NRL REPORT NO. R-3244

A FINITE DIFFERENCE SOLUTION OF RECURRENT NETWORKS

B. N. Navid

★

Approved by:

Mr. F. M. Gager, Head, Special Research Section
Dr. R. M. Page, Superintendent, Radio Division III

Problem No. 36R08-38 March 1, 1948

NAVAL RESEARCH LABORATORY
CAPTAIN H. A. SCHADE, USN, DIRECTOR
WASHINGTON, D.C.

APPROVED FOR PUBLIC RELEASE - DISTRIBUTION UNLIMITED

CONTENTS

ABSTRACT

Electrical network analysis from associated system matrices generally results in determinants which are awkward to handle because computation becomes laborious with an appreciable number of meshes. This paper is a study of several ladder-type networks where recursion formulas for the system determinants are solved by the method of finite differences. It appears that a broad class of networks are amenable to this type of analysis and certain generalizations of the subject method are set forth.

PROBLEM STATUS

This is an interim report on this problem; work is continuing.

A FINITE DIFFERENCE SOLUTION OF RECURRENT NETWORKS

INTRODUCTION

It is well known that electrical networks can be analyzed from the standpoint of their associated system matrices. In general, however, the resulting determinants become awkward to handle. Should a circuit have an appreciable number of meshes, the computations become extremely laborious. The present paper is a study of several ladder-type networks, uniform and otherwise where recursion formulas for the system determinants are obtained which are then solved by the method of finite differences. It appears that a broad class of networks are amenable to this analysis. With the ladder-type circuits as points of departure, certain generalizations of the method are set forth.

THEORY

A basic recurrent ladder network is illustrated in Figure 1.

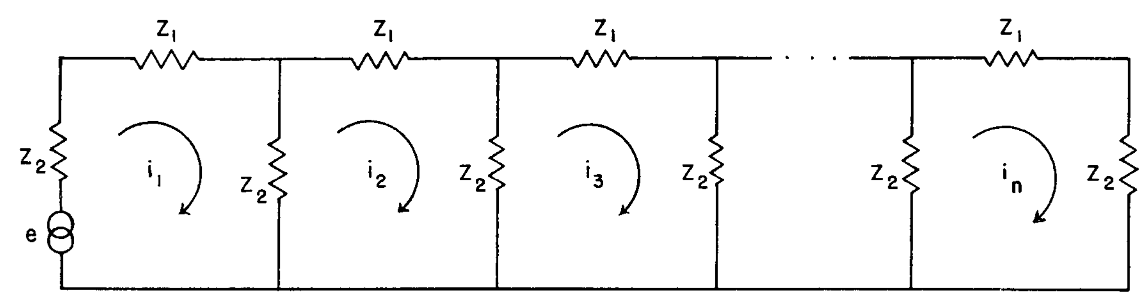

Figure 1

By definition,

$$Z_1 + 2Z_2 = S \tag{1}$$

and

$$Z_2 = B \tag{2}$$

Then the loop equations become, in matrix notation,

$$\begin{bmatrix} e \\ o \\ \cdot \\ \cdot \\ \cdot \\ o \end{bmatrix} = \begin{bmatrix} S & -B & & 0 \\ -B & \ddots & \ddots & \\ & \ddots & \ddots & -B \\ 0 & & -B & S \end{bmatrix} \begin{bmatrix} i_1 \\ i_2 \\ \cdot \\ \cdot \\ \cdot \\ i_n \end{bmatrix} \tag{3}$$

where designating the first matrix on the right hand side as $\left[M_{ij}\right]$,

$$M_{ii} = S \tag{4}$$

$$M_{i,\, i+1} = M_{i+1,\, i} = -B \tag{5}$$

$$M_{ij} = 0 \;\; (j \neq i,\; j \neq i+1) \tag{6}$$

Equation (3) could also be written

$$\left[E\right] = \left[M(n)\right] \left[I\right] \tag{7}$$

the argument denoting the number of meshes in the system.

With $S \neq 0$, let

$$\frac{B}{S} = \rho \tag{8}$$

The case of S = 0 is considered later.

It follows, therefore, that

$$\left[E\right] = S \left[Q(n)\right] \left[I\right] \tag{9}$$

where

$$Q(n) = \begin{bmatrix} 1 & -\rho & & 0 \\ -\rho & \ddots & \ddots & \\ & \ddots & \ddots & -\rho \\ 0 & & -\rho & 1 \end{bmatrix} \tag{10}$$

A complete analysis of the circuit, Figure 1, requires evaluation of the determinant of $\left[M(n)\right]$, its zeros, and its first cofactors.

Defining the determinant of $\left[M(n)\right] = D(n)$ (11)

and the determinant of $\left[Q(n)\right] = \Delta(n)$ (12)

then $D(n) = S^n \Delta(n)$ (13)

SOLUTION OF Δ (n)

Consider the determinant Δ (n).

Expansion by the first row (column) results in

$$\Delta(n) = \Delta(n-1) - \rho^2 \; \Delta(n-2) \tag{14}$$

Where Δ (n−1), Δ (n−2) are determinants of the same form as Δ (n) but of dimensions (n−1), (n−2), respectively.

Assume a solution

$$\Delta(n) = CA^n \tag{15}$$

where C and A are constants.

Direct substitution into (14) yields

$$A^n = A^{n-1} - \rho^2 A^{n-2} \tag{16}$$

or

$$A^2 - A + \rho^2 = 0 \tag{17}$$

yielding

$$A = \frac{1 \pm \sqrt{1-4\rho^2}}{2} \tag{18}$$

Defining A_1 and A_2 as

$$A_1 = \frac{1 + \sqrt{1-4\rho^2}}{2} \text{ and } A_2 = \frac{1 - \sqrt{1-4\rho^2}}{2} \tag{19}$$

The complete solution is then

$$\Delta(n) = C_1 A_1^n + C_2 A_2^n \tag{20}$$

since (14) is a linear finite difference equation and (16) through (18) justify choice of bases A_1 and A_2. Where C_1 and C_2 are arbitrary constants, it should be noted that

$$A_1 + A_2 = 1 \tag{21}$$

$$A_1 A_2 = \rho^2 \tag{22}$$

To evaluate the constants C_1 and C_2, two boundary conditions are required.

By definition of Δ (n)

$$\Delta(1) = 1 \tag{23}$$

$$\Delta(2) = 1-\rho^2 \tag{24}$$

Using (20) through (24), it follows that

$$C_1 = \frac{-A_1}{A_2-A_1} \text{ and } C_2 = \frac{A_2}{A_2-A_1} \tag{25}$$

Hence the complete solution becomes

$$\Delta(n) = \frac{1}{A_2-A_1}\left[A_2^{n+1} - A_1^{n+1}\right] \tag{26}$$

The solution (26) can be transformed into a more convenient form by the following substitutions:

Let

$$A_1 = Re^{\phi}$$
$$A_2 = Re^{-\phi} \qquad (27)$$

Hence $R = \pm \sqrt{A_1 A_2}$ (28)

Choose $R = + \sqrt{A_1 A_2}$ (29)

By (22) R also equals ρ

Consequently, $\cosh \phi = \dfrac{1}{2\rho}$ (30)

or, referring ρ back to the circuit parameters,

$$\cosh \phi = \frac{Z_1 + 2Z_2}{2Z_2} = 1 + \frac{Z_1}{2Z_2} \qquad (31)$$

The angle ϕ is recognized as the propagation constant of the network.

By means of the transformation (27)

$$\Delta(n) = \rho^{n+1} \frac{\left[e^{(n+1)\phi} - e^{-(n+1)\phi}\right]}{\rho \left[e^{\phi} - e^{-\phi}\right]} \qquad (32)$$

or

$$\Delta(n) = \rho^n \frac{\sinh (n+1)\phi}{\sinh \phi} \qquad (33)$$

$$\text{and } D(n) = B^n \frac{\sinh (n+1)\phi}{\sinh \phi} \qquad (34)$$

Since $B = Z_2$

$$D(n) = Z_2^{\,n} \frac{\sinh (n+1)\phi}{\sinh \phi} \qquad (35)$$

where

$$\phi = \cosh^{-1} \left[1 + \frac{Z_1}{2Z_2}\right] \qquad (36)$$

Equations (35) and (36) were obtained by assuming

$$S \equiv Z_1 + 2Z_2 \neq 0$$

If $S = 0$, it follows that

$$D(n) = -Z_2^{\,2}\, D(n-2) \qquad (37)$$

$$D(1) = 0 \qquad (38)$$

$$D(2) = -Z_2^{\,2} \qquad (39)$$

Solving (37), and using the boundary conditions (38) and (39),

$$D(n) = -j^n \frac{Z_2{}^n}{2} \left[(-1)^{n+1} - 1\right] \tag{40}$$

Hence

$$D(n) = 0 \text{ if } S = 0 \text{ and n is odd, and} \tag{41}$$

$$D(n) = (-1)^{n/2}\ Z_2{}^n \text{ when } S = 0 \text{ and n is even.} \tag{42}$$

ZEROS OF D(n)

To obtain the natural modes of vibration of the system, which are required in analyzing the transient behavior, the zeros of D(n) must be evaluated.

If $B = Z_2 = 0$ then from (3) $D(n) = S^n = Z_1{}^n$

Thus, unless Z_1 and Z_2 are zero simultaneously, $Z_2 = 0$ does not yield a zero of D(n).

The preceding analysis shows that only zeros of $\frac{\sinh (n+1)\phi}{\sinh\phi}$ need be considered.

If $\sinh (n+1)\phi = 0, \phi = j\frac{k\pi}{n+1}$.

However, since $\sin k\pi = 0$, $k = 0$ and $k = n+1$ must be excluded.

Hence zeros occur when $k = 1, 2, \ldots, n$

Specifically, then, since $\cosh j\theta = \cos\theta$, zeros are determined by $\rho = 1/2 \sec\frac{k\pi}{n+1}$ (43)

It should be noted that equation (41) yields a possible source of zeros, namely if n is odd and S = 0.

This case corresponds to choosing $k = n+1/2$ in (43), and solving for ρ. The solution is $\rho = \infty$, i.e. S = 0.

The circuit of Figure 2 may be considered as an example.

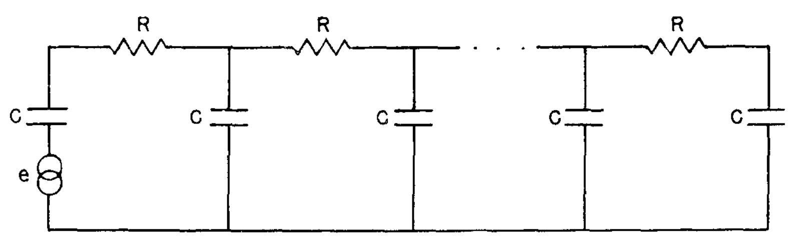

Figure 2

$Z_1 = R_1, Z_2 = 1/Cp$ where p is the required angular frequency mode.

Zeros occur when $\cos\frac{k\pi}{n+1} = 1 + \frac{RCp}{2}$, or $p = \frac{2}{RC}\left[\cos\frac{k\pi}{n+1} - 1\right]$

GENERAL STEADY STATE BEHAVIOR

To determine the steady state currents, it is necessary to evaluate the first cofactors of D(n). If the driving voltage is placed in the first mesh, the relevant cofactors are $D_{1j}(n)$ ($j = 1, 2, \ldots, n$) and for applied voltages in the k-th mesh, $D_{kj}(n)$ ($j=1, 2, \ldots, n$).

By a Laplacian expansion about the first (r−1) rows of $D_{r,\,r+k}$ (n), it is apparent that

$$D_{r,\,r+k}(n) = B^k\, D(r-1)\; D(n-r-k) \qquad (44)$$

with the convention that D(0) = 1

Since D(n) is symmetric

$$D_{kj}(n) = D_{jk}(n) \qquad (45)$$

or

$$D_{r+k,r}(n) = B^k\, D(r-1)\; (D-n-r-k) \qquad (46)$$

Hence, for a voltage e_1 applied to the first mesh, the current in the k-th mesh is given by

$$i_{k1} = \frac{D_{1k}(n)}{D(n)}\, e_1 = B^{k-1}\, \frac{D(n-k)}{D(n)}\, e_1 \qquad (47)$$

and substituting the appropriate values from (2) and (35),

$$i_1 = \frac{\sinh(n-k+1)\phi}{Z_2 \sinh(n+1)\phi}\, e_1 \qquad (48)$$

Similarly for voltage in the jth mesh

$$i_{kj} = \frac{B^{j-k}\, D(k-1)\, D(n-j)}{D(n)}\, e_j \quad (k<j) \qquad (49)$$

or

$$i_{kj} = \frac{\sinh k\phi \sinh(n-j)\phi}{Z_2 \sinh(n+1)\phi \sinh\phi}\, e_j \quad (k<j) \qquad (50)$$

and

$$i_{kj} = \frac{B^{k-j}\, D(j-1)\, D(n-k)}{D(n)}\, e_j \quad (k>j) \qquad (51)$$

which is equivalent to

$$i_{kj} = \frac{\sinh j\phi \sinh(n-k)\phi}{Z_2 \sinh(n+1)\phi \sinh\phi}\, e_j \quad (k>j) \qquad (52)$$

Finally, for voltages $e_1, e_2, \ldots e_n$ in all the loops

$$i_k = \frac{1}{Z_2 \sinh(n+1)\phi \sinh\phi}\left[\sum_{j-1}^{k} e_j \sinh j\phi \sinh(n-k)\phi + \sum_{t=k+1}^{n} e_t \sinh k\phi \sinh(n-t)\phi\right] \qquad (53)$$

GENERAL TRANSIENT BEHAVIOR

The preceding analysis carries over to the transient case as well. For the steady state, Z_2 and ϕ are functions of the impressed angular velocities, ω_i for the transient case they are the same functions of the natural modes p_i. The amplitudes of the transient currents are proportional to the cofactors D_{jk} (n). If $j \neq k$, the cases of $j>k$ and $j<k$ must be distinguished as in equations (50) and (51).

TAPERED LADDER STRUCTURE

The preceding analysis applies to a uniform ladder structure. In this section the applicability of the method to a nonuniform structure will be indicated.

Consider the network of Figure 3.

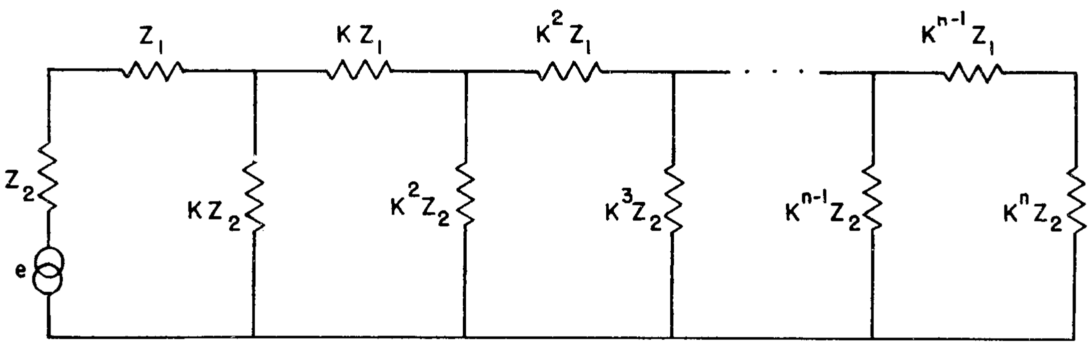

Figure 3

Let S_j be the self impedance of the j-th loop

$$S_1 = Z_1 + (1+k)\, Z_2 \tag{54}$$

$$S_j = K^{j-1} S_1 \tag{55}$$

The system determinant, D' (n), becomes

$$D'(n) = \begin{vmatrix} S_1 & -kZ_2 & & 0 \\ -kZ_2 & \ddots & \ddots & \\ & \ddots & \ddots & -k^{n-1}Z_2 \\ 0 & & -k^{n-1}Z_2 & S_1 \end{vmatrix} \tag{56}$$

Removing the factor k^{j-1} from the j-th row yields

$$D'(n) = k.k^2 \ldots k^{n-1} \begin{vmatrix} S_1 & -kZ_2 & & 0 \\ -Z_2 & \ddots & \ddots & \\ & \ddots & \ddots & -kZ_2 \\ 0 & & -Z_2 & S_1 \end{vmatrix} \tag{57}$$

and expanding by the first row of the determinant results in

$$D'(n) = k^{n^2/2}\left[S_1\, D'(n-1) - k\, Z_2^2\, D'(n-2)\right] \tag{58}$$

which can be solved by the same method as that for the uniform ladder structure.

Let

$$\rho' = Z_2/S_1 \tag{59}$$

Then

$$D'(n) = k^{n^2/2}\, Z_2^{\,n}\, Q'(n) \tag{60}$$

Where

$$Q'(n) = \begin{vmatrix} 1 & -k\rho' & & 0 \\ -\rho' & \ddots & \ddots & \\ & \ddots & \ddots & -k\rho' \\ 0 & & -\rho' & 1 \end{vmatrix} \tag{61}$$

Expanding $Q'(n)$ by its first row (or column)

$$Q'(n) = Q'(n-1) - k\rho'^2\, Q'(n-2) \tag{62}$$

Hence, $\rho'\sqrt{k}$ replaces ρ of the uniform ladder structure.

The solution of (62) becomes

$$D'(n) = k^{n^2/2}\, Z_2^{\,n}\, \frac{\sinh (n+1)\phi'}{\sinh \phi'} \tag{63}$$

where

$$\cosh \phi' = \frac{1}{2\rho'\sqrt{k}} = \frac{1}{\sqrt{k}}\left[\frac{Z_1}{2Z_2} + \frac{(1+k)Z_2}{2Z_2}\right] \tag{64}$$

For $k = 1$, the solution reduces to that of the uniform ladder. The currents are obtained in a similar manner.

CONCLUSIONS

The preceding analysis applies to networks actuated by ideal generators. The case for generators and loads of arbitrary impedance can be readily obtained for the steady state; the transient case involves the solution of a transcendental equation which cannot, in general, be expressed in a closed form but requires design curves. With the more common types of terminations, however, transient solutions are obtainable.

PRNC-5405-4-22-48-100