NRL Formal Report 7012

UNCLASSIFIED

# A Fortran Computer Program for Calculating the Prolate Spheroidal Radial Functions of the First and Second Kind and Their First Derivatives

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**March 30, 1970**

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# A FORTRAN COMPUTER PROGRAM FOR CALCULATING THE PROLATE SPHEROIDAL RADIAL FUNCTIONS OF THE FIRST AND SECOND KIND AND THEIR FIRST DERIVATIVES

## INTRODUCTION

The solutions $\psi = R(\xi)S(\eta)\Phi(\varphi)$ of the Helmholtz wave equation $(\nabla^2 + k^2)\psi(\xi,\eta,\varphi) = 0$ in prolate spheroidal coordinates $\xi$, $\eta$, $\varphi$ can be obtained by the technique of the separation of variables. The computer program PRAD calculates numerical values to the solutions of the resulting ordinary differential equation for the "radial" coordinate $\xi$. Specifically, radial functions of the first and second kinds $R_{m,\ell}^{(1),(2)}(h,\xi)$, along with their first derivatives $d(R_{m,\ell}^{(1),(2)}(h,\xi))/d\xi$ are numerically calculated from eigenfunction expansions in spherical Bessel and associated Legendre functions.

Prolate spheroidal wave functions have been applied to the following problems: spheroidal antennas, scattering of acoustic and electromagnetic waves, and the electromagnetic resonant behavior of certain spheroidal cavities. For an extensive list of references on applications of spheroidal wave functions, the reader is directed to Ref. 1.

The two independent solutions of the radial equation $R^{(1)}(\xi)$ and $R^{(2)}(\xi)$ are characterized by four parameters, $\xi$ (called X), M, H, and L. M is the separation constant integer relating to the solution for the rotational angle $\varphi$. H is equal to $kd/2$, where $d$ is the interfocal distance, and $k$ is the propagation constant or wave vector magnitude $2\pi\lambda$. For each choice of M, X, and H there will be a set of solutions to the radial equation, each solution characterized by a separation constant or eigenvalue $A$. These eigenvalues are ordered in an ascending sequence and labeled with integers L, beginning with L = M for the smallest eigenvalue and continuing with L = M + 1, L = M + 2, etc. For each choice of M, X, H, and L there will then be two independent radial functions.

Operationally the program PRAD first sets M and X and calculates Legendre functions if they are needed. Then H is chosen and for each L desired, the eigenvalue $A$, the expansion constants, and finally the two radial functions and their first derivatives are calculated. Spherical Bessel functions are calculated when needed.

## INPUT

The input consists of three data cards.

Card No. 1 has format 6I5. It provides the six integer variables M1, IDM, NM, L1, IDL, and NL, where

M1 = the initial value of M desired

IDM = the increment on M used to generate other values of M

NM = the total number of M's desired

L1 = the initial value of L desired

IDL = the increment on L used to generate other values of L

NL = the total number of L's desired.

Each variable must be set flush right in its five spaces.

Card No. 2 has format 2D20.10, I5. This card provides the variables Z1, DX, and NX, where

Z1 is a double-precision, floating-point number placed in columns 1 to 20 and equals the initial value of X desired minus 1

DX is a double-precision, floating-point number placed in columns 21 to 40 and equals the increment on X

NX is an integer and is placed flush right in columns 41 to 45 and equals the total number of X's desired.

Card No. 3 has format identical with that of Card No. 2 and provides the variables H1, DH, and NH, where

H1 = the initial value of H desired

DH = the increment on H

NH = the total number of H's desired (NH $\leq$ 40).

Incrementation of the various parameters is accomplished with nested loops in PRAD. L is the innermost loop, H is next, X is outside H, and M is the outermost loop.

## OUTPUT

The output is in the form of the tabulation contained in Appendix A. Each page provides numerical values for particular choices of M, H, and X. The numerical values consist of the radial functions of the first and second kind R1 and R2, their first derivatives R1D and R2D, and the eigenvalue for all choices of L requested. Only 18 significant figures are printed in the tables, although 26 figures are used in the computation.

The column headed by ACC represents an accuracy check for the radial functions and their derivatives. This check is obtained by comparing the theoretical value of the Wronskian $1/[H(X+1)(X-1)]$ to the calculated value R1*R2D - R1D*R2. The accuracy number ACC defined in line 4980 of PRAD (see Appendix B) is the number of digits (±1 digit) that agree in the two methods of calculating the Wronskian. The number ACC indicates in terms of significant digits the accuracy of the least accurate of the four numbers $R^{(1)}$, $dR^{(1)}/d\xi$, $R^{(2)}$, $dR^{(2)}/d\xi$ when the exponents of the two products $R^{(1)}(dR^{(2)}/d\xi)$ and $R^{(2)}(dR^{(1)}/d\xi)$ in the calculated Wronskian are equal and in turn equal the exponent of the theoretical Wronskian. When the exponents of the two products differ, the calculated value of the Wronskian will be more sensitive to the product with the larger exponent. In this case, the accuracy of the least accurate of the two numbers forming the other product may be less than ACC by the difference in exponents of the two products. However, ACC still represents a lower bound on the accuracy of both numbers in the product with the larger exponent.

Since the Wronskian check is independent of the eigenvalue, the value of ACC is meaningless if $A_{m,\ell}(h)$ is incorrect. Examination of successive eigenvalues for monotonicity and uniform spreading is a helpful check on their correctness. Checks on all computations to date show that for ranges discussed in the next section, PRAD always delivers the correct eigenvalue accurate to at least 21 significant figures.

## PARAMETER RANGES AND ACCURACY

This program has been run for values of the parameters within the following ranges:

M = 0 through 12

X = 1.00000001 through 500 and the special case X = 1, M = 0

H = 0.1 through 80

L = M through M + 49.

Appendix C gives various degrees of accuracy (i.e., values of ACC) for particular values of the parameters. In general, there is little effect on accuracy with increasing M except at high H (H > 30), where accuracy increases for low eigenvalues as M increases.

The value of X has almost no effect on accuracy except when X is very close to 1. In this case, as can be seen from Appendix C, lower accuracy is obtained for high L.

The effect of H on accuracy is quite significant as H becomes large. Again referring to Appendix C, it is seen that at large H (H > 40), subtraction errors introduced in calculations of the radial functions result in low accuracy for low values of L, whereas for high values of L good accuracy is obtained.

## COMPUTATION TIME

When using the NRL CDC 3800, the compilation time of PRAD is about 1 min. The execution time varies, but the average time will be about 0.5 second for each line of printout if all 50 values of L are requested.

## SOLUTION OF THE HELMHOLTZ EQUATION

### Separation in Spheroidal Coordinates

The prolate spheroidal coordinates, shown in Fig. 1, are related to rectangular coordinates by the transformation

$$x = \frac{d}{2}\sqrt{(1-\eta^2)(\xi^2-1)}\,\cos\varphi$$

$$y = \frac{d}{2}\sqrt{(1-\eta^2)(\xi^2-1)}\,\sin\varphi \qquad (1)$$

$$z = \frac{d}{2}\eta\xi\,,$$

where $-1 \le \eta \le 1$, $1 \le \xi < \infty$, and $0 \le \varphi \le 2\pi$ as on Fig. 1, and $d$ denotes the interfocal distance.

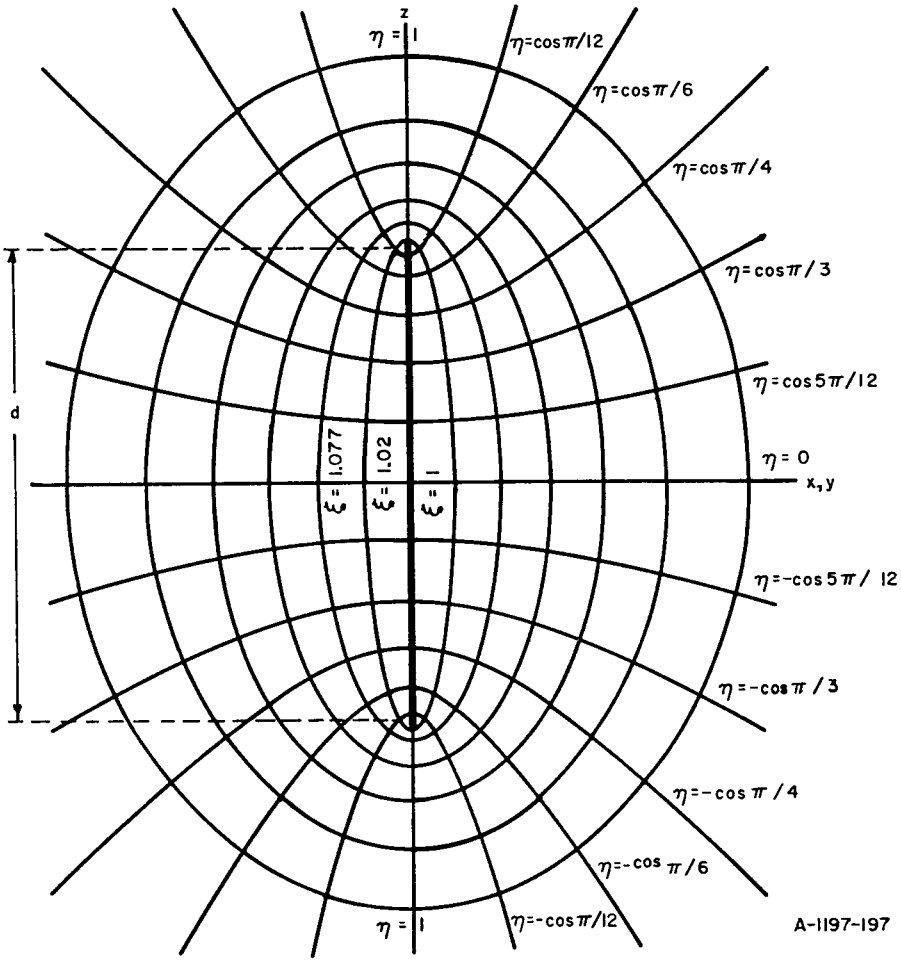


Fig. 1 - The prolate spheroidal coordinate system

The surface $\xi = \xi_0$, $\xi_0 > 1$, is an elongated ellipsoid of revolution having major axis $d\xi$ and minor axis $d\sqrt{\xi^2 - 1}$. $\xi = 1$ represents a straight line along the $z$ axis from $-d/2$ to $d/2$. The surface $|\eta| = \eta_0$, $\eta_0 < 1$ is a hyperboloid of revolution of two sheets with an asymptotic cone whose generating line passes through the origin. $|\eta| = 1$ is that part of the $z$ axis for which $|z| > d/2$. The surface $\varphi = \text{constant}$ is a plane through the $z$ axis making the angle $\varphi$ with the $xz$ plane.

The Helmholtz equation $(\nabla^2 + k^2)\psi = 0$ written in prolate spheroidal coordinates is

$$\left[\frac{\partial}{\partial\eta}(1-\eta^2)\frac{\partial}{\partial\eta} + \frac{\partial}{\partial\xi}(\xi^2-1)\frac{\partial}{\partial\xi} + \frac{\xi^2-\eta^2}{(\xi^2-1)(1-\eta^2)}\frac{\partial^2}{\partial\varphi^2} + h^2(\xi^2-\eta^2)\right]\psi(\eta,\xi,\varphi) = 0 , \tag{2}$$

where $h = kd/2$, $k = 2\pi/\lambda$, and $\lambda$ = wavelength of sound in the medium. Separation of variables yields

$$\psi_{m,\ell}(h/\eta,\xi,\varphi) = R_{m,\ell}(h,\xi)\, S_{m,\ell}(h,\eta)\, \Phi_m(\varphi) , \tag{3}$$

where

$$\frac{d}{d\xi}\left[(\xi^2-1)\frac{dR}{d\xi}\right]-\left[A-h^2\xi^2+\frac{m^2}{\xi^2-1}\right]R=0 \tag{4a}$$

$$\frac{d}{d\eta}\left[(1-\eta^2)\frac{dS}{d\eta}\right]+\left[A-h^2\eta^2-\frac{m^2}{1-\eta^2}\right]S=0 \tag{4b}$$

$$\frac{d^2\Phi}{d\varphi^2}+m^2\Phi=0\ . \tag{4c}$$

If $h \to 0$, Eq. (4b) reduces to the standard differential equation for associated Legendre polynomials whose eigenvalues are $A = \ell(\ell+1)$, $\ell = m, m+1, \ldots$, where $m$ is a nonnegative integer. When $h = 0$, the separation constant $A$ depends on $m$, $\ell$, and $h$ and Eqs. (4a) and (4b) have nontrivial convergent solutions for eigenvalues $A = A_{m,\ell}(h)$. For each choice of $m$, $h$, $\xi$, and $\eta$ there will be a set of solutions to the radial and angle equations (Eqs. (4a) and (4b)), each solution characterized by a particular value of $A$.

The two independent solutions of Eq. (4a) and Eq. (4b) define the radial functions of the first and second kind $R^{(1)}$ and $R^{(2)}$ and the angle functions of the first and second kind $S^{(1)}$ and $S^{(2)}$, respectively.

## Radial Functions of the First Kind

The solutions $S^{(1)}$ (the angle function of the first kind) can be expressed in the form of an expansion in terms of associated Legendre functions as follows:

$$S^{(1)}(h,\eta) = \sum_{n=0,1}^{\infty}{}' \, d_n(h|m,\ell)\, P^m_{m+n}(\eta)\ , \tag{5}$$

where $d_n$ is an expansion coefficient and $P^m_{m+n}(\eta)$ is an associated Legendre function of the first kind. The prime sign on the sum means that $n = 0,2,4,\ldots$ if $(\ell - m)$ is even and that $n = 1,3,5,\ldots$ if $(\ell - m)$ is odd.

Calculations of the $d_n$ constants are discussed in the fourth section.

It can be shown that Eqs. (4a) and (4b) are identical except for range of argument. Using the general principle that one solution of the scalar wave equation in a given coordinate system is a suitable kernel for the integral representation of a second solution and making necessary substitutions (Ref. 2), we show that $R^{(1)}$ the radial function of the first kind can be written in the form

$$R^{(1)}_{m,\ell}(h,\xi) = \frac{(\ell-m)!}{(\ell+m)!}\left(\frac{\xi^2-1}{\xi^2}\right)^{m/2}\sum_{n=0,1}^{\infty}{}' \, i^{n+m-\ell}\, d_n(h|m,\ell)\,\frac{(n+2m)!}{n!}\, j_{n+m}(h\xi)\ , \tag{6}$$

where $j_{n+m}(h\xi)$ is a spherical Bessel function and $i = \sqrt{-1}$ (Ref. 3).

The first derivative of $R^{(1)}$ with respect to the radial coordinate $\xi$ is evaluated using the following standard recursion formula for spherical Bessel functions:

$$(2r+1)\,\frac{d}{d\xi}\,j_r(\xi) = r\,j_{r-1}(\xi) - (r+1)\,j_{r+1}(\xi)\ . \tag{7}$$

The result is

$$\frac{d}{d\xi}\,R^{(1)}_{m,\ell}(h,\xi) = \frac{(\ell-m)!}{(\ell+m)!}$$

$$\left\{h\left(\frac{\xi^2-1}{\xi^2}\right)^{m/2}\sum_{n=0,1}^{\infty}{}' \; i^{n+m-\ell}\,d_n(h|m,\ell)\,\frac{(n+2m)!}{n!}\left[\frac{n+m}{2(n+m)+1}\,j_{n+m-1}(h\xi)\right.\right.$$

$$\left.-\frac{n+m+1}{2(n+m)+1}\,j_{n+m+1}(h\xi)\right]$$

$$\left.+\frac{m}{\xi^3}\left(\frac{\xi^2-1}{\xi^2}\right)^{(m-2)/2}\sum_{n=0,1}^{\infty}{}' \; i^{n+m-\ell}\,d_n(h|m,\ell)\,\frac{(n+2m)!}{n!}\,j_{n+m}(h\xi)\right\}. \tag{8}$$

When X = 1, $R^{(1)}_{0,\ell}(h,\xi)$ and $d(R^{(1)}_{0,\ell}(h,\xi))/d\xi$ are calculated by special formulas found in Ref. 1a. $R^{(2)}_{0,\ell}(h,\xi)$ is undefined at X = 1.

## The Radial Function of the Second Kind

The radial function of the second kind $R^{(2)}(h,\xi)$ may similarly be expanded but this time in terms of spherical Neumann functions. Thus we have (Ref. 2)

$$R^{(2)}_{m,\ell}(\xi) = \frac{(\ell-m)!}{(\ell+m)!}\left(\frac{\xi^2-1}{\xi^2}\right)^{m/2}\sum_{n=0,1}^{\infty}{}' \; i^{n+m-\ell}\,d_n(h|m,\ell)\,\frac{(n+2m)!}{n!}\,y_{n+m}(h\xi)\ , \tag{9}$$

where $y_{n+m}(h\xi)$ is a spherical Neumann function (Ref. 3). The $d_n$ constants are the same for $R^{(1)}$ and $R^{(2)}$.

The first derivative of $R^{(2)}$ with respect to $\xi$ is evaluated using an equation analogous to Eq. (7). The result is

$$\frac{d}{d\xi}\,R^{(2)}_{m,\ell}(h,\xi) = \frac{(\ell-m)!}{(\ell+m)!}$$

$$\left\{h\left(\frac{\xi^2-1}{\xi^2}\right)^{m/2}\sum_{n=0,1}^{\infty}{}' \; i^{n+m-\ell}\,d_n(h|m,\ell)\,\frac{(n+2m)!}{n!}\left[\frac{n+m}{2(n+m)+1}\,y_{n+m-1}(h\xi)\right.\right. \tag{10 (Cont.)}$$

$$- \frac{n+m+1}{2(n+m)+1} y_{n+m+1}(h\xi) \Bigg]$$

$$+ \frac{m}{\xi^3} \left( \frac{\xi^2 - 1}{\xi^2} \right)^{(m-2)/2} \sum_{n=0,1}^{\infty}{}' i^{n+m-\ell} d_n(h|m,\ell) \frac{(n+2m)!}{n!} y_{n+m}(h\xi) \Bigg\} . \tag{10}$$

Since Eq. (9) contains Neumann functions which are singular at the origin, its usefulness is limited to large values of $h\xi$. An alternate method is necessary to obtain the radial function of the second type when $h\xi$ is small. It is shown (Ref. 1b) that $R^{(2)}$ can be represented in terms of associated Legendre functions as follows:

$$R^{(2)}_{m,\ell}(\xi) = \frac{1}{K^{(2)}_{m,\ell}(h)} \left[ \sum_{\substack{n=-2m,\\ -2m+1}}^{\infty}{}' d_n Q^m_{m+n}(\xi) + \sum_{\substack{n=2m+2\\ 2m+1}}^{\infty}{}' d_{\rho/n} P^m_{n-m-1}(\xi) \right] . \tag{11}$$

Here $P^m_{n-m-1}$ and $Q^m_{m+n}$ are associated Legendre functions of the first and second kinds, respectively (Ref. 4 and 5). The $d_n$ constants are the same as those used in $R^{(1)}$. Calculation of the $d_n$ constants for $n < 0$ and the $d_{\rho/n}$ constants is given in Ref. 6.

$K^{(2)}_{m,\ell}(h)$ is a constant of proportionality between $R^{(2)}$ and $S^{(2)}$. Its formula can be obtained from formulas given in Ref. 1c.

The derivative of $R^{(2)}_{m,\ell}(h,\xi)$ with respect to $\xi$ is obtained by the use of recursion relations for derivatives of the associated Legendre functions. The result is

$$\frac{d}{d\xi} R^{(2)}_{m,\ell}(\xi) = \frac{1}{K^{(2)}_{m,\ell}(h)} \Bigg\{ \sum_{\substack{n=-2m,\\ -2m+1}}^{\infty}{}' d_n \left[ \begin{array}{ll} \dfrac{Q^1_n(\xi)}{\sqrt{\xi^2 - 1}}, & \text{if } m = 0 \\[2ex] -\dfrac{m\xi Q^m_n}{\xi^2 - 1} + \dfrac{(m+n)(n-m+1)}{\sqrt{\xi^2 - 1}} Q^{m-1}_n, & \text{if } m > 0 \end{array} \right]$$

$$+ \sum_{\substack{n=2m+2,\\ 2m+1}}^{\infty}{}' d_{\rho/n} \left[ \frac{P^{m+1}_n(\xi)}{\sqrt{\xi^2 - 1}} + m\xi \frac{P^m_n(\xi)}{\xi^2 - 1} \right] \Bigg\} . \tag{12}$$

## Calculation of Eigenvalues and $d_n$ Constants

Substitution of Eq. (5) into Eq. (4b) with use of the associated Legendre differential equation and recursion formulas for the associated Legendre functions results in the following recursion formula between three different $d$'s:

$$\frac{(2m+n+2)(2m+n+1)}{(2m+2n+3)(2m+2n+5)}\, h^2\, d_{n+2}(h|m,\ell)$$

$$+\left[(m+n)(m+n+1) - A_{m,\ell}(h) + \frac{2(m+n)(m+n+1) - 2m^2 - 1}{(2m+2n-1)(2m+2n+3)}\, h^2\right] d_n(h|m,\ell)$$

$$+\frac{n(n-1)\,h^2}{(2m+2n-3)(2m+2n-1)}\, d_{n-2}(h|m,\ell) = 0\,, \quad n \geq 0\,. \tag{13}$$

Examination of Eq. (13) shows that as $n \to \infty$ either $d_n/d_{n-2}$ increases as $-4n^2/h^2$ or goes to zero as $-h^2/(4n^2)$. To ensure a convergent series we choose the latter, i.e., we require that

$$\lim_{n\to\infty} \frac{d_n}{d_{n-2}} \to 0\,.$$

The $d_n$ constants are normalized in this report by the equation

$$\sum_{n=0,1}^{\infty}{}' \frac{(n+2m)!}{n!}\, d_n(h|m,\ell) = \frac{(\ell+m)!}{(\ell-m)!}\,. \tag{14}$$

The requirement

$$\lim_{n\to\infty} \frac{d_n}{d_{n-2}} \to 0$$

enables us to obtain the following equations:

$$N_n^m = \frac{-\beta_n^m}{\gamma_n^m - A_{m,\ell}(h) + N_{n+2}^m}\,, \quad (n \geq 2) \tag{15}$$

$$N_n^m = -\gamma_{n-2} + A_{m,\ell}(h) - \frac{\beta_{n-2}^m}{N_{n-2}^m}\,, \quad (n \geq 4)\,, \tag{16}$$

with

$$N_2^m = -\gamma_0^m + A_{m,\ell}(h)\,, \quad N_3^m = -\gamma_1^m + A_{m,\ell}(h) \tag{17}$$

where

$$\gamma_n^m = (m+n)(m+n+1) + \frac{1}{2}\, h^2\left[1 - \frac{4m^2-1}{(2m+2n-1)(2m+2n+3)}\right], \quad n \geq 0\,, \tag{18a}$$

$$\beta_n^m = \frac{n(n-1)(2m+n)(2m+n-1)\,h^4}{(2m+2n-1)^2(2m+2n-3)(2m+2n+1)}\,, \quad n \geq 2\,, \tag{18b}$$

$$N_n^m = \frac{(2m+n)(2m+n-1)\,h^2}{(2m+2n-1)(2m+2n+1)}\, \frac{d_n(h|m,\ell)}{d_{n-2}(h|m,\ell)}\,, \quad n \geq 2\,. \tag{18c}$$

The correct value of $A_{m,\ell}(h)$ is calculated using Eqs. (15) through (17) by a variational method due to Bouwkamp (Ref. 1d) which, however, requires an initial approximation for $A_{m,\ell}(h)$.

An accurate first approximation for $A_{m,\ell}(h)$ for a given $m$, $\ell$, and $h$ is obtained by a matrix method whose detailed description is given in Ref. 7. Briefly, an equation similar to Eq. (13) is derived using normalized associated Legendre functions. Written in matrix form the equation is

$$[B][d] = A[d] \quad , \tag{19}$$

where $[B]$ is an infinite square symmetric matrix depending on $m$ and $h$, $[d]$ is a vector representation of the $d_n$ constants with respect to normalized associated Legendre functions, and $A$ is an unknown eigenvalue. The eigenvalues of $[B]$ when arranged in a monotonically increasing sequence are the eigenvalues $A_{m,\ell}(h)$, $\ell = m, m+1, \ldots$ used in Eqs. (15) through (17).

In actual practice matrices of modest proportions yield starting values quite sufficient for use in the Bouwkamp method. A matrix of order 50 provides good starting values for $A_{m,\ell}(h)$, $\ell = m, m+1, \ldots m+49$ when $h < 10$. As $h$ increases, the order of the matrix must also be increased to maintain good starting values for the 50 lowest eigenvalues. The largest matrix used in PRAD is of order 90, which provides 50 good starting values for $h = 80$.

## THE COMPUTER PROGRAM PRAD

The Fortran IV computer program PRAD used to calculate the prolate radial functions of the first and second kinds, their derivatives, and the associated eigenvalues is listed in Appendix B. PRAD was written in Fortran for the CDC 3800 computer at NRL and uses double-precision arithmetic. Double-precision numbers in this machine have 26 decimal digits, and the magnitude of numbers permitted is $10^{-307}$ to $10^{307}$.

### Symbols Used in PRAD

The symbols used in the CDC 3800 computer for the PRAD program are as follows:

M = parameter $m$

L = parameter $\ell$

H = parameter $h$

X = variable $\xi$

CL = eigenvalue $A_{m,\ell}(h)$

R1 = radial function of the first kind

R2 = radial function of the second kind

R1D = derivative of the radial function of the first kind

R2D = derivative of the radial function of the second kind.

Special Functions

PRAD requires these special functions: (a) integer factorials, (b) associated Legendre functions, and (c) spherical Bessel functions.

The first 295 integer factorials are calculated in lines 180 through 230 of PRAD (see Appendix B). Scaling is necessary to prevent overflow.

Associated Legendre functions of the first and second kinds with nonnegative subscripts are obtained from subroutine CALCPQ which is called in lines 410 through 460. Given a value of M and X, CALCPQ calculates $P_n^m(X)$ and $Q_n^m(X)$ for $n = 0, 1, \ldots, 140$. $P_n^m(X)$ are calculated by means of a hypergeometric series (Ref. 5), while $Q_n^m(X)$ are calculated by an expansion in $P_n^m(X)$ (Ref. 5) for $X \leq 1.001$ and by a hypergeometric series (Ref. 4) for $X > 1.001$. Associated Legendre functions of the second kind with negative subscripts are calculated by means of a standard recursion relation in subroutine CALNQ which is called in lines 2640 through 2660. The number of functions calculated in CALNQ depends upon the value of M. If M = 0, CALNQ is not called. All necessary derivatives of these functions are calculated in lines 2670 through 2720.

Spherical Bessel functions of the first kind are calculated in subroutine SBESF, which is called in line 4430. This subroutine calculates the 141 Bessel functions $j_n(HX)$, $n = 0, 1, \ldots, 140$ by a series expansion in HX when $HX < 0.4$, by a backward-recursion relation when $0.4 \leq HX < 100$, and by a forward-recursion relation when $HX \geq 100$. Spherical Bessel functions of the second kind, i.e., spherical Neumann functions, are calculated in the main program in lines 3730 through 3800. A forward-recursion relation is used for the Neumann functions. They are scaled by $10^{-150}$. Scaling is done in order that more terms can be taken in Eqs. (9) and (10).

All special functions are accurate to at least 20 significant figures where they are used, with the possible exception of the $P_n^m(X)$ and $Q_n^m(X)$ for high values of M and N and low values of X.

Eigenvalues and Expansion Constants

The eigenvalues $A_{m,\ell}(h)$ must be obtained before the $d_n$ constants can be calculated. This is accomplished by the matrix method (see Eq. (19)) in conjunction with Bouwkamp's method. The matrix $[B]$ is calculated in lines 680 through 770, the order of the matrix being determined in line 600. Subroutine EIGEN is called in line 780 to obtain the eigenvalues of $[B]$. This subroutine calculates the eigenvalues by means of the Givens tridiagonalization method followed by an iterative procedure based upon the Sturm sequence property. The eigenvalues are ordered from smallest to largest for use as starting values in Bouwkamp's method.

Bouwkamp's method is programmed in lines 820 through 1350.

The $d_n$ constants are calculated in lines 1360 through 1650 and are stored in array DLIST. These same $d_n$ constants scaled by $10^{150}$ are stored in array DLAST. The scaled $d_n$ constants are supplied with the scaled spherical Neumann functions in calculating terms of Eqs. (9) and (10).

The $d_{p/n}$ constants and $d_n$ constants with negative subscripts are calculated in lines 1980 through 2570 and are only used if $R^{(2)}$ and $dR^{(2)}/d\xi$ are calculated by Eqs. (11) and (12).

Radial Functions

The radial function of the first kind $R^{(1)}$ and its derivative are calculated from Eqs. (6) and (8) in lines 4390 through 4950. $R^{(2)}$ and its derivative are calculated either

from Eqs. (9) and (10) in lines 3810 through 4320 or from Eqs. (11) and (12) in lines 2580 through 3620. The special formulas used to calculate $R^{(1)}$ and $dR^{(1)}/d\xi$ when X = 1 and M = 0 are located in lines 1670 through 1920.

Experience has shown that Eqs. (9) and (10) yield greater or lesser accuracy than Eqs. (11) and (12) depending on the values of H and X. The following table indicates the method used to calculate $R^{(2)}$ and its derivative in PRAD.

| Range of Parameters | Method |
|---|---|
| $1 < X \leq 1.05$ | Eqs. (11) and (12) |
| $1.05 < X \leq 1.3$ or $H < 0.1$ | Both (select better result) |
| $1.3 < X$ and $10 \leq H$ | Eqs. (9) and (10) |
| $2 < X$ and $1 \leq H$ | Eqs. (9) and (10) |
| $5 < X$ and $0.1 \leq H$ | Eqs. (9) and (10) |
| X and H outside the above ranges | Eqs. (11) and (12) |

## REFERENCES


1. Flammer, C., "Spheroidal Wave Functions," Stanford University Press, Stanford, Calif., 1957: (a) pp. 41, 86, (b) pp. 32, 33, (c) pp. 33, 86, (d) p. 20.

2. Hanish, S., Baier, R.V., Buckler, W.H., and Carmichael, B.K., "Tables of Spheroidal Wave Functions Part 2 - Prolate Radial Functions of the First and Second Type," NRL Report 6502, Jan. 1967

3. Abramowitz, M., and Stegun, I.A., eds., "Handbook of Mathematical Functions," National Bureau of Standards, Applied Mathematical Series 55, Washington, D.C.: U.S. Government Printing Office, 1964

4. "Tables of Associated Legendre Functions," Mathematical Tables Project No. 41, New York:Columbia University Press, 1945, p. xx, Eq. (16)

5. Robin, L., "Fonctions Sphériques de Legendre et Fonctions Sphéroïdales, Volume 1," Gauthier-Villars, Paris, France 1957:p. 79, Eq. 19; p. 123, Eq. 76

6. Sternberg, D., "A Computation Procedure for the Scattering of Sound by a Prolate Spheroid," Columbia Univ., Hudson Lab., Tech. Report 61, 1958, p. 18

7. Hanish, S., and King, B., "Tables of Eigenvalues of the Wave Equation in Prolate Spheroidal Coordinates," NRL Report 6856, Apr. 4, 1969

PROLATE RADIAL FUNCTIONS

H = 10.00 X = 1.00000001 M = 0

| L | R1 | R1D | R2 | R2D | EIGENVALUE | ACC |
|---|---|---|---|---|---|---|
| 0 | 3,96332541144880451-001 | -1,79878926011562997+001 | -1,69845566939076024+000 | 1,26157456367187221+007 | 9,22830429724994515+000 | 22 |
| 1 | 3,96331947409645346-001 | -1,42415048218550202+001 | -1,72871075368650868+000 | 1,26157495683146412+007 | 2,81334637328267278+001 | 22 |
| 2 | 3,96311365213364565-001 | -1,07263762339061690+001 | -1,76624204651338679+000 | 1,26163904426127687+007 | 4,58689526502349138+001 | 23 |
| 3 | 3,95916533147670201-001 | -7,47140106084509387+000 | -1,81810539125524472+000 | 1,26289587328994624+007 | 6,22577004507793381+001 | 23 |
| 4 | 3,91238335599079709-001 | -4,50055412638430523+000 | -1,91602981809245413+000 | 1,27799557028063011+007 | 7,69932888221748565+001 | 23 |
| 5 | 3,60018988471992423-001 | -1,84702953297294992+000 | -2,19840125982576106+000 | 1,38881675567538190+007 | 8,97392672388856581+001 | 24 |
| 6 | 2,62942233497887198-001 | 1,36129101633423735-001 | -3,08424161595359631+000 | 1,90155818205033079+007 | 1,01035430728085458+002 | 24 |
| 7 | 1,32830506467537723-001 | 8,55499152026807289-001 | -5,86090999774503814+000 | 3,76419174628262770+007 | 1,12881065848800060+002 | 24 |
| 8 | 4,84113230072513256-002 | 6,54783048476222817-001 | -1,52460340122567158+001 | 1,03281416031667741+008 | 1,27050825284769665+002 | 23 |
| 9 | 1,43699423513082160-002 | 3,15219069833039731-001 | -4,94532858110881152+001 | 3,47947421370556589+008 | 1,43872008037477157+002 | 23 |
| 10 | 3,72243762910086600-003 | 1,17436656199169611-001 | -1,85738887005908079+002 | 1,34319998362307034+009 | 1,63096652717099588+002 | 23 |
| 11 | 8,65281533412438519-004 | 3,65787381821773607-002 | -7,81710371096111204+002 | 5,77843301624900523+009 | 1,84547618588524576+002 | 22 |
| 12 | 1,83085352250407033-004 | 9,89927473543131265-003 | -3,62639667023108743+003 | 2,73094707733067488+010 | 2,08138389347061202+002 | 22 |
| 13 | 3,56054331117458742-005 | 2,38241123623870616-003 | -1,83458070252180673+004 | 1,40426778465272203+011 | 2,33822950869860936+002 | 22 |
| 14 | 6,41155995090167732-006 | 5,17969779608453511-004 | -1,00404472355798918+005 | 7,79833302161422399+011 | 2,61573781917784598+002 | 22 |
| 15 | 1,07547968746734295-006 | 1,02908904629242638-004 | -5,90680450789597027+005 | 4,64903172694616525+012 | 2,91373126086383561+002 | 21 |
| 16 | 1,68892736409823213-007 | 1,88491745225711544-005 | -3,71572367789256044+006 | 2,96041738855094408+013 | 3,23208950487775780+002 | 20 |
| 17 | 2,49374413604820614-008 | 3,20536691496791947-006 | -2,48818446935398918+007 | 2,00498525381257617+014 | 3,57072805277785220+002 | 21 |
| 18 | 3,47489198048704089-009 | 5,08999348496610743-007 | -1,76682436540919365+008 | 1,43886776102148723+015 | 3,92958588886523294+002 | 20 |
| 19 | 4,58458228632798673-010 | 7,58430925456868576-008 | -1,32589438826438246+009 | 1,09058996061206396+016 | 4,30861793512772513+002 | 19 |
| 20 | 5,74365266132074647-011 | 1,06481196854780991-008 | -1,04841181612526661+010 | 8,70506737380826179+016 | 4,70779023926351161+002 | 20 |
| 21 | 6,85069847163300388-012 | 1,41366646722961673-009 | -8,71173621618748075+010 | 7,29834606034642135+017 | 5,12707679999041401+002 | 19 |
| 22 | 7,79752134901577015-013 | 1,78035041319217965-010 | -7,58903858932850698+011 | 6,41212077495782396+018 | 5,56645741326563651+002 | 18 |
| 23 | 8,48740582048503503-014 | 2,13284681556979493-011 | -6,91570804579854252+012 | 5,89090775119250931+019 | 6,02591617619163844+002 | 19 |
| 24 | 8,85176373266710531-015 | 2,43663952462444500-012 | -6,57957820679688376+013 | 5,64841065057479694+020 | 6,50544042637081502+002 | 17 |
| 25 | 8,86114416007387706-016 | 2,66056337284062855-013 | -6,52361370674161335+014 | 5,64241628372452575+021 | 7,00501997641327649+002 | 17 |
| 26 | 8,52827998609677283-017 | 2,78219607417677006-014 | -6,72959092559865945+015 | 5,86262734424339256+022 | 7,52464655263808502+002 | 17 |
| 27 | 7,90308259509634734-018 | 2,79148765715502816-015 | -7,21169243716033537+016 | 6,32639049552384397+023 | 8,06431337761561114+002 | 16 |
| 28 | 7,06152142638571983-019 | 2,69185206525803694-016 | -8,01717807882319340+017 | 7,08032145274582298+024 | 8,62401485568416499+002 | 16 |
| 29 | 6,09153598475759906-020 | 2,49866569804406625-017 | -9,23365468804947 13+018 | 8,20773162849601397+025 | 9,20374633326325596+002 | 15 |
| 30 | 5,07929650623597770-021 | 2,23577539932652407-018 | -1,10043697399329964+020 | 9,84339846250397721+026 | 9,80350391421683412+002 | 15 |
| 31 | 4,09840869224434734-022 | 1,93101895671907031-019 | -1,35551126838238672+021 | 1,21992182787084229+028 | 1,04232843162232797+003 | 15 |
| 32 | 3,20344531514765801-023 | 1,61182302156694013-020 | -1,72395772125467419+022 | 1,56073277726459725+029 | 1,10630847580303271+003 | 16 |
| 33 | 2,42793336058158744-024 | 1,30172118276595825-021 | -2,26153657558500080+023 | 2,05924333310193883+030 | 1,17229028702094204+003 | 14 |
| 34 | 1,78597298085336433-025 | 1,01824606722954788-022 | -3,05723213573093209+024 | 2,79942010721298699+031 | 1,24027366239593246+003 | 13 |
| 35 | 1,27617276748707320-026 | 7,72247082284773169-024 | -4,25519231776958071+025 | 3,91770730138656077+032 | 1,31025842738947402+003 | 15 |
| 36 | 8,86536976616355090-028 | 5,68376725370996861-025 | -6,09278664631538277+026 | 5,63953202975253338+033 | 1,38224443117596094+003 | 13 |
| 37 | 5,99202586996468580-029 | 4,06327344513535625-026 | -8,96765255699844903+027 | 8,34381510236175357+034 | 1,45623154287398274+003 | 13 |
| 38 | 3,94326821059730878-030 | 2,82381730047988246-027 | -1,35576992776161920+029 | 1,26788022161575263+036 | 1,53221964845935686+003 | 12 |
| 39 | 2,52844373527987933-031 | 1,90923157863464150-028 | -2,10393802353402045+030 | 1,97734211552726640+037 | 1,61020864822230530+003 | 12 |
| 40 | 1,58068212887740089-032 | 1,25679413587446012-029 | -3,34911850833266504+031 | 3,16292502358830081+038 | 1,69019845466169254+003 | 11 |
| 41 | 9,64061161450046095-034 | 8,06042856135761259-031 | -5,46517464999816458+032 | 5,18593597421992352+039 | 1,77218899073241628+003 | 11 |
| 42 | 5,73974445869264910-035 | 5,03999059661076261-032 | -9,13676025880962808+033 | 8,71038688046997555+040 | 1,85618018837976615+003 | 10 |
| 43 | 3,33776178904228070-036 | 3,07435145831388839-033 | -1,56403394814799308+035 | 1,49786577112772470+042 | 1,94217198730821849+003 | 11 |
| 44 | 1,89683003445103295-037 | 1,83058799640116628-034 | -2,73986101274492287+036 | 2,63571238807435592+043 | 2,03016433394272441+003 | 9 |
| 45 | 1,05399040912668652-038 | 1,06460776425369602-035 | -4,90922237876342674+037 | 4,74338029195539358+044 | 2,12015718054881547+003 | 9 |
| 46 | 5,72921353466596341-040 | 6,05044858924874212-037 | -8,99253496511694934+038 | 8,72625161626411394+045 | 2,21215048448434351+003 | 10 |
| 47 | 3,04795913395000104-041 | 3,36210013349272367-038 | -1,68317645000073679+040 | 1,64025626760648987+047 | 2,30614420756080120+003 | 9 |
| 48 | 1,58772776204617884-042 | 1,82757392640557125-039 | -3,21778369044461205+041 | 3,14878408590568869+048 | 2,40213831549624478+003 | 8 |
| 49 | 8,10185651650276095-044 | 9,72270722482915651-041 | -6,28020149684782055+042 | 6,17067137248885263+049 | 2,50013277744509482+003 | 8 |

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PROLATE RADIAL FUNCTIONS

H = 40,00 X = 1,10000000 M = 1

| L | R1 | R1D | R2 | R2D | EIGENVALUE | ACC |
|---|---|---|---|---|---|---|
| 1 | -3,5331209083147757 1-002 | 3,4274246249783387 3-001 | -2,4612650925421431 9-003 | -3,3455984581816383 7+000 | 4,0258151803713306 0+001 | 13 |
| 2 | -3,3443470686984687 4-002 | -1,0818804818930941 3+000 | 1,2764576729568795 9-002 | -3,1467389737403670 3+000 | 1,1926550449219127 6+002 | 13 |
| 3 | -2,9115040273702584 5-002 | -2,2897722070669210 1+000 | 2,6059988982231976 9-002 | -2,3641682400175140 3+000 | 1,9724472047833323 2+002 | 11 |
| 4 | -1,1595269481773540 5-002 | -3,0481374660945377 2+000 | 3,4720158840649905 5-002 | -1,1397554490858130 0+000 | 2,7417462012734134 7+002 | 11 |
| 5 | 4,7267201816285779 0-003 | -3,2030942449588718 2+000 | 3,6733543545524497 1-002 | 2,9335714134345430 3-001 | 3,5003251982674545 8+002 | 11 |
| 6 | 2,0677762885548596 5-002 | -2,7152335036951457 5+000 | 3,1268802591758874 0-002 | 1,6513158999460346 0+000 | 4,2479404931869347 0+002 | 13 |
| 7 | 3,2869531193753222 1-002 | -1,6752356411084718 0+000 | 1,8988093865190585 4-002 | 2,6540715452412841 3+000 | 4,9843293713977072 5+002 | 11 |
| 8 | 3,8398724336986728 8-002 | -2,9392988292544863 2-001 | 2,0900686623111740 9-003 | 3,0843026325428664 3+000 | 5,7092075682441105 2+002 | 11 |
| 9 | 3,5551615632812492 7-002 | 1,1343267197637551 4+000 | -1,5976355778176169 8-002 | 2,8388358352987865 2+000 | 6,4222662441912313 4+002 | 11 |
| 10 | 2,4357051712735438 5-002 | 2,2905210514250004 9+000 | -3,1127734909852273 9-002 | 1,9603721959694211 1+000 | 7,1231683502593896 5+002 | 13 |
| 11 | 6,8299719905456770 8-003 | 2,9020143253139661 7+000 | -3,9516990916459484 3-002 | 6,3964322553969460 2-001 | 7,8115442226932986 2+002 | 11 |
| 12 | -1,3219086676765819 6-002 | 2,8110840358752891 3+000 | -3,8508009603685522 3-002 | -8,1687700941643715 4-001 | 8,4869861938218979 4+002 | 13 |
| 13 | -3,0860688378194200 5-002 | 2,0241800040240588 2+000 | -2,7537431212947363 0-002 | -2,0513768405062169 0+000 | 9,1490419353239910 7+002 | 13 |
| 14 | -4,1154363121911292 8-002 | 7,2646707947265090 6-001 | -8,6162876665991567 5-003 | -2,7406126873221116 8+000 | 9,7972061547009715 9+002 | 13 |
| 15 | -4,0504310273995300 4-002 | -7,4825846015993613 2-001 | 1,3755194587227187 3-002 | -2,6850272869947659 9+000 | 1,0430910141066246 9+003 | 14 |
| 16 | -2,7954770227649858 9-002 | -1,9927133164835286 2+000 | 3,3398380779098589 2-002 | -1,8778269501997191 1+000 | 1,1049508508114444 6+003 | 14 |
| 17 | -6,0297770535094954 0-003 | -2,6345560260420913 3+000 | 4,3972528599814497 6-002 | -5,3062141705873935 7-001 | 1,1652262359719966 3+003 | 14 |
| 18 | 1,9286956572754647 7-002 | -2,4537602799994177 4+000 | 4,0959118095327608 1-002 | 9,6146646525322085 8-001 | 1,2238318224871403 8+003 | 15 |
| 19 | 3,9727391652115463 0-002 | -1,4750605936991470 7+000 | 2,3635121962094853 5-002 | 2,1190513273875326 3+000 | 1,2806683342020813 9+003 | 15 |
| 20 | 4,7110085759597656 8-002 | 9,8564741107273274 3-004 | -3,6500254480510932 6-003 | 2,5269328104594501 4+000 | 1,3356203580113996 5+003 | 17 |
| 21 | 3,6542975113395599 5-002 | 1,4568614030878921 0+000 | -3,1466331251820642 1-002 | 1,9952990429265141 4+000 | 1,3885572578308979 6+003 | 15 |
| 22 | 9,5757393676152680 9-003 | 2,3152792805363396 9+000 | -4,8609021124497192 4-002 | 6,7923314753396062 5-001 | 1,4393475240505281 7+003 | 15 |
| 23 | -2,4160484671324770 8-002 | 2,1720470890523961 0+000 | -4,4717687881776952 5-002 | -9,0721256435995003 9-001 | 1,4879154239988836 1+003 | 17 |
| 24 | -4,8882432166030464 2-002 | 1,0384176550324930 2+000 | -1,8377239773935756 7-002 | -2,0449958370213964 0+000 | 1,5343889164074971 9+003 | 17 |
| 25 | -5,0083807579519401 3-002 | -5,5013102442270893 6-001 | 1,9557273509542898 9-002 | -2,1621470365440260 6+000 | 1,5793339881700892 9+003 | 17 |
| 26 | -2,2305378304260172 0-002 | -1,7728576009326799 8+000 | 4,9473457995515194 4-002 | -1,2420751078106177 7+000 | 1,6238668705254205 0+003 | 15 |
| 27 | 1,2746843189391755 4-002 | -2,0880316815878582 4+000 | 5,6232685994553226 2-002 | 1,2803088576552077 -001 | 1,6693648640271802 9+003 | 15 |
| 28 | 4,5760003982450845 2-002 | -1,5792117124311653 8+000 | 3,9321052784696305 8-002 | 1,2445658041163305 8+000 | 1,7169352554043875 3+003 | 15 |
| 29 | 6,3190394798434168 4-002 | -7,2257882442067484 5-001 | 9,1420576712241280 2-003 | 1,7794122369614656 2+000 | 1,7671482444733152 2+003 | 15 |
| 30 | 6,4683938826862787 1-002 | 4,0138279326250832 9-002 | -2,3052977334248448 1-002 | 1,8261459389751693 9+000 | 1,8201576966502940 0+003 | 15 |
| 31 | 5,2659058500335933 8-002 | 5,0299674281527537 4-001 | -5,1166002880248412 0-002 | 1,6764797819103230 4+000 | 1,8759258349654291 2+003 | 19 |
| 32 | 4,2372132670324961 7-002 | 6,6619460247496395 2-001 | -7,4985990962019992 5-002 | 1,6306084271248698 9+000 | 1,9343610953402981 5+003 | 19 |
| 33 | 2,9324166485929115 6-002 | 6,2789376777194316 3-001 | -9,8780168116279987 0-002 | 1,9446134205310762 2+000 | 1,9953705826403492 7+003 | 20 |
| 34 | 1,8759354751153041 3-002 | 4,9634908777335002 1-001 | -1,3035915585323791 1-001 | 2,8968997962787278 7+000 | 2,0588736044705816 4+003 | 20 |
| 35 | 1,1218727932952845 0-002 | 3,4841360688822188 9-001 | -1,8214528046995380 8-001 | 4,9547261715959184 6+000 | 2,1248027948339263 0+003 | 22 |
| 36 | 6,3229483036720967 2-003 | 2,2334532357928818 4-001 | -2,7566586631330237 5-001 | 9,0905277373102091 3+000 | 2,1931022795588603+003 | 21 |
| 37 | 3,3790955556170916 5-003 | 1,3291691905057660 2-001 | -4,5236612143909480 7-001 | 1,7436768784218501 9+001 | 2,2637255370331633 0+003 | 22 |
| 38 | 1,7205370197518271 6-003 | 7,4233198846824819 8-002 | -7,9789664657274110 2-001 | 3,4766586207227962 1+001 | 2,3366335693504096 3+003 | 22 |
| 39 | 8,3788411607068862 0-004 | 3,9205985624820690 6-002 | -1,4975080927734124 0+000 | 7,2010361733843488 7+001 | 2,4117934338460846 0+003 | 22 |
| 40 | 3,9150726447383032 7-004 | 1,9693573172896402 7-002 | -2,9461449047438449 5+000 | 1,5487229184909610 8+002 | 2,4891770992399231 1+003 | 22 |
| 41 | 1,7599100611225704 9-004 | 9,4502879443095143 0-003 | -6,1629461294555618 0+000 | 3,4550631243900259 2+002 | 2,5687605480696768 1+003 | 22 |
| 42 | 7,6282497362904432 1-005 | 4,3477368355413828 6-003 | -1,3372673439357071 2+001 | 7,9843681383298420 1+002 | 2,6505230708385733 3+003 | 22 |
| 43 | 3,1944680206302605 4-005 | 1,9233300467831097 5-003 | -3,0199197059085019 8+001 | 1,9084428313405436 4+003 | 2,7344467062096217 5+003 | 22 |
| 44 | 1,2946723056019338 7-005 | 8,2013771873492857 9-004 | -7,0783346758662945 4+001 | 4,7112714351441294 0+003 | 2,8205157931738086 5+003 | 22 |
| 45 | 5,0859284105259794 3-006 | 3,3781231278945075 3-004 | -1,7180745953737444 0+002 | 1,1995635577676004 2+004 | 2,9087166096846060 6+003 | 21 |
| 46 | 1,9391850707485616 5-006 | 1,3465124410821432 7-004 | -4,3101685053014556 3+002 | 3,1462011961849228 5+004 | 2,9990370786291316 9+003 | 22 |
| 47 | 7,1851601388541358 7-007 | 5,2021610267624780 4-005 | -1,1157512339604811 3+003 | 8,4903387037608632 -004 | 3,0914665267030149 4+003 | 21 |
| 48 | 2,5900054661089916 9-007 | 1,9507798888260844 0-005 | -2,9759805632729516 5+003 | 2,3549289573850032 3+005 | 3,1859954852133491 0+003 | 21 |
| 49 | 9,0917519429281014 2-008 | 7,1093605468319717 4-006 | -8,1481330585118657 8+003 | 6,7068939543155153 5+005 | 3,2826155243909698 8+003 | 21 |
| 50 | 3,1108250512593601 7-008 | 2,5208231257558922 0-006 | -2,3043305146293655 9+004 | 1,9595934043529633 1+006 | 3,3813191146976272 5+003 | 23 |

PROLATE RADIAL FUNCTIONS

H = 0.10000 X = 1.1000000000 M = 5

| L | R1 | R1D | R2 | R2D | EIGENVALUE | ACC |
|---|---|---|---|---|---|---|
| 5 | 1,943908492510108 30-011 | 5,091024421970059 52-010 | -4,582781229142741 18+010 | 1,249441347394579 53+012 | 3,000076919435872 80+001 | 19 |
| 6 | 1,644802370680539 27-013 | 4,457222600198550 09-012 | -5,209053503091462 40+012 | 1,483530000587786 16+014 | 4,200199993725754 38+001 | 19 |
| 7 | 1,129506952685928 62-015 | 3,177465013612396 44-014 | -7,275446439454900 38+014 | 2,169227116398433 93+016 | 5,600276013941935 48+001 | 19 |
| 8 | 6,514750722400682 50-018 | 1,907335409735370 61-016 | -1,207559798663477 61+017 | 3,774024983161554 68+018 | 7,200326313487746 97+001 | 20 |
| 9 | 3,236899875825015 00-020 | 9,880904660100004 02-019 | -2,323947207302575 95+019 | 7,617272011634540 46+020 | 9,000361343441218 56+001 | 19 |
| 10 | 1,412448521631421 73-022 | 4,501106628509576 24-021 | -5,089473260757349 20+021 | 1,749499922368055 27+023 | 1,100038672732240 40+002 | 20 |
| 11 | 5,495513610830371 10-025 | 1,829590254004891 87-023 | -1,249870751107479 11+024 | 4,503952846254071 93+025 | 1,320040571435087 42+002 | 20 |
| 12 | 1,929801449503270 13-027 | 6,714180690697651 09-026 | -3,401567387678054 12+026 | 1,284085964023460 32+028 | 1,560042029014697 60+002 | 20 |
| 13 | 6,177245688217880 12-030 | 2,245936328753081 39-028 | -1,016063541316768 59+029 | 4,014557405203946 37+030 | 1,820043172459128 26+002 | 19 |
| 14 | 1,817313694657972 42-032 | 6,902787070613351 77-031 | -3,304462442621854 00+031 | 1,365149399782587 23+033 | 2,100044086074039 95+002 | 20 |
| 15 | 4,947933515684342 04-035 | 1,962409556382281 61-033 | -1,162181418181413 22+034 | 5,014677001520842 88+035 | 2,400044827641869 28+002 | 19 |
| 16 | 1,254107956498893 92-037 | 5,190187245031326 65-036 | -4,394648808471013 90+036 | 1,978298383194343 99+038 | 2,720045437844339 96+002 | 19 |
| 17 | 2,974151982041157 65-040 | 1,283379397996108 33-038 | -1,777783570856338 39+039 | 8,339634181268476 18+040 | 3,060045946001447 33+002 | 19 |
| 18 | 6,628550167002499 17-043 | 2,979758794735370 82-041 | -7,660270018733716 56+041 | 3,740379123039802 35+043 | 3,420046373680191 60+002 | 19 |
| 19 | 1,393713065727663 69-045 | 6,520947508060346 29-044 | -3,502311779963160 13+044 | 1,778031429454122 60+046 | 3,800046737032540 56+002 | 19 |
| 20 | 2,773978123681025 76-048 | 1,349602402794446 81-046 | -1,693310094119589 57+047 | 8,928006205465134 87+048 | 4,200047048349798 10+002 | 20 |
| 21 | 5,242256841545229 44-051 | 2,649543880600814 19-049 | -8,631264072934774 66+049 | 4,721274722101974 42+051 | 4,620047317119952 87+002 | 19 |
| 22 | 9,431722911168139 01-054 | 4,947408170156987 53-052 | -4,625824659375842 10+052 | 2,622342189066138 90+054 | 5,060047550761779 77+002 | 19 |
| 23 | 1,619477746831364 49-056 | 8,808114563111503 24-055 | -2,600261808801743 88+055 | 1,526149016952929 38+057 | 5,520047755143952 08+002 | 18 |
| 24 | 2,659638776921651 11-059 | 1,498462321338804 11-057 | -1,529656635986636 08+058 | 9,286117910912369 03+059 | 6,000047934958257 62+002 | 21 |
| 25 | 4,185999806754246 81-062 | 2,440845531736991 89-060 | -9,398139006407162 83+060 | 5,895758040598606 37+062 | 6,500048093991987 78+002 | 20 |
| 26 | 6,325486367428983 97-065 | 3,813870901452921 07-063 | -6,019467501500071 91+063 | 3,898763548020855 33+065 | 7,020048235329483 10+002 | 18 |
| 27 | 9,192419542986952 81-068 | 5,726078393701592 81-066 | -4,012414703982987 24+066 | 2,680866159421662 21+068 | 7,560048361503144 63+002 | 19 |
| 28 | 1,286683524286139 00-070 | 8,273532935652497 76-069 | -2,779114703238654 66+069 | 1,913908908059512 29+071 | 8,120048474607894 98+002 | 18 |
| 29 | 1,737132760075675 48-073 | 1,152107064940200 47-071 | -1,997252526260807 19+072 | 1,416619427419479 05+074 | 8,700048576388864 30+002 | 19 |
| 30 | 2,265068823407207 64-076 | 1,548256856092713 34-074 | -1,487319066128310 20+075 | 1,085685695357829 06+077 | 9,300048668309229 32+002 | 19 |
| 31 | 2,855906593880627 12-079 | 2,010393118954302 46-077 | -1,146251189903821 77+078 | 8,604935688004006 22+079 | 9,920048751603179 95+002 | 20 |
| 32 | 3,485872351218443 24-082 | 2,525280666311905 87-080 | -9,131804173629423 64+080 | 7,045203213551604 95+082 | 1,056004882731762 73+003 | 18 |
| 33 | 4,123270309301739 16-085 | 3,071842154500694 44-083 | -7,512151445234782 89+083 | 5,952290851078921 78+085 | 1,122004889634530 80+003 | 19 |
| 34 | 4,731114221120909 38-088 | 3,622336201124917 28-086 | -6,374745376276046 25+086 | 5,184312938400812 77+088 | 1,190004895945125 58+003 | 18 |
| 35 | 5,270818973263973 58-091 | 4,144691238208874 73-089 | -5,574926197480094 26+089 | 4,650643472870246 60+091 | 1,260004901729411 68+003 | 18 |
| 36 | 5,706414242035563 23-094 | 4,605719243229000 39-092 | -5,020014032561069 03+092 | 4,293111461754411 66+094 | 1,330004907044342 57+003 | 18 |
| 37 | 6,008615414271307 72-097 | 4,974739637973597 98-095 | -4,650446007356949 11+095 | 4,074863815354084 81+097 | 1,406004911939369 45+003 | 18 |
| 38 | 6,158098714705662 19-100 | 5,227042425030241 46-098 | -4,428566249430605 29+098 | 3,973749867435571 24+100 | 1,482004916457596 93+003 | 18 |
| 39 | 6,147474429810018 37-103 | 5,346637104648901 53-101 | -4,331960592984831 35+101 | 3,978483628678371 03+103 | 1,560004920636736 10+003 | 18 |
| 40 | 5,981696887459346 99-106 | 5,327865419351850 70-104 | -4,349607142937290 82+104 | 4,086620668739520 51+106 | 1,640004924509894 74+003 | 18 |
| 41 | 5,676933016572819 24-109 | 5,175668349538771 89-107 | -4,479888294375893 11+107 | 4,303843553032994 27+109 | 1,722004928106235 68+003 | 18 |
| 42 | 5,258168778622558 31-112 | 4,904539848954251 41-110 | -4,729968148121195 77+110 | 4,644341286948985 80+112 | 1,806004931451527 74+003 | 19 |
| 43 | 4,756012745496575 92-115 | 4,536417202887153 89-113 | -5,116336701438203 08+113 | 5,132284351843525 11+115 | 1,892004934568609 08+003 | 18 |
| 44 | 4,203230683924763 63-118 | 4,097907344909317 63-116 | -5,666548479523639 78+116 | 5,804596230664480 52+118 | 1,980004937477778 26+003 | 18 |
| 45 | 3,631513720980372 70-121 | 3,617307341491069 90-119 | -6,422398324771144 03+119 | 6,715452806354636 20+121 | 2,070004940197125 84+003 | 18 |
| 46 | 3,068868271021983 38-124 | 3,121846238896016 74-122 | -7,445031819792317 59+122 | 7,943254933470940 69+124 | 2,162004942748164 4+003 | 17 |
| 47 | 2,537854972705607 67-127 | 2,635474752496401 85-125 | -8,822839671951943 00+125 | 9,601287969705108 25+127 | 2,256004945129329 72+003 | 18 |
| 48 | 2,054735709690179 74-130 | 2,177390399343803 07-128 | -1,048351401131886 32+129 | 1,185401444316142 16+131 | 2,352004947369666 82+003 | 17 |
| 49 | 1,629444950473943 94-133 | 1,761342080928585 32-131 | -1,321247350203367 98+132 | 1,494212003890042 26+134 | 2,450004949475527 98+003 | 18 |
| 50 | 1,266204311274005 48-136 | 1,395637250950753 97-134 | -1,668119499216298 21+135 | 1,922134546605965 28+137 | 2,550004951457465 62+003 | 17 |
| 51 | 9,645537859502378 13-140 | 1,083693373475095 85-137 | -2,149115922013266 02+138 | 2,522329095408974 12+140 | 2,652004953325016 93+003 | 17 |
| 52 | 7,205742196158517 18-143 | 8,249383491427891 09-141 | -2,824269922183100 27+141 | 3,375164040786756 60+143 | 2,756004955086818 86+003 | 16 |
| 53 | 5,281106328584887 22-146 | 6,158670990033864 69-144 | -3,784401574104572 26+144 | 4,603611803530175 32+146 | 2,862004956750708 19+003 | 16 |
| 54 | 3,798595103180025 52-149 | 4,510925116207063 51-147 | -5,168570070330157 95+147 | 6,398158901845814 59+149 | 2,970004958323808 92+003 | 17 |

ASSIFIED

## Appendix B

## COMPUTER LISTING OF PRAD

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      PROGRAM PRAD
      TYPE DOUBLE   ARR,ARRAY,BAY,BLIST,BOOK,BRRAY,C101,CFAC,
     1CL,CORA,CORB,CWRON,DE,DH,DL,DLAST,DLIST,DM,DN,DNDO,DNEG,DNEGF,
     2DNFREE,DNUM,DR,DRFREE,DRRATIO,DX,EA,EIGL,EM,ENR,ENRC,EYE,FACT,
     3GLIST,H,H1,HSQ,PAPER,PL1,PL3,PL4,PL8,PL12,PL13,PL17,PL18,R1,
     4R1D,R2,R2D,RATIO,RR2,RR2D,SAIL3,SAIL4,SAIL18,
     5SAIL19,SAIL22,SAIL23,SAIL181,SAIL191,SHIP1,SHIP2,SHIP3,SHIP4,
     6SHIP5,SHIP7,SHIP8,SHIP9,SHIP10,SHIP11,SHIP12,SHIP13,SHORE,SILAND,
     7SSUM1,SUBSUM,TWRON,UN,VN,W,W26,W28,WN,X,X1SQ,XTH,Z,Z1
      DIMENSION A(100,100),ARRAY(300),BAY(100),BLIST(150),BOOK(300),
     1BRRAY(250),DLAST(150),DLIST(150),DM(150),DN(150),DNDO(150),
     2DNEG(30),DR(100),DRRATIO(100),EIG(100),EIGL(40,80),ENR(150),
     3GLIST(150),PAPER(300),RATIO(30),SHIP1(100),SHIP2(100),
     4SHIP3(100),SHIP4(150),SHIP5(150),SHIP7(150),SHIP8(150),SHIP9(150),
     5SHIP10(150),SHIP11(150),SHIP12(150),SHIP13(150),SHORE(175),
     6SILAND(175)
      COMMON/BLK1/FACT(300)
      FACT(1)=1.D
      DO 2945 J=2,171
 2945 FACT(J)=(J-1)*FACT(J-1)
      FACT(171)=FACT(171)*1.D-300
      DO 2950 J=172,296
 2950 FACT(J)=(J-1)* FACT(J-1)
      READ 90, M1, IDM, NM, L1, IDL, NL
   90 FORMAT (6I5)
      READ 91, Z1, DX, NX
   91 FORMAT (2D20.10,I5)
      READ 91, H1, DH, NH
      DO 1 IM = 1,NM
      M = M1 + (IM-1)*IDM
      IF (M.GT.L1) L1 = M
      NBL = L1 + (NL-1)*IDL+1-M
      EM = M
      LDK = 0
      DO 1 INX = 1,NX
      Z = Z1 + (INX-1)*DX
      X = Z + 1.D
      IF (X.EQ.1.D) GO TO 3
      IF ((X.GT.1.3D.AND.H1.GE.10.D).OR.(X.GT.2.D.AND.H1.GE.1.D).OR.(X.G
     1T.5.D.AND.H1.GE..1D)) GO TO 3
      CALL CALCPQ(M,140,Z,SHIP10,SHIP4)
      IV2 = M + 1
      CALL CALCPQ(IV2,140,Z,SHIP11,SHIP5)
      IF (M.EQ.0) GO TO 3
      IV1 = M - 1
      CALL CALCPQ(IV1,140,Z,SHIP13,SHIP8)
    3 DO 2 JNH = 1,NH
      H = H1 + (JNH-1)*DH
      LRK = 0
      HSQ = H*H
      PRINT 92
   92 FORMAT (1H155X*PROLATE RADIAL FUNCTIONS*//)
      PRINT 93, H, X, M
```

```
93 FORMAT (47X3HH =F6.2,9X3HX =F12.8,3X3HM =I3//)
   PRINT 94
94 FORMAT (3X1HL13X2HR123X3HR1D22X2HR223X3HR2D18X10HEIGENVALUE9X3HACC
  1/)
   LCK = LDK
   IH = H
   N = XMINOF(50+IH/2,100)
   DO 2 IL = 1,NL
   L = L1 + (IL-1)*IDL
   IF (LCK.EQ.1) GO TO 4
   LCK = 1
   DO 31 J = 1,N
   DO 31 I = 1,N
31 A(I,J) = 0.
   DO 32 I = 1,N
   W = I-1.+EM
32 A(I,I) = W*(W+1.)+HSQ*(2.*W*(W+1.)-2*EM*EM-1.)/((2.*W-1.)*(2.*W+3.
  1))
   NM2 = N-2
   DO 33 I = 1,NM2
   W = I-1.+EM
   A(I,I+2) = (HSQ/(2.*W+3.))*DSQRT(((W+2.+EM)*(W+1.+EM)*(W+2.-EM)*(W
  1+1.-EM))/((2.*W+5.)*(2.*W+1.)))
33 A(I+2,I) = A(I,I+2)
   CALL EIGEN (A,EIG,N,NBL)
   DO 34 I = 1,NBL
34 EIGL(JNH,I) = EIG(I)
 4 CL = EIGL(JNH,L+1-M)
   IW6 = (L-M)/2
   IX = L-M - 2*IW6
   ISC = 2 + IX
   LIM1 = 278
   J=1
   DO 13 I=ISC,LIM1,2
   EYE =I
   EA = 2.D*EM+2.D*EYE
   BLIST(J) = (EYE*(EYE-1.D)*(EA-EYE)*(EA-EYE-1.D)*HSQ*HSQ)/((EA-1.D)
  1*(EA-1.D)*(EA-3.D)*(EA+1.D))
13 J=J+1
   J=1
   ID21 = ISC-1
   LIM11=LIM1+1
   DO 14 I=ID21,LIM11,2
   EYE=I-1.D
   EA = EM + EYE
   GLIST(J) = EA*(EA+1.D)+0.5D*HSQ*((1.D)-(4.D*EM*EM-1.D)/((2.D*EA-1.
  1D)*(2.D*EA+3.D)))
14 J=J+1
   IFC=0
   IBLIM = 139 - IX
   IGLIM=IBLIM+1
   IRIO=IW6+1
   IW1 = IW6+2
17 ENR(1)=CL-GLIST(1)
   DO 18 I=1,IW6
18 ENR(I+1)=-BLIST(I)/ENR(I)-GLIST(I+1)+CL
   ENR(IBLIM)=-BLIST(IBLIM)/(GLIST(IGLIM)-CL)
   IW15=IBLIM-1
   IP=IW1  +IW15
```

```
      DO 19 I=IW1  ,IW15
      IPI=IP-I
   19 ENR(IPI)=-BLIST(IPI)/(GLIST(IPI+1)-CL+ENR(IPI+1))
      ENRC=-BLIST(IRIO)/(GLIST(IRIO+1)-CL+ENR(IRIO+1))
      DE = ENRC*ENRC/BLIST(IRIO)
      CORB = DE
      DO 20 I = IW1,IBLIM
      DE = ENR(I)*ENR(I)/BLIST(I)*DE
      CORB = CORB + DE
      IF (DABS(DE/CORB).LT.1.D-27) GO TO 23
   20 CONTINUE
   23 CORA = 1.D
      DE = 1.D
      DO 26   I = 1,IW6
      DE = BLIST(IRIO-I)/(ENR(IRIO-I)*ENR(IRIO-I))*DE
      CORA = CORA + DE
      IF (DABS(DE/CORA).LT.1.D-27) GO TO 27
   26 CONTINUE
   27 DL = (ENRC-ENR(IRIO))/(CORA+CORB)
      CL=DL+CL
      IF (DABS(DL/CL).LT.1.0D-24) GO TO 22
      IFC=IFC+1
      IF (IFC.LT.20) GO TO 17
   22 EIGL(JNH,L+1-M) = CL
      DO 118 I = 1,IBLIM
      ARR = IX + 2.D*I
      EA = 2.D*EM + 2.D*ARR
  118 DNDO(I) = (EA-1.D)*(EA+1.D)*ENR(I)/((EA-ARR)*(EA-ARR-1.D)*HSQ)
      DN(1) = DNDO(1)
      DM(1) = DNDO(1)*1.D150
      DO 119 J = 2,IBLIM
      DN(J) = DNDO(J)*DN(J-1)
  119 DM(J) = DNDO(J)*DM(J-1)
      IW21 = 2*IBLIM
      W28 = 0.D
      IWW = 0
      IF (M.EQ.0) GO TO 165
      M2 = 2*M
      DO 120 J = ISC,IW21,2
      W26 = 1.D
      DO 160 K = 1,M2
  160 W26 = W26*(J+K)
      IWW = IWW + 1
  120 W28 = W28 + W26*DN(IWW)
      GO TO 167
  165 DO 166 J = ISC,IW21,2
      IWW = IWW + 1
  166 W28 = W28 + DN(IWW)
  167 DLIST(1) = FACT(L+M+1)/FACT(L-M+1)/(W28+FACT(M+M+IX+1))
      DLAST(1) = DLIST(1)*1.D150
      DO 121 J = 1, IBLIM
      DLAST(J+1) = DM(J)*DLIST(1)
  121 DLIST(J+1) = DN(J)*DLIST(1)
      IF (X.NE.1.D) GO TO 299
      IF (M.NE.0) GO TO 59
      IF (IX.NE.0) GO TO 51
      DNUM = FACT(M+M+1)/FACT(M+1)*DLIST(1)
      DO 56  J = 1,48
   56 DNUM = FACT(J+J+M+M+1)/(4.D**J*FACT(J+1)*FACT(J+M+1))*DLIST(J+1)
```

```
     1-DNUM
      C101 = FACT(L+M+1)/(2.D**(L-M)*FACT(IW6+1)*FACT((L+M)/2+1)*DNUM)
      IF (2*(IW6/2).NE.IW6) C101 = -C101
      GO TO 58
   51 DNUM = FACT(M+M+2)/FACT(M+1)*DLIST(1)
      DO 57  J = 1,48
   57 DNUM = FACT(J+J+M+M+3)/(2.D**(J+J+1)*FACT(J+1)*FACT(J+M+2))*DLIST(
     1J+1) - DNUM
      C101 = FACT(L+M+2)/(2.D**(L-M)*FACT(IW6+1)*FACT((L+M+1)/2+1)*DNUM)
      IF (2*(IW6/2).NE.IW6) C101 = -C101
   58 DLIST(1) = DLIST(1)*C101
      IF (IX.NE.0) GO TO 40
      R1 = 2.D**L/FACT(L+1)*FACT(L/2+1)**2*DLIST(1)
      GO TO 41
   40 R1 = 2.D**L*H/(3.D*FACT(L+2))*FACT((L-1)/2+1)*FACT((L+1)/2+1)*
     1DLIST(1)
   41 SUBSUM = 0.D
      J = 1
      DO 53 I = IX,LIM1,2
      SUBSUM = SUBSUM + I*(I+1.D)*DLIST(J)
   53 J = J + 1
      R1D = R1*SUBSUM/2.D
      PRINT 50, L,R1,R1D,CL
   50 FORMAT (1XI3,1X2D25.17,50XD25.17)
      GO TO 59
  299 IF ((X.GT.1.3D.AND.H.GE.10.D).OR.(X.GT.2.D.AND.H.GE.1.D).OR.(X.GT.
     15.D.AND.H.GE..1D)) GO TO 499
  300 DO 333 I = 1,100
      BAY(I) = 0.0D
  333 DR(I) = 0.0D
  303 IF (M .EQ. 0) 305, 306
  305 DNEG (1) = DLIST (1)
      GO TO 307
  306 N = 2 - 2*M + IX
      CALL USN (N, M, UN)
      N = -2*M + IX
      CALL VSN (N, M, H, CL, VN)
      RATIO (1) = - UN/VN
      IT = 0
      DO 308  I = 2, 40, 2
      ITD2 = I - 2*M + IX
      IF (ITD2) 309, 310, 310
  309 N = ITD2 + 2
      CALL USN (N, M, UN)
      N = ITD2
      CALL VSN (N, M, H, CL, VN)
      N = ITD2 - 2
      CALL WSN (N, M, WN)
      IT = I/2
  308 RATIO ( IT+1) = - UN/(WN*RATIO(IT)+VN)
  310 IAR = IT + 1
      DNFREE = DLIST(1)
      DO 311 I = 1, IAR
      J = IAR + 1 - I
      DNEG (J) = RATIO (J) * DNFREE
  311 DNFREE = DNEG (J)
  307 IS1 = 4 + M + M
      DRFREE = 0.D
      IK = IX
```

```
      DO 319 K = IS1, 140, 2
      J = 140 - IK
      N = 2 - J
      IY = 1 + (J-1)/2
      CALL USN (N, M, UN)
      N = -J
      CALL VSN (N, M, H, CL, VN)
      N = -J -2
      CALL WSN (N, M, WN)
      DRRATIO (IY) = -UN/(WN*DRFREE + VN)
      DRFREE = DRRATIO (IY)
  319 IK = IK + 2
      N = -2*M -2 + IX
      CALL VSN (N, M, H, CL, VN)
      N = -2*M -4 + IX
      CALL WSN (N, M, WN)
      IA = IY - 1
      DRRATIO(IA) = 1.D/((M+M+1-IX-IX)*(M+M-1-IX-IX)*(WN*DRRATIO(IY)+
     1VN))
      IF (IX.EQ.1) DRRATIO(IA) = -DRRATIO(IA)
      DNEGF = DNEG (1)
      IAA = 0
      DO 320 I = IA, 70
      IAA = IAA + 1
      DR (IAA) = DRRATIO (I) * DNEGF
      DNEGF = DR (IAA)
      IF (DABS(DNEGF).LE.1.D-290) 323,320
  320 CONTINUE
  323 SAIL 3 = Z * (Z + 2.D)
      SAIL4 = DSQRT (SAIL3)
      IF (M.NE.0) GO TO 401
      DO 407 I = 1, 141
  407 SHIP7 (I) = SHIP5 (I)/SAIL4
      GO TO 411
  401 CALL CALNQ (M, Z, SHIP1)
      IM1 = M + 1
      CALL CALNQ(IM1,Z,SHIP2)
      DO 403 I = 1, M
  403 SHIP3(I) = (SAIL4*SHIP2(I+1) + EM*X*SHIP1(I))/SAIL3
      DO 410 I = 1, 141
  410 SHIP9(I) = -EM*X*SHIP4(I)/SAIL3 + SHIP8(I)*(I+EM-1.D)*(I-EM)/SAIL4
  411 DO 414 I = 1, 141
  414 SHIP12(I) = (SAIL4*SHIP11(I) +  EM*X*SHIP10(I))/SAIL3
      DO 415 I = 1, M
  415 SILAND (I) = SHIP1 (I)
      IM1 = M + 1
      DO 416 I = 1, 141
  416 SILAND(I+M) = SHIP4(I)
      DO 417 I = 1, M
  417 BAY (I) = DNEG (I)
      DO 418 I = 1, 70
  418 BAY(I+M) = DLIST(I)
      IS16 = IX - 1
      SAIL18 = 0.D
      IB = 1
      IFJ = 20
  471 DO 422 I = IB,IFJ
      II = I
      IS16 = IS16 + 2
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422 SAIL 18 = BAY(I)*SILAND(IS16) + SAIL18
    I = II
    IF (DABS(BAY(I)*SILAND(IS16)/SAIL18).LT.1.D-26) GO TO 470
    IB = I + 1
    IFJ = IFJ + 5
    GO TO 471
470 SAIL19 = 0.D
    IS20 = 0
    IM1 = M + 2 - IX
    IFJ = 40
474 DO 426 I = IM1, IFJ, 2
    II = I
    IS20 = IS20 + 1
426 SAIL19 = SHIP10(I)*DR(IS20) + SAIL19
    I = II
    IF (DABS(SHIP10(I)*DR(IS20)/SAIL19).LT.1.D-26) GO TO 473
    IFJ = IFJ + 10
    IM1 = II + 2
    GO TO 474
473 SAIL22 = SAIL18 + SAIL 19
    DO 427 I = 1, M
427 SHORE (I) = SHIP3 (I)
    IM1 = M + 1
    IF (M .EQ. 0) 428, 430
428 DO 429 I = 1, 141
429 SHORE (I+M) = SHIP7 (I)
    GO TO 440
430 DO 431 I = 1, 141
431 SHORE (I+M) = SHIP9 (I)
440 IS16 = IX - 1
    SAIL181 = 0.D
    IB = 1
    IFJ = 20
475 DO 435 I = IB,IFJ
    II = I
    IS16 = IS16 + 2
435 SAIL181 = BAY(I)*SHORE(IS16) + SAIL181
    I = II
    IF (DABS(BAY(I)*SHORE(IS16)/SAIL181).LT.1.D-26) GO TO 476
    IB = I + 1
    IFJ = IFJ + 5
    GO TO 475
476 IS21 = 0
    SAIL191 = 0.D
    IM1 = M + 2 - IX
    IFJ = 40
478 DO 439 I = IM1, IFJ, 2
    II = I
    IS21 = IS21 +1
439 SAIL191 = DR(IS21)*SHIP12(I) + SAIL191
    I = II
    IF (DABS(DR(IS21)*SHIP12(I)/SAIL191).LT.1.D-26) GO TO 477
    IFJ = IFJ + 10
    IM1 = II + 2
    GO TO 478
477 SAIL23 = SAIL181 + SAIL191
    CFAC = FACT(M+M+1)*FACT(L+M+1+IX)*DNEG(1)/(H**(M-1-IX)*FACT(L-M+1)
   1*(M+M-1)*FACT(M+1))
    IF (IX.EQ.0) 70,71
```

```
 70 DNUM = FACT(M+M+1)/FACT(M+1)*DLIST(1)
    DO 717 J = 1,48
717 DNUM = FACT(J+J+M+M+1)/(4.D**J*FACT(J+1)*FACT(J+M+1))*DLIST(J+1)
   1-DNUM
    CFAC = CFAC/DNUM
    IF (2*(IW6/2).NE.IW6) CFAC = -CFAC
    GO TO 72
 71 DNUM = FACT(M+M+2)/FACT(M+1)*DLIST(1)
    DO 718 J = 1,48
718 DNUM = FACT(J+J+M+M+3)/(2.D**(J+J+1)*FACT(J+1)*FACT(J+M+2))*DLIST(
   1J+1) - DNUM
    CFAC = -CFAC/(DNUM*(M+M-3)*(L+M+1))
    IF (2*(IW6/2).NE.IW6) CFAC = -CFAC
 72 R2 = SAIL22/CFAC
    R2D = SAIL23/CFAC
    IF ((X.LE.1.05D.OR.X.GT.1.3D).AND.H.GT..1D) GO TO 599
    RR2 = R2
    RR2D = R2D
499 PL1 = FACT(L-M+1)/FACT(L+M+1)
    PL3 = (Z + 2.D) * Z
    X1SQ = X*X
    PL4 = PL1*DSQRT((PL3/X1SQ)**M)
    XTH = X*H
    NYMN = 2*IBLIM + M
    IF (LRK.EQ.1) GO TO 63
    DO 550 I = 1,300
550 ARRAY(I) = 0.0D
    ARRAY(1) = -DCOS(XTH)/XTH*1.D-150
    ARRAY(2) = ARRAY(1)/XTH-DSIN(XTH)/XTH*1.D-150
    DO 62 K = 1,NYMN
    ARRAY(K+2) = (K+K+1.D)*ARRAY(K+1)/XTH-ARRAY(K)
    IF (DABS(ARRAY(K+2)).GT.1.D300) GO TO 63
 62 CONTINUE
 63 IA = 1 + IX
    IC = 41 + IX
    SUBSUM = 0.D
503 DO 505 K = IA, IC, 2
    KK = K
    IBOX3 = IABS(K+M-L-1)/2
    PL8 = 1.D
    KPK = K+M+M-1
    DO 700  J = K,KPK
700 PL8 = PL8*J
    BOOK(K) = DLAST((K+1)/2)*PL8*ARRAY(K+M)
    IF (2*(IBOX3/2).NE.IBOX3) BOOK(K) = -BOOK(K)
    SUBSUM = BOOK (K) + SUBSUM
505 CONTINUE
    K = KK
    IF (DABS(BOOK(K)/SUBSUM).GE.1.D-26) 506,507
506 IA = IC + 2
    IC = IC + 10
    GO TO 503
507 R2 = PL4 * SUBSUM
    PL17 = PL3/X1SQ
    IF (M .EQ. 0) 508, 509
508 PL12 = H
    PL18 = 0.D
    GO TO 513
509 PL18 = DSQRT(PL17**(M-2))*EM/(X*X1SQ)
```

```
      PL12 = H*DSQRT(PL17**M)
  513 IBB = 1 + IX
      ID = 41 + IX
      SSUM1 = 0.D
  517 DO 522 K = IBB, ID, 2
      KK = K
      IBOX3 = IABS(K+M-L-1)/2
      PL8 = 1.D
      KPK = K+M+M-1
      DO 701  J = K,KPK
  701 PL8 = PL8*J
      IF (2*(IBOX3/2).NE.IBOX3)  PL8 = -PL8
      IF ((K+M-1).EQ.0) 518,519
  518 PAPER(K) = -DLAST((K+1)/2)*PL8*ARRAY(2)
      GO TO 521
  519 PL13 = K + M -1
  520 PAPER(K) = (PL13*ARRAY(K+M-1)-(PL13+1.D)*ARRAY(K+M+1))/(2.D*PL13
     1+1.D)*DLAST((K+1)/2)*PL8
  521 SSUM1 = PAPER (K) + SSUM1
  522 CONTINUE
      K = KK
      IF (DABS(PAPER(K)/SSUM1).GE.1.D-26) 523,524
  523 IBB = ID + 2
      ID = ID + 10
      GO TO 517
  524 R2D = PL1*(PL12*SSUM1 + SUBSUM*PL18)
      GO TO 597
  599 PL1 = FACT(L-M+1)/FACT(L+M+1)
      PL3 = (Z + 2.D) * Z
      X1SQ = X*X
      PL4 = PL1*DSQRT((PL3/X1SQ)**M)
      XTH = X*H
  597 DO 598 I = 1,300
      BOOK (I) = 0.D
  598 PAPER (I) = 0.D
      IF (LRK.EQ.1) GO TO 460
      CALL SBESF(XTH, BRRAY)
  460 IA = 1 + IX
      IC = 41 + IX
      SUBSUM = 0.D
  603 DO 605 K = IA, IC, 2
      KK = K
      IBOX3 = IABS(K+M-L-1)/2
      PL8 = 1.D
      KPK = K+M+M-1
      DO 702  J = K,KPK
  702 PL8 = PL8*J
      BOOK(K) = DLIST((K+1)/2)*PL8*BRRAY(K+M)
      IF (2*(IBOX3/2).NE.IBOX3)  BOOK(K) = -BOOK(K)
      SUBSUM = BOOK (K) + SUBSUM
  605 CONTINUE
      K = KK
      IF (DABS(BOOK(K)/SUBSUM).GE.1.D-26) 606,607
  606 IA = IC + 2
      IC = IC + 10
      GO TO 603
  607 R1 = PL4*SUBSUM
      PL17 = PL3/X1SQ
      IF (M .EQ. 0) 608, 609
```

```
608 PL12 = H
    PL18 = 0.D
    GO TO 613
609 PL18 = DSQRT(PL17**(M-2))*EM/(X*X1SQ)
    PL12 = H*DSQRT(PL17**M)
613 IBB = 1 + IX
    ID = 41 + IX
    SSUM1 = 0.D
617 DO 622 K = IBB, ID, 2
    KK = K
    IBOX3 = IABS(K+M-L-1)/2
    PL8 = 1.D
    KPK = K+M+M-1
    DO 703  J = K,KPK
703 PL8 = PL8*J
    IF (2*(IBOX3/2).NE.IBOX3)  PL8 = -PL8
    IF ((K+M-1).EQ.0) 618,619
618 PAPER(K) = -DLIST((K+1)/2)*PL8*BRRAY(2)
    GO TO 621
619 PL13 = K+M-1
620 PAPER(K) = (PL13*BRRAY(K+M-1)-(PL13+1.D)*BRRAY(K+M+1))/(2.D*PL13
   1+1.D)*DLIST((K+1)/2)*PL8
621 SSUM1 = PAPER (K) + SSUM1
622 CONTINUE
    K = KK
    IF (DABS(PAPER(K)/SSUM1).GE.1.D-26) 623,624
623 IBB = ID + 2
    ID = ID + 10
    GO TO 617
624 R1D = PL1*(PL12*SSUM1 + SUBSUM*PL18)
    TWRON = 1.D/(H*Z*(Z+2.D))
    CWRON = R1*R2D - R1D*R2
    IAC = -DLOG10(DABS((TWRON-CWRON)/TWRON) + 1.D-26)
    IF (IAC.LT.0) IAC = 0
    IF ((X.LE.1.05D.OR.X.GT.1.3D).AND.H.GT..1D) GO TO 55
    CWRON = R1*RR2D - R1D*RR2
    IBC = -DLOG10(DABS((TWRON-CWRON)/TWRON) + 1.D-26)
    IF (IAC.GE.IBC) GO TO 55
    R2 = RR2
    R2D = RR2D
    IAC = IBC
 55 PRINT 60, L,R1,R1D,R2,R2D,CL,IAC
 60 FORMAT (1X13,1X5D25.17,I5)
 59 LRK = 1
  2 LCK = 1
  1 LDK = 1
    STOP
    END
    SUBROUTINE CALCPQ(M,LN,XX,P,Q)
    DIMENSION P(150),Q(150),AP(25,150)
    COMMON/BLK1/FACT(300)
    TYPE DOUBLE P,Y,Y1,S,X,D2M,DJ,DM,DN,FACT,TERM,SUM,CWRON,DK,Z,ZZ,
   1XA,HAL,AL,BE,XY,COEF,GA,REALJ,REALJH,Q,FIRST, HM,  XZ,TWRON,B,
   2XX,RC1,RC2,RC3,RC4,PC1,PC2,PC3,PC4,AP,FACTR1,FACTR2,FACTR3
    INITIAL=0
    NN=0
 16 X=XX+1.D
    KAPPA=0
    IF(X.GT. 1.001D) KAPPA=M
```

```
      DO 65  MN=KAPPA,M
     *DM = MN
      D2M=2.D**DM
      Z=XX**(DM/2.0D)
      ZZ=(X+1.D)**(DM/2.0D)
      Y=Z*ZZ
      DO 65   N=NN,LN
      IF (MN.GT.N) 30,35
   30 AP(MN+1,N+1) = 0.0D
      GO TO 65
   35 DN = N
      NLESSM=N-MN
      TERM = FACT(N+MN+1)/FACT(N-MN+1)/(FACT(MN+1)*D2M)
      SUM = TERM
      DO 40 J = 1,NLESSM
      DJ = J
      TERM = TERM*(DN+DM+DJ)*XX*(DN-DM-DJ+1)/(DJ*(DM+DJ)*2.D)
      SUM = SUM + TERM
   40 IF (DABS(TERM/SUM).LT.1.D-30) GO TO 45
   45 AP(MN+1,N+1)=Y*SUM
   65 P(N+1) = AP(MN+1,N+1)
      IF (X.GT. 1.001D) 300,50
   50 FIRST= 0.5D * DLOG( (XX+2.0D)/XX)
      MONE=M+1
      DO 145 N=NN,LN
      DN = N
      Q(N+1) = FIRST*P(N+1)
      IF(N.EQ.0) GO TO70
      SUM=0.D
      KRAP=(N-1)/2
      DO 60 J=INITIAL,KRAP
       RJ=J
      DJ=DBLE(RJ)
      S=((2.D*DN-4.D*DJ-1.D)/(( (1.D) +2.D*DJ)*(DN-DJ)))*P(N-2*J)
   60 SUM=SUM+S
      Q(N+1) = Q(N+1)-SUM
   70 DO 140 J=2,MONE
      RJ=J-1
      REALJ=DBLE(RJ)
      REALJH= REALJ/2.0D
      Y= (XX**REALJ-((X+1.D)**REALJ))/(((XX*(X+1.D))**REALJH) *2.0D)
      S=FACT(M+1)/(REALJ  *FACT(M-J+2))
      TERM=Y*S*AP(M-J+2,N+1)
      IF (2*(J/2).NE.J) TERM = -TERM
  140 Q(N+1) = Q(N+1) + TERM
  145 CONTINUE
      GO TO 350
  300 EPS=1.0E-30
      XY=XX*(XX+2.0D)
      DM=M
      XZ=1.0D/(X+DSQRT(XY))
      Z=XZ*XZ
      HM=DM/2.0D
      XA=XY**HM
      DO 340 N=NN,LN
      DN = N
      AL=DN+DM+1.0D
      BE=DM+0.5D
      GA=DN+1.5D
```

```
      HAL=AL/2.0D
      SUM=1.0D
      TERM=1.0D
      K=0
  330 K=K+1
      DK = K-1
      IF(K.GT.2500)GO TO 335
  331 TERM=TERM*(AL+DK)*(BE+DK)*Z/(K*(GA+DK))
      SUM=SUM+TERM
      EPC=TERM/SUM
      IF(EPC-EPS)335,335,330
  335 IF(M+N .GE.131) 410,420
  410 FACTR1=2.D**(AL+DN)*(FACT(N+M+1)*1.D-300)
      GO TO 425
  420 FACTR1=2.0D**(AL+DN)*FACT(N+M+1)
  425 FACTR2= (Z**HAL)*XA
      FACTR3 = FACTR1*FACTR2
      COEF=FACTR3*(FACT(N+1)/FACT(2*N+2))
      IF(N.LT.85) GO TO 1338
      IF ( M+N.LT.131) COEF=COEF*(1.D-300)
 1338 JM=M/2
      IF(2*JM.EQ.M) 340,338
  338 COEF=-1.D*COEF
  340 Q(N+1) = COEF*SUM
  350 END
      SUBROUTINE USN(N,M,S)
      TYPE DOUBLE S,A
      A = N+M+M
      S = A*(A-1.D)/((N+A+1.D)*(N+A-1.D))
      END
      SUBROUTINE VSN(N,M,H,CL,S)
      TYPE DOUBLE H,CL,S,A
      A = N+M
      S = (2.D*A*(A+1.D)-2*M*M-1.D)/((2.D*A+3.D)*(2.D*A-1.D))
     1+(A*(A+1.D)-CL)/(H*H)
      END
      SUBROUTINE WSN(N,M,S)
      TYPE DOUBLE S
      S = (N+2.D)*(N+1.D)/((N+N+M+M+3.D)*(N+N+M+M+1.D))
      END
      SUBROUTINE SBESF(XH,RAY)
      DIMENSION RAY(250)
      TYPE DOUBLE CP,FACT,RAY,SUM,TERM,TK,TM,XH,XI,Z2H
      COMMON /BLK1/FACT(300)
      L = 0
    1 IF (XH.GE..4D) GO TO 4
      Z2H=XH*XH/2.D
      DO 3 N = L,140
      TM=FACT(N+1)*(XH +XH)**N/FACT(N+N+2)
      IF(N.GT.84) TM=TM*1.D-300
      IF(TM.EQ.0.D) GO TO 8
      SUM=1.D
      TERM = 1.D
      DO 2 I=1,50
      XI=I*(N+N+I+I+1)
      TERM=-TERM*Z2H/XI
      SUM = SUM + TERM
    2 IF(DABS(TERM/SUM).LT.1.D-26) GO TO 3
    3 RAY(N+1) =TM*SUM
```

```
      RETURN
    4 N = 170
      IF(XH.LT.100.D) GO TO 20
      RAY(1)=DSIN(XH)/XH
      RAY(2)=(RAY(1)-DCOS(XH))/XH
      NM1=N-1
      DO 11 K = 1,138
   11 RAY(K+2)=(K+K+1)*RAY(K+1)/XH-RAY(K)
      RETURN
   20 IF (XH.GT.10.0D) N = 200
      RAY(N+1) = 1.D-200
      RAY(N+2)=0.D
      I = -N
      M = -1
      DO 5 KK=I,M
      K = -KK
      TK=K+K+1
    5 RAY(K)=TK*RAY(K+1)/XH-RAY(K+2)
      CP=DSIN(XH)/(XH*RAY(1))
      IF(DSIN(XH).LT.1.D-2) CP=(DSIN(XH)/XH-DCOS(XH))/(XH*RAY(2))
      DO 6 L=1,140
    6 RAY(L)=CP*RAY(L)
    8 DO 9 J=N,139
    9 RAY(J+1)=0.D
      DO 12 I = 141,250
   12 RAY(I) = 0.D
      END
      SUBROUTINE CALNQ(IY,ZB,Q)
      TYPE DOUBLE ZB,Q,A,CV,P,FACT,XI
      COMMON/BLK1/FACT(300)
      DIMENSION CV(100), P(100), Q(100)
      XI = ZB + 1.D
      IY1 = IY + 1
      A = DSQRT((ZB+2.D)*ZB)
      P(1) = -1.D/A
      DO 5 I = 1,IY
      CV(I+1) = -FACT(I+I-1)*XI*(A/2.D)**(I-1)/(FACT(I)*A)-(I+I-2)*
     1CV(I)/A
    5 P(I+1) = ((I+I-1)*P(I)+XI*CV(I+1))*P(1)
      NT = 2-IY-IY
      Q(1) = CV(IY1)
      Q(2) = P(IY)
      DO 6 I = 1,IY1
      IF (IABS(NT).LE.IY) GO TO 7
      Q(I+2) = ((I+I+1-IY-IY)*XI*Q(I+1)-I*Q(I))/NT
    6 NT = NT + 1
    7 END
      SUBROUTINE EIGEN(A,VALU,N,NBL)
      DIMENSION A(100,100),VALU(100),DIAG(100),Q(100),VALL(100)
      ANORM = A(1,1)*A(1,1) + A(2,2)*A(2,2)
      DO 6 I = 3,N
    6 ANORM = ANORM + A(I,I)*A(I,I)+ 2*A(I-2,I)*A(I-2,I)
      ANORM = SQRTF(ANORM)
      NN=N-2
      DO 160 I=1,NN
      II=I+2
      DO 160 J=II,N
      T1=A(I,I+1)
      T2=A(I,J)
```

```
      IF (T2.EQ.0.) GO TO 160
      T=1./SQRT(T1*T1+T2*T2)
      SIN=T2*T
      COS=T1*T
      DO 105 K=I,N
      T2=COS*A(K,I+1)+SIN*A(K,J)
      A(K,J)=COS*A(K,J)-SIN*A(K,I+1)
  105 A(K,I+1)=T2
      DO 125 K=I,N
      T2=COS*A(I+1,K)+SIN*A(J,K)
      A(J,K)=COS*A(J,K)-SIN*A(I+1,K)
  125 A(I+1,K)=T2
  160 CONTINUE
      DO15I=1,N
      DIAG(I)=A(I,I)
      Q(I)=A(I,I-1)*A(I,I-1)
      VALL(I)=0.
   15 VALU(I) = ANORM
      I=1
   18 TAU=(VALL(I)+VALU(I))/2.
      MATCH=0
      T0 = 0.
      T1 = 1.
      DO20J=1,N
      T2=(DIAG(J)-TAU)*T1-Q(J)*T0
      IF (ABS(T2).LT.1.E130) GO TO 26
      T2 = T2*1.E-200
      T1 = T1*1.E-200
   26 IF((T1.NE.0.).AND.((T2*T1).LE.0.))  MATCH=MATCH+1
      T0=T1
   20 T1=T2
      DO 25 K = I, MATCH
   25 VALU(K) = TAU
      NATCH = MATCH + 1
      DO 30 K = NATCH, NBL
   30 IF (VALL(K).LT.TAU) VALL(K) = TAU
   40 IF ((VALU(I)-VALL(I))/VALU(I).GT.1.E-7)  GO TO 18
      I=I+1
      IF(I.LE.NBL)  GO TO 40
      END
```

Appendix C

VALUES OF ACC FOR SOME PARAMETERS

X = 1.00000001 M = 0

| L | H = 0.1 | H = 1 | H = 10 | H = 20 | H = 40 | H = 60 |
|---|---|---|---|---|---|---|
| 0 | 23 | 23 | 21 | 18 | 11 | 0 |
| 3 | 21 | 23 | 24 | 19 | 11 | 4 |
| 9 | 20 | 23 | 23 | 23 | 17 | 8 |
| 18 | 19 | 19 | 20 | 23 | 21 | 14 |
| 26 | 16 | 16 | 17 | 18 | 23 | 16 |
| 34 | 13 | 13 | 14 | 15 | 21 | 14 |
| 42 | 10 | 10 | 11 | 11 | 18 | 12 |
| 48 | 8 | 8 | 8 | 9 | 13 | 10 |

X = 1.2 M = 0

| L | H = 0.1 | H = 1 | H = 10 | H = 20 | H = 40 | H = 60 |
|---|---|---|---|---|---|---|
| 0 | 24 | 24 | 23 | 17 | 8 | 0 |
| 3 | 21 | 23 | 22 | 19 | 12 | 3 |
| 9 | 20 | 22 | 23 | 22 | 17 | 11 |
| 18 | 19 | 22 | 23 | 21 | 20 | 15 |
| 26 | 19 | 22 | 21 | 23 | 22 | 19 |
| 34 | 21 | 19 | 20 | 21 | 19 | 22 |
| 42 | 18 | 18 | 18 | 19 | 20 | 22 |
| 48 | 17 | 17 | 19 | 19 | 19 | 18 |

X = 500 M = 0

| L | H = 0.1 | H = 1 | H = 10 | H = 20 | H = 40 | H = 60 |
|---|---|---|---|---|---|---|
| 0 | 24 | 24 | 21 | 17 | 9 | 0 |
| 3 | 23 | 24 | 22 | 20 | 12 | 4 |
| 9 | 21 | 24 | 24 | 23 | 17 | 8 |
| 18 | 20 | 24 | 24 | 24 | 22 | 15 |
| 26 | 20 | 24 | 24 | 24 | 24 | 20 |
| 34 | 19 | 24 | 24 | 24 | 24 | 23 |
| 42 | 19 | 24 | 24 | 24 | 24 | 26 |
| 48 | 19 | 22 | 24 | 24 | 24 | 24 |

Security Classification

## DOCUMENT CONTROL DATA - R & D

*(Security classification of title, body of abstract and indexing annotation must be entered when the overall report is classified)*

| | |
|---|---|
| 1. ORIGINATING ACTIVITY *(Corporate author)*<br>Naval Research Laboratory<br>Washington, D.C. 20390 | 2a. REPORT SECURITY CLASSIFICATION<br>Unclassified<br>2b. GROUP |

3. REPORT TITLE

A FORTRAN COMPUTER PROGRAM FOR CALCULATING THE PROLATE SPHEROIDAL RADIAL FUNCTIONS OF THE FIRST AND SECOND KIND AND THEIR FIRST DERIVATIVES

4. DESCRIPTIVE NOTES *(Type of report and inclusive dates)*

An interim report on one phase of the problem; work on other phases is continuing.

5. AUTHOR(S) *(First name, middle initial, last name)*

B.J. King, R.V. Baier, and S. Hanish

| | | |
|---|---|---|
| 6. REPORT DATE<br>March 30, 1970 | 7a. TOTAL NO. OF PAGES<br>34 | 7b. NO. OF REFS<br>6 |
| 8a. CONTRACT OR GRANT NO.<br>NRL Problem S01-29<br>b. PROJECT NO.<br>RR 102-08-41-5226<br>c.<br>d. | 9a. ORIGINATOR'S REPORT NUMBER(S)<br>NRL Report 7012<br>9b. OTHER REPORT NO(S) *(Any other numbers that may be assigned this report)* | |

10. DISTRIBUTION STATEMENT

This document has been approved for public release and sale; its distribution is unlimited.

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|---|---|
| 11. SUPPLEMENTARY NOTES | 12. SPONSORING MILITARY ACTIVITY<br>Department of the Navy<br>(Office of Naval Research),<br>Washington, D.C. 20360 |

13. ABSTRACT


The solutions $\psi = R(\xi)\,S(\eta)\,\Phi(\varphi)$ of the Helmholtz wave equation $(\nabla^2 + k^2)\psi(\xi,\eta,\varphi) = 0$ in prolate spheroidal coordinates $\xi,\eta,\varphi$ can be obtained by separation of variables. The subject of this report is a Fortran computer program called PRAD which calculates numerical values to the solutions of the resulting ordinary differential equation for the "radial" coordinate $\xi$. The printed output of PRAD consists of radial functions of the first and second types $R^{(1),(2)}_{m,\ell}(h,\xi)$, their first derivatives $d(R^{(1),(2)}_{m,\ell}(h,\xi))/d\xi$, the separation constants or eigenvalues $A_{m,\ell}(h)$, and an accuracy check.

This report describes the computer program PRAD and briefly reviews the theory of prolate spheroidal wave functions. A computer listing of PRAD along with some sample output is included in an appendix.


DD FORM 1 NOV 65 1473 (PAGE 1)

S/N 0101-807-6801



Security Classification

Security Classification

| 14. KEY WORDS | LINK A ROLE | LINK A WT | LINK B ROLE | LINK B WT | LINK C ROLE | LINK C WT |
|---|---|---|---|---|---|---|
| Tables (data) | | | | | | |
| Helmholtz wave equations | | | | | | |
| Wave equations | | | | | | |
| Underwater acoustics | | | | | | |
| Acoustic fields | | | | | | |
| Sonar | | | | | | |

DD FORM 1 NOV 65 1473 (BACK)
(PAGE 2)



Security Classification

☆U.S. GOVERNMENT PRINTING OFFICE: 1970-397-174/65