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Scattering from a Periodic Corrugated Surface Part 3 - Finite-Depth Alternately Filled Plates with Soft Boundaries

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Solutions of the Helmholtz equations, with unknown amplitude coefficients, are assumed in the various geometric regions. By requiring that the pressure and velocity be continuous functions at the boundaries, sets of linear equations are obtained that relate the amplitudes for arbitrary incident angles. Equations for normal incidence are solved using a variation of the modified residue calculus technique involving two zero shifts, and the results yield the amplitudes as values or residues of a meromorphic function. With the exception of the finite depth, this paper is similar to NRL Report 7320.

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SCATTERING FROM A PERIODIC CORRUGATED SURFACE

Part 3—Finite-Depth Alternately Filled Plates with Soft Boundaries

1. INTRODUCTION

The problem considered in this report is the calculation of the scattered field which results when plane waves are incident on a periodic corrugated surface, such as that illustrated in Fig. 1. The surface consists of infinitesimally thin finite-depth parallel plates with lossless bottoms. The periodicity interval $2l$ is further divided by an additional parallel plate into a "homogeneous" or "free-space" region of width $(2a)$ specified by wavenumber k and density ρ_A, and a region of "inhomogeneity"(of width $2(l-a)$) whose density and wavenumber structure each differ by a constant amount from the surrounding free-space regions. The plates and bottoms have soft (i.e., Dirichlet) boundary conditions in terms of the velocity potential ψ. This report is one of a series of papers and reports (1-4) in which additional references to the literature can be found.

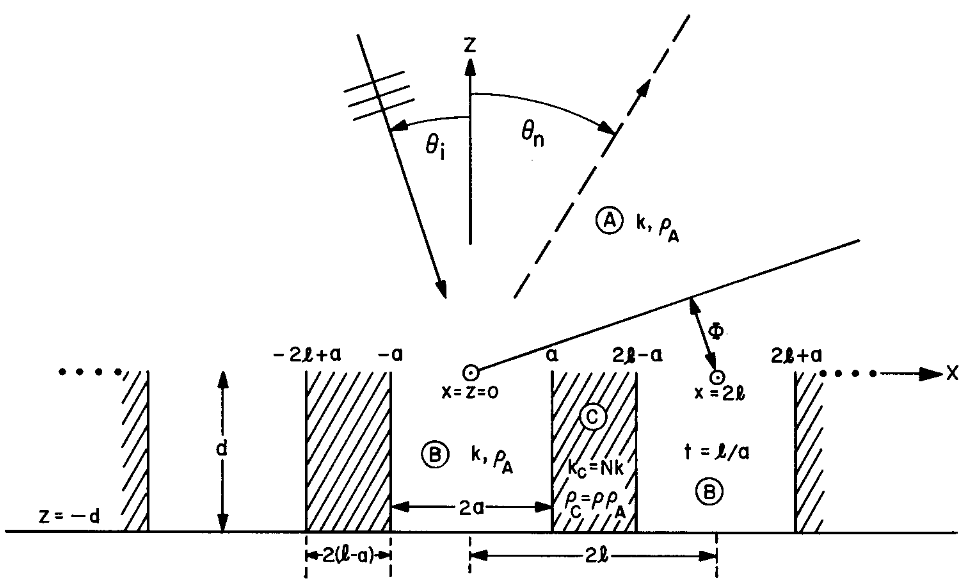

Fig. 1—Plane wave incident at an angle θ_i on a finite-depth (d) corrugated surface which is periodic (period $2l$). Shading indicates regions of density and wavenumber inhomogeneity (region C) as distinct from the free-space regions A and B. The discrete scattering angles are indicated by θ_n.

The basic formalism for the problem is presented in Sec. 2. Forms for the potential ψ are assumed in the various geometric regions. Each contains unknown amplitude coefficients.

In Sec. 3, the requirement that the pressure and velocity be continuous across the interfaces of the regions leads to sets of linear equations relating the various amplitude coefficients. The most general sets of linear equations, those for arbitrary incident angle, are derived.

In Sec. 4 a special case of the equations, that of normal incidence, is solved using a variation of the modified residue calculus method (5) which consists in the simultaneous use of two sets of zero shifts. The amplitudes are shown to be related to either values or residues of a meromorphic function.

A summary is presented in Sec. 5. The present report is confined to analytic results only. The numerical procedure used to construct the two sets of zero shifts, and the numerical results on reflection coefficients, are being worked on and will be published at a later date.

2. BASIC FORMALISM

The problem is to calculate the scalar wave function or velocity potential ψ_γ which satisfies the two-dimensional Helmholtz equation*

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial z^2} + k_\gamma^2\right)\psi_\gamma(x,z) = 0 \tag{2.1}$$

for the case of a plane waves ψ_i incident at angles θ_i on an infinite number of periodically spaced (period $2l$) parallel plates with finite depth d. The region between each pair of adjacent plates is further divided into "homogeneous" and "inhomogeneous" regions by a parallel plate located at a distance $2a$ from the left-most adjacent plate. Thus there are three geometric regions of the problem which are indicated by the label $\gamma = A$, B, or C in Eq. (2.1).

Region A is the region for which $z \geqslant 0$, B is the region where $-d \leqslant z \leqslant 0$ and $-a \leqslant x + 2ml \leqslant a$ ($m = 0, 1, 2, \ldots$), and C is where $-d \leqslant z \leqslant 0$ and $a \leqslant x + 2ml \leqslant 2l-a$. In the homogeneous regions A and B (see Fig. 1), $k_A = k_B \equiv k = 2\pi/\lambda$ where k is the incident (free-space) wavenumber (λ is the wavelength), and the densities are related by $\rho_A = \rho_B$. We define the inhomogeneous region C by the wavenumber $k_C = Nk_A$ and the density $\rho_C = \rho\rho_A$ where N and ρ are constants. Hence, "inhomogeneous" is here taken to mean a region whose wavenumber and density differ from the free-space values. The surface $z = S(x)$ is given by

$$S(x) = \begin{Bmatrix} 0, & x = a + 2ml \\ -d, & x \neq a + 2ml \; (m = 0, \pm 1, \ldots) \end{Bmatrix} \tag{2.2}$$

and the wave functions ψ_B and ψ_C satisfy the soft (i.e., Dirichlet) boundary condition

$$\psi[x,S(x)] = 0. \tag{2.3}$$

*The factor $e^{-i\omega t}$ is suppressed throughout this report.

In addition, ψ_γ satisfies the following restrictions:

a. ψ_γ and $\nabla\psi_\gamma$ are finite in each region, except at the plate edges where $|\nabla\psi_\gamma| = O\,(r^{-(1/2)+\epsilon})$, with r, the radial coordinate, being centered at an edge. The form for ϵ will be given later.

b. ψ_γ and $\nabla\psi_\gamma$ are continuous in each region, and the pressure $p_\gamma = -i\omega\rho_\gamma\psi_\gamma$ and the normal velocity $v_\gamma = -\partial\psi_\gamma/\partial z$ are continuous across the interface $z = 0$.

c. The quantity $\psi_A - \psi_i$ represents upgoing waves as $z \to \infty$.

The field representations, with notation similar to that used in Ref. 3, are given by

a.
$$\psi_A(x,z) = e^{ik(\alpha_0 x - \beta_0 z)} + \sum_{n=-\infty}^{\infty} A_n^s e^{ik(\alpha_n x + \beta_n z)} \tag{2.4}$$

where the first term is the incident plane wave with

$$\alpha_n = \alpha_0 + n\Lambda \quad \text{and } \Lambda = \frac{\lambda}{2l} ,$$

and the superscript s stands for soft;

b.
$$\psi_B(x,z) = \sum_{j=1}^{\infty} B_j^s \sin\left(\frac{j\pi(x+a)}{2a}\right) \sin\ q_j k(z+d) \tag{2.5}$$

where

$$q_j^2 = 1 - \left(\frac{j\Lambda t}{2}\right)^2 \quad \text{and} \quad t = \frac{l}{a} ;$$

and

c
$$\psi_C(x,z) = \sum_{j=1}^{\infty} C_j^s \sin\,[jku(x-a)]\ \sin\,[r_j k(z+d)] \tag{2.6}$$

where

$$u = \left(\frac{\Lambda t}{2(t-1)}\right) \quad \text{and} \quad r_j^2 = N^2 - (ju)^2 .$$

These field representations satisfy the boundary conditions at $x = a$, $x = 2l - a$, and at $z = -d$. Field representations outside the above regions are given by the Floquet conditions in Ref. (4), Eq. (2.10).

3. GENERAL LINEAR EQUATIONS AND FLUX CONSERVATION

To find the linear equations relating the A_n^s and B_n^s amplitudes, we require the continuity of pressure and velocity across the interface $z = 0$, $|x| \leqslant a$. This yields

$$\rho_A \psi_A(x,0) = \rho_B \psi_B(x,0) \tag{3.1}$$

and

$$\frac{\partial \psi_A}{\partial z}(x,0) = \frac{\partial \psi_B}{\partial z}(x,0) \; . \tag{3.2}$$

Substituting ψ_A and ψ_B from Eqs. (2.4) and (2.5) into Eqs. (3.1) and (3.2), solving for the B_n^s coefficients in terms of the A_n^s, and manipulating the resulting equations as in Refs. 1 and 3 yields the set of linear equations

$$\sum_{n=-\infty}^{\infty} A_n^s I_{nm}\left(\frac{e^{-iq_m kd}}{\beta_n - q_m} \pm \frac{e^{iq_m kd}}{\beta_n - q_m}\right) - I_{0m}\left(\frac{e^{-iq_m kd}}{\beta_0 + q_m} \pm \frac{e^{iq_m kd}}{\beta_0 - q_m}\right) \tag{3.3}$$

$$= -\left(\frac{2\pi i q_m B_m^s}{\Lambda t \rho_m}\right)\begin{Bmatrix}1\\0\end{Bmatrix}$$

where

$$I_{nm} = e^{-\pi i \alpha_n/\Lambda t} - (-)^m e^{\pi i \alpha_n/\Lambda t} \tag{3.4}$$

and $p_m = m\Lambda t/2$.

The linear equations relating A_n^s and C_m^s result from applying continuity of pressure and velocity at $z = 0$, for the condition $a \leqslant x \leqslant 2l - a$:

$$\rho_A \psi_A(x,0) = \rho_C \psi_C(x,0) \tag{3.5}$$

and

$$\frac{\partial \psi_A}{\partial z}(x,0) = \frac{\partial \psi_C}{\partial z}(x,0) \; . \tag{3.6}$$

Substituting ψ_A and ψ_C from Eqs. (2.4) and (2.6) into Eqs. (3.5) and (3.6), and solving for the C_j^s coefficients in terms of the A_n^s, yields the set of equations

$$\sum_{n=-\infty}^{\infty} \frac{A_n^s J_{nm}}{\beta_n \pm u_m} - \frac{J_{0m}}{\beta_0 \mp u_m} + \frac{\pi(t-1)\sigma_m^{s(\pm)} C_m^s}{\Lambda t m u} = 0 \tag{3.7}$$

where

$$J_{nm} = e^{\pi i\alpha_n/\Lambda t}\left(1 - (-)^m e^{\pi i\alpha_n/u}\right) \tag{3.8}$$

and

$$\sigma_m^{s(\pm)} = ir_m \cos(r_m kd) \pm \rho u_m \sin(r_m kd) \tag{3.9}$$

and $u_m = r_m | N = 1$.

It is not convenient to further manipulate these equations into a different form as was done with similar equations in Ref. 1.

The flux conservation relation follows from a similar derivation in Ref. 3 and from the remark that in the present report there is no transmission loss down the wells as was the case in that reference. The result is written in terms of the reflection coefficient R as

$$R \equiv \sum_n R_n = \sum_n |A_n^s|^2 \left(\frac{\beta_n}{\beta_0}\right) = 1 \tag{3.10}$$

where the sum is over integers n such that β_n is real. The R_n terms are the individual spectral reflection coefficients.

The most general case of Eqs. (3.3) and (3.7) (i.e., arbitrary $\alpha_0 = \sin\theta_i$) is not discussed here. Instead, the special case of normal incidence ($\alpha_0 = 0$), with t, ρ, and the wavenumber parameter N arbitrary, is discussed in the next section. The case of arbitrary α_0 and $t = 1$ was previously discussed in Ref. 1.

4. CASE OF NORMAL INCIDENCE ($\alpha_0 = 0$)

The general equations I have derived apparently cannot be solved by the methods outlined here. Instead, as in Refs. 3 and 4 the restriction $\alpha_0 = 0$ is used. It is obvious from Fig. 1 that the geometry of the problem is symmetric about the plane $x = 0$. The restriction that $\alpha_0 = 0$ induces an additional field symmetry and a resulting simplification of the linear equations. The problem is thus a generalization of some earlier problems by Deryugin (6) which can also be found in a book by Beckmann and Spizzichino (7). In the limit $\alpha_0 = 0$, Eqs. (3.3) and (3.7) reduce to

$$B_m^s = C_m^s = 0 \quad (m \text{ even}) \tag{4.1}$$

and, excluding values of t for which $\cos(\pi n/t) = 0$,

$$\sum_{n=0}^{\infty} A_n^s \epsilon_n \cos\left(\frac{\pi n}{t}\right)\left(\frac{e^{-iq_m kd}}{\beta_n - q_m} \pm \frac{e^{iq_m kd}}{\beta_n + q_m}\right)$$

$$- \left(\frac{e^{-iq_m kd}}{1+q_m} \pm \frac{e^{iq_m kd}}{1-q_m}\right) + \frac{\pi i q_m B_m^s}{\Lambda t p_m} \begin{Bmatrix} 1 \\ 0 \end{Bmatrix} = 0 \text{ } (m \text{ odd}) \tag{4.2}$$

and

$$\sum_{n=0} \frac{A_n^s \epsilon_n \cos\left(\frac{\pi n}{t}\right)}{\beta_n \pm u_m} - \frac{1}{\beta_0 \pm u_m} + \frac{\pi(t-1)}{2\Lambda t m u} \sigma_m^{s(\pm)} C_m^s = 0 \text{ } (m \text{ odd}) \tag{4.3}$$

where

$$\epsilon_n = \begin{Bmatrix} 1, & n = 0 \\ 2, & n > 0 \end{Bmatrix} .$$

These equations are of such form that they can be treated by the modified residue calculus technique due to Mittra (5). Consider a meromorphic function $F(\omega)$ which has the following properties:

a. $F(\omega)$ has simple poles at $\omega = \beta_n$ $(n = 0, 1, 2, \ldots)$ and $\omega = -\beta_0 = -1$.

b. $F(\omega)$ has simple zeroes at $\omega = \overline{q}_m \equiv q_m + \delta_m$ $(m = 1, 3, 5, \ldots)$ where the δ_m zero shifts (from the known q_m values) are found from the functional symmetry relation

$$F(q_m) = e^{2iq_m kd} F(-q_m) \quad (m \text{ odd}) . \tag{4.4}$$

This condition is similar to Eq. (3.1) in Ref. 1. The asymptotic value of δ_m ($\delta = \lim_{m\to\infty} = 0$) plays a role in property (*d*) below. The value $\delta = 0$ can be derived directly from Eq. (4.4). The calculation was illustrated in Ref. 3.

c. $F(\omega)$ has simple zeroes at $\omega = \overline{u}_m \equiv u_m + v_m$ $(m = 1, 3, 5, \ldots)$ where the v_m "zero shifts" (from the known u_m values) are found from the functional symmetry relation

$$F(u_m) = \sigma_m^s F(-u_m) \quad (m \text{ odd}) \tag{4.5}$$

where

$$\sigma_m^s \equiv \frac{\sigma_m^{s-}}{\sigma_m^{s+}} . \tag{4.6}$$

The asymptotic values of σ_m and v_m defined by

$$\sigma \equiv \lim_{m\to\infty} \sigma_m^s = \begin{Bmatrix} 1, & N = \infty \\ \\ \dfrac{1-\rho}{1+\rho}, & N = \text{finite} \end{Bmatrix}$$

and

$$v \equiv \lim_{m\to\infty} v_m = \frac{2iu}{\pi} \sin^{-1}\left(\frac{\sigma}{2}\right)$$

play a role in property (d) below. The value of v is derived from the limit as $m \to \infty$ of Eq. (4.5), and a similar calculation was performed in Ref. 3. The limit for $\rho = 0$ (or $N = \infty$) is the Deryugin (6,7) case.

d. $F(\omega) = O\left(\omega^{-(3/2)-\epsilon}\right)$ as $|\omega| \to \infty$

where

$$\epsilon = \left(\frac{\delta}{i\Lambda t}\right) + \left(\frac{v}{2iu}\right) = \pi^{-1} \sin^{-1}\left(\frac{\sigma}{2}\right), \tag{4.7}$$

e. As an edge is approached ($r \to 0$), $\partial\psi/\partial r = O\left(r^{-(1/2)+\epsilon}\right)$.

Notice that unlike previous papers (1-4) where the zero shifting was employed only once, here there are two different sets of shifts to calculate by means of Eqs. (4.4) and (4.5). The reason there are now two different shifts is as follows. In Refs. 1 and 2 there was a zero shift corresponding physically to the finite depth d, and in Refs. 3 and 4 there was a zero shift corresponding to the "thickness" parameter t. This paper contains both finite depth and "thick" plates, so that both zero shifts are necessary.

Integrals of the form

$$(2\pi i)^{-1} \oint_C F(\omega)\left(\frac{e^{-iq_m kd}}{\omega - q_m} \pm \frac{e^{iq_m kd}}{\omega + q_m}\right) d\omega \quad (m \text{ odd}),$$

where C is a circle at infinity, yield residue series matching Eq. (4.2) if Eq. (4.4) is used and the following identifications are made ($R(\beta)$ is the residue of $F(\omega)$ at $\omega = \beta$):

$$R(\beta_n) = A_n^s \epsilon_n \cos\left(\frac{\pi n}{t}\right), \tag{4.7}$$

$$R(-1) = 1, \tag{4.8}$$

and

$$B_m^s = \left(\frac{2\Lambda t p_m}{\pi i q_m}\right) e^{ikdq_m} F(-q_m). \tag{4.9}$$

Similarly, integrals of the form

$$(2\pi i)^{-1}\oint_C \frac{F(\omega)}{\omega \pm u_m}\, d\omega \ (m \text{ odd})$$

yield residue series matching Eq. (4.3) if use is made of Eq. (4.5) and the additional identification

$$C_m^s = \left(\frac{4mu^2}{\pi\sigma_m^{s+}}\right)F(-u_m) \ . \tag{4.10}$$

Thus, the three sets of amplitude coefficients A_n^s, B_m^s, and C_m^s are known when $F(\omega)$ is known be means of Eqs. (4.1), (4.7), (4.9), and (4.10). The function $F(\omega)$ can be constructed by using the methods given in Refs. 1-4. The result is

$$F(\omega) = \frac{2e^{iL(1+\omega)}}{1-\omega^2} \ \frac{\Pi_0(\omega,\bar{q})}{\Pi_0(-1,\bar{q})} \ \frac{\Pi_0(\omega,\bar{u})}{\Pi_0(-1,\bar{u})} \ \frac{\Pi_1(-1,\beta)}{\Pi_1(\omega,\beta)} \tag{4.11}$$

where the infinite products are defined as

$$\Pi_1(\omega,\beta) = \prod_{n=1}^{\infty}\left(1-\frac{\omega}{\beta_n}\right)\left(\frac{\beta_n}{in\Lambda}\right)e^{\omega/in\Lambda} \ ,$$

$$\Pi_0(\omega,\bar{q}) = \prod_{m=1}^{\infty}\left(1-\frac{\omega}{\bar{q}_{2m-1}}\right)\left[\frac{\bar{q}_{2m-1}}{\left(i\left(m-\frac{1}{2}\right)\Lambda t + \delta_{2m-1}\right)}\right]e^{\omega/i(m-½)\Lambda t} \ ,$$

and

$$\Pi_0(\omega,\bar{u}) = \prod_{m=1}^{\infty}\left(1-\frac{\omega}{\bar{u}_{2m-1}}\right)\left[\frac{\bar{u}_{2m-1}}{(i(2m-1)u + v_{2m-1})}\right]e^{\omega/i(2m-1)u} \ .$$

Also,

$$L = \left[\frac{2t\ln(2) + (t-1)\ln(t-1) - t\ln(t)}{\Lambda t}\right] . \tag{4.12}$$

These products have been discussed in Refs. 1-4, and it is clear that $F(\omega)$ satisfies properties (a)-(e). Its similarity to the meromorphic function in Ref. 3 is noted. The amplitudes are thus given by Eqs. (4.7), (4.9), and (4.10) and are

$$A_n^s = \frac{(-)^{n+1} e^{iL(1+\beta_n)}}{\beta_n \cos \frac{\pi n}{t}} \quad \frac{\Pi_0(\beta_n, \bar{q})}{\Pi_0(-1, \bar{q})} \quad \frac{\Pi_0(\beta_n, \bar{u})}{\Pi_0(-1, \bar{u})} \quad \frac{\Pi_1(-\beta_n, \beta)}{\Pi_1(1, \beta)}, \tag{4.13}$$

$$B_m^s = \frac{4\Lambda t}{\pi i p_m q_m} \quad \frac{\Pi_0(-q_m, \bar{q})}{\Pi_0(-1, \bar{q})} \quad \frac{\Pi_0(-q_m, \bar{u})}{\Pi_0(-1, \bar{u})} \quad \frac{\Pi_1(-1, \beta)}{\Pi_1(-q_m, \beta)} \; e^{i(L-Lq_m-kdq_m)}, \tag{4.14}$$

and

$$C_m^s = \frac{8}{\pi m \sigma_m^{s+}} \quad \frac{\Pi_0(-u_m, \bar{q})}{\Pi_0(-1, \bar{q})} \quad \frac{\Pi_0(-u_m, \bar{u})}{\Pi_0(-1, \bar{u})} \quad \frac{\Pi_1(-1, \beta)}{\Pi_1(-u_m, \beta)} \; e^{iL(1-u_m)} . \tag{4.15}$$

Thus, these amplitudes are known when the zero shifts δ_m and v_m are known. The numerical calculation of these shifts is different from the calculations given in Refs. 1-4. Both the calculation and the numerical results for the reflection coefficients R_n will be developed in a future paper.

5. SUMMARY

General linear equations relating field amplitude coefficients for plane waves reflected from periodic, inhomogeneously loaded, finite-depth parallel plates with soft boundaries have been presented. Solving for the special case of normal incidence ($\alpha_0 = 0$) leads to simplifications in the linear equations which enable them to be solved using a variation of the modified residue claculus scheme. The edge behavior of the fields was presented and is similar to the results given in Ref. 3. Only the analytic results are presented in this report. The numerical procedures used, and the numerical results, are being worked on and will be published at a later date.

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