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An Extension of Ghazala's Method to Incompletely Specified Multiple-Output Functions

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AN EXTENSION OF GHAZALA'S METHOD TO INCOMPLETELY SPECIFIED MULTIPLE-OUTPUT FUNCTIONS

NOTATION

Symbols from Boolean Algebra and Set Theory

$+$	Boolean "or"
$\bigvee$	Iterated Boolean "or"
lub	Boolean "or" of the members of a set ("least upper bound")
$\cdot$	Boolean "and" (also indicated by the concatenation of expressions)
$\bigwedge$	Iterated Boolean "and"
glb	Boolean "and" of the members of a set ("greatest lower bound")
$'$	Boolean "not" (e.g., a$'$ means "not a")
$\rightarrow$	Boolean implication
E/T	Boolean "ratio" of the expression E to the term T (E/T is equivalent to E evaluated with T set equal to 1; see Ref. 1)
$\{x_1, x_2, \ldots, x_n\}$	The set of objects $x_1, x_2, \ldots, x_n$
$\{x \mid C\}$	The set of objects x such that condition C is satisfied
Φ	The empty set
$\subseteq$	"(which) is a subset of"
$\in$	"(which) is a member of"
$\notin$	"(which) is not a member of"
$-$	Set "difference" $\left(A - B = \{x \mid x \in A \text{ and } x \notin B\}\right)$

Meta-Language Symbols

$\Rightarrow$	"implies"
$\Leftrightarrow$	"if and only if"
$\equiv$	"is identically equal to"
$\forall$	"for every"
$\ni:$	"such that"
$\exists$	"there exists (a)"
$\blacksquare$	"end of proof"

AN APPROACH USING QUINE'S DISPENSABILITY TEST AS ITS BASIS

Once the prime implicants of a Boolean function have been found, the implication relations among them may be used to find all of the irredundant covers for that function (1, 2). It is desirable that the method used to find the implication relations not require a prime implicant table, because the use of a prime implicant table requires that the function be expanded into canonical form. Ghazala has presented such a method for the completely specified case (1). The extension of his method to the incompletely specified case requires an extension of Quine's test for dispensability (3):

Definitions: Let f be an incompletely specified Boolean function of n Boolean variables $x_n, x_{n-1}, \ldots, x_1$. Let f_{max} be a Boolean function such that $f_{max}(x_n, x_{n-1}, \ldots, x_1) = 1$ if and only if $(x_n, x_{n-1}, \ldots, x_1)$ is either a "1" vertex or a don't care vertex of f, and let $\{\phi_1, \phi_2, \ldots, \phi_p\}$ be the set of all prime implicants of f_{max}. Let f_{dc} be a Boolean function such that $f_{dc}(x_n, x_{n-1}, \ldots, x_1) = 1$ if and only if $(x_n, x_{n-1}, \ldots, x_1)$ is a don't care vertex of f, and let $\{\Delta_1, \Delta_2, \ldots, \Delta_q\}$ be the set of all prime implicants of f_{dc}. Let $A \subseteq \{\phi_1, \phi_2, \ldots, \phi_p\}$ such that A covers all of the "1" vertices of f, let $\bigvee_A \phi_i$ be the disjunction of those and only those ϕ_i that are in A, and let $B = A - \{\phi_j\}$

for a given $\phi_j \in A$. Then ϕ_j is said to be dispensable with respect to A if and only if B covers all the "1" vertices of f. Let $\bigvee_B \phi_i$ be the disjunction of those and only those ϕ_i which are in B.

Theorem (a generalization of Quine's dispensability theorem): *$\phi_j \in A$ is dispensable with respect to A if and only if*

$$\phi_j \to \bigvee_B \phi_i + \Delta_1 + \Delta_2 + \cdots + \Delta_q, \text{ where } B = A - \{\phi_j\}.$$

Proof: Suppose $\phi_j \to \bigvee_B \phi_i + \Delta_1 + \Delta_2 + \cdots + \Delta_q$; then (using $g \to h \Leftrightarrow h = g + h$),

$$\bigvee_B \phi_i + \Delta_1 + \Delta_2 + \cdots + \Delta_q = \bigvee_B \phi_i + \phi_j + \Delta_1 + \Delta_2 + \cdots + \Delta_q = \bigvee_A \phi_i + \Delta_1 + \Delta_2 + \cdots + \Delta_q.$$

But A covers all of the "1" vertices of f, whereas $\{\Delta_1, \Delta_2, \ldots, \Delta_q\}$ covers none of them. Thus the above equation implies that B covers all the "1" vertices of f. This completes the sufficiency part of the proof.

Now suppose that B covers all the "1" vertices of f (i.e., suppose ϕ_j is dispensable). Then

$$\bigvee_B \phi_i + \Delta_1 + \Delta_2 + \cdots + \Delta_q = f_{max} = \bigvee_A \phi_i + \Delta_1 + \Delta_2 + \cdots + \Delta_q = \bigvee_B \phi_i + \phi_j + \Delta_1 + \Delta_2 + \cdots + \Delta_q.$$

Thus (from $h = g + h \Leftrightarrow g \to h$),

$$\phi_j \to \bigvee_B \phi_i + \Delta_1 + \Delta_2 + \cdots + \Delta_q. \blacksquare$$

The application of this theorem to the finding of implication relations among the prime implicants makes use of a ratio chart like that of Ghazala (1) except that q additional columns, corresponding to $\Delta_1, \Delta_2, \ldots, \Delta_q$ are added. Just as in Ghazala's method for the completely specified case, "presence factors" $\sigma_1, \sigma_2, \ldots, \sigma_p$ are defined corresponding to $\phi_1, \phi_2, \ldots, \phi_p$ respectively. The presence factor σ_1 is 1 if ϕ_i is present in a given cover and is 0 otherwise. In addition, new quantities $\delta_1, \delta_2, \ldots, \delta_q$, which might be called "pseudo presence factors," are defined corresponding respectively to $\Delta_1, \Delta_2, \ldots, \Delta_q$. The pseudo presence factors are handled exactly as the presence factors are in deriving implication relations from rows of the ratio chart. For example, suppose that in row i of the ratio chart, the only irredundant disjunctions of ratios which are equal to 1 are $\frac{\phi_k}{\phi_i} + \frac{\phi_m}{\phi_i}$ and $\frac{\phi_\ell}{\phi_i} + \frac{\Delta_j}{\phi_i}$. Then the implication relation obtained from this row of the ratio chart would be

$$\sigma_i' \to \sigma_k \sigma_m + \sigma_\ell \delta_j.$$

This relation states that if ϕ_i is absent from a given cover, then either both ϕ_k and ϕ_m must be present or ϕ_ℓ must be present. Note that ϕ_ℓ does not cover all of the "1" vertices of f_{max} covered by ϕ_i, but those which it fails to cover correspond to don't care conditions covered by Δ_j.

When the implication relations have been found, all the δ_i's ($i = 1, 2, \ldots, q$) should be set equal to 1, and all terms which subsume other terms should be deleted. The resulting implication relations may be treated by one of Gaines' exact or approximate procedures for finding a minimal sum of products (2), or they may be treated by Ghazala's "presence function" method for finding all irredundant sums of products (1).

For a simple example, consider a function f of three variables a, b, and c specified such that the prime implicants of f_{max} are

$$\phi_1 = ab,$$
$$\phi_2 = ac,$$
$$\phi_3 = bc',$$
$$\phi_4 = b'c$$

and the prime implicants of f_{dc} are

$$\Delta_1 = ab,$$
$$\Delta_2 = a'b'c.$$

As an aid to visualization, the Karnaugh map is drawn:

			b	b
a	0	don't care	0	1
a	0	1	don't care	don't care
		c	c	

The ratio chart is as follows:

		ϕ_1 ab	ϕ_2 ac	ϕ_3 bc′	ϕ_4 b′c	Δ_1 ab	Δ_2 a′b′c
ϕ_1	ab	•	c	c′	0	1	0
ϕ_2	ac	b	•	0	b′	b	0
ϕ_3	bc′	a	0	•	0	a	0
ϕ_4	b′c	0	a	0	•	0	a′

The chart yields the following implication relations:

$$\sigma_1' \to \sigma_2\sigma_3 + \delta_1,$$
$$\sigma_2' \to \sigma_1\sigma_4 + \sigma_4\delta_1,$$
$$\sigma_3' \to 0,$$
$$\sigma_4' \to \sigma_2\delta_2.$$

Setting all δ_i's equal to 1 gives

$$\sigma_1' \to 1,$$
$$\sigma_2' \to \sigma_4,$$
$$\sigma_3' \to 0,$$
$$\sigma_4' \to \sigma_2.$$

Using $g \to h \Leftrightarrow g' + h = 1$ yields

$$\sigma_1 + 1 \;\;= 1,$$
$$\sigma_2 + \sigma_4 = 1,$$
$$\sigma_3 + 0 \;\;= 1,$$
$$\sigma_4 + \sigma_2 = 1,$$

all of which must be satisfied simultaneously (1). Thus

$$1 = (\sigma_1 + 1)(\sigma_2 + \sigma_4)(\sigma_3 + 0)(\sigma_4 + \sigma_2) = \sigma_2\sigma_3 + \sigma_3\sigma_4.$$

This indicates that there are two irredundant covers for f:

$$f = \phi_2 + \phi_3 = ac + bc',$$

$$f = \phi_3 + \phi_4 = bc' + b'c.$$

Note that ϕ_1, which consists entirely of don't cares, does not appear.

A second example shows the application of this method to a more complicated situation and also illustrates that an irredundant cover is not necessarily minimal. Let f, a function of the four variables a, b, c, and d, be specified such that the prime implicants of f_{max} are

$$\phi_1 = ab, \qquad \phi_5 = b'c,$$
$$\phi_2 = a'b', \qquad \phi_6 = b'd,$$
$$\phi_3 = ac, \qquad \phi_7 = c'd$$
$$\phi_4 = ad,$$

and the prime implicants of f_{dc} are

$$\Delta_1 = abc', \qquad \Delta_4 = c'd,$$
$$\Delta_2 = a'b'c', \qquad \Delta_5 = b'cd'.$$
$$\Delta_3 = a'b'd',$$

As an aid to visualization, the Karnaugh map is drawn:

			c	c	
	don't care	don't care	1	don't care	
	0	don't care	0	0	b
a	don't care	don't care	1	1	b
a	0	don't care	1	don't care	
		d	d		

The ratio chart is as follows:

		ϕ_1 ab	ϕ_2 a'b'	ϕ_3 ac	ϕ_4 ad	ϕ_5 b'c	ϕ_6 b'd	ϕ_7 c'd	Δ_1 abc'	Δ_2 a'b'c'	Δ_3 a'b'd'	Δ_4 c'd	Δ_5 b'cd'
ϕ_1	ab	·	0	c	d	0	0	c'd	c'	0	0	c'd	0
ϕ_2	a'b'	0	·	0	0	c	d	c'd	0	c'	d'	c'd	cd'
ϕ_3	ac	b	0	·	d	b'	b'd	0	0	0	0	0	b'd'
ϕ_4	ad	b	0	c	·	b'c	b'	c'	bc'	0	0	c'	0
ϕ_5	b'c	0	a'	a	ad	·	d	0	0	0	a'd'	0	d'
ϕ_6	b'd	0	a'	ac	a	c	·	c'	0	a'c'	0	c'	0
ϕ_7	c'd	ab	a'b'	0	a	0	b'	·	ab	a'b'	0	1	0

The following implication relations are obtained (if this were programmed for a computer, Ghazala's "cracking" method could be used (1)):

$$\sigma_1' \to \sigma_3\delta_1,$$

$$\sigma_2' \to \sigma_5\delta_2 + \sigma_6\delta_3 + \sigma_6\delta_2\delta_5 + \sigma_5\delta_3\delta_4 + \sigma_5\sigma_7\delta_3,$$

$$\sigma_3' \to \sigma_1\sigma_5 + \sigma_1\sigma_4\delta_5 + \sigma_1\sigma_6\delta_5,$$

$$\sigma_4' \to \sigma_1\sigma_6 + \sigma_3\sigma_7 + \sigma_3\delta_4 + \sigma_1\sigma_5\sigma_7 + \sigma_1\sigma_5\delta_4 + \sigma_3\sigma_6\delta_1,$$

$$\sigma_5' \to \sigma_2\sigma_3 + \sigma_6\delta_5 + \sigma_3\sigma_6\delta_3 + \sigma_2\sigma_4\delta_5,$$

$$\sigma_6' \to \sigma_2\sigma_4 + \sigma_5\sigma_7 + \sigma_5\delta_4 + \sigma_4\sigma_5\delta_2 + \sigma_2\sigma_3\sigma_7 + \sigma_2\sigma_3\delta_4,$$

$$\sigma_7' \to \delta_4.$$

Setting all δ_i's equal to 1, deleting terms which subsume other terms, and constructing the presence function gives

$$\begin{aligned} I &= (\sigma_1 + \sigma_3)(\sigma_2 + \sigma_5 + \sigma_6)(\sigma_3 + \sigma_1\sigma_5 + \sigma_1\sigma_4 + \sigma_1\sigma_6) \\ &\quad \cdot (\sigma_4 + \sigma_3 + \sigma_1\sigma_5 + \sigma_1\sigma_6)(\sigma_5 + \sigma_6 + \sigma_2\sigma_3 + \sigma_2\sigma_4) \\ &\quad \cdot (\sigma_6 + \sigma_5 + \sigma_2\sigma_3 + \sigma_2\sigma_4)(\sigma_7 + 1) \\ &= \sigma_1\sigma_5 + \sigma_1\sigma_6 + \sigma_2\sigma_3 + \sigma_3\sigma_5 + \sigma_3\sigma_6 + \sigma_1\sigma_2\sigma_4. \end{aligned}$$

Therefore, the six irredundant covers are

$$\begin{aligned} f &= \phi_1 + \phi_5 &&= ab + b'c, \\ f &= \phi_1 + \phi_6 &&= ab + b'd, \\ f &= \phi_2 + \phi_3 &&= a'b' + ac, \\ f &= \phi_3 + \phi_5 &&= ac + b'c, \\ f &= \phi_3 + \phi_6 &&= ac + b'd, \\ f &= \phi_1 + \phi_2 + \phi_4 &&= ab + a'b' + ad. \end{aligned}$$

The last of these is not minimal.

In general, the minimal covers form a subset of the irredundant covers. Thus this method yields all minimal covers plus possibly some additional nonminimal irredundant covers.

AN APPROACH USING PETRICK'S ALGORITHM AS ITS BASIS

Outline of the Approach

This section will approach the covering problem from a different viewpoint than the preceding section. Instead of starting with a generalization of Quine's test for dispensability, this section will take as its starting point a covering problem defined in terms of a generalized basic cell matrix. First, a matrix A, which can be *any* matrix whose elements are taken from the set $\{0, 1\}$, will be defined together with a function f_1 and a function f_2. Lemma 1 to follow will show that f_1, which is computed by Petrick's (4, 5) method (expansion by columns) is identical to f_2, which is computed by expansion by rows. This result does not require that A be a basic cell matrix. Then, however, A will be specialized to be a generalized basic cell matrix. A generalized basic cell matrix is less restricted than a conventional basic cell matrix in that the terms associated with the columns are not required to be vertices but may be cells consisting of several vertices. With A reinterpreted in this manner, f_1 will be shown to be the presence function computed by Petrick's algorithm. Next a modified generalized basic cell matrix W, which contains A as a submatrix, will be defined, and Lemma 4 will show that a function g_1 can be defined on W such that g_1 (with certain arguments set equal to 1) equals f_1. This g_1 is the Petrick algorithm expansion for W. Then a function g_2 will be defined on W, and the proof of Lemma 5 will show that $g_1 = g_2$. Finally, a modified ratio chart will be defined together with a function h_2 which the modified ratio chart generates. The proofs of Lemma 7 and Lemma 8 will show that h_2 (with certain of its arguments put equal to 1) is equal to g_2 (if the corresponding arguments of g_2 are set equal to 1).

Putting all these results together, one obtains the following: Given any generalized basic cell matrix, it is possible to construct a modified ratio chart such that the presence function generated by a modification of Ghazala's algorithm is identical to the presence function generated by the application of Petrick's algorithm to the original basic cell matrix.

Notation

If $S = \{s_1, s_2, \ldots, s_n\}$, then glb(S) means $s_1 \cdot s_2 \cdots s_n$ and lub (S) means $s_1 + s_2 + \ldots + s_n$. If the glb and lub are taken over some index set I, then the symbols $\bigwedge_{i\in I}$ and $\bigvee_{i\in I}$ are used respectively. The symbol $\bigwedge_{i=1}^{m}$ means $\bigwedge_{i\in\{1,2,\ldots,m\}}$ and the symbol $\bigvee_{i=1}^{m}$ means $\bigvee_{i\in\{1,2,\ldots,m\}}$.

Let A be any m by n matrix consisting of elements a_{ij}, where $a_{ij} = 1$ or $a_{ij} = 0$ ($1 \leqslant i \leqslant m$, $1 \leqslant j \leqslant n$), and let

$$F_A = \{k | 1 \leqslant k \leqslant n \text{ and, for some } i\ (1 \leqslant i \leqslant m),\ a_{ik} = 1\},$$

$$G_{A_j} = \{k | a_{kj} = 1,\ 1 \leqslant k \leqslant m\},$$

$$f_1(\sigma_1, \sigma_2, \ldots, \sigma_m) = \bigwedge_{j\in F_A}\left(\bigvee_{i\in G_{A_j}} \sigma_i\right),$$

$$S = \{\sigma_1, \sigma_2, \ldots, \sigma_m\},$$

$$\mathcal{P}(S) = \text{set of all subsets of a given set } S,$$

$$H_i = \{\text{glb}(X) | X \in \mathcal{P}(S) \text{ and, for } j = 1, 2, \ldots, n,$$
$$i \in G_{A_j} \Rightarrow \exists \sigma_k \in X \ni: k \in G_{A_j}\},$$

$$f_2(\sigma_1, \sigma_2, \ldots, \sigma_n) = \bigwedge_{i=1}^{m} \text{lub}(H_i).$$

It should be noted that f_1 is an expansion by columns and f_2 is an expansion by rows.

The Approach

Lemma 1: $f_1 = f_2$.

Proof:

(Part I) Suppose $f_1 = 0$, then $\exists j$ such that G_{A_j} is nonvoid and

$$\bigvee_{i\in G_{A_j}} \sigma_i = 0.$$

This implies that if $i \in G_{A_j}$, then $\sigma_i = 0$. Since $G_{A_j} \neq \Phi$ then $\exists\, \ell$, $1 \leqslant \ell \leqslant m$, $\ni: \ell \in G_{A_j}$. Then $H_\ell = \{0\}$, for, by the definition of H_ℓ, every "X" must contain a σ_k such that $k \in G_{A_j}$, and this means that every "X" must contain at least one element which is zero. Therefore glb(X) = 0 $\forall$ X. Thus lub $(H_\ell) = 0$ which $\Rightarrow f_2 = 0$. Therefore

$$f_1 = 0 \Rightarrow f_2 = 0.$$

(Part II) Suppose $f_1 = 1$, then for $j = 1, 2, \ldots, n$

$$\bigvee_{i \in G_{A_j}} \sigma_i = 1.$$

Thus for $j = 1, 2, \ldots, n$ $\exists\, \ell \in G_{A_j} \ni: \sigma_\ell = 1$. Then for every i $(i = 1, 2, \ldots, m)$ it is possible to construct an "X" (See the definition of H_i) such that glb(X) = 1. The construction proceeds as follows: Start with $X = \Phi$. Then for $j = 1, 2, \ldots, n$ add σ_ℓ to X, where $\ell \in G_{A_j}$, such that $\sigma_\ell = 1$ (it having been shown that such an ℓ exists for every j). This construction automatically satisfies the requirement that $i \in G_{A_j} \Rightarrow \exists\, \sigma_k \in X \ni: k \in G_{A_j}$. Then since X consists entirely of σ's which are 1, glb(X) = 1.

Thus lub $(H_i) = 1$ for $i = 1, 2, \ldots, m$, and this implies $f_2 = 1$. Therefore

$$f_1 = 1 \Rightarrow f_2 = 1.$$

So, from Part I and Part II, $f_1 = f_2$. ∎

Let A be an m by n generalized basic cell matrix with terms $\phi_1, \phi_2, \ldots \phi_m$ associated with each row in the order given and with terms $\mho_1, \mho_2, \ldots, \mho_n$ associated with each column in the order given. The elements of A are defined as follows:

$$(a_{ij} = 1) \Leftrightarrow (\mho_j \to \phi_i),\ a_{ij} = 0 \text{ otherwise.}$$

The terms must conform to the following restriction: $\forall\ i \in \{1, 2, \ldots, m\}$ and $\forall\ j \in \{1, 2, \ldots, n\}$ *exactly one* of the following two statements must be true:

$$\mho_j \to \phi_i,$$
$$\mho_j \to \phi_i'.$$

With A defined in this manner, define $f_1(\sigma_1, \sigma_2, \ldots, \sigma_n)$ exactly as before. This f_1 is the presence function generated by Petrick's algorithm (4, 5).

Let W be a "generalized" basic cell matrix, defined as follows:

$$W = \left(\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right)$$

W is a p by q matrix with elements w_{ij}, where $p = m + r$ and $q = n + s$ and where $w_{ij} = 0$ or 1. A is an m by n submatrix with elements a_{ij}, B is an m by s submatrix with elements b_{ij}, C is an r by n submatrix with elements c_{ij}, and D is an r by s submatrix with elements d_{ij}.

The rows of W are associated with terms $\phi_1, \phi_2, \ldots, \phi_m, \Delta_1, \Delta_2, \ldots, \Delta_r$ in the order given. The columns of W are associated with terms $\mho_1, \mho_2, \ldots, \mho_n, V_1, V_2, \ldots, V_s$ in the order given. These terms must conform to the following restrictions:

R1: $\mho_1 + \mho_2 + \cdots + \mho_n + V_1 + V_2 + \cdots + V_s \equiv \phi_1 + \phi_2 + \cdots + \phi_m$,

R2: $V_1 + V_2 + \cdots + V_s \equiv \Delta_1 + \Delta_2 + \cdots + \Delta_r$,

R3: $(\mho_1 + \mho_2 + \cdots + \mho_n) \cdot (V_1 + V_2 + \cdots + V_n) \equiv 0$,

R4: let $x \in \{\mho_1, \mho_2, \ldots, \mho_n, V_1, V_2, \ldots, V_s\}$,

$y \in \{\phi_1, \phi_2, \ldots, \phi_m, \Delta_1, \Delta_2, \ldots, \Delta_r\}$.

Then *exactly one* of the following two statements must be true for given x and y:

$$x \to y,$$

$$x \to y'.$$

(I.e., either y covers *all* of the vertices of x, or it covers *none* of them. In the usual case, where each column term is a minterm, this is a trivial restriction.)

The w_{ij} are defined as follows: Let x be the term corresponding to the jth column of W and y be the term corresponding to the ith row of W. Then

$$(w_{ij} = 1) \Leftrightarrow (x \to y),$$

$$w_{ij} = 0 \text{ otherwise}.$$

Define

$$g_1(\nu_1, \ldots, \nu_m, \nu_{m+1}, \ldots, \nu_p) = \left[\bigwedge_{j \in F_w} \left(\bigvee_{i \in G_{w_j}} \nu_i\right)\right],$$

where

$$F_w = \{k \mid 1 \leqslant k \leqslant p \text{ and, for some } i, w_{ik} = 1\},$$

$$G_{w_j} = \{k \mid 1 \leqslant k \leqslant q \text{ and } w_{kj} = 1\}.$$

Lemma 2: *All elements of submatrix C are zero.*
Proof: Suppose $c_{ij} = 1$ for some i, j $(1 \leqslant i \leqslant r, 1 \leqslant j \leqslant n)$. Then

$$\mho_j \to \Delta_i \Rightarrow \mho_j \cdot \Delta_i = \mho_j.$$

But restrictions R2 and R3 require

$$(\mho_1 + \mho_2 + \cdots + \mho_n) \cdot (\Delta_1 + \Delta_2 + \cdots + \Delta_r) \equiv 0,$$

which is inconsistent with the above, since $\mho_j = 1$ would imply that this product = 1. Therefore there would be a contradiction. ■

Lemma 3: *Every column in submatrix* D *contains at least one "1".*
Proof: Suppose there is a j $(1 \leqslant j \leqslant s)$ such that $d_{ij} = 0$ for $i = 1, 2, \ldots, r$. Then from restriction R4

$$V_j \to \Delta_1', \quad V_j \to \Delta_2', \ldots, V_j \to \Delta_r'$$

or equivalently

$$V_j \rightarrow \Delta_1'\Delta_2' \cdots \Delta_r' = (\Delta_1 + \Delta_2 + \cdots + \Delta_r)'.$$

But this says that

$$V_j = 1 \Rightarrow \Delta_1 + \Delta_2 + \cdots + \Delta_r = 0,$$

contradicting R2. ■

Lemma 4:

$$g_1(\sigma_1, \sigma_2, \ldots, \sigma_m, 1, 1, \ldots, 1) = f_1(\sigma_1, \sigma_2, \ldots, \sigma_m).$$

Proof: F_w can be partitioned into two sets F_A and $F_{\bar{A}}$ where F_A is defined as before and where $F_{\bar{A}} = \{k | k \in F_w \text{ but } k \notin F_A\}$. Then

$$g_1(\nu_1, \ldots, \nu_m, \nu_{m+1}, \ldots, \nu_p) = \left[\bigwedge_{j \in F_A} \left(\bigvee_{i \in G_{w_j}} \nu_i\right)\right] \cdot \left[\bigwedge_{j \in F_{\bar{A}}} \left(\bigvee_{i \in G_{w_j}} \nu_i\right)\right].$$

But from Lemma 2, if $\nu_i = \sigma_i$ $(i = 1, 2, \ldots, m)$, then

$$\bigwedge_{j \in F_A} \left(\bigvee_{i \in G_{w_j}} \nu_i\right) = \bigwedge_{j \in F_A} \left(\bigvee_{i \in G_{A_j}} \sigma_i\right) = f_1(\sigma_1, \sigma_2, \ldots, \sigma_m),$$

and from Lemma 3, if $\nu_i = 1$ $(m + 1 \leqslant i \leqslant p)$, then

$$\bigwedge_{j \in F_{\bar{A}}} \left(\bigvee_{i \in G_{w_j}} \nu_i\right) = 1.$$

So,

$$g_1(\sigma_1, \sigma_2, \ldots, \sigma_m, 1, 1, \ldots, 1) = f_1(\sigma_1, \sigma_2, \ldots, \sigma_m) \cdot 1 = f_1(\sigma_1, \sigma_2, \ldots, \sigma_m). \blacksquare$$

Let $T = \{\nu_1, \nu_2, \ldots, \nu_p\}$ and let $\mathcal{P}(T)$ be the power set of T (the set of all subsets of T). Define

$$J_i = \{Y | Y \in \mathcal{P}(T) \text{ and, for } j = 1, 2, \ldots, q, i \in G_{w_j} \Rightarrow \exists\ \nu_k \in Y \ni: k \in G_{w_j}\},$$

$$K_i = \{\mathrm{glb}(Y) | Y \in J_i\},$$

$$g_2(\nu_1, \nu_2, \ldots, \nu_m, \nu_{m+1}, \ldots, \nu_p) = \bigwedge_{i=1}^{p} \mathrm{lub}\,(K_i).$$

Lemma 5:

$$g_1(\nu_1, \nu_2, \ldots, \nu_p) = g_2(\nu_1, \nu_2, \ldots, \nu_p).$$

Proof: The proof follows directly from Lemma 1 with g_1 substituted for f_1, g_2 for f_2, p for m, ν_i for σ_i, G_{w_j} for G_{A_j}, and K_i for H_i. ■

Define

$$\beta_i = \begin{cases} \phi_i, 1 \leqslant i \leqslant m, \\ \Delta_{i-m}, m+1 \leqslant i \leqslant p, \end{cases}$$

$$\theta_j = \begin{cases} \mho_j, 1 \leqslant j \leqslant n, \\ V_{j-n}, n+1 \leqslant j \leqslant q, \end{cases}$$

$$Q(Z) = \{\beta_i | \nu_i \in Z\},$$

$$L_i = \{Z | Z \in \mathscr{P}(T) \text{ and } \beta_i \to \text{lub } [Q(Z)]\}.$$

Lemma 6: $J_i = L_i$.

Proof:

(Part I) Let $y \in J_i$. Then $y \in \mathscr{P}(T)$. Suppose $\beta_i = 1$. Then by restriction R1 or R2, $\exists\ j\ (1 \leqslant j \leqslant q) \ni: \theta_j = 1$. This j cannot be such that $w_{ij} = 0$, for if it were, restriction R4 together with $\theta_j = 1$ would require that $\beta_i' = 1$ or $\beta_i = 0$. Therefore j must be such that $i \in G_{W_j}$. But, from the definition of J_i, this requires that y contain a ν_k such that $k \in G_{W_j}$, i.e., such that $w_{kj} = 1$. Thus Q(y) must contain a β_k such that $\theta_j \to \beta_k$. But since $\theta_j = 1$, then β_k must equal 1. So, lub $[Q(y)] = 1$. Hence, $\beta_i = 1$ requires that lub $[Q(y)] = 1$. Thus $\beta_i \to$ lub $[Q(y)]$. Then $y \in L_i$. Therefore

$$y \in J_i \Rightarrow y \in L_i.$$

(Part II) Now let $z \in L_i$. Then $z \in \mathscr{P}(T)$. Further, $\beta_i \to$ lub $[Q(z)]$. Suppose that for some j, $w_{ij} = 1$, that is, $i \in G_{W_j}$. Then $\theta_j \to \beta_i$. Hence, by the transitivity of the implication relation, $\theta_j \to$ lub $[Q(z)]$. Then Q(z) must contain a β_k such that $\theta_j \to \beta_k$, for if no such $\beta_k \in Q(z)$ existed, then restriction R4 would require that $\theta_j \to$ the complement of every member of Q(z), that is, that $\theta_j \to$ the conjunction of the complements of the members of Q(z), or in other words (using de Morgan's law) that $\theta_j \to (\text{lub } [Q(z)])'$, which would be a contradiction. Thus, since $\theta_j \to \beta_k \in Q(z)$, then $\nu_k \in z$ and $w_{kj} = 1$, i.e., $k \in G_{W_j}$. Thus, $z \in L_i$ and $i \in G_{W_j} \Rightarrow \exists\ \nu_k \in z \ni: k \in G_{W_j}$. Therefore

$$z \in L_i \Rightarrow z \in J_i.$$

Combining this result with the result of Part I yields

$$J_i = L_i. \ \blacksquare$$

Let Ghazala's ratio chart be constructed for the terms β_i, $1 \leqslant i \leqslant p$. Ghazala has shown (1) that the presence function which this ratio chart yields is

$$h_1(\nu_1, \nu_2, \ldots, \nu_p) = \bigwedge_{i=1}^{p} \text{lub } (\{\text{glb}(Z) | Z \in L_i\}).$$

Lemma 7:

$$h_1(\nu_1, \nu_2, \ldots, \nu_p) = g_2(\nu_1, \nu_2, \ldots, \nu_p).$$

Proof: The proof follows directly from $J_i = L_i$ (Lemma 6). $\blacksquare$

Define

$$h_2(\nu_1, \nu_2, \ldots, \nu_p) = \bigwedge_{i=1}^{m} \text{lub}\ \left(\{\text{glb}(Z) | Z \in L_i\}\right).$$

This is the presence function generated by a "modified" ratio chart, constructed as follows: Start with a conventional ratio chart for $\beta_i = \phi_i$, $1 \leqslant i \leqslant m$. Then without adding any more rows, add columns corresponding to $\beta_i = \Delta_{i-m}$, $m + 1 \leqslant i \leqslant p = m + r$.

Lemma 8:

$$h_1(\nu_1, \nu_2, \ldots, \nu_m, 1, 1, \ldots, 1) = h_2(\nu_1, \nu_2, \ldots, \nu_m, 1, 1, \ldots, 1).$$

Proof:

$$h_1(\nu_1, \nu_2, \ldots, \nu_p) = \left[\bigwedge_{i=1}^{m} \text{lub}\ \left(\{\text{glb}(Z) | Z \in L_i\}\right)\right] \cdot \left[\bigwedge_{i=m+1}^{p} \text{lub}\ \left(\{\text{glb}(Z) | Z \in L_i\}\right)\right]$$

$$= h_2(\nu_1, \nu_2, \ldots, \nu_p) \cdot \left[\bigwedge_{i=m+1}^{p} \text{lub}\ \left(\{\text{glb}(Z) | Z \in L_i\}\right)\right].$$

Since $\beta_i \to \beta_i$ for every i, then $\{\nu_i\} \in L_i$ for every i. Thus if $\nu_{m+1} = \nu_{m+2} = \cdots = \nu_p = 1$, then lub $\left(\{\text{glb}(Z) | Z \in L_i\}\right) = 1$ for $i = m + 1, m + 2, \ldots, p$. This means that

$$\bigwedge_{i=m+1}^{p} \text{lub}\ \left(\{\text{glb}(Z) | Z \in L_i\}\right) = 1.$$

Therefore,

$$h_1(\nu_1, \nu_2, \ldots, \nu_m, 1, 1, \ldots, 1) = h_2(\nu_1, \nu_2, \ldots, \nu_m, 1, 1, \ldots, 1) \cdot 1 = h_2(\nu_1, \nu_2, \ldots, \nu_m, 1, 1, \ldots, 1). \blacksquare$$

Theorem:

$$f_1(\nu_1, \nu_2, \ldots, \nu_m) = h_2(\nu_1, \nu_2, \ldots, \nu_m, 1, 1, \ldots, 1).$$

Proof:

$$\begin{aligned} f_1(\nu_1, \nu_2, \ldots, \nu_m) &= g_1(\nu_1, \nu_2, \ldots, \nu_m, 1, 1, \ldots, 1) \quad \text{(Lemma 4)} \\ &= g_2(\nu_1, \nu_2, \ldots, \nu_m, 1, 1, \ldots, 1) \quad \text{(Lemma 5)} \\ &= h_1(\nu_1, \nu_2, \ldots, \nu_m, 1, 1, \ldots, 1) \quad \text{(Lemma 7)} \\ &= h_2(\nu_1, \nu_2, \ldots, \nu_m, 1, 1, \ldots, 1) \quad \text{(Lemma 8).} \ \blacksquare \end{aligned}$$

This theorem says that given any generalized basic cell matrix constructed for terms ϕ_i, $1 \leqslant i \leqslant m$, a modified ratio chart which will yield a presence function identical to that produced by the application of Petrick's algorithm to the given basic cell matrix can be constructed, where ν_i is interpreted to be the presence factor corresponding to ϕ_i.

This modified ratio chart is constructed as follows: Start with a conventional ratio chart for ϕ_i, $1 \leq i \leq m$. Then add r columns corresponding to $\Delta_1, \Delta_2, \ldots, \Delta_r$ such that the following conditions (based on restrictions R1 through R4) hold:

$$\text{C1:}\quad \mho_1 + \mho_2 + \cdots + \mho_n + \Delta_1 + \Delta_2 + \cdots + \Delta_r \equiv \phi_1 + \phi_2 + \cdots + \phi_m,$$

$$\text{C2:}\quad (\mho_1 + \mho_2 + \cdots + \mho_n) \cdot (\Delta_1 + \Delta_2 + \cdots + \Delta_r) \equiv 0.$$

Note that one need not concern oneself with the terms $V_1, V_2, \ldots, V_s$, since, given the terms $\Delta_1, \Delta_2, \ldots, \Delta_r$, one can always find a set of $V_1, V_2, \ldots, V_s$ such that restrictions R2 and R4 are satisfied. (R4 is automatically satisfied if each of $V_1, V_2, \ldots, V_s$ is a single vertex, provided that R4 is also satisfied by the original generalized basic cell matrix. The original basic cell matrix will automatically satisfy R4 if each of the $\mho_1, \mho_2, \ldots, \mho_n$ is a single vertex.)

Application to an Incompletely Specified Multiple-Output Problem

Bartee's (5) method involves setting up a multiple-output table of combinations. For each row in which the output section of the row is not all zeros, an "ϵ" term is formed by writing each input variable primed or unprimed depending on whether a 0 or 1 respectively appears in the corresponding position in the row and by writing a primed output variable if a zero appears in the corresponding row position or omitting the output variable entirely if a 1 or I (don't care) appears. Each row may also yield "c" terms (the "care" vertices) constructed as follows: Choose a particular 1 in the output section of the row and leaving that 1 as it is, change all other entries in the output section of the row to 0. Then write each input and output variable either primed or unprimed depending on whether the corresponding row entry is 0 or 1 respectively. This is done once for each 1 in the output section of the row. Rows which contain no 1 in their output sections generate no c terms.

If a basic cell matrix is used to find the cover, the ϵ and c terms (and the ϕ's generated by iterated consensus on the ϵ's) are the only terms needed. But if a modified ratio chart is used, then an additional term called a "d" term is constructed for each row not having all zeros in its output section as follows: Take the glb of the set consisting of the ϵ term for that row and the complements of all c terms generated by that row. (If the row generates no c terms, then d is identical to the ϵ for that row.) The d terms may be used directly as the Δ's or their number may be reduced and their length shortened by iterated consensus. It should be noted that d terms will always be generated in nontrivial multiple output problems, even if the problem is completely specified. Thus the d terms cannot be regarded strictly as conventional "don't care" terms.

To conclude with an example, let $\lambda(x, y)$ and $\mu(x, y)$ be specified as follows:

x y	λ μ
0 0	1 0
0 1	1 I
1 0	1 1
1 1	0 1

The multiple-output prime implicants are

$$\phi_1 = y'\mu',$$

$$\phi_2 = x'y,$$

$$\phi_3 = y\lambda',$$

$$\phi_4 = x\lambda',$$

$$\phi_5 = xy',$$

$$\phi_6 = x'\mu'.$$

An acceptable set of Δ's is

$$\Delta_1 = \lambda'\mu',$$
$$\Delta_2 = x'y'\mu',$$
$$\Delta_3 = x'y\mu,$$
$$\Delta_4 = xy'\lambda\mu.$$

The modified ratio chart is as follows:

	ϕ_1	ϕ_2	ϕ_3	ϕ_4	ϕ_5	ϕ_6	Δ_1	Δ_2	Δ_3	Δ_4
ϕ_1	1	0	0	$x\lambda'$	x	x'	λ'	x'	0	0
ϕ_2	0	1	λ'	0	0	μ'	$\lambda'\mu'$	0	μ	0
ϕ_3	0	x'	1	x	0	$x'\mu'$	μ'	0	$x'\mu$	0
ϕ_4	$y'\mu'$	0	y	1	y'	0	μ'	0	0	0
ϕ_5	μ'	0	0	λ'	1	0	$\lambda'\mu'$	0	0	$\lambda\mu$
ϕ_6	y'	y	$y\lambda'$	0	0	1	λ'	y'	0	0

Let σ_i be the presence factor for ϕ_i and δ_i be the presence factor for Δ_i. Then

$$\begin{aligned} h_2(\sigma_1, \sigma_2, \sigma_3, \sigma_4, \sigma_5, \sigma_6, \delta_1, \delta_2, \delta_3, \delta_4) = {} & (\sigma_1 + \sigma_5\sigma_6 + \sigma_5\delta_2) \cdot (\sigma_2 + \sigma_6\delta_3) \\ & \cdot (\sigma_3 + \sigma_2\sigma_4 + \sigma_4\sigma_6\delta_3 + \sigma_4\delta_1\delta_3) \cdot (\sigma_4 + \sigma_3\sigma_5) \\ & \cdot (\sigma_5 + \sigma_1\sigma_4\delta_4) \cdot (\sigma_6 + \sigma_1\sigma_2 + \sigma_2\delta_2). \end{aligned}$$

Therefore

$$h_2(\sigma_1, \sigma_2, \sigma_3, \sigma_4, \sigma_5, \sigma_6, 1, 1, 1, 1) = \sigma_1\sigma_2\sigma_4 + \sigma_2\sigma_3\sigma_5 + \sigma_2\sigma_4\sigma_5 + \sigma_1\sigma_4\sigma_6 + \sigma_3\sigma_5\sigma_6 + \sigma_4\sigma_5\sigma_6.$$

For purposes of comparison, the basic cell matrix is constructed. The c terms are

$$\mho_1 = x'y\lambda\mu',$$
$$\mho_2 = xy'\lambda\mu',$$
$$\mho_3 = xy'\lambda'\mu,$$
$$\mho_4 = xy\lambda'\mu.$$

The basic cell matrix is therefore

$$A = \begin{array}{c} \\ \phi_1 \\ \phi_2 \\ \phi_3 \\ \phi_4 \\ \phi_5 \\ \phi_6 \end{array} \begin{array}{c} \begin{array}{cccc} \mho_1 & \mho_2 & \mho_3 & \mho_4 \end{array} \\ \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \end{array}$$

So,

$$f_1(\sigma_1, \sigma_2, \sigma_3, \sigma_4, \sigma_5, \sigma_6) = (\sigma_2 + \sigma_6) \cdot (\sigma_1 + \sigma_5) \cdot (\sigma_4 + \sigma_5) \cdot (\sigma_3 + \sigma_4)$$
$$= \sigma_1\sigma_2\sigma_4 + \sigma_2\sigma_3\sigma_5 + \sigma_2\sigma_4\sigma_5 + \sigma_1\sigma_4\sigma_6 + \sigma_3\sigma_5\sigma_6 + \sigma_4\sigma_5\sigma_6$$
$$= h_2(\sigma_1, \sigma_2, \sigma_3, \sigma_4, \sigma_5, \sigma_6, 1, 1, 1, 1).$$

A TECHNIQUE FOR COMPUTING THE INCOMPLETELY SPECIFIED MULTIPLE-OUTPUT FUNCTION WITHOUT RECOURSE TO A TABLE OF COMBINATIONS

Since the main virtue of the method using the modified ratio chart is that it does not require a canonical expansion, the calculation of ϵ and d terms from a table of combinations largely offsets any advantage one might hope to obtain by using the modified ratio chart method, especially if the incompletely specified multiple-output problem is initially stated in the form of several incompletely specified Boolean functions which separately have already been partially or completely "digested" by the use of iterated consensus or some other means of deriving prime implicants.

The modified ratio chart method depends directly on the ϕ's and Δ's, not on the ϵ's and d's; so one would like a technique for going from the separate partially digested incompletely specified Boolean functions to the multiple-output ϕ's and Δ's without having to pass through an intermediate step involving the construction of a multiple-output table of combinations. The following is such a technique.

Construct

$$G_{2_{max}}(x_1, x_2, \ldots, x_n, f_1, f_2, \ldots, f_m) = \bigvee_{a_1,\ldots,a_m} \left[\left(\bigwedge_{i=1,\ldots,m} [F_{i_{max}}(x_1, \ldots, x_n) + h(a_i)]\right)\left(f_1^{a_1} \cdots f_m^{a_m}\right)\right]$$

and

$$G_{2_{dc}}(x_1, x_2, \ldots, x_n, f_1, f_2, \ldots, f_m) = \bigvee_{j=1,\ldots,m} [F_{j_{dc}}(x_1, \ldots, x_n) f_1' \cdots f_{j-1}' f_j f_{j+1}' \cdots f_m']$$
$$+ \bigvee_{\substack{a_1,\ldots,a_m \\ \text{where not exactly} \\ \text{one } a \text{ is blank.}}} \left[\left(\bigwedge_{i=1,\ldots,m} [F_{i_{max}}(x_1, \ldots, x_n) + h(a_i)]\right)\left(f_1^{a_1} \cdots f_m^{a_m}\right)\right],$$

where the $F_i(x_1, \ldots, x_n)$, $i = 1, \ldots, m$, are the m separate incompletely specified Boolean functions (of the variables $x_1, \ldots, x_n$) which are being treated together as a multiple output problem. $F_{i_{max}}$ covers those and only those vertices of F_i which are either 1's or don't cares. $F_{i_{dc}}$ covers those and only those vertices of F_i which are don't cares. The f_i, $i = 1, \ldots, m$, are the formal symbols used in Bartee's method to stand for the F_i, $i = 1, \ldots, m$, respectively.

The symbol a_i, $i = 1, \ldots, m$, represents either a blank or a prime ('). The function $h(a_i)$ is a two-valued function defined as follows:

$$h(a_i) = \begin{cases} 0, & a_i = \text{blank}, \\ 1, & a_i = \text{prime}. \end{cases}$$

$\bigvee_{a_1,\ldots,a_m}$ means the iterated disjunction performed over all possible combinations of blanks and primes among the a's.

$G_2(x_1, \ldots, x_n, f_1, \ldots, f_m)$ is the incompletely specified multiple-output function. That is, the prime implicants of $G_{2_{max}}$ are the ϕ's, which are the multiple-output prime implicants, and any set of terms whose lub yields $G_{2_{dc}}$ constitutes an acceptable set of Δ's.

Instead of defining G_2 by specifying $G_{2_{max}}$ and $G_{2_{dc}}$, one could define G_2 by specifying $G_{2_{max}}$ and $G_{2_{min}}$, where $G_{2_{min}}$ is that function which covers those and only those vertices of G_2 which are 1's.

The incompletely specified multiple-output function which Bartee's method yields is defined as follows:

$$G_{1_{max}}(x_1, x_2, \ldots, x_n, f_1, f_2, \ldots, f_m) = \bigvee_{\beta_1,\ldots,\beta_n}\left(\left[\bigwedge_{i=1,\ldots,m}\left(F_{i_{max}}[h'(\beta_1), \ldots, h'(\beta_n)] + f_i'\right)\right]\left(x_1^{\beta_1} \cdots x_n^{\beta_n}\right)\right),$$

$$G_{1_{min}}(x_1, x_2, \ldots, x_n, f_1, f_2, \ldots, f_m) = \bigvee_{\beta_1,\ldots,\beta_n}\left[\bigwedge_{i=1,\ldots,m}\left(F_{i_{min}}[h'(\beta_1), \ldots, h'(\beta_n)]\right.\right.$$

$$\left.\left.\cdot\, x_1^{\beta_1} \cdots x_n^{\beta_n} \cdot f_1' f_2' \cdots f_{i-1}' f_i f_{i+1}' \cdots f_m'\right)\right].$$

In the next subsection it will be proved that G_1 and G_2 are identical and that $G_{2_{dc}}$ is what it was previously stated to be. Following the next subsection an example of the application of the technique to an incompletely specified multiple-output problem will be given.

Proofs Concerning G_1, G_2, and $G_{2_{dc}}$

Lemma I:

$$G_{2_{max}} = 0 \Rightarrow G_{1_{max}} = 0.$$

Proof:

$$G_{2_{max}} = 0 \;\Rightarrow\; \left[\left(\bigwedge_{i=1,\ldots,m}[F_i(x_1, \ldots, x_n) + h(\alpha_i)]\right)\left(f_1^{\alpha_1} \cdots f_m^{\alpha_m}\right)\right] = 0$$

$$\forall\,(\alpha_1, \ldots, \alpha_m).$$

In particular,

$$\left[\left(\bigwedge_{i=1,\ldots,m}[F_i(x_1, \ldots, x_n) + h(\zeta_i)]\right)\left(f_1^{\zeta_1} \cdots f_m^{\zeta_m}\right)\right] = 0$$

where $\zeta_1, \ldots, \zeta_m$ are such that $\left(f_1^{\zeta_1} \cdots f_m^{\zeta_m}\right) = 1$. Then

$$\exists\, k,\ 1 \leqslant k \leqslant m,\ \ni\!:\ [F_k(x_1, \ldots, x_n) + h(\zeta_k)] = 0.$$

This $\Rightarrow$ $F_k(x_1, \ldots, x_n) = 0$ and $h(\zeta_k) = 0$.

Now suppose $\beta_1, \ldots, \beta_n$ are such that $x_1^{\beta_1} \cdots x_n^{\beta_n} = 0$. Then the term of $G_{1_{max}}$ containing $x_1^{\beta_1} \cdots x_n^{\beta_n}$ is 0 and therefore need not be considered further from the standpoint of this lemma. The terms of G_1 which should be considered are those for which $\beta_1, \ldots, \beta_n$ are such that $x_1^{\beta_1} \cdots x_n^{\beta_n} = 1$.

If $x_1^{\beta_1} \cdots x_n^{\beta_n} = 1$, then

$$[h'(\beta_1), \ldots, h'(\beta_n)] = (x_1, \ldots, x_n),$$

because

$$x_j = 0 \Rightarrow \beta_j = \text{prime} \Rightarrow h'(\beta_j) = 0$$

and

$$x_j = 1 \Rightarrow \beta_j = \text{blank} \Rightarrow h'(\beta_j) = 1.$$

In a term of $G_{1_{max}}$ for which $x_1^{\beta_1} \cdots x_n^{\beta_n} = 1$ consider the factor $F_{k_{max}}[h'(\beta_1), \ldots, h'(\beta_n)] + f_k'$. From the immediately preceding argument,

$$F_{k_{max}}[h'(\beta_1), \ldots, h'(\beta_n)] = F_{k_{max}}(x_1, \ldots, x_n).$$

But it has already been shown that $F_{k_{max}}(x_1, \ldots, x_n) = 0$. Moreover, it has been shown that $h(\zeta_k) = 0$, and this $\Rightarrow f_k' = 0$.

Therefore, every term of $G_{1_{max}}$ which is not rendered = 0 by $x_1^{\beta_1} \cdots x_n^{\beta_n}$ *is* rendered = 0 by $F_{k_{max}}[h'(\beta_1), \ldots, h'(\beta_n)] + f_k'$. Therefore

$$G_{2_{max}} = 0 \Rightarrow G_{1_{max}} = 0. \blacksquare$$

Lemma II:

$$G_{2_{max}} = 1 \Rightarrow G_{1_{max}} = 1.$$

Proof:

$$G_{2_{max}} = 1 \Rightarrow \exists (\zeta_1, \ldots, \zeta_m) \ni$$

$$\left[\left(\bigwedge_{i=1,\ldots,m} [F_{i_{max}}(x_1, \ldots, x_n) + h(\zeta_i)]\right)\left(f_1^{\zeta_1} \cdots f_m^{\zeta_m}\right)\right] = 1.$$

This $\Rightarrow \left(f_1^{\zeta_1} \cdots f_m^{\zeta_m}\right) = 1$ and

$$\left(\bigwedge_{i=1,\ldots,m} [F_{i_{max}}(x_1, \ldots, x_n) + h(\zeta_i)]\right) = 1.$$

But $\left(f_1^{\zeta_1} \cdots f_m^{\zeta_m}\right) = 1 \Rightarrow \zeta_j = \begin{cases} \text{prime}, f_j = 0, \\ \text{blank}, f_j = 1, \end{cases}$

$$\Rightarrow h(\zeta_j) = f_j'.$$

Therefore

$$\left(\bigwedge_{i=1,\ldots,m}\left[F_{i_{max}}(x_1,\ldots,x_n)+f_i'\right]\right)=1.$$

Now let $(\rho_1,\ldots,\rho_n)$ be such that

$$\rho_j=\begin{cases}\text{prime}, & x_j=0,\\ \text{blank}, & x_j=1.\end{cases}$$

Then $x_1^{\rho_1}\cdots x_n^{\rho_n}=1$ and $h'(\rho_j)=x_j$. Therefore

$$F_{i_{max}}[h'(\rho_1),\ldots,h'(\rho_n)]=F_{i_{max}}(x_1,\ldots,x_n).$$

But it was shown that

$$\bigwedge_{i=1,\ldots,m}\left[F_{i_{max}}(x_1,\ldots,x_n)+f_i'\right]=1;$$

therefore it follows that

$$\bigwedge_{i=1,\ldots,m}\left(F_{i_{max}}[h'(\rho_1),\ldots,h'(\rho_n)]+f_i'\right)=1.$$

So, there is at least one term in $G_{1_{max}}$ which = 1, namely,

$$\left[\bigwedge_{i=1,\ldots,m}\left(F_{i_{max}}[h'(\rho_1),\ldots,h'(\rho_n)]+f_i'\right)\right]\left(x_1^{\rho_1}\cdots x_n^{\rho_n}\right).$$

Therefore $G_{2_{max}}=1\Rightarrow G_{1_{max}}=1$. ■

Lemma III:

$$G_{2_{max}}\equiv G_{1_{max}}.$$

Proof: The proof follows directly from Lemmas I and II. ■

Define

$$G_{2_{min}}(x_1,\ldots,x_n,f_1,\ldots,f_m)=\bigvee_{j=1}^{m}\left[F_{j_{min}}(x_1,\ldots,x_n)f_1'f_2'\cdots f_{j-1}'f_jf_{j+1}'\cdots f_{m-1}'f_m'\right].$$

Now

$$\begin{aligned}G_{2_{dc}}(x_1,x_2,\ldots,x_n,f_1,f_2,\ldots,f_m)&=G_{2_{max}}(x_1,x_2,\ldots,x_n,f_1,f_2,\ldots,f_m)\\&\quad\cdot G_{2_{min}}'(x_1,x_2,\ldots,x_n,f_1,f_2,\ldots,f_m)\end{aligned}$$

$$= \bigvee_{\alpha_1,\ldots,\alpha_m} \left[\left(\bigwedge_{i=1,\ldots,m} [F_{i_{\max}}(x_1, \ldots, x_n) + h(\alpha_i)]\right)\left(f_1^{\alpha_1} \cdots f_m^{\alpha_m}\right)\right]$$

$$\cdot\left(\bigvee_{j=1,\ldots,m} [F_{j_{\min}}(x_1, \ldots, x_n) f_1' f_2' \cdots f_{j-1}' f_j f_{j+1}' \cdots f_{m-1}' f_m']\right)'$$

$$= \bigvee_{\alpha_1,\ldots,\alpha_m} \left[\left(\bigwedge_{i=1,\ldots,m} [F_{i_{\max}}(x_1, \ldots, x_n) + h(\alpha_i)]\right)\left(f_1^{\alpha_1} \cdots f_m^{\alpha_m}\right)\right]$$

$$\cdot\left(\bigwedge_{j=1,\ldots,m} [F_{j_{\min}}(x_1, \ldots, x_n) f_1' f_2' \cdots f_{j-1}' f_j f_{j+1}' \cdots f_{m-1}' f_m']'\right)$$

$$= \bigvee_{\alpha_1,\ldots,\alpha_m} \left[\left(\bigwedge_{i=1,\ldots,m} [F_{i_{\max}}(x_1, \ldots, x_n) + h(\alpha_i)]\right)\left(f_1^{\alpha_1} \cdots f_m^{\alpha_m}\right)\right]$$

$$\cdot\left(\bigwedge_{j=1,\ldots,m} [F_{j_{\min}}'(x_1, \ldots, x_n) + f_1 + f_2 + \cdots + f_{j-1} + f_j' + f_{j+1}\right.$$

$$\left. + \cdots + f_{m-1} + f_m]\right)$$

$$= \bigvee_{\alpha_1,\ldots,\alpha_m} \left(\bigwedge_{i=1,\ldots,m} [F_{i_{\max}}(x_1, \ldots, x_n) + h(\alpha_i)]\right.$$

$$\bigwedge_{j=1,\ldots,m} [F_{j_{\min}}'(x_1, \ldots, x_n) + h'(\alpha_1) + \cdots + h'(\alpha_{j-1})$$

$$\left. + h(\alpha_j) + h'(\alpha_{j+1}) + \cdots + h'(\alpha_m)]\left(f_1^{\alpha_1} \cdots f_m^{\alpha_m}\right)\right)$$

$$= \bigvee_{\alpha_1,\ldots,\alpha_m} \left[\bigwedge_{i=1,\ldots,m} \bigwedge_{j=1,\ldots,m} \left(F_{i_{\max}}(x_1, \ldots, x_n) F_{j_{\min}}'(x_1, \ldots, x_n)\right.\right.$$

$$+ F_{i_{\max}}(x_1, \ldots, x_n)[h'(\alpha_1) + \cdots + h'(\alpha_{j-1}) + h(\alpha_j) + h'(\alpha_{j+1})$$

$$+ \cdots + h'(\alpha_m)] + h(\alpha_i) F_{j_{\min}}'(x_1, \ldots, x_n) + h(\alpha_i)[h'(\alpha_1)$$

$$\left.\left. + \cdots + h'(\alpha_{j-1}) + h(\alpha_j) + h'(\alpha_{j+1}) + \cdots + h'(\alpha_m)]\right)\left(f_1^{\alpha_1} \cdots f_m^{\alpha_m}\right)\right]$$

$$= \bigvee_{\substack{\alpha_1,\ldots,\alpha_m \\ \text{where exactly} \\ \text{one } \alpha \text{ is blank}}} \{\cdots\} + \bigvee_{\substack{\alpha_1,\ldots,\alpha_m \\ \text{where not} \\ \text{exactly one} \\ \alpha \text{ is blank}}} \{\cdots\}$$

$$= \bigvee_{\substack{\alpha_1,\ldots,\alpha_m \\ \text{where exactly} \\ \text{one } \alpha \text{ is blank}}} \Bigg(\bigwedge_{i=1,\ldots,m} \; \bigwedge_{j=1,\ldots,m} \Big[F_{i_{max}}(x_1, \ldots, x_n) F'_{j_{min}}(x_1, \ldots, x_n)$$

$$+ F_{i_{max}}(x_1, \ldots, x_n) h(\alpha_j) + h(\alpha_i) F'_{j_{min}}(x_1, \ldots, x_n)$$

$$+ h(\alpha_i) h(\alpha_j) \Big] \left(f_1^{\alpha_1} \cdots f_m^{\alpha_m} \right) \Bigg)$$

$$+ \bigvee_{\substack{\alpha_1,\ldots,\alpha_m \\ \text{where not} \\ \text{exactly one} \\ \alpha \text{ is blank}}} \Bigg(\bigwedge_{i=1,\ldots,m} \; \bigwedge_{j=1,\ldots,m} \Big[F_{i_{max}}(x_1, \ldots, x_n) F'_{j_{min}}(x_1, \ldots, x_n)$$

$$+ F_{i_{max}}(x_1, \ldots, x_n) + h(\alpha_i) F'_{j_{min}}(x_1, \ldots, x_n) + h(\alpha_i) \Big] \left(f_1^{\alpha_1} \cdots f_m^{\alpha_m} \right) \Bigg)$$

$$= \bigvee_{\substack{\alpha_1,\ldots,\alpha_m \\ \text{where exactly} \\ \text{one } \alpha \text{ is blank}}} \Bigg(\bigwedge_{i=1,\ldots,m} \; \bigwedge_{j=1,\ldots,m} \Big[F_{i_{max}}(x_1, \ldots, x_n) F'_{j_{min}}(x_1, \ldots, x_n)$$

$$+ F_{i_{max}}(x_1, \ldots, x_n) h(\alpha_j) + h(\alpha_i) F'_{j_{min}}(x_1, \ldots, x_n)$$

$$+ h(\alpha_i) h(\alpha_j) \Big] \left(f_1^{\alpha_1} \cdots f_m^{\alpha_m} \right) \Bigg)$$

$$+ \bigvee_{\substack{\alpha_1,\ldots,\alpha_m \\ \text{where not} \\ \text{exactly one} \\ \alpha \text{ is blank}}} \Bigg(\bigwedge_{i=1,\ldots,m} \Big[F_{i_{max}}(x_1, \ldots, x_n) + h(\alpha_i) \Big] \left(f_1^{\alpha_1} \cdots f_m^{\alpha_m} \right) \Bigg)$$

Lemma IV:

$$\bigvee_{\substack{\alpha_1,\ldots,\alpha_m \\ \text{where exactly} \\ \text{one } \alpha \text{ is blank}}} \Bigg(\bigwedge_{i=1,\ldots,m} \; \bigwedge_{j=1,\ldots,m} \Big[F_{i_{max}}(x_1, \ldots, x_n) F'_{j_{min}}(x_1, \ldots, x_n)$$

$$+ F_{i_{max}}(x_1, \ldots, x_n) h(\alpha_j) + h(\alpha_i) F'_{j_{min}}(x_1, \ldots, x_n) + h(\alpha_i) h(\alpha_j) \Big] \left(f_1^{\alpha_1} \cdots f_m^{\alpha_m} \right) \Bigg)$$

$$= \bigvee_{j=1,\ldots,m} \Big[F_{j_{dc}}(x_1, \ldots, x_n) f'_1 \cdots f'_{j-1} f_j f'_{j+1} \cdots f'_m \Big].$$

Proof:

(Part I) Suppose

$$\bigvee_{\substack{\alpha_1,\ldots,\alpha_m\\ \text{where exactly}\\ \text{one } \alpha \text{ is blank}}} \left(\bigwedge_{i=1,\ldots,m} \bigwedge_{j=1,\ldots,m} \left[F_{i_{max}}(x_1,\ldots,x_n)F'_{j_{min}}(x_1,\ldots,x_n) + F_{i_{max}}(x_1,\ldots,x_n)h(\alpha_j) + h(\alpha_i)F'_{j_{min}}(x_1,\ldots,x_n) + h(\alpha_i)h(\alpha_j) \right] \left(f_1^{\alpha_1} \cdots f_m^{\alpha_m} \right) \right) = 1.$$

Then $\exists$ an integer k, $1 \leqslant k \leqslant m$, $\ni$

$$\bigwedge_{i=1,\ldots,m} \bigwedge_{j=1,\ldots,m} \left[F_{i_{max}}(x_1,\ldots,x_n)F'_{j_{min}}(x_1,\ldots,x_n) + F_{i_{max}}(x_1,\ldots,x_n)h(\alpha_j) + h(\alpha_i)F'_{j_{min}}(x_1,\ldots,x_n) + h(\alpha_i)h(\alpha_j) \right] \left(f_1^{\alpha_1} \cdots f_m^{\alpha_m} \right) = 1$$

and α_k = blank, all other α's = prime.

But this implies that the expression in brackets must = 1 $\forall$ i, j, where $i = 1, \ldots, m$ and $j = 1, \ldots, m$. In particular, the expression in brackets must = 1 for $i = j = k$. Furthermore, $f'_1 \cdots f'_{k-1}f_kf'_{k+1} \cdots f'_m = 1$.

Thus

$$\begin{aligned}
1 &= \left[F_{k_{max}}(x_1,\ldots,x_n)F'_{k_{min}}(x_1,\ldots,x_n) + F_{k_{max}}(x_1,\ldots,x_n)h(\alpha_k) + h(\alpha_k)F'_{k_{min}}(x_1,\ldots,x_n) \right.\\
&\quad \left. + h(\alpha_k)h(\alpha_k)\right] \left(f'_1 \cdots f'_{k-1}f_kf'_{k+1} \cdots f'_m\right)\\
&= \left[F_{k_{max}}(x_1,\ldots,x_n)F'_{k_{min}}(x_1,\ldots,x_n) + F_{k_{max}}(x_1,\ldots,x_n) \cdot 0 \right.\\
&\quad \left. + 0 \cdot F'_{k_{min}}(x_1,\ldots,x_n) + 0 \cdot 0\right] \left(f'_1 \cdots f'_{k-1}f_kf'_{k+1} \cdots f'_m\right)\\
&= \left[F_{k_{max}}(x_1,\ldots,x_n)F'_{k_{min}}(x_1,\ldots,x_n)\right] \left(f'_1 \cdots f'_{k-1}f_kf'_{k+1} \cdots f'_m\right)\\
&= F_{k_{dc}}(x_1,\ldots,x_n) \left(f'_1 \cdots f'_{k-1}f_kf'_{k+1} \cdots f'_m\right)
\end{aligned}$$

Since k is an integer such that $1 \leqslant k \leqslant m$, this $\Rightarrow \bigvee_{j=1,\ldots,m} \left[F_{j_{dc}}(x_1,\ldots,x_n)f'_1 \cdots f'_{j-1}f_jf'_{j+1} \cdots f'_m\right] = 1$.

Thus

$$\bigvee_{\substack{\alpha_1,\ldots,\alpha_m\\ \text{where exactly}\\ \text{one } \alpha \text{ is blank}}} \left(\bigwedge_{i=1,\ldots,m} \bigwedge_{j=1,\ldots,m} \left[F_{i_{max}}(x_1,\ldots,x_n)F'_{j_{min}}(x_1,\ldots,x_n) + F_{i_{max}}(x_1,\ldots,x_n)h(\alpha_j) + h(\alpha_i)F'_{j_{min}}(x_1,\ldots,x_n) + h(\alpha_i)h(\alpha_j) \right] \left(f_1^{\alpha_1} \cdots f_m^{\alpha_m} \right) \right) = 1$$

$$\Rightarrow \bigvee_{j=1,\ldots,m} \left[F_{j_{dc}}(x_1,\ldots,x_n)f'_1 \cdots f'_{j-1}f_jf'_{j+1} \cdots f'_m\right] = 1.$$

This completes Part I of the proof.

(Part II) Suppose

$$\bigvee_{j=1,\ldots,m} \left[F_{j_{dc}}(x_1, \ldots, x_n) f'_1 \cdots f'_{j-1} f_j f'_{j+1} \cdots f'_m\right] = 1.$$

Then $\exists$ an integer k, $1 \leqslant k \leqslant m$, $\ni$:

$$\begin{aligned} 1 &= F_{k_{dc}}(x_1, \ldots, x_n) f'_1 \cdots f'_{k-1} f_k f'_{k+1} \cdots f'_m \\ &= F_{k_{max}}(x_1, \ldots, x_n) F'_{k_{min}}(x_1, \ldots, x_n)(f'_1 \cdots f'_{k-1} f_k f'_{k+1} \cdots f'_m) \end{aligned}$$

Since k is an integer such that $1 \leqslant k \leqslant m$, this $\Rightarrow$

$$\bigvee_{\substack{\alpha_1,\ldots,\alpha_m \\ \text{where exactly} \\ \text{one } \alpha \text{ is blank}}} \left(\bigwedge_{i=1,\ldots,m} \bigwedge_{j=1,\ldots,m} \left[F_{i_{max}}(x_1, \ldots, x_n) F'_{j_{min}}(x_1, \ldots, x_n) \right.\right.$$

$$\left.\left. + F_{i_{max}}(x_1, \ldots, x_n) h(\alpha_j) + h(\alpha_i) F'_{j_{min}}(x_1, \ldots, x_n) + h(\alpha_i) h(\alpha_j)\right] \left(f_1^{\alpha_1} \cdots f_m^{\alpha_m}\right)\right) = 1.$$

The preceding statement may be seen to be true as follows: First note that

$$\begin{aligned} 1 &= F_{k_{max}}(x_1, \ldots, x_n) F'_{k_{min}}(x_1, \ldots, x_n) f'_1 \cdots f'_{k-1} f_k f'_{k+1} \cdots f'_m \\ &\Rightarrow F_{k_{max}}(x_1, \ldots, x_n) = F'_{k_{min}}(x_1, \ldots, x_n) = f'_1 \cdots f'_{k-1} f_k f'_{k+1} \cdots f'_m = 1. \end{aligned}$$

Then consider the term such that α_k = blank, all other α's = prime. One factor in this term is $f_1^{\alpha_1} \cdots f_m^{\alpha_m} = f'_1 \cdots f'_{k-1} f_k f'_{k+1} \cdots f'_m = 1$. The remaining factors may be partitioned into four cases. T. is $i = j = k$, in which case

$$F_{i_{max}}(x_1, \ldots, x_n) F'_{j_{min}}(x_1, \ldots, x_n) = F_{k_{max}}(x_1, \ldots, x_n) F'_{k_{min}}(x_1, \ldots, x_n) = 1.$$

So, the factor belonging to this case equals 1. The second case is $i = k, j \neq k$. In this case, α_j = prime, so $h(\alpha_j) = 1$. Then

$$F_{i_{max}}(x_1, \ldots, x_n) h(\alpha_j) = F_{k_{max}}(x_1, \ldots, x_n) \cdot 1 = 1.$$

Thus all factors belonging to this case equal 1. The third case is $i \neq k, j = k$. In this case, α_i = prime, so $h(\alpha_i) = 1$. Then

$$h(\alpha_i) F'_{j_{min}}(x_1, \ldots, x_n) = 1 \cdot F'_{k_{min}}(x_1, \ldots, x_n) = 1.$$

Thus all factors belonging to this case equal 1. The fourth case is $i \neq k, j \neq k$. In this case, $\alpha_i = \alpha_j$ = prime, so $h(\alpha_i) = h(\alpha_j) = 1$. Then $h(\alpha_i)h(\alpha_j) = 1$. Thus all factors belonging to this case equal 1. Since all factors belonging to all cases equal 1, then the term under consideration equals 1. Thus, the expression of which this term is a part equals 1.

Thus,

$$\bigvee_{j=1,\ldots,m}\left[F_{j_{dc}}(x_1,\ldots,x_n)f_1'\cdots f_{j-1}'f_jf_{j+1}'\cdots f_m'\right]=1$$

$$\Rightarrow \bigvee_{\substack{a_1,\ldots,a_m\\ \text{where exactly}\\ \text{one } a \text{ is blank}}}\Bigg(\bigwedge_{i=1,\ldots,m}\ \bigwedge_{j=1,\ldots,m}\big[F_{i_{max}}(x_1,\ldots,x_n)F_{j_{min}}'(x_1,\ldots,x_n)$$

$$+F_{i_{max}}(x_1,\ldots,x_n)h(a_j)+h(a_i)F_{j_{min}}'(x_1,\ldots,x_n)+h(a_i)h(a_j)\big]\left(f_1^{a_1}\cdots f_m^{a_m}\right)\Bigg)=1.$$

This completes Part II of the proof.

The lemma then follows directly from Part I and Part II. ■

So,

$$G_{2_{dc}}(x_1,x_2,\ldots,x_n,f_1,f_2,\ldots,f_m)=\bigvee_{j=1,\ldots,m}\left[F_{j_{dc}}(x_1,\ldots,x_n)f_1'\cdots f_{j-1}'f_jf_{j+1}'\cdots f_m'\right]$$

$$+\bigvee_{\substack{a_1,\ldots,a_m\\ \text{where not}\\ \text{exactly one}\\ a \text{ is blank}}}\left[\left(\bigwedge_{i=1,\ldots,m}\left[F_{i_{max}}(x_1,\ldots,x_n)+h(a_i)\right]\right)\left(f_1^{a_1}\cdots f_m^{a_m}\right)\right].$$

Lemma V:

$$G_{2_{min}}(x_1,\ldots,x_n,f_1,\ldots,f_m)\equiv G_{1_{min}}(x_1,\ldots,x_n,f_1,\ldots,f_m).$$

Proof:

(Part I) Suppose $G_{1_{min}}(x_1,\ldots,x_n,f_1,\ldots,f_m)=1$. Then $\exists$ an integer k, $1\leqslant k\leqslant m$, and a set $\{a_1,\ldots,a_n\}\ni$:

$$F_{k_{min}}[h'(a_1),\ldots,h'(a_n)]\left(x_1^{a_1}\cdots x_n^{a_n}\right)f_1'f_2'\cdots f_{k-1}'f_kf_{k+1}'\cdots f_m'=1.$$

But this means $x_1^{a_1}\cdots x_n^{a_n}=1$, which $\Rightarrow (h'(a_1),\ldots,h'(a_n))=(x_1,\ldots,x_n)$. So

$$F_{k_{min}}(x_1,\ldots,x_n)\left(x_1^{a_1}\cdots x_n^{a_n}\right)f_1'f_2'\cdots f_{k-1}'f_kf_{k+1}'\cdots f_m'=1.$$

It follows that

$$F_{k_{min}}(x_1,\ldots,x_n)f_1'\cdots f_{k-1}'f_kf_{k+1}'\cdots f_m'=1,$$

where k is an integer such that $1\leqslant k\leqslant m$. But this $\Rightarrow$

$$1=\bigvee_{j=1}^{m}\left[F_{j_{min}}(x_1,\ldots,x_n)f_1'\cdots f_{j-1}'f_jf_{j+1}'\cdots f_m'\right]$$

$$=G_{2_{min}}(x_1,\ldots,x_n,f_1,\ldots,f_m).$$

So $G_{1_{min}} = 1 \Rightarrow G_{2_{min}} = 1$.

This completes Part I of the proof.

(Part II) Let $\zeta_1, \ldots, \zeta_n$ be such that

$$\zeta_i = \begin{cases} \text{prime}, & x_i = 0, \\ \text{blank}, & x_i = 1. \end{cases}$$

Then $x_1^{\zeta_1} \cdots x_n^{\zeta_n} = 1$ and $(x_1, \ldots, x_n) = [h'(\zeta_1), \ldots, h'(\zeta_n)]$. Suppose $G_{2_{min}}(x_1, \ldots, x_n, f_1, \ldots, f_m) = 1$. Then $\exists$ an integer k, $1 \leqslant k \leqslant m$, $\ni$:

$$\begin{aligned} 1 &= F_{k_{min}}(x_1, \ldots, x_n) f_1' \cdots f_{k-1}' f_k f_{k+1}' \cdots f_m' \\ &= F_{k_{min}}(x_1, \ldots, x_n)\left(x_1^{\zeta_1} \cdots x_n^{\zeta_n}\right) f_1' \cdots f_{k-1}' f_k f_{k+1}' \cdots f_m' \\ &= F_{k_{min}}[h'(\zeta_1), \ldots, h'(\zeta_n)]\left(x_1^{\zeta_1} \cdots x_n^{\zeta_n}\right) f_1' \cdots f_{k-1}' f_k f_{k+1}' \cdots f_m' . \end{aligned}$$

But this is one of the terms in $G_{1_{min}}(x_1, \ldots, x_n, f_1, \ldots, f_m)$. So

$$G_{2_{min}} = 1 \Rightarrow G_{1_{min}} = 1.$$

This completes Part II of the proof.

The lemma follows directly from Part I and Part II. ■

Theorem: G_1 *and* G_2 *are identical.*
Proof: The proof follows from Lemmas III and V. ■

Example of Application of the Technique

The remainder of this section is an example of the application of the technique for computing the incompletely specified multiple-output function without recourse to a table of combinations.

Let the functions K, L, and M of the variables x, y, and z be specified as follows:

$$\begin{aligned} K_{max}(x, y, z) &= x'z + xy'z' + x'yz', \\ K_{dc}(x, y, z) &= x'y + x'y'z, \\ L_{max}(x, y, z) &= xy' + x'yz', \\ L_{dc}(x, y, z) &= xy'z + x'yz', \\ M_{max}(x, y, z) &= xy'z' + y'z + x'y + xyz, \\ M_{dc}(x, y, z) &= xy'z' + x'y'z + xyz + x'yz'. \end{aligned}$$

Let κ, λ, and μ be the Bartee's method symbols used to represent K, L, and M respectively.

The multiple-output function $G_2(x, y, z, \kappa, \lambda, \mu)$ is constructed using the formulas developed previously:

$$
\begin{aligned}
G_{2_{max}}(x, y, z, \kappa, \lambda, \mu) = {} & \kappa'\lambda'\mu' \\
& + M_{max}(x, y, z)\kappa'\lambda'\mu \\
& + L_{max}(x, y, z)\kappa'\lambda\mu' \\
& + L_{max}(x, y, z)M_{max}(x, y, z)\kappa'\lambda\mu \\
& + K_{max}(x, y, z)\kappa\lambda'\mu' \\
& + K_{max}(x, y, z)M_{max}(x, y, z)\kappa\lambda'\mu \\
& + K_{max}(x, y, z)L_{max}(x, y, z)\kappa\lambda\mu' \\
& + K_{max}(x, y, z)L_{max}(x, y, z)M_{max}(x, y, z)\kappa\lambda\mu,
\end{aligned}
$$

$$
\begin{aligned}
G_{2_{dc}}(x, y, z, \kappa, \lambda, \mu) = {} & \kappa'\lambda'\mu' \\
& + M_{dc}(x, y, z)\kappa'\lambda'\mu \\
& + L_{dc}(x, y, z)\kappa'\lambda\mu' \\
& + L_{max}(x, y, z)M_{max}(x, y, z)\kappa'\lambda\mu \\
& + K_{dc}(x, y, z)\kappa\lambda'\mu' \\
& + K_{max}(x, y, z)M_{max}(x, y, z)\kappa\lambda'\mu \\
& + K_{max}(x, y, z)L_{max}(x, y, z)\kappa\lambda\mu' \\
& + K_{max}(x, y, z)L_{max}(x, y, z)M_{max}(x, y, z)\kappa\lambda\mu.
\end{aligned}
$$

The prime implicants of $G_{2_{max}}$, that is, the multiple-output prime implicants, are

$$
\begin{aligned}
\phi_1 &= x'z\lambda', \\
\phi_2 &= x'y\lambda', \\
\phi_3 &= x'yz', \\
\phi_4 &= z\kappa'\lambda', \\
\phi_5 &= xy'\kappa', \\
\phi_6 &= xy'z', \\
\phi_7 &= \kappa'\lambda'\mu'.
\end{aligned}
$$

A cover for $G_{2_{dc}}$ is provided by the terms

$$
\begin{aligned}
\Delta_1 &= x'y'z\lambda', \\
\Delta_2 &= x'y\kappa\lambda', \\
\Delta_3 &= x'yz', \\
\Delta_4 &= xyz\kappa'\lambda', \\
\Delta_5 &= xy'z\kappa'\lambda, \\
\Delta_6 &= xy'z'\mu, \\
\Delta_7 &= \kappa'\lambda'\mu', \\
\Delta_8 &= xy'z'\kappa\lambda.
\end{aligned}
$$

The modified ratio chart is as follows:

	ϕ_1	ϕ_2	ϕ_3	ϕ_4	ϕ_5	ϕ_6	ϕ_7	Δ_1	Δ_2	Δ_3	Δ_4	Δ_5	Δ_6	Δ_7	Δ_8
ϕ_1	1	y	0	κ'	0	0	$\kappa'\mu'$	y'	$y\kappa$	0	0	0	0	$\kappa'\mu'$	0
ϕ_2	z	1	z'	$z\kappa'$	0	0	$\kappa'\mu'$	0	κ	z'	0	0	0	$\kappa'\mu'$	0
ϕ_3	0	λ'	1	0	0	0	$\kappa'\lambda'\mu'$	0	$\kappa\lambda'$	1	0	0	0	$\kappa'\lambda'\mu'$	0
ϕ_4	x'	$x'y$	0	1	xy'	0	μ'	$x'y'$	0	0	xy	0	0	μ'	0
ϕ_5	0	0	0	$z\lambda'$	1	z'	$\lambda'\mu'$	0	0	0	0	$z\lambda$	$z'\mu$	$\lambda'\mu'$	0
ϕ_6	0	0	0	0	κ'	1	$\kappa'\lambda'\mu'$	0	0	0	0	0	μ	$\kappa'\lambda'\mu'$	$\kappa\lambda$
ϕ_7	$x'z$	$x'y$	$x'yz'$	z	xy'	$xy'z'$	1	$x'y'z$	0	$x'yz'$	xyz	0	0	1	0

Let $p(\sigma_1, \sigma_2, \sigma_3, \sigma_4, \sigma_5, \sigma_6, \sigma_7, \delta_1, \delta_2, \delta_3, \delta_4, \delta_5, \delta_6, \delta_7, \delta_8)$ denote the presence function for $G_2(x, y, z, \kappa, \lambda, \mu)$. Then, from the modified ratio chart,

$$p(\sigma_1, \ldots, \sigma_7, \delta_1, \ldots, \delta_8) = (\sigma_1 + \sigma_2\delta_1 + \sigma_4\delta_1\delta_2) \cdot (\sigma_2 + \sigma_1\sigma_3 + \sigma_1\delta_3 + \sigma_3\sigma_4\delta_2 + \sigma_4\delta_2\delta_3) \cdot (\sigma_3 + \delta_3)$$
$$\cdot (\sigma_4 + \sigma_1\sigma_5\delta_4 + \sigma_2\sigma_5\delta_1\delta_4) \cdot (\sigma_5 + \sigma_4\sigma_6\delta_5) \cdot \sigma_6 \cdot (\sigma_7 + \delta_7).$$

So

$$p(\sigma_1, \sigma_2, \sigma_3, \sigma_4, \sigma_5, \sigma_6, \sigma_7, 1, 1, 1, 1, 1, 1, 1, 1) = (\sigma_1 + \sigma_2 + \sigma_4) \cdot (\sigma_1 + \sigma_2 + \sigma_4) \cdot 1$$
$$\cdot (\sigma_4 + \sigma_1\sigma_5 + \sigma_2\sigma_5) \cdot (\sigma_5 + \sigma_4\sigma_6) \cdot \sigma_6 \cdot 1$$
$$= \sigma_1\sigma_5\sigma_6 + \sigma_2\sigma_5\sigma_6 + \sigma_4\sigma_6 .$$

As a check, the basic cell matrix is constructed: The c terms are

$$\mho_1 = x'yz\kappa'\lambda'\mu, \qquad \mho_3 = xy'z'\kappa'\lambda\mu',$$
$$\mho_2 = xy'z\kappa'\lambda'\mu, \qquad \mho_4 = xy'z'\kappa\lambda'\mu'.$$

Then the basic cell matrix is

$$\begin{array}{c} \\ \phi_1 \\ \phi_2 \\ \phi_3 \\ \phi_4 \\ \phi_5 \\ \phi_6 \\ \phi_7 \end{array}
\begin{array}{c} \begin{array}{cccc} \mho_1 & \mho_2 & \mho_3 & \mho_4 \end{array} \\
\begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \end{array}$$

The presence function obtained by Petrick's algorithm is then

$$(\sigma_1 + \sigma_2 + \sigma_4) \cdot (\sigma_4 + \sigma_5) \cdot (\sigma_5 + \sigma_6) \cdot \sigma_6 = \sigma_1\sigma_5\sigma_6 + \sigma_2\sigma_5\sigma_6 + \sigma_4\sigma_6.$$

REFERENCES

1. M. J. Ghazala, "Irredundant Disjunctive and Conjunctive Forms of a Boolean Function," IBM Journal of Research and Development 1, 171-176 (Apr. 1967)

2. R. S. Gaines, "Implication Techniques for Boolean Functions," Proceedings of the Fifth Annual Conference on Switching Circuit Theory and Logical Design, October 1964, IEEE Publication S-164, pp. 174-182

3. W. V. Quine, "On Cores and Prime Implicants of Truth Functions," American Mathematical Monthly **66**, 755-760 (Nov. 1959)

4. S. R. Petrick, "A Direct Determination of the Irredundant Forms of a Boolean Function from the Set of Prime Implicants," Air Force Cambridge Research Center Report AFCRC-TR-56-110, Cambridge, Mass., 1956

5. T. C. Bartee, I. R. Lebow, and I. S. Reed, "Theory and Design of Digital Machines," New York, McGraw-Hill, 1962