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Two-Dimensional Point Resolution

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ABSTRACT

A hypothesis test has been designed for a situation which arises when multiple sensors observe the positions and only the positions of vessels in a specified region of the ocean. Given that two sensors each observe the position of a vessel, the problem is to determine whether the sensors observed the position of the same vessel or the positions of different vessels. Assuming the observations occur at the same time and the observation errors are normally distributed with known covariance matrices, it is a simple matter to design a test to test the null hypothesis that the observations refer to the same vessel against the alternative hypothesis that the observations refer to different vessels. This test is designed to have a fixed but arbitrary probability of the type I error (concluding that the observations refer to different vessels when they actually do not). Remaining is the question of the probability of the type II error (concluding that the observations refer to the same vessel when they actually do not.) With out a complete statistical description of the processes, one cannot answer this question exactly. The authors have only assumed that the number of vessels in any subregion is a Poisson random variable having for the parameter the product of a constant and the area of the subregion, the constant being the expected number of vessels per unit area. Under this condition the least upper bound for the probability of the type II error is expressed as an explicit function of the error covariance matrices, the probability of the type I error, and the expected number of vessels per unit area.

PROBLEM STATUS

This is an interim report; work on the problem is continuing.

AUTHORIZATION

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TWO-DIMENSIONAL POINT RESOLUTION

INTRODUCTION

The problem of two-dimensional point resolution arose in the context of ocean surveillance. Suppose that the vessels in a region of the ocean are uniformly and independently distributed with a specified density (expected number of vessels per unit area). Further, suppose that at the same moment in time each of two sensors, which are operating independently, observes the position of one of the vessels in the region. Do the observed positions refer to the same vessel? As the density in the region increases and/or the observational errors grow larger, the confidence with which one can answer this question decreases. In this report, this question is put into the form of a hypothesis test in which the null hypothesis is that the observations refer to the same vessel. Under the assumption that the density and the observational errors are known, the least upper bound for the probability of the type II error (concluding that the observations refer to the same vessel when they actually do not) is found as a function of the probability of the type I error (concluding that the observations refer to different vessels when they actually do not). The least upper bound is the original contribution of this report.

PRELIMINARY DISCUSSION

In this section, a model of the experiment will be developed, and the hypothesis to be tested will be given. Here the word "experiment" refers to the entire process which generates the positions of the vessels, determines the vessel or vessels whose positions are to be observed and then generates the observed positions.

Let the random vectors $\overline{\mathbf{X}}_n$, $n = 1,2,\ldots$, be independently distributed in the plane in such a manner that if Q is any region of the plane with finite area a_Q then the number of these vectors in Q is a Poisson random vector with parameter $\rho \cdot a_Q$*. The random vector $\overline{\mathbf{X}}_n$ can be considered to be the position of vessel n, while ρ is the density per unit area. Let **I** and **J** be random variables which assume positive integral values. The values of **I** and **J** in the experiment determine the vessel or vessels whose positions are to be observed; therefore, the joint and marginal distributions of **I** and **J** are assumed to be unknown.

*The theoretical aspects of this statement are questionable. To give the statement a firm foundation, the reader can replace the words "the plane" by "a bounded convex subset of the plane," and he can assume that this subset is large enough to be considered infinite in area.

Suppose in the experiment

$$\overline{\mathbf{X}}_n = \overline{\mathbf{x}}_n$$

for $n = 1,2,\ldots$, and then

$$\mathbf{I} = \mathbf{i},$$

$$\mathbf{J} = \mathbf{j},$$

Now define the conditional random vectors

$$\overline{\mathbf{Y}}_1 = \overline{\mathbf{x}}_\mathbf{i} + \overline{\mathbf{Z}}_1$$

and

$$\overline{\mathbf{Y}}_2 = \overline{\mathbf{x}}_\mathbf{j} + \overline{\mathbf{Z}}_2,$$

where

$$\overline{\mathbf{Z}}_k \sim N[\tbinom{0}{0}, \Sigma_k]$$

for $k = 1,2$. Of course, $\mathbf{Y}_1$ and $\mathbf{Y}_2$ are the random vectors corresponding to the position observations, and $\mathbf{Z}_1$ and $\mathbf{Z}_2$ which are assumed to be independent correspond to the errors in these observations. The values assumed by $\overline{\mathbf{Y}}_1$ and $\overline{\mathbf{Y}}_2$ in the experiment will be denoted by $\overline{\mathbf{y}}_1$ and $\overline{\mathbf{y}}_2$ respectively.

Given that $\overline{\mathbf{y}}_1, \overline{\mathbf{y}}_2, \Sigma_1, \Sigma_2$, and ρ are the only known quantities, a hypothesis test will be given to test the null hypothesis

$$H_0 : \mathbf{i} = \mathbf{j}$$

against the composite alternative hypothesis

$$H_1 : \mathbf{i} \neq \mathbf{j}.$$

The test is based on the statistic

$$\mathbf{R} = |D(\overline{\mathbf{Y}}_1 - \overline{\mathbf{Y}}_2)| \tag{1}$$

where D is the matrix such that

$$D(\Sigma_1 + \Sigma_2)D^\mathsf{T} = I^*. \tag{2}$$

*Here I is the 2 x 2 identity matrix.

Specifically if

$$\Sigma_1 + \Sigma_2 = \begin{pmatrix} \sigma_1{}^2 & \sigma_{12} \\ \sigma_{12} & \sigma_2{}^2 \end{pmatrix},$$

then

$$D = \frac{1}{\sigma_2 \sqrt{\sigma_1{}^2 \sigma_2{}^2 - \sigma_{12}{}^2}} \begin{pmatrix} \sigma_2{}^2 & -\sigma_{12} \\ 0 & \sqrt{\sigma_1{}^2 \sigma_2{}^2 - \sigma_{12}{}^2} \end{pmatrix}.$$

Given that

$$\overline{\mathbf{X}}_I = \overline{\mathbf{x}}_i$$

and

$$\overline{\mathbf{X}}_J = \overline{\mathbf{x}}_j,$$

it follows from the relations expressed in Eqs. (1) and (2) that the random vector

$$D(\overline{\mathbf{Y}}_1 - \overline{\mathbf{Y}}_2)$$

has the conditional distribution

$$N[D(\overline{\mathbf{x}}_i - \overline{\mathbf{x}}_j), \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)].$$

Define the quantity

$$\mathbf{a} = |D(\overline{\mathbf{x}}_i - \overline{\mathbf{x}}_j)]|$$

and note that a is a sample of the random variable*

$$\mathbf{A} = |D(\overline{\mathbf{X}}_I - \overline{\mathbf{X}}_J)|. \tag{3}$$

If

$$\mathbf{A} = \mathbf{a}$$

it is known† that the random variable R has the probability density function‡

$$f_{\mathbf{R}|\mathbf{X}}\,(\mathbf{r}|A = \mathbf{a}) = \mathbf{r} \cdot \exp\,[-(\mathbf{a}^2 + \mathbf{r}^2)/2] \cdot I_0\,(\mathbf{ar}) \tag{4}$$

for r positive§. Observe that Eq. (4) defines the conditional distribution of the test statistic R.

*In justifying that A is in fact a random variable, the reader should restrict his thinking to a convex set as mentioned in the previous footnote.

†K. S. Miller, "Multidimensional Gaussian Distributions," New York: John Wiley and Sons, Inc., 1964.

‡Given that A = a, the random variable R^2 is distributed according to the noncentral chi-square distribution.

§Here I_0 is the modified Bessel function of order zero.

THE DISTRIBUTION OF R WHEN H_0 IS TRUE

When the null hypothesis

$$H_0 : \mathbf{i} = \mathbf{j}$$

is true, Eq. (3) yields

$$\mathrm{A} = 0.$$

If F_0 denotes the distribution function of the random variable R when H_0 is true, it follows from Eq. (4) that

$$F_0(\mathbf{r}) = 1 - \exp(-\mathbf{r}^2/2) \tag{5}$$

for r positive.

THE LEAST UPPER BOUND FOR THE DISTRIBUTION OF R WHEN H_1 IS TRUE

The distribution of the random variable A when H_1 is true must be examined before anything can be said about R. Recall

$$\mathrm{A} = |D(\overline{\mathbf{X}}_\mathrm{I} - \overline{\mathbf{X}}_\mathrm{J})|,$$

and when H_1 is true, the random variables I and J are such that

$$\mathbf{I} \neq \mathbf{J}.$$

Given

$$\mathbf{I} = \mathbf{i},$$

define the conditional random variable $\mathbf{I}'$ to be the positive integer $\mathbf{i}'$, $\mathbf{i} \neq \mathbf{i}'$, which has the property

$$|D(\overline{\mathbf{x}}_\mathrm{i} - \overline{\mathbf{x}}_\mathrm{i}')| \leqq |D(\overline{\mathbf{x}}_\mathrm{i} - \overline{\mathbf{x}}_\mathrm{k})|$$

for $\mathbf{k} = 1,2,...$ and $\mathbf{k} \neq \mathbf{i}$. In other words, $D\overline{\mathbf{x}}_\mathrm{i}'$ is the point which is nearest to $D\overline{\mathbf{x}_\mathrm{i}}$.

The linear transformation D does not alter the Poisson property, which was discussed previously, but under this transformation the unit density is

$$\rho' = \rho/|\det D|.$$

It follows directly from the Poisson property that

$$P[\,|D(\overline{\mathrm{X}}_{\mathrm{I}} - \overline{\mathrm{X}}_{\mathrm{I}}')| \leqslant \mathrm{a}] = 1 - \exp(-\mathrm{a}^2 \pi \rho')$$

for positive a. Note that this probability is independent of the value the random variable **I** assumes in the experiment.

When H_1 is tiue, the distribution of the random variable A is

$$\begin{aligned} F_{\mathrm{A}}(\mathrm{a}) &= P(\mathrm{A} \leqslant \mathrm{a}) \\ &\leqslant P[\,|D(\overline{\mathrm{X}}_{\mathrm{I}} - \overline{\mathrm{X}}_{\mathrm{I}}')| \leqslant \mathrm{a}] \\ &= 1 - \exp(-\mathrm{a}^2\, \pi \rho') \end{aligned} \tag{6}$$

for positive **a**. If **I**$'$ = **J**, the inequality in Eq. (6) becomes equality; therefore, the upper bound on F_{A} is the least upper bound.

Now if F_1 denotes the distribution function of **R** when H_1 is true, then

$$F_1(\mathrm{r}) = \int_0^{\mathrm{r}} \int_0^{\infty} f_{\mathrm{R}|\mathrm{A}}(\mathrm{x}|\mathrm{A} = \mathrm{a}) dF_{\mathrm{A}}(\mathrm{a}) d\mathrm{x}$$

$$\leqslant \int_0^{\mathrm{r}} \int_0^{\infty} \left\{ \mathrm{x} \cdot \exp[-(\mathrm{a}^2 + \mathrm{x}^2)/2] \cdot I_0(\mathrm{ax}) \right\}$$

$$\cdot\ [2\mathrm{a}\pi\rho' \cdot \exp(-\mathrm{a}^2 \pi\rho')]\, d\mathrm{a} d\mathrm{x}$$

$$= \int_0^{\mathrm{r}} 2\pi\rho'\mathrm{x} \cdot \exp(-\mathrm{x}^2/2) \int_0^{\infty} \mathrm{a} \cdot \exp[-\mathrm{a}^2(½ + \pi\rho')] I_0(\mathrm{ax}) d\mathrm{a} d\mathrm{x}$$

$$= \int_0^{\mathrm{r}} 2\pi\rho'\mathrm{x} \cdot \exp(-\mathrm{x}^2/2) \int_0^{\infty} \mathrm{a} \cdot \exp[-\mathrm{a}^2(½ + \pi\rho')] J_0(\mathrm{ax}\sqrt{-1}) d\mathrm{a} d\mathrm{x}$$

$$= \int_0^{\mathrm{r}} 2\pi\rho'\mathrm{x} \cdot \exp(-\mathrm{x}^2/2) \left\{ \frac{1}{I + 2\pi\rho'} \cdot \exp[\mathrm{x}^2/(2 + 4\pi\rho')] \right\} d\mathrm{x}$$

$$= 1 - \exp(-\mathrm{r}^2/2\sigma^2),$$

where

$$\sigma^2 = 1 + 1/(2\pi\rho').$$

Or

$$F_1(\mathbf{r}) \leqslant 1 - \exp[-\mathbf{r}^2/(2\sigma^2)]$$

for $\mathbf{r}$ positive.

HYPOTHESIS TEST

For a test of the null hypothesis

$$H_0 : \mathbf{i} = \mathbf{j}$$

against the alternative hypothesis

$$H_1 : \mathbf{i} \neq \mathbf{j}$$

with the probability of the type I error of α, the critical region is

$$C_\alpha = (\mathbf{r} : \mathbf{r} > \sqrt{-2\ln\alpha}).$$

The reader should note that the critical region C_α is the uniformly most powerful for the test of H_0 against H_1.

The least upper bound for the probability of the type II error for the critical region C_α is

$$\begin{aligned} \beta_{LUB} &= 1 - \exp(\ln\alpha/\sigma^2) \\ &= 1 - (\alpha)^{1/\sigma^2}. \end{aligned} \tag{7}$$

For a graph of α and β_{LUB} as a function of the lower point of the critical region, see Fig. 1. Observe that in the limiting case, when $\sigma = 1$,

$$\alpha = 1 - \beta_{LUB}.$$

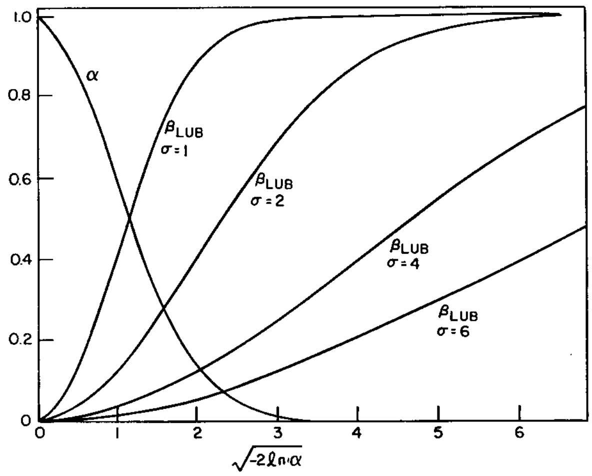

Fig. 1 – Graphs of α (the type I error) and β_{LUB} (the least upper bound of the type II error) as functions of the lower end point of the critical region

If the hypothesis H_0 is accepted, the minimum variance unbiased estimator $\overline{\mathbf{Y}}$ of $\overline{\mathbf{x}}_\mathbf{i}$ is given by

$$\overline{\mathbf{Y}} = (\Sigma_1^{-1} + \Sigma_2^{-1})^{-1}(\Sigma_1^{-1}\, \overline{\mathbf{Y}}_1 + \Sigma_2^{-1}\, \overline{\mathbf{Y}}_2),$$

and if H_0 is true, $\overline{Y}$ is distributed according to

$$N[\overline{\mathbf{x}}_\mathbf{i}, (\Sigma_1^{-1} + \Sigma_2^{-1})^{-1}].$$

EXAMPLE

For the sake of illustration, suppose that the errors in the position observations are circular, and

$$\Sigma_2 = k\Sigma_1$$

with

$$\Sigma_1 = \begin{pmatrix} \sigma_1{}^2 & 0 \\ 0 & \sigma_1{}^2 \end{pmatrix}.$$

Now

$$\Sigma_1 + \Sigma_2 = (k+1) \cdot \Sigma_1$$

and

$$D = \frac{1}{\sqrt{k+1} \cdot \sigma_1} \cdot I.$$

It follows that

$$|\det. D| = \frac{1}{(k+1)\,\sigma_1^{\,2}}$$

and

$$\begin{aligned}\sigma^2 &= 1 + (2\pi\rho')^{-1}\\ &= 1 + [2\pi(k+1)\sigma_1^{\,2}\rho]^{-1}.\end{aligned}$$

If

$$\alpha = \beta_{LUB},$$

Eq. (7) yields

$$\sigma_1^{\,2} = \frac{1}{2\pi(k+1)\rho} \cdot \frac{\ln(1-\alpha)}{\ln \dfrac{\alpha}{1-\alpha}} \, .$$

For

$$k = 3$$

the relationships between ρ and σ_1 are plotted in Fig. 2 for four different values of α.

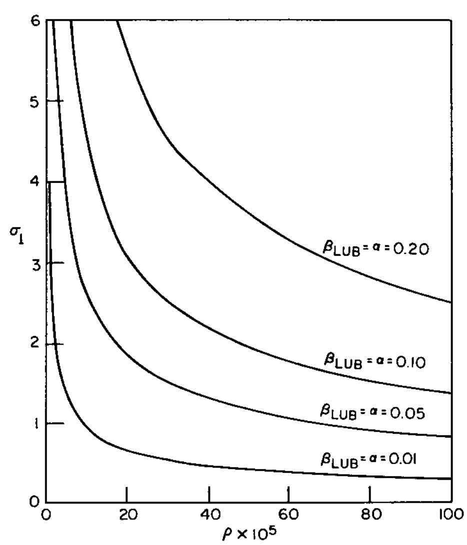

Fig. 2 – The required standard deviation of the errors of the more accurate sensor as a function of density, α, and β_{LUB}

The results obtained in this report have strong implications concerning the ocean-surveillance problem, which was discussed in the Introduction. To depict a special case relating to this example, suppose that two sensors are viewing a region of the ocean in which the expected density is 40 vessels per 100,000 square nautical miles, and let the circular errors of the two sensors be in the ratio of $\sqrt{3} : 1$. If the probability of the type I error and the probability of the type II error of the aforementioned decision is to be less than or equal to 5 percent, then Fig. 2 indicates that the standard deviation of the more accurate sensor must be less than or equal to 1.315 naut mi.

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13. ABSTRACT

A hypothesis test has been designed for a situation which arises when multiple sensors observe the positions and only the positions of vessels in a specified region of the ocean. Given that two sensors each observe the position of a vessel, the problem is to determine whether the sensors observed the position of the same vessel or the positions of different vessels. Assuming the observations occur at the same time and the observation errors are normally distributed with known covariance matrices, it is a simple matter to design a test to test the null hypothesis that the observations refer to the same vessel against the alternative hypothesis that the observations refer to different vessels. This test is designed to have a fixed but arbitrary probability of the type I error (concluding that the observations refer to different vessels when they actually do not). Remaining is the question of the probability of the type II error (concluding that the observations refer to the same vessel when they actually do not). Without a complete statistical description of the processes, one cannot answer this question exactly. The authors have only assumed that the number of vessels in any subregion is a Poisson random variable having for the parameter the product of a constant and the area of the subregion, the constant being the expected number of vessels per unit area. Under this condition the least upper bound for the probability of the type II error is expressed as an explicit function of the error covariance matrices, the probability of the type I error, and the expected number of vessels per unit area.

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