^2)] , \quad (38)$$

as is shown in Appendix D.

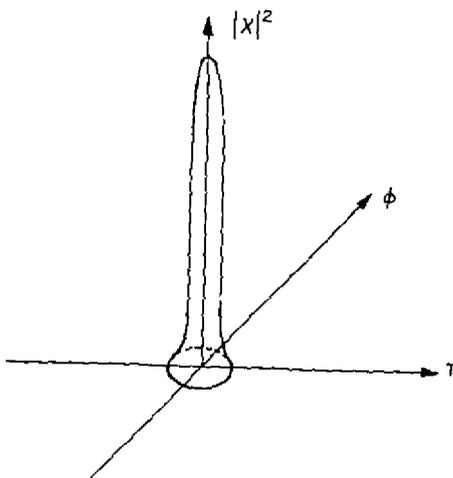
Note that in this case

$$|R(\tau)|^2 |\kappa(\Phi)|^2 = \exp \left[ - \left( a + \frac{b^2}{a} \right) \tau^2 \right] \exp (-\pi^2 \Phi^2 / a) = |X(\tau, \Phi)|^2$$

### The Ideal Ambiguity Function and its Approximation

We saw in Eq. (27) and in Fig. 1 that the parameter  $a$  suffices to reduce the ambiguity in either the  $\tau$  direction or the  $\Phi$  direction but not in both. Equation (37) shows that with two parameters it is possible to reduce the ambiguity both along the  $\tau$  axis and the  $\Phi$  axis but not simultaneously along a line  $\Phi = \tau \tan \theta$ . One would surmise that with additional parameters we might do better. Ideally, we would like the ambiguity function to be a delta function at the origin, but this we cannot achieve. We might hope to obtain as an approximation a narrow spike in both directions as shown in Fig. 3.

Fig. 3 - Approximation to ideal ambiguity function



Equation (35) and Woodward's Fig. 19 show that with a train of pulses we can reduce the ambiguity in all directions in the vicinity of the origin to an arbitrarily low level. We do this, however, at the expense of having additional peaks appear elsewhere. This is a consequence of Theorem 2; if we reduce the ambiguity in one place it must pop up elsewhere so that the total area of the ambiguity surface remains constant.

The closest to the ideal ambiguity function we can expect is the so-called "thumbtack" ambiguity function. This consists of a narrow spike surrounded by a uniformly low pedestal, with most of the volume lying under the pedestal. We do not know of any waveform which produces the "thumbtack" ambiguity function. For many applications, an ambiguity free region near the origin, as obtainable with a train of pulses, is sufficient.

### Pseudo-Random Sequences

We recall that the ambiguity function along the  $\tau$  axis is the square of the autocorrelation function of the signal (see Eq. (13)). Thus, to approximate the "thumbtack" ambiguity function, a necessary but not sufficient condition is that the autocorrelation function

of the signal be small near the origin. One practical signal known to have such an auto-correlation function is the signal generated by maximum length pseudo-random sequences. Such a sequence,  $\{a_i\}$ , may be specified by its recursion formula

$$a_i = \sum_{j=1}^n c_j a_{i-j}, \quad i = 0, 1, 2, \dots, \quad (39)$$

where  $c_j = 0$  or  $1$  and  $\sum$  indicates modulo 2 summation and by the initial conditions given by the values of

$$a_{-n}, \quad a_{-n+1}, \quad \dots, \quad a_{-1}.$$

A new sequence formed by the modulo 2 addition of a maximal-length pseudo-random sequence and a nontrivial shift of itself will be a shifted version of the original sequence. That is, if

$$b_i = a_i \oplus a_{i+T}, \quad \text{where } T \neq 0 \pmod{2^n - 1}$$

$$= \sum_{j=1}^n c_j a_{i-j} \oplus \sum_{j=1}^n c_j a_{i+T-j},$$

then

$$\begin{aligned} b_i &= \sum_{j=1}^n c_j (a_{i-j} \oplus a_{i+T-j}) \\ &= \sum_{j=1}^n c_j b_{i-j}. \end{aligned}$$

Thus,  $\{b_i\}$  obeys the same recursion relation as  $\{a_i\}$ . Since  $\{a_i\}$  contains all  $n$ -tuples (except the all zero  $n$ -tuple), these sequences are identical to within a phase shift.

This so-called "shift-and-add" property can be used to obtain the narrowband ambiguity function of signals generated by such sequences for time differences of  $\tau = k t_0$ , where  $k$  is an integer and  $t_0$  is the shifting period of the generator.

The additive group of integers modulo 2 is isomorphic to the group consisting of  $-1$  and  $1$  with multiplication as the group operation. If we let  $s(t)$  represent a signal obtained from a pseudo-random sequence of  $-1$ 's and  $1$ 's, the "shift-and-add" property becomes a "shift-and-multiply" property, as shown in Appendix E,

$$s(t) s(t - k t_0) = s(t - k' t_0),$$

where  $k$  and  $k'$  are integers. Thus from Eq. (11)

$$\begin{aligned}
 \chi(k t_0, \Phi) &= \int s(t) s(t - k t_0) \exp(-2\pi i \Phi t) dt \\
 &= \int s(t - k' t_0) \exp(-2\pi i \Phi t) dt \\
 &= \exp(-2\pi i k' \Phi t_0) S(\Phi) .
 \end{aligned}$$

Therefore,

$$|\chi(k t_0, \Phi)|^2 = |S(\Phi)|^2 .$$

That is, for  $\tau = k t_0$ , the narrowband ambiguity function of maximal-length pseudo-random signals is proportional to the power spectrum and is independent of  $k$ .

## GENERALIZATIONS OF NARROWBAND AMBIGUITY FUNCTIONS

### Cross-Ambiguity Function

An extension of the concept of ambiguity functions to two waveforms was defined by Stutt (7).

Let  $u_1(t)$  and  $u_2(t)$  be two complex waveforms. We define the generalized cross-correlation function as

$$\chi_{12}(\tau, \Phi) \triangleq \int u_1(t) u_2^*(t - \tau) \exp(-2\pi i \Phi t) dt \quad (40)$$

$$= \int U_2^*(f) U_1(f + \Phi) \exp(2\pi i f \tau) df , \quad (41)$$

and the cross-ambiguity function as

$$A_{12}(\tau, \Phi) = |\chi_{12}(\tau, \Phi)|^2. \quad (42)$$

These functions may be useful when it is desired to identify one of many possible waveforms. Some, but not all, of the properties of the autoambiguity function apply to the cross-ambiguity function. For example, by Parseval's theorem we have

$$\iint |\chi_{12}(\tau, \Phi)|^2 d\Phi = \int |u_1(t) u_2^*(t - \tau)|^2 dt .$$

so that

$$\iint |\chi_{12}(\tau, \Phi)|^2 d\Phi d\tau = \int |u_1(t)|^2 \int |u_2(t - \tau)|^2 d\tau dt = 1 ,$$

if the energy in both waveforms is normalized to unity. Furthermore,

$$\chi_{12}(\tau, 0) = \int u_1(t) u_2^*(t - \tau) dt ,$$

$$\chi_{12}(0, \Phi) = \int U_2^*(f) U_1(f + \Phi) df ,$$

but

$$\begin{aligned} |X_{12}(0,0)|^2 &= \left| \int u_1(t) u_2^*(t) dt \right|^2 \\ &\leq \int |u_1(t)|^2 dt \int |u_2(t)|^2 dt = 1 \end{aligned}$$

by the Cauchy-Schwarz inequality.

Therefore, if  $u_1(t) = k u_2(t)$ , where  $k$  is a constant,

$$|X_{12}(0,0)|^2 < 1, \quad (43)$$

in contrast with Eq. (15). Also,

$$\begin{aligned} |X_{12}(\tau, \Phi)|^2 &= \left| \int u_1(t) u_2^*(t - \tau) \exp(-2\pi i \Phi t) dt \right|^2 \\ &\leq \int |u_1(t)|^2 dt \int |u_2^*(t - \tau) \exp(-2\pi i \Phi t)|^2 dt = 1. \end{aligned} \quad (44)$$

It does not follow from this relationship that  $|X_{12}(\tau, \Phi)|^2 \leq |X_{12}(0,0)|^2$ .

In fact, if  $u_1(t) = u_2(t - \tau_0) \exp(2\pi i \Phi_0 t)$ ,  $\tau_0, \Phi_0 \neq 0$ , then equality holds in Eq. (44):  $|X_{12}(\tau_0, \Phi_0)|^2 = 1$ . If, in addition,

$$u_2(t - \tau) \exp(2\pi i \Phi t) = k u_2(t)$$

then from Eq. (43),

$$|X_{12}(0,0)|^2 < 1,$$

so that in this case  $|X_{12}(0,0)|^2 < |X_{12}(\tau_0, \Phi_0)|^2$ . In this case,

$$|X_{12}(0,0)|^2 = \left| \int u_2(t - \tau_0) \exp(2\pi i \Phi_0 t) u_2^*(t) dt \right|^2 = |X_u(\tau_0, \Phi_0)|^2.$$

By proof similar to that for Theorem 4 it can be shown that the generalization becomes

$$\iint |X_{12}(\tau, \Phi)|^2 \exp[-2\pi i (f\tau - \Phi t)] d\tau d\Phi = X_{11}(t, f) X_{22}^*(t, f). \quad (45)$$

In place of the symmetry relation, Eq. (20), we have

$$\begin{aligned} X_{12}(-\tau, -\Phi) &= \int u_1(t) u_2^*(t + \tau) \exp(2\pi i \Phi t) dt \\ &= \exp(-2\pi i \Phi \tau) \int u_1(t - \tau) u_2^*(t) \exp(2\pi i \Phi t) dt \\ &= \exp(-2\pi i \Phi \tau) X_{21}^*(\tau, \Phi). \end{aligned}$$

Hence,

$$|X_{12}(-\tau, -\Phi)|^2 = |X_{21}(\tau, \Phi)|^2. \quad (46)$$

## Convolution Theorems

Theorem 7

Let  $u_1(t) = v_1(t) * w_1(t)$ ,  $u_2(t) = v_2(t) * w_2(t)$ . Then

$$\begin{aligned} \chi_{u_1 u_2}(\tau, \Phi) &= \chi_{v_1 v_2}(\tau, \Phi) *_{\tau} \chi_{w_1 w_2}(\tau, \Phi) \\ &= \chi_{v_1 w_2}(\tau, \Phi) *_{\tau} \chi_{w_1 v_2}(\tau, \Phi) . \end{aligned} \quad (47)$$

The proof is similar to that of Theorem 5.

Similarly, if  $u_1(t) = v_1(t) w_1(t)$  and  $u_2(t) = v_2(t) w_2(t)$ , then

$$\begin{aligned} \chi_{u_1 u_2}(\tau, \Phi) &= \chi_{v_1 v_2}(\tau, \Phi) *_{\Phi} \chi_{w_1 w_2}(\tau, \Phi) \\ &= \chi_{v_1 w_2}(\tau, \Phi) *_{\Phi} \chi_{w_1 v_2}(\tau, \Phi) . \end{aligned} \quad (48)$$

Invariance Relations for the Real and Imaginary Parts of Ambiguity  
Functions of Analytic Waveforms (Ref. 7)

Let  $\chi_{12}(\tau, \Phi) = \zeta_{12}(\tau, \Phi) + i\xi_{12}(\tau, \Phi)$  be the generalized cross-correlation function of analytic waveforms  $u_1(t)$  and  $u_2(t)$ . Then

$$\iint \zeta_{12}^2(\tau, \Phi) d\tau d\Phi = \iint \xi_{12}^2(\tau, \Phi) d\tau d\Phi = 1/2 , \quad (49)$$

if  $u_1(t)$  and  $u_2(t)$  are both normalized.

Proof

$$\begin{aligned} \iint |\chi_{12}(\tau, \Phi)|^2 d\tau d\Phi &= 1 \\ &= \iint [\zeta_{12}^2(\tau, \Phi) + \xi_{12}^2(\tau, \Phi)] d\tau d\Phi . \end{aligned}$$

Since

$$\begin{aligned} \zeta_{12} &= (1/2)(\chi_{12} + \chi_{12}^*) , \\ \iint \zeta_{12}^2(\tau, \Phi) d\tau d\Phi &= (1/4) \iint (\chi_{12}^2 + 2|\chi_{12}|^2 + \chi_{12}^{*2}) d\tau d\Phi \\ &= 1/2 + \iint (\chi_{12}^2 + \chi_{12}^{*2}) d\tau d\Phi . \end{aligned}$$

But

$$\begin{aligned}
\iint \chi_{12}^{*2}(\tau, \Phi) d\tau d\Phi &= \iiint U_2(f) U_1^*(f + \Phi) U_2(\nu) U_1^*(\nu + \Phi) \\
&\quad \times \exp[-2\pi i (f + \nu) \tau] df d\nu d\tau d\Phi \\
&= \iint U_2(f) U_1^*(f + \Phi) U_2(-f) U_1^*(-f + \Phi) df d\Phi = 0,
\end{aligned}$$

since  $u_2(t)$  is analytic, so that

$$U_2(f) = 0, \quad f < 0$$

and

$$U_2(-f) = 0, \quad f > 0.$$

Similarly,

$$\iint \chi_{12}^2(\tau, \Phi) d\tau d\Phi = 0,$$

so that

$$\iint \zeta_{12}^2(\tau, \Phi) d\tau d\Phi = 1/2;$$

hence,

$$\iint \xi_{12}^2(\tau, \Phi) d\tau d\Phi = 1/2;$$

Notice that the autoambiguity function is a special case of the cross-ambiguity function, so that in general the real and imaginary parts of the ambiguity function of analytic signals contribute equally to the invariant volume under the ambiguity surface.

### The Most General Ambiguity Function

Before considering other ambiguity functions, it is of interest to reformulate the problem in complete generality. From this generalization we will rederive the previous results and then define an angular ambiguity function. The generalization is used in the companion report to obtain a wideband ambiguity function.

Let  $s_1(t)$  and  $s_2(t)$  be functions, square integrable on  $(-\infty, \infty)$ , representing signals which we wish to resolve. Obviously, if  $s_1$  and  $s_2$  are to be resolved at all, they must differ in some respect. As in Eq. (1), we use a mean-square criterion to maximize their difference. That is, we wish to maximize

$$\begin{aligned}
d^2 &= \int |s_1(t) - s_2(t)|^2 dt \\
&= \int |s_1(t)|^2 dt + \int |s_2(t)|^2 dt - 2 \operatorname{Re} \left[ \int s_1(t) s_2^*(t) dt \right].
\end{aligned} \tag{50}$$

Equivalently (as in Eq. 2), we can achieve maximum resolution by minimizing  $|X_{s_1 s_2}|$  or  $|X_{s_1 s_2}|^2$ , where

$$X_{s_1 s_2} \triangleq \int s_1(t) s_2^*(t) dt. \tag{51}$$

$|X_{s_1 s_2}|^2$  may be termed the "most general ambiguity function." It is too general to be useful in itself; its only value lies in the fact that from it we can obtain less general but more useful ambiguity functions. The argument and functional form of  $X_{s_1 s_2}$  depend, of course, on the characteristics of  $s_1$  and  $s_2$  that are, in fact, distinguishable.

1. If  $s_1(t) = u(t)$  (which may be the low-frequency part of a high-frequency waveform  $\psi(t) = u(t) \exp(i\omega t)$  but need not be so restricted) and if  $s_2(t) = u(t - \tau)$ , a time-shifted version of the same waveform, we have

$$X_{s_1 s_2} = X_u(\tau, 0) = \int u(t) u^*(t - \tau) dt = R(\tau) ,$$

the complex autocorrelation function, Eq. (5). In this case  $|X_{s_1 s_2}|^2 = |R(\tau)|^2$ , the range ambiguity function.

2. If  $s_1(t) = u(t)$  and if  $s_2(t) = u(t) \exp(2\pi i \Phi t)$ , a frequency-shifted version of the same waveform, we have

$$X_{s_1 s_2} = X_u(0, \Phi) = \int |u(t)|^2 \exp(-2\pi i \Phi t) dt = \kappa(\Phi) ,$$

the frequency autocorrelation function, Eq. (8).

3. If  $s_1(t) = u(t)$  and if  $s_2(t) = u(t - \tau) \exp[2\pi i \Phi (t - \tau)]$ , a time- and frequency-shifted version of the same waveform, we have (neglecting  $\exp(-2\pi i \Phi \tau)$ )

$$X_{s_1 s_2} = X_u(\tau, \Phi) = \int u(t) u^*(t - \tau) \exp(-2\pi i \Phi t) dt ,$$

as in Eq. (11).

4. If  $s_1(t) = u_1(t)$  and if  $s_2(t) = u_2(t - \tau) \exp[2\pi i \Phi (t - \tau)]$ , a time- and frequency-shifted version of a different waveform, we have

$$X_{s_1 s_2} = X_{u_1 u_2}(\tau, \Phi) = \int u_1(t) u_2^*(t - \tau) \exp(-2\pi i \Phi t) dt ,$$

as in Eq. (40).

### Angular Ambiguity Function

The distinguishing features between  $s_1$  and  $s_2$  need not be temporal. By consideration of spatial differences, Urkowitz et al. (13) define an angular ambiguity function.

We consider two plane waves from distant sources incident on a linear aperture. Let  $x$  be the distance along the aperture from a reference point (Fig. 4) and let  $\theta_1$  and  $\theta_2$  be the angles between the directions of arrival of the wavefronts and the normal to the aperture. Let  $s(t)$  represent a signal transmitted with a propagation velocity  $c$ , assumed constant. If the incident waves are echoes from stationary targets at the same range, then

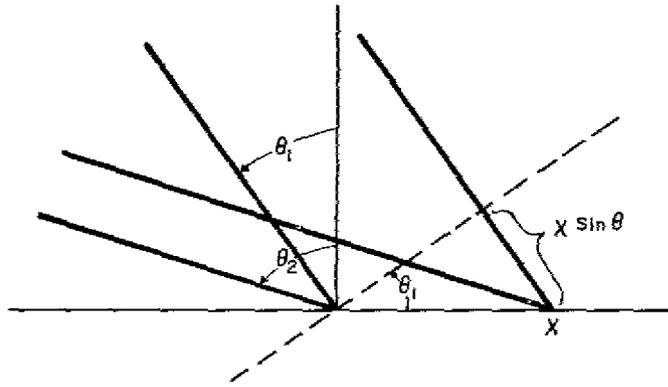


Fig. 4 - Wavefront geometry

$$s_1(t) = s_1(t, x, \theta_1) = s \left( t - \tau - \frac{x \sin \theta_1}{c} \right),$$

$$s_2(t) = s_2(t, x, \theta_2) = s \left( t - \tau - \frac{x \sin \theta_2}{c} \right),$$

and from Eq. (51)

$$\begin{aligned} X_{s_1 s_2}(x, \theta_1, \theta_2) &= \int s \left( t - \tau - \frac{x \sin \theta_1}{c} \right) s^* \left( t - \tau - \frac{x \sin \theta_2}{c} \right) dt \\ &= \int s(t) s^* \left[ t + x \left( \frac{\sin \theta_1 - \sin \theta_2}{c} \right) \right] dt. \end{aligned}$$

If  $v \triangleq (\sin \theta_1 - \sin \theta_2)/c$ , then  $X_{s_1 s_2}(x, v) = \int s(t) s^*(t + xv) dt = R_s(xv)$ , where  $R_s(\cdot)$  is the complex autocorrelation function of the transmitted signal.

Urkowitz et al. define the angular ambiguity function  $J(v)$  as

$$J(v) \triangleq \int |I(x)|^2 R_s(xv) dx, \quad (52)$$

where  $I(x)$  is the "illumination function" of the aperture. A detailed discussion of the illumination function and its spatial Fourier transform, the "space pattern" of the aperture, is beyond the scope of the present treatment.

If resolution in both azimuth and elevation is considered, a two-dimensional illumination function  $I(x, y)$  is required, and a two-dimensional angular ambiguity function is defined. If  $I(x, y) = I_1(x) I_2(y)$ , the two-dimensional angular ambiguity function becomes the product of the individual ambiguity functions.

If the targets are at different ranges and are moving with different radial components of velocity, a four-dimensional ambiguity function in azimuth, elevation, range, and "range rate" (Doppler shift) is required. For narrowband signals this four-dimensional ambiguity function is separable into the product of the range-Doppler ambiguity function and the azimuth-elevation ambiguity function. The same is true, in general, of a six-dimensional formulation which includes angular velocity as well.

If the illumination function is complex, Eq. (52) cannot account for the effect of the aperture phase function on angular resolution. To avoid this difficulty and to avoid dependence of resolution on the orientation of a receiving array, Procopio et al. (15) re- Eq. (50) with the integrated squared difference criterion

$$\epsilon^2 \triangleq \iiint |\tilde{s}_1(t, \theta_1, \Phi_1) - \tilde{s}_2(t, \theta_2, \Phi_2)|^2 \sin \theta \, d\theta d\Phi dt ,$$

where

$$\tilde{s}_1(t, \theta_1, \Phi_1) = \iint I(x, y) s_1(t, x, y, \theta_1, \Phi_1) \, dx dy$$

and

$$\tilde{s}_2(t, \theta_2, \Phi_2) = \iint I(x, y) s_2(t, x, y, \theta_2, \Phi_2) \, dx dy$$

are the signals received by the entire array, and the angular orientation of the array is given by the coordinates  $\theta, \Phi$ . An angular ambiguity function may now be defined as

$$\epsilon^2(t, \theta_1, \Phi_1, \theta_2, \Phi_2) = \iiint \tilde{s}_1^*(t, \theta_1, \Phi_1) \tilde{s}_2(t, \theta_2, \Phi_2) \sin \theta \, d\theta d\Phi dt .$$

## REFERENCES

1. Woodward, P.M., "Probability and Information Theory, with Applications to Radar," 2nd edition, London:Pergamon, 1964
2. Elspas, B., "A Radar System Based on Statistical Estimation and Resolution Considerations," Stanford Electronics Labs., Stanford University, Calif., Tech. Report. 381-1, Aug. 1, 1955
3. Siebert, W.M., "A Radar Detection Philosophy," IRE Trans. Inform. Theory IT-2: 204-221 (1956)
4. Siebert, W.M., "Studies of Woodward's Uncertainty Function," Quart. Progr. Rept., Massachusetts Institute of Technology, Cambridge, Mass., pp 90-94, Apr. 1958
5. Lerner, R.M., "Signals with Uniform Ambiguity Functions," IRE Natl Conv. Rec., 4:27-36 (1958)
6. Stewart, J.L., and Westerfield, E.C., "A Theory of Active Sonar Detection " IRE 47:872-881 (1959)
7. Stutt, C.A., "A Note on Invariant Re..." IRE Trans. Inform. Theor...
8. ...

16. Levin, M.J., "Instantaneous Spectra and Ambiguity Functions," IEEE Trans. Inform. Theory IT-10:95-97 (1964)
17. Rihaczek, A.W., "Radar Resolution Properties of Pulse Trains," Proc. IEEE 52: 153-164 (1964)
18. Cahlander, D.A., "Echolocation with Wideband Waveforms: Bat Sonar Signals," Lincoln Lab., Massachusetts Institute of Technology, Tech. Rept. 271, May 1964
19. Stutt, C.A., "Some Results on Real-Part/Imaginary-Part and Magnitude-Phase Relations in Ambiguity Functions," IEEE Trans. Inform. Theory IT-10:321-327 (1964)
20. Kelly, E.J., and Wishner, R.P., "Matched-Filter Theory for High-Velocity Accelerating Targets," IEEE Trans. on Military Electronics Mil-9:56-69 (1965)
21. Rihaczek, A.W., "Radar Signal Design for Target Resolution," Proc. IEEE 53:116-128 (1965)
22. Price, R., and Hofstetter, E.M., "Bounds on the Volume and Height Distributions of the Ambiguity Function," IEEE Trans. Inform. Theory IT-11:207-214 (1965)
23. Berkowitz, R.S., ed., "Modern Radar," New York:Wiley, 1965
24. Remley, W.R., "Doppler Dispersion Effects in Matched Filter Detection and Resolution," Proc. IEEE 54:33-39 (1966)
25. Titlebaum, E.L., "A Generalization of a Two-Dimensional Fourier Transform Property for Ambiguity Functions," IEEE Trans. Inform. Theory IT-12:80-81 (1966)
26. Helstrom, C.W., "An Expansion of a Signal in Gaussian Elementary Signals," IEEE Trans. Inform. Theory IT-12:81-82 (1966)
27. Titlebaum, E.L., and DeClaris, N., "Linear Transformations of the Ambiguity Function," IEEE Trans. Inform. Theory IT-12:120-125 (1966)
28. Persons, C.E., "Ambiguity Function of Pseudo-Random Sequencies," Proc. IEEE 54:1946-1947 (1966)
29. Rihaczek, A.W., "Radar Resolution of Moving Targets," IEEE Trans. Inform. Theory IT-13:51-56 (1967)
30. de Buda, R., "An Extension of Green's Condition to Cross-Ambiguity Functions," IEEE Trans. Inform. Theory IT-13:75-81 (1967)
31. Speiser, J.M., "Wideband Ambiguity Function," IEEE Trans. Inform. Theory IT-13: 122-123 (1967)
32. Gassner, R.L., and Cooper, G.R., "Note on a Generalized Ambiguity Function," IEEE Trans. Inform. Theory IT-13:126 (1967)
33. Woodward, P.M., "Radar Ambiguity Analysis," Royal Radar Establishment Tech. Note 731, Malvern, Worcester, Feb. 1967
34. Rihaczek, A.W., "Delay-Doppler Ambiguity Function for Wideband Signals," Aerospace Corp. Report TR-1001 (2230-46)-4 Air Force Systems Command, Los Angeles, Calif., AD813707, Mar. 1967

35. Montgomery, L.K., Jr., and Reed, I.S., "A Generalization of the Gabor-Helstrom Transform," *IEEE Trans. Inform. Theory* IT-13:344-345 (1967)
36. Cook, C.E., and Bernfeld, M., "Radar Signals," New York:Academic Press, 1967
37. Kramer, S.A., "Sonar Signal Design," M.S. Thesis, Brooklyn Polytechnic Institute, 1967
38. Purdy, R.J., and Cooper, G.R., "A Note on the Volume of Generalized Ambiguity Functions," *IEEE Trans. Inform. Theory* IT-14:153-154 (1968)
39. Burdic, W.S., "Radar Signal Analysis," Englewood Cliffs:Prentice-Hall, 1968
40. Stutt, C.A., and Spafford, L.J., "A "Best" Mismatched Filter Response for Radar Clutter Discrimination," *IEEE Trans. Inform. Theory* IT-14:280-287 (1968)
41. Rihaczek, A.W., "Signal Energy Distribution in Time and Frequency," *IEEE Trans. Inform. Theory* IT-14:369-374 (1968)

## Appendix A

### PROOFS OF THEOREMS 1 THROUGH 4

#### Theorem 1

$$\begin{aligned}
 X(\tau, \Phi) &\stackrel{\Delta}{=} \int u(t) u^*(t-\tau) \exp(-2\pi i \Phi t) dt \\
 &= \iiint U(\nu) U^*(f) \exp\{2\pi i [\nu t - f(t-\tau) - \Phi t]\} d\nu df dt \\
 &= \iint U(\nu) U^*(f) \delta[\nu - (f + \Phi)] \exp(2\pi i f \tau) d\nu df \\
 &= \int U(f + \Phi) U^*(f) \exp(2\pi i f \tau) df,
 \end{aligned}$$

using the well-known properties of the Dirac delta function  $\delta(\cdot)$ .

#### Theorem 2

$$X(\tau, \Phi) = \mathcal{F}_{t\Phi} [u(t) u^*(t-\tau)].$$

Hence,

$$\mathcal{F}_{\Phi t}^{-1} [X(\tau, \Phi)] = u(t) u^*(t-\tau),$$

and by Parseval's theorem,

$$\int |X(\tau, \Phi)|^2 d\Phi = \int |u(t)|^2 |u(t-\tau)|^2 dt.$$

Thus

$$\iint |X(\tau, \Phi)|^2 d\Phi d\tau = \int |u(t)|^2 \int |u(\tau)|^2 d\tau dt = 1.$$

#### Theorem 3

$$\begin{aligned}
 |X(\tau, \Phi)|^2 &= \left| \int u(t) u^*(t-\tau) \exp(-2\pi i \Phi t) dt \right|^2 \\
 &\leq \int |u(t)|^2 dt \int |u(t-\tau)|^2 dt = 1,
 \end{aligned}$$

by the Cauchy-Schwarz inequality.

#### Theorem 4

$$\mathcal{F}_{\Phi t}^{-1} \mathcal{F}_{\tau f} [ |X(\tau, \Phi)|^2 ] = \iint |X(\tau, \Phi)|^2 \exp[-2\pi i (f\tau - \Phi t)] d\tau d\Phi$$

$$\begin{aligned}
\mathcal{F}_{\Phi t}^{-1} \mathcal{F}_{\tau f} \left[ |\chi(\tau, \Phi)|^2 \right] &= \iiint u(\eta) u^*(\eta - \tau) \exp(-2\pi i \Phi \eta) u^*(\zeta) u(\zeta - \tau) \\
&\quad \times \exp(2\pi i \Phi \zeta) \exp[-2\pi i(f\tau - \Phi t)] d\eta d\zeta d\tau d\Phi \\
&= \iint u(\eta) u^*(\eta - \tau) u^*(\eta - t) u(\eta - t - \tau) \exp(-2\pi i f \tau) d\eta d\tau \\
&= \iint u(\eta) u^*(\eta - t) u^*(\xi) u(\xi - t) \exp[-2\pi i f(\eta - \xi)] d\eta d\xi \\
&= |\chi(t, f)|^2 .
\end{aligned}$$

## Appendix B

### GACF OF A SINGLE GAUSSIAN PULSE

$$u(t) = (2a/\pi)^{1/4} \exp(-at^2), \quad a > 0$$

$$\begin{aligned} \chi(\tau, \Phi) &= (2a/\pi)^{1/2} \int \exp[-at^2 - a(t-\tau)^2 - 2\pi i\Phi t] dt \\ &= (2a/\pi)^{1/2} \exp(-a\tau^2/2) \mathcal{F}_{t\Phi}\{\exp[-2a(t-\tau/2)^2]\}. \end{aligned} \quad (\text{B1})$$

From Woodward (1), page 28, "pair 3," we have  $\mathcal{F}_{tf}[\exp(-\pi t^2)] = \exp(-\pi f^2)$ . This can be shown as follows:

$$\begin{aligned} \mathcal{F}_{tf}[\exp(-\pi t^2)] &= \int_{-\infty}^{\infty} \exp(-\pi t^2 - 2\pi i f t) dt \\ &= \exp(-\pi f^2) \int_{-\infty}^{\infty} \exp[-\pi(t + if)^2] dt \\ &= \exp(-\pi f^2) \int_{-\infty + if}^{\infty + if} \exp(-\pi t^2) dt \\ &= \exp(-\pi f^2) \int_{-\infty}^{\infty} \exp(-\pi t^2) dt = \exp(-\pi f^2). \end{aligned}$$

Then

$$\mathcal{F}_{t\Phi}[\exp(-2at^2)] = (\pi/2a)^{1/2} \exp(-\pi^2 f^2/2a)$$

by Woodward's "Rule 8," and

$$\mathcal{F}_{t\Phi}\{\exp[-2a(t-\tau/2)^2]\} = (\pi/2a)^{1/2} \exp[-\pi^2 f^2/2a - 2\pi i f \tau/2]$$

by "Rule 6." Applying this result to Eq. (B1) we have

$$\chi(\tau, \Phi) = \exp(-a\tau^2/2 - \pi^2 \Phi^2/2a - \pi i \Phi \tau).$$

Appendix C

GACF OF A TRAIN OF DELTA FUNCTIONS

$$w(t) = \sum_{-\infty}^{\infty} \delta(t - nT)$$

$$X_w(\tau, \Phi) = \sum_{n, m} \int \delta(t - nT) \delta(t - \tau - mT) \exp(-2\pi i \Phi t) dt$$

$$= \sum_{n, m} \delta[(n - m)T - \tau] \exp(-2\pi i n \Phi T) = \sum_{n, m} \delta(\tau - nT) \exp(-2\pi i m \Phi T) .$$

Now consider the formal Fourier series expansion

$$\sum_n \delta(\Phi - n/T) = \sum_n C_n \exp(-2\pi i n \Phi T) .$$

Then

$$\begin{aligned} \sum_n \int_{-1/2T}^{1/2T} \delta(\Phi - n/T) \exp(2\pi i m \Phi T) d\Phi &= 1 \\ &= \sum_n C_n \int_{-1/2T}^{1/2T} \exp[-2\pi i (n - m) \Phi T] d\Phi \\ &= \sum_n C_n \delta_{nm}/T = C_m/T , \end{aligned}$$

so that  $C_n = T, \forall n$ . Hence,

$$\sum_m \exp(-2\pi i m \Phi T) = T^{-1} \sum_m \delta(\Phi - m/T) ,$$

and

$$X_w(\tau, \Phi) = T^{-1} \sum_{n, m} \delta(\tau - nT) \delta(\Phi - m/T) .$$

## Appendix D

### ROTATION OF AXES OF THE AMBIGUITY DIAGRAM OF A SINGLE GAUSSIAN PULSE WITH LINEAR FREQUENCY MODULATION

Equation (37) will be in the standard form for an ellipse if we define new axes,  $\tau'$ ,  $\Phi'$  rotated by an angle  $\theta$  with respect to the  $\tau$ ,  $\Phi$  axes, where

$$\tau = \tau' \cos \theta - \Phi' \sin \theta$$

and

$$\Phi = \tau' \sin \theta + \Phi' \cos \theta .$$

If this is substituted in Eq. (37), rewritten in the form  $A\tau^2 + B\tau\Phi + C\Phi^2 = K$ , and if the coefficient of  $\tau'\Phi'$  is required to vanish, we obtain  $\tan 2\theta = B/(A-C)$  from which Eq. (38) follows.

## Appendix E

### MULTIPLICATIVE PROPERTY OF PSEUDO-RANDOM SIGNALS

#### Theorem

Let  $s(t)$  represent a signal obtained from a maximal-length pseudo-random sequence of -1s and 1s, of infinite duration. Let  $k$  be an integer,  $k \neq 0 \pmod{p}$ , where  $p$  is the period of the pseudo-random sequence, and let  $t_0$  be the shifting period of the generator producing the signal. Then  $s(t) s(t - kt_0) = s(t - k't_0)$ , where  $k'$  is also an integer.

#### Proof

We may write

$$s(t) = \sum_{-\infty}^{\infty} a_i \Delta_i(t) ,$$

where  $\{a_i\}$  is the pseudo-random sequence, and†

$$\Delta_i(t) \triangleq \begin{cases} 1, & (i-1)t_0 < t < it_0 \\ 0, & \text{otherwise} \end{cases} . \quad (\text{E1})$$

Then

$$s(t - kt_0) = \sum_i a_i \Delta_i(t - kt_0) = \sum_j a_{j-k} \Delta_j(t) ,$$

since

$$\begin{aligned} \Delta_i(t - kt_0) &= \begin{cases} 1, & (i-1)t_0 < t - kt_0 < it_0 \\ 0, & \text{otherwise} \end{cases} = \begin{cases} 1, & (i+k-1)t_0 < t < (i+k)t_0 \\ 0, & \text{otherwise} \end{cases} \\ &= \Delta_{i+k}(t) = \Delta_j(t) , \quad \text{if } j = i+k . \end{aligned}$$

Thus

$$s(t) s(t - kt_0) = \sum_i \sum_j a_i a_{j-k} \Delta_i(t) \Delta_j(t) . \quad (\text{E2})$$

Now

†This notation is similar to that used on p. 18 of J. L. Lawson and G. E. Uhlenbeck, "Threshold Signals," New York:McGraw Hill, 1950. I am grateful to Dr. H. L. Saxton for calling my attention to it.

$$a_i a_{j-k} = a_i a_{i-(k-j+i)} = a_{i-k'} \quad (\text{E3})$$

where  $k'$  is an integer by the "shift-and-multiply" property of pseudo-random sequences of 1s and -1s.†

Using Eq. (E1), we have

$$\begin{aligned} \Delta_i(t) \Delta_j(t) &= \begin{cases} 1, & [(i-1)t_0 < t < it_0] \cap [(j-1)t_0 < t < jt_0] \\ 0, & \text{otherwise} \end{cases} \\ &= \begin{cases} 1, & t_0 [\max(i, j) - 1] < t < t_0 \min(i, j) \\ 0, & \text{otherwise} \end{cases} \end{aligned}$$

If  $i \leq j$ , this product vanishes unless  $(j-1)t_0 < t < it_0$ ; this means  $j-1 < i \leq j$ , which can only be satisfied for  $i = j$ . Similarly, if  $i \geq j$ ,  $\Delta_i(t) \Delta_j(t) = 0$  unless  $(i-1)t_0 < t < jt_0$ , for which we get  $i-1 < j \leq i$ , and again  $j = i$ .

Thus, we can write  $\Delta_i(t) \Delta_j(t) = \Delta_j(t) \delta_{ij}$  and from Eqs. (E2) and (E3),

$$\begin{aligned} s(t) s(t-kt_0) &= \sum_i \sum_j a_{i+k} \Delta_j(t) \delta_{ij} \\ &= \sum_i a_{i+k} \Delta_i(t) \\ &= s(t-k't_0) \end{aligned}$$

Q.E.D.

†See, for example, C. McCoy, Jr., "Power Spectrum Estimates of Sampled Pseudo-Random Sequences," NRL Report 6673, p. 68, Dec. 29, 1967; Thesis, The George Washington University, Feb. 22, 1968.