NRL Report 7202

CROSEC Computational Manual I (NRL APT System): Equations for Intersections Between Planes and Cylinders (Infinite and Bounded), Planes and Spheres

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PREVIOUS REPORTS IN THIS SERIES

"Cross-Sectional Plots of Plane Intersections: An Adaptation of the APT System," K.P. Thompson, NRL Report 7025, Jan. 27, 1970

CONTENTS

ABSTRACT

This report is the second in a series devoted to the development of a cross-sectional plotting capability for the numerical control APT system in use at NRL. The report covers the development of equations in a computational form for the plotting of intersections between a plane and a cylinder, and a plane and a sphere. Initially, a local coordinate system for a cylinder is established in one of twelve possible ways. Subsequently, the intersection of the cylinder with a plane is considered in the general (infinite) case, in a bounded situation, and in the degenerate condition when the cross-sectional plane is parallel to the axis of the cylinder. The development for the sphere concludes the equation presentation and is obviously less complicated than the cylinder case because of the symmetry, but similar to the extent that a circular intersection and one degenerate case, a tangent point, can be obtained. Computer-generated plots illustrating the types of intersections developed in the report are included. These plots were made using a program that was written to implement the equations developed in this report. The computer program is described in detail in "CROSEC, A Fortran IV - APT Program to give Orthographic, Section and Definable Perspective Views of a Planar-Curved Surface," NRL Report 7228.

PROBLEM STATUS

This is an interim report on a continuing problem.

AUTHORIZATION

NRL Problem Z00-01

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CROSEC COMPUTATIONAL MANUAL I
(NRL APT SYSTEM)

INTRODUCTION

Background

The objective of this NRL problem is to develop the facility for producing cross-sectional plots of a part defined in the APT program. NRL Report 7025 (Ref. 1) describes a procedure for obtaining cross-sectional plots of plane intersections, and it is recommended that the reader be familiar with its contents. Briefly, it presents the several types of plane equations and a method for defining a coordinate system in a cross-sectional plane, and describes the computer program, CROSEC, which computes points of intersection between any two planes and the cross-sectional plane. CROSEC then, from these points, defines and plots the lines of intersection.

The next consideration in this problem, the subject of this report, is the development of the ability to plot cylindrical and spherical cross sections. The scope of this report is the exposition of the computational forms needed to obtain all types of cylindrical and spherical cross sections and the presentation of some computer-generated plots demonstrating the correctness of the forms.

Overview

The report commences with the presentation of some basic equations, then specifies twelve situations designed to cover all possible cylinder orientations. These result in a set of direction cosines, thereby establishing an orthogonal coordinate system within a cylinder from the information provided in the APT canonical form. For some of these situations the direction cosines are constant, but for others they must be computed. Only one of these twelve possible solutions applies for any given cylinder. The cylinder's coordinate system and its set of direction cosines (which are relative to the APT part-programmer's XYZ coordinate system) are next used to describe the cylinder in the XYZ system. It is then possible to solve the cylinder's equation simultaneously with the equation of the cross-sectional plane (called the HOPE plane), obtaining as a result the equation of the intersection expressed in the XYZ system. This intersection is then transformed into the HOPE plane coordinate system (1) for two-dimensional plotting purposes. This is a complicated step involving many terms. Once accomplished, however, a simplification is possible because all z' terms in the HOPE system can be set to zero.

The resulting two-variable, second-degree, intersection polynomial equation in the HOPE plane must now be analyzed to determine if it is a circle, an ellipse, or a degenerate form, namely a pair of parallel lines or a single line. Tests to make these determinations are in terms of the invariants of a two-dimensional, second-degree polynominal. If the cylinder is considered to be finite in length, additional computations are necessary. If an ellipse of intersection is present, it must be determined whether or not it is totally within the cylinder. If not, only a part of the ellipse is valid as the curve of intersection. The possible curves of intersection for the bounded cylinder are a circle, an ellipse, a truncated ellipse, a rectangle, or a line segment. The rectangle and line segment cases arise when the HOPE plane is parallel to the cylinder's axis but still close enough to intersect it. It is necessary to create a discrete set of points for plotting. The number of points created should be sufficient to present a plot that is pleasing to the eye. For a long, thin ellipse, for example, as high as 400 line segments are necessary to produce a satisfactory curve. The method followed for the circle and ellipse computes the discrete point coordinates relative to the center of the curve, followed by the addition of the center coordinates and rotation, as required, to properly locate the curve. In order to plot the finite line segments or rectangles, it is necessary to determine the HOPE plane coordinates of the points where the lines pierce the end planes of the cylinder.

A similar process is also followed for the case of a plane intersecting a sphere but with the great simplification of a singly oriented local reference system because of the symmetry. For the sphere there is neither an infinite case nor a parallel case, only a tangent point, a circle, or no intersection.

Equations for all of the steps described above are presented in this report. The report is concluded with some illustrative cross-sectional plots obtained by use of a computer program written to implement these equations. This program is called CROSEC MOD 2.0 and is described in Ref. 2. The basic material used in the development of these equations has been Ref. 3, Korn and Korn's "*Mathematical Handbook for Scientists and Engineers*," in the author's opinion, an excellent reference book.

CYLINDER COORDINATE SYSTEM

To begin the development of the cylinder coordinate system, consider the canonical form of the cylinder and the base plane.

The APT Cylinder

The APT canonical form for the cylinder contains a point, a vector, and a radius. The point is on the axis which has the direction of the vector, and the radius is the radius of a circular cross section. Thus in the APT form

$$\text{CYL 1 = CYLINDER/CANON,}\ x_c,\ y_c,\ z_c,\ u_x,\ u_y,\ u_z,\ r,$$

where

x_c, y_c, z_c are the point coordinates,
u_x, u_y, u_z are the vector's direction cosines, and
r is the radius.

Cylinder's Base Plane

A plane through the cylinder's defined point, with the vector as its normal, can be defined in normal form as

$$u_x\, x + u_y\, y + u_z\, z = u_x\, x_c + u_y\, y_c + u_z\, z_c\,. \tag{1}$$

This is conveniently considered, by definition, to be the base plane. Thus, for example, if

$$(x_c,\ y_c,\ z_c) = (0,\ 0,\ 1),$$

and

$$(u_x,\ u_y,\ u_z) = (0,\ 0,\ 1),$$

substitution into Eq. (1) gives

$$0 \cdot x + 0 \cdot y + 1 \cdot z = 0 \cdot 0 + 0 \cdot 0 + 1 \cdot 1,$$

which yields $z = 1$. A second example where

$$(x_c,\ y_c,\ z_c) = \left(\frac{1}{3},\ \frac{1}{3},\ \frac{1}{3}\right)$$

and

$$(u_x,\ u_y,\ u_z) = \left(\frac{1}{\sqrt{3}},\ \frac{1}{\sqrt{3}},\ \frac{1}{\sqrt{3}}\right)$$

yields, upon substitution into Eq. (1),

$$\frac{1}{\sqrt{3}}\,x + \frac{1}{\sqrt{3}}\,y + \frac{1}{\sqrt{3}}\,z = \frac{1}{3} \cdot \frac{1}{\sqrt{3}} + \frac{1}{3} \cdot \frac{1}{\sqrt{3}} + \frac{1}{3} \cdot \frac{1}{\sqrt{3}}$$

Or, simplifying, $x + y + z = 1$, the much-used equiangular plane of Ref. 1.

Cylinder Equation

An infinite cylinder can be mathematically defined by an equation of the form

$$\overset{c}{x}{}^2 + \overset{c}{y}{}^2 = r^2. \tag{2}$$

This is an equation of a circle in which no z value is specified, and therefore z can have an infinite number of values. The little letter c centered above the x and y designates a coordinate system which is local to the cylinder and whose z axis is identical with the axis of the cylinder with direction cosines u_x, u_y, u_z. It follows, naturally, to equate the cylinder's base plane with the $\overset{c}{X}\overset{c}{Y}$ plane. Then if $\overset{c}{X}$ is found by means next presented in this report, $\overset{c}{Y}$ can be defined by taking the cross product of $\overset{c}{Z}$ and $\overset{c}{X}$.

Thus, to summarize the method of obtaining the cylinder coordinate system: $\overset{c}{Z}$ is known because of the cylinder's axis, $\overset{c}{X}$ must be tested for and computed if it is not a standard case, and $\overset{c}{Y}$ is computed from $\overset{c}{Z}$ and $\overset{c}{X}$'s direction cosines.

The $\overset{c}{X}$ Axis

Table 1 lists twelve $\overset{c}{X}$ orientations. The first six represent vector orientations parallel to the plus and minus directions of one of the major axes (X, Y, Z). Each "standard" solution has a predetermined direction cosine matrix that preserves right-handedness. Numbers 7, 8, and 9 are intersection solutions for the situation in which the c point is on one of the major axes. Numbers 10, 11, and 12 are intercept solutions for the situation wherein the c point is not on an axis and at least one component of the axis's direction cosines is nonzero. The last two columns of the table give some sample APT definitions and vector sketches for clarity, one for each of the twelve cases.

The Intersection Situations

If the vector U is not oriented in one of the six directions along the major axes, then the scheme is to use the axis point to aid in defining the X_c axis. Consider the three situations when the point is *on* one of the three major axes X, Y, or Z (for these purposes consider the origin to be on the X axis). For example, if the axis point is on the X axis $(x_c, y_c, z_c) = (x_c, 0, 0)$ then the intersection of the base plane (Eq. (1)) with the XY plane will define the X_c axis in this fashion.

Case 7

General base plane: $u_x x + u_y y + u_z z = u_x x_c + u_y y_c + u_z z_c$ (1)

Base Plane for $(x_c, 0, 0)$: $u_x x + u_y y + u_z z = u_x x_c$ (3)

XY plane: $z = 0$ (4)

Intersection of
base plane with XY plane: $u_x x + u_y y = u_x x_c$ (5)

Expressed as a line: $y = (-u_x/u_y)\, x + (u_x/u_y) x_c$ (6)

X_c's direction cosines:

$$t_{11} = \cos\left(\tan^{-1}\left(-\frac{u_x}{u_y}\right)\right),$$
$$t_{21} = \sin\left(\tan^{-1}\left(-\frac{u_x}{u_y}\right)\right), \qquad (7)$$
$$\tan_{31} = 0.$$

A restriction $(u_y \neq 0)$ is imposed to prevent an undefined tangent. This means the axis vector must not be parallel to the Y axis. See Fig. 1 for an illustration of Case 7.

When the axis point is on the Y axis, the intersection of the base plane with the YZ plane is used to define the X_c axis. This forms Case 8. Its equations are given next.

Case 8

Base Plane for $(0, y_c, 0)$: $u_x x + u_y y + u_z z = u_y y_c$ (8)

YZ plane: $x = 0$ (9)

Intersection: $u_y y + u_z z = u_y y_c$ (10)

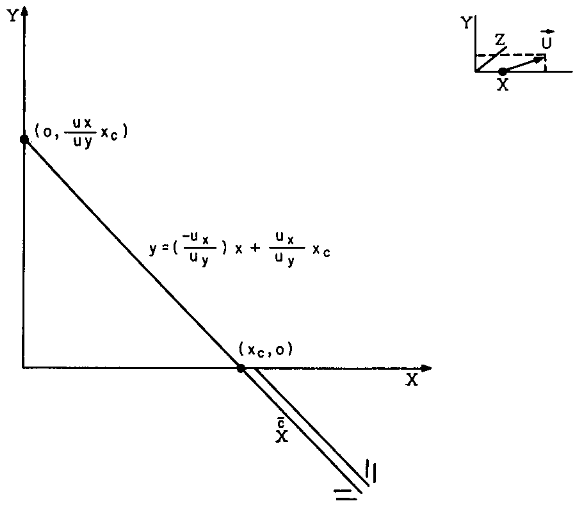


Fig. 1 - Case 7 occurs when the cylinder's axis vector is not directed along a major axis (X, Y, or Z) and the axis point is on the X axis. The cylinder system's X_c axis is defined by the intersection of the cylinder's base plane with the XY plane. Small sketch shows orientation of cylinder's axis vector.

Expressed as a line: $z = (-u_y/u_z)\,y + (u_y/u_z)\,y_c$ (11)

X_c's direction cosines:

$$\begin{aligned} t_{11} &= 0, \\ t_{21} &= \cos(\tan^{-1}(-u_y/u_z)), \\ t_{31} &= \sin(\tan^{-1}(u_y/u_z)) \end{aligned} \qquad (12)$$

In Case 8 the vector component u_z must be nonzero.

Case 9 occurs when the axis point is on the Z axis. In this case the intersection of the base plane with the XZ plane is used. Its equations are given next.

Case 9

Base plane for $(0, 0, z_c)$: $u_x\,x + u_y\,y + u_z\,z = u_z\,z_c$ (13)

XZ plane: $y = 0$ (14)

Intersection: $u_x\,x + u_z\,z = u_z\,z_c$ (15)

Expressed as a line: $x = (-u_z/u_x)\,z + (u_z/u_x)z_c$ (16)

X_c's direction cosines:

$$\begin{aligned} t_{11} &= \sin(\tan^{-1}(-u_z/u_x)), \\ t_{21} &= 0, \\ t_{31} &= \cos(\tan^{-1}(-u_z/u_x)). \end{aligned} \qquad (17)$$

In Case 9, u_x must be nonzero.

The Intercept Situations

The intercept situation is characterized by the point c being *off* any major axis and the cylinder axis's vector being in a skewed position, not oriented with any major axis. Figure 2 illustrates Case 10. The symbol x_I, called x *intercept*, is the x value at which the cylinder's base plane is pierced by the x axis. It is obtained by setting y and z to zero in Eq. (1) and solving for x. Therefore,

$$x_I = \frac{u_x x_c + u_y y_c + u_z z_c}{u_x}. \tag{18}$$

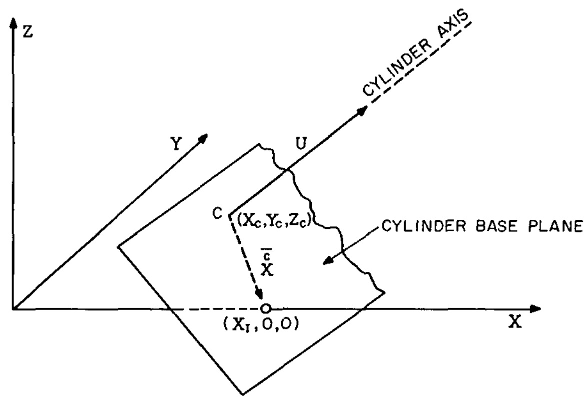


Fig. 2 - Intercept solution on x axis.

If u_x were equal to zero, the plane would be paralled to the x axis and x_I would be undefined. This is the reason the intercept cases have a qualification that one of the direction cosines of the cylinder axis vector must be nonzero. The local axis x is equated to the directed line segment from the c point (x_c, y_c, z_c) to the intercept point $(x_I, 0, 0)$. Its direction cosines are $(\Delta x/x_L, -y_c/x_L, -z_c/x_L)$,

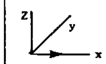
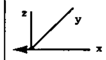
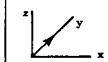
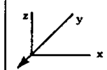
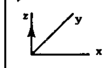
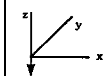
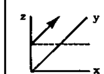
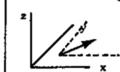
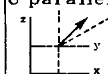
where,

$$\Delta x = x_I - x_c, \tag{19}$$

$$x_L = \sqrt{\Delta x^2 + (0 - y_c)^2 + (0 - z_c)^2}. \tag{20}$$

Cases 11 and 12 are interpreted in a similar manner for intercepts with the Y axis and the Z axis, respectively. The results are listed in Table 1, which summarizes all twelve cases.

Table 1
Solutions for Cylinder Coordinate Systems

Number	Point X_c, Y_c, Z_c	Vector U_x, U_y, U_z	Standard Matrix of Direction Cosines for $\overset{c}{X}, \overset{c}{Y}, \overset{c}{Z}$ axes or Computation Form for $\overset{c}{X}$-axis Direction Cosines	A Sample APT Definition for a Cylinder Meeting the Specifications	Point and Vector Sketch for APT Sample Cylinder
1	Any set of values	Positive X direction	$\begin{bmatrix} 0, & 1, & 0 \\ 0, & 0, & 1 \\ 1, & 0, & 0 \end{bmatrix} = \begin{bmatrix} t_{11}, & t_{21}, & t_{31} \\ t_{12}, & t_{22}, & t_{32} \\ t_{13}, & t_{23}, & t_{33} \end{bmatrix}$	CYL 1 = CYLNDR/CANON, 0, 0, 0, 1, 0, 0, 1	
2	Any set of values	Negative X direction	$\begin{bmatrix} 0, & 0, & 1 \\ 0, & 1, & 0 \\ -1, & 0, & 0 \end{bmatrix}$	CYL 2 = CYLNDR/CANON, 0, 0, 0, –1, 0, 0, 1	
3	Any set of values	Positive Y direction	$\begin{bmatrix} 0, & 0, & 1 \\ 1, & 0, & 0 \\ 0, & 1, & 0 \end{bmatrix}$	CYL 3 = CYLNDR/CANON, 0, 0, 0, 0, 1, 0, 1	
4	Any set of values	Negative Y direction	$\begin{bmatrix} 1, & 0, & 0 \\ 0, & 0, & 1 \\ 0, & -1, & 0 \end{bmatrix}$	CYL 4 = CYLNDR/CANON, 0, 0, 0, 0, –1, 0, 1	
5	Any set of values	Positive Z direction	$\begin{bmatrix} 1, & 0, & 0 \\ 0, & 1, & 0 \\ 0, & 0, & 1 \end{bmatrix}$	CYL 5 = CYLNDR/CANON, 0, 0, 0, 0, 0, 1, 1	
6	Any set of values	Negative Z direction	$\begin{bmatrix} 0, & 1, & 0 \\ 1, & 0, & 0 \\ 0, & 0, & -1 \end{bmatrix}$	CYL 6 = CYLNDR/CANON, 0, 0, 0, 0, 0, –1, 1	
7	Must be at the origin or on the X axis	$\mu_y \neq 0$	XY Intersection $\phi = \tan^{-1}\left(-\frac{u_x}{u_y}\right)$, $t_{11} = \cos\phi$, $t_{21} = \sin\phi$, $t_{31} = 0$	CYL 7 = CYLNDR/CANON, 0, 0, 0, 1/2, $\frac{\sqrt{3}}{2}$, 0, 1	 U in XY
8	Must be on the Y axis	$\mu_z \neq 0$	YZ Intersection $\phi = \tan^{-1}\left(-\frac{u_y}{u_z}\right)$, $t_{11} = 0$, $t_{21} = \cos\phi$, $t_{31} = \sin\phi$	CYL 8 = CYLNDR/CANON, 0, 1, 0, 0, 1/2, $\frac{\sqrt{3}}{2}$, 1	 U in YZ
9	Must be on the Z axis	$\mu_x \neq 0$	XZ Intersection $\phi = \tan^{-1}\left(-\frac{u_z}{u_x}\right)$, $t_{11} = \sin\phi$, $t_{21} = 0$, $t_{31} = \cos\phi$	CYL 9 = CYLNDR/CANON, 0, 0, 1, 1/2, 0, $\frac{\sqrt{3}}{2}$, 1	 U in XZ
10	Must *not* be on the X axis	$\mu_x \neq 0$	X Intercept $t_{11} = \frac{\Delta x}{x_L}$, $t_{21} = \frac{-y_c}{x_L}$, $t_{31} = \frac{-z_c}{x_L}$, where $\Delta x = [(x_c u_x + y_c u_y + z_c u_y)/u_x] - x_c$ $x_L = (\Delta x^2 + y_c^2 + z_c^2)^{1/2}$	CYL 10 = CYLNDR/CANON, 1, 1, 0, 1/2, $\frac{\sqrt{3}}{2}$, 0, 1	U in XY 
11	Must *not* be on the Y axis	$\mu_y \neq 0$	Y Intercept $t_{11} = \frac{-x_c}{y_L}$, $t_{21} = \frac{\Delta y}{y_L}$, $t_{31} = \frac{-z_c}{y_L}$, where $\Delta y = [(x_c u_x + y_c u_y + z_c u_z)/u_y] - y_c$ $y_L = (x_c^2 + \Delta y^2 + y_c^2)^{1/2}$	CYL 11 = CYLNDR/CANON, 0, 1, 1, 0, 1/2, $\frac{\sqrt{3}}{2}$, 1	U in YZ 
12	Must *not* be on the Z axis	$\mu_z \neq 0$	Z Intercept $t_{11} = \frac{-x_c}{z_L}$, $t_{21} = \frac{-y_c}{z_L}$, $t_{31} = \frac{\Delta z}{z_L}$ where $\Delta Z = [(x_c u_x + y_c u_y + z_c u_z)/u_z] - z_c$ $z_L = (x_c^2 + y_c^2 + \Delta z^2)^{1/2}$	CYL 12 = CYLNDR/CANON, 1, 0, 1, 0, 1/2, $\frac{\sqrt{3}}{2}$, 1	U parallel to YZ 

INTERSECTION BETWEEN HOPE PLANE AND INFINITE CYLINDER

Cylinder Defined in XYZ System

After a cylinder has been identified in the defined symbol table and its coordinate system established in one of twelve possible ways, its equation can be expressed in the form of a circle, Eq. (2), in the cylinder coordinate system.

To relate this cylinder to the XYZ coordinate system, it is necessary to express its equation in that system. The conversion matrix is the vehicle designed to accomplish this task. With respect to the XYZ system, the $\overset{c}{x}$ axis has direction cosines t_{11}, t_{21}, and t_{31}. Similarly the $\overset{c}{y}$ axis has direction cosines t_{12}, t_{22}, and t_{32}. Finally, the z_c axis has direction cosines t_{13}, t_{23}, and t_{33}. The equations which allow for both translational and rotational conversion from the cylinder coordinate system to the XYZ coordinate system are

$$\overset{c}{x} = t_{11}x + t_{21}y + t_{31}z - K_1 , \tag{21}$$

$$\overset{c}{y} = t_{12}x + t_{22}y + t_{32}z - K_2 , \tag{22}$$

and

$$\overset{c}{z} = t_{13}x + t_{23}y + t_{33}z - K_3 ; \tag{23}$$

where

$$K_1 = t_{11}x_c + t_{21}y_c + t_{31}z_c , \tag{24}$$

$$K_2 = t_{12}x_c + t_{22}y_c + t_{32}z_c , \tag{25}$$

$$K_3 = t_{13}x_c + t_{23}y_c + t_{33}z_c . \tag{26}$$

This point (x_c, y_c, z_c) is part of the canonical form. It is on the axis of the cylinder and is in the XYZ system. It should be noted that K_1, K_2, and K_3 are constants.

To express the cylinder in the XYZ system it is necessary to square Eqs. (21) and (22), substitute them into Eq. (2), and regroup the resulting terms. One grouping that provides a balanced appearance and has some computational benefits takes the following form:

$$\begin{aligned} &(a_{11}x + a_{12}y + a_{13}z + a_{14})\,x \\ &+ (a_{21}x + a_{22}y + a_{23}z + a_{24})\,y \\ &+ (a_{31}x + a_{32}y + a_{33}z + a_{34})\,z \\ &+ (a_{41}x + a_{42}y + a_{43}z + a_{44}) = 0 . \end{aligned} \tag{27}$$

These a_{ik} $(i, k = 1, 2, 3, 4)$ coefficients are computed as follows:

$$a_{11} = t_{11}^2 + t_{12}^2 \quad (28)$$

$$a_{12} = t_{12}\, t_{21} + t_{12}\, t_{22} \quad (29)$$

$$a_{13} = t_{11}\, t_{31} + t_{12}\, t_{32} \quad (30)$$

$$a_{14} = -(K_1\, t_{11} + K_2\, t_{12}) \quad (31)$$

$$a_{21} = a_{12} \quad (32)$$

$$a_{22} = t_{21}^2 + t_{22}^2 \quad (33)$$

$$a_{23} = t_{21}\, t_{31} + t_{22}\, t_{32} \quad (34)$$

$$a_{24} = -(K_1\, t_{21} + K_2\, t_{22}) \quad (35)$$

$$a_{31} = a_{13} \quad (36)$$

$$a_{32} = a_{23} \quad (37)$$

$$a_{33} = t_{31}^2 + t_{32}^2 \quad (38)$$

$$a_{34} = -(K_1\, t_{31} + K_2\, t_{32}) \quad (39)$$

$$a_{41} = a_{14} \quad (40)$$

$$a_{42} = a_{24} \quad (41)$$

$$a_{43} = a_{34} \quad (42)$$

$$a_{44} = K_1^2 + K_2^2 - r^2 \quad (43)$$

Intersection in the XYZ System

The simultaneous solution of the HOPE plane equation and the cylinder equation is accomplished by adding the first-degree coefficients and the constant terms of their respective equations, The HOPE plane in the XYZ system is expressed as,

$$A * x + B * y + C * z = D, \text{ (See Ref. 1, p. 1)} \quad (44)$$

The cylinder has just been developed as Eq. (27). Expressing the simultaneous solution of these two equations in the form of replacement statements,

$$a_{14} \leftarrow a_{14} + \frac{A}{2}, \quad a_{24} \leftarrow a_{24} + \frac{B}{2}, \quad a_{34} \leftarrow a_{34} + \frac{C}{2}, \quad a_{44} \leftarrow a_{44} - D, \quad (45)$$

is a computer-oriented convenience and should be easily understood. The remaining coefficients are unchanged. The intersection in the XYZ system has been found.

Intersection in HOPE Plane Coordinate System

Once the coefficients that define the curve of intersection, expressed in the XYZ system, have been obtained, the next step is to compute a similiar set of coefficients in the HOPE plane coordinate system, another coordinate conversion. The creation of the HOPE plane coordinate system is quite thoroughly discussed in Ref. 1. It should suffice here merely to define the terms and give the equations.

If with respect to the XYZ system the HOPE plane coordinate system has an X_h axis with direction cosines h_{11}, h_{21}, h_{31}, a Y_h axis with direction cosines h_{12}, h_{22}, h_{32}, and a Z_h axis with direction cosines h_{13}, h_{23}, h_{33}; then the equations which allow for both translational and rotational conversion are

$$x = h_{11} x_h + h_{12} y_h + h_{13} z_h + x_0, \quad (46)$$

$$y = h_{21} x_h + h_{22} y_h + h_{23} z_h + y_0, \quad (47)$$

and

$$z = h_{31} x_h + h_{32} y_h + h_{33} z_h + z_0. \quad (48)$$

In these conversion equations x_0, y_0, and z_0 are the XYZ coordinates of the HOPE plane coordinate system's origin. These are computable from the the HOPE constants as

$$x_0 = D * A, \quad y_0 = D * B, \quad z_0 = D * C. \tag{49}$$

Substituting Eqs. (46), (47), and (48) into Eq. (27), followed by regrouping, creates a set of $p_{i,k}$ (i, k = 1, 2, 3, 4) coefficients similiar in positional definition to the $a_{i,k}$ series of Eq. (27), which are defined as follows:

$$\begin{aligned} p_{11} = \big(&a_{11} h_{11}^2 + a_{22} h_{21}^2 + a_{33} h_{31}^2 + 2a_{12} h_{11} h_{21} \\ &+ 2a_{13} h_{11} h_{31} + 2a_{23} h_{21} h_{31}\big) \end{aligned} \tag{50}$$

$$\begin{aligned} p_{12} = (&a_{11} h_{11} h_{12} + a_{22} h_{21} h_{22} + a_{33} h_{31} h_{32} \\ &+ a_{12} (h_{11} h_{22} + h_{21} h_{12}) + a_{13} (h_{11} h_{32} + h_{31} h_{12}) \\ &+ a_{23} (h_{21} h_{32} + h_{22} h_{31})) \end{aligned} \tag{51}$$

$$\begin{aligned} p_{13} = (&a_{11} h_{11} h_{13} + a_{22} h_{21} h_{23} + a_{33} h_{31} h_{33} + a_{12} (h_{11} h_{23} + h_{21} h_{13}) \\ &+ a_{13} (h_{11} h_{33} + h_{31} h_{13}) + a_{23} (h_{21} h_{33} + h_{23} h_{31})) \end{aligned} \tag{52}$$

$$\begin{aligned} p_{14} = (&a_{11} h_{11} x_0 + a_{22} h_{21} y_0 + a_{33} h_{31} z_0 + a_{12} (h_{21} x_0 + h_{11} y_0) \\ &+ a_{13} (h_{31} x_0 + h_{11} z_0) + a_{23} (h_{31} y_0 + h_{21} z_0) \\ &+ a_{14} h_{11} + a_{24} h_{21} + a_{34} h_{31}) \end{aligned} \tag{53}$$

$$p_{21} = p_{12} \tag{54}$$

$$\begin{aligned} p_{22} = \big(&a_{11} h_{12}^2 + a_{22} h_{22}^2 + a_{33} h_{32}^2 + 2a_{12} h_{12} h_{22} \\ &+ 2a_{13} h_{12} h_{32} + 2a_{23} h_{22} h_{32}\big) \end{aligned} \tag{55}$$

$$\begin{aligned} p_{23} = (&a_{11} h_{12} h_{13} + a_{22} h_{22} h_{23} + a_{33} h_{32} h_{33} \\ &+ a_{12} (h_{12} h_{23} + h_{22} h_{13}) + a_{13} (h_{12} h_{33} + h_{13} h_{32}) \\ &+ a_{23} (h_{22} h_{33} + h_{32} h_{23})) \end{aligned} \tag{56}$$

$$\begin{aligned} p_{24} = (&a_{11} h_{12} x_0 + a_{22} h_{22} y_0 + a_{33} h_{32} z_0 + a_{12} (h_{22} x_0 + h_{12} y_0) \\ &+ a_{13} (h_{32} x_0 + h_{12} z_0) + a_{23} (h_{32} y_0 + h_{22} z_0) \\ &+ a_{14} h_{12} + a_{24} h_{22} + a_{34} h_{32}) \end{aligned} \tag{57}$$

$$p_{31} = p_{13} \tag{58}$$

$$p_{32} = p_{23} \tag{59}$$

$$p_{33} = a_{11} h_{13}^2 + a_{22} h_{23}^2 + a_{33} h_{33}^2 + 2a_{12} h_{13} h_{23} + 2a_{13} h_{13} h_{33} + 2a_{23} h_{23} h_{33} \tag{60}$$

$$\begin{aligned} p_{34} = {} & (a_{11} h_{13} x_0 + a_{22} h_{23} y_0 + a_{33} h_{33} z_0 + a_{12} (h_{23} x_0 + h_{13} y_0) \\ & + a_{13} (h_{33} x_0 + h_{13} z_0) + a_{23} (h_{33} y_0 + h_{23} z_0) \\ & + a_{14} h_{13} + a_{24} h_{23} + a_{34} h_{33}) \end{aligned} \tag{61}$$

$$p_{41} = p_{14} \tag{62}$$

$$p_{42} = p_{24} \tag{63}$$

$$p_{43} = p_{34} \tag{64}$$

$$\begin{aligned} p_{44} = {} & a_{11} x_0^2 + a_{22} y_0^2 + a_{33} z_0^2 + 2a_{12} x_0 y_0 + 2a_{13} x_0 z_0 \\ & + 2a_{23} y_0 z_0 + 2a_{14} x_0 + 2a_{24} y_0 + 2a_{34} z_0 + a_{44} \end{aligned} \tag{65}$$

Simplification, Reduction to Two Variables

With the curve of intersection defined in the HOPE plane coordinate system, a simplification is achieved by eliminating all z_h terms, since by definition $z_h = 0$ in the HOPE plane. The result is a second-order equation in two variables of the form

$$(e_{11} x_h + e_{12} y_h + e_{13}) x_h + (e_{21} x_h + e_{22} y_h + e_{23}) y_h + (e_{31} x_h + e_{32} y_h + e_{33}) = 0 \tag{66}$$

The following relationships exist between the $e_{i,j}$ $(i, j = 1, 2, 3)$ and the $p_{i,j}$ $(i, j = 1, 2, 3, 4)$ coefficients:

$$\begin{array}{lll} e_{11} = p_{11} & e_{21} = p_{21} & e_{31} = p_{41} \\ e_{12} = p_{12} & e_{22} = p_{22} & e_{32} = p_{42} \\ e_{13} = p_{14} & e_{23} = p_{24} & e_{33} = p_{44} \end{array} \tag{67}$$

Analysis Using Invariants

The computer is now asked to interpret the intersection. Is it a circle or an ellipse? This analysis is accomplished by use of the invariants of a plane second-order curve. These three invariants are defined as follows:

$$w_1 = e_{11} + e_{12}, \tag{68}$$

$$w_2 = \begin{vmatrix} e_{11} & e_{12} \\ e_{21} & e_{22} \end{vmatrix}, \tag{69}$$

$$w_3 = \begin{vmatrix} e_{11} & e_{12} & e_{13} \\ e_{21} & e_{22} & e_{23} \\ e_{31} & e_{32} & e_{33} \end{vmatrix} . \tag{70}$$

If $w_3 = 0$, a proper conic section is *not* present. This means that the HOPE plane is parallel to the axis of the cylinder. Three possible situations can exist for a parallel plane. (a) If the plane is parallel and also outside the cylinder wall there is *no* intersection *at all*. (b) If the plane is tangent to the cylinder wall the intersection is a single, straight line. (c) If the parallel plane is inside the cylinder wall, the intersection consists of two parallel lines. The equations for the parallel case are developed in the fifth section of this report.

The intersection is an ellipse if the following conditions are true:

$$1) \quad w_3 \neq 0 , \tag{71}$$

$$2) \quad w_2 > 0 , \tag{72}$$

$$3) \quad w_3/w_1 < 0 . \tag{73}$$

A circle is present if, in addition to the above three conditions, the following two conditions are also true:

$$4) \quad e_{11} = e_{22} , \tag{74}$$

$$5) \quad e_{12} = 0 . \tag{75}$$

The center (c_x, c_y) of either a circle or an ellipse of intersection can be determined in an identical manner from Eqs. (76) and (77):

$$c_x = -(e_{13}\, e_{22} - e_{12}\, e_{23})/w_2 , \tag{76}$$

$$c_y = -(e_{11}\, e_{23} - e_{13}\, e_{21})/w_2 . \tag{77}$$

The invariants w_1 and w_2 can be used to form a second-degree equation, called the characteristic equation, using the symbol λ as the variable. This is shown in Eq. (78):

$$\lambda^2 - w_1 \lambda + w_2 = 0 . \tag{78}$$

The roots of the characteristic equation, λ_1 and λ_2, are called the eigenvalues of the $e_{i,j}$ matrix. If the roots are equal the presence of a circle is confirmed and its radius r is computed as

$$r = \sqrt{\frac{-w_3}{\lambda_1^3}} . \tag{79}$$

If $\lambda_1 > \lambda_2$ the presence of an ellipse is confirmed. Its constants are computed as follows:

Semimajor axis $$a = \sqrt{\frac{-w_3}{\lambda_1\, \lambda_2^2}} \tag{80}$$

Semiminor axis
$$b = \sqrt{\frac{-w_3}{\lambda_1^2 \lambda_2}} \tag{81}$$

Eccentricity
$$e = \sqrt{1 - \frac{b^2}{a^2}}. \tag{82}$$

Elliptical Rotation

Another constant related to the ellipse which it is important to know is the amount of rotation (if any) from a horizontal position the major axis exhibits when it is plotted. This quantity is represented by ϕ. Now rotation other than 0° and 90° is present if the xy term (i.e., coefficient e_{12}) is nonzero. To distinguish between 0° and 90° when e_{12} is zero, set $x = c_x$ and solve $e_{22} y^2 + 2e_{23} y + (e_{33} + e_{11} c_x^2 + 2e_{23} c_x) = 0$ for y. The solution to this quadratic should be a perfect square with the two roots y_1 and y_2 equal to each other in magnitude. If such is not the case an error exists. If $y_1 = y_2 = b$, then $\phi = 0°$. If $y_1 = y_2 = a$, then $\phi = 90°$.

The general solution for the rotation involves an arc tangent and has two possible solutions in the range 0° to 360°, the first of which is identified as ϕ_1 and is defined as

$$\phi_1 = \frac{1}{2} \arctan \left(\frac{2 * e_{12}}{e_{11} - e_{22}} \right). \tag{83}$$

Assuming that the arc tangent function returns the principal value, then the second possible solution, identified as ϕ_2, is defined as

$$\phi_2 = \phi_1 + 90°. \tag{84}$$

A special case must be accounted for if $(e_{11} - e_{22}) = 0$; then Eq. (83) is unsolvable. In such a case $\phi_1 = 45°$.

It is possible to select the proper rotation between ϕ_1 and ϕ_2 by a series of tests. If the xy term is negative ($e_{12} < 0$), then the major axis of the ellipse lies in the first and third quadrants. Conversely, if the xy term is positive ($e_{12} > 0$), the major axis lies in the second and fourth quadrants. Since the arc tangent function returns a principle value $-45° \leq \phi_1 \leq +45°$, and because of Eq. (84), $+45° \leq \phi_2 \leq 135°$. Therefore one of four possible situations exists;

if $e_{12} < 0$ and $\phi_1 > 0$ then $\phi = \phi_1$, or

if $e_{12} < 0$ and $\phi_2 < 90°$ then $\phi = \phi_2$, or

if $e_{12} > 0$ and $\phi_1 < 0$ then $\phi = \phi_1$, or

if $e_{12} > 0$ and $\phi_2 > 90°$ then $\phi = \phi_2$.

Discrete Set of Points

In order for a plot to be made discrete, perimeter points must first be computed. These perimeter point computations are conveniently made using the intersection curve constants relative to the center.

For a circle of intersection,

$$x = r \cos \theta,$$

and (85)

$$y = r \sin \theta,$$

where r is obtained from Eq. (79) and θ, the polar angle, is illustrated in Fig. 3.

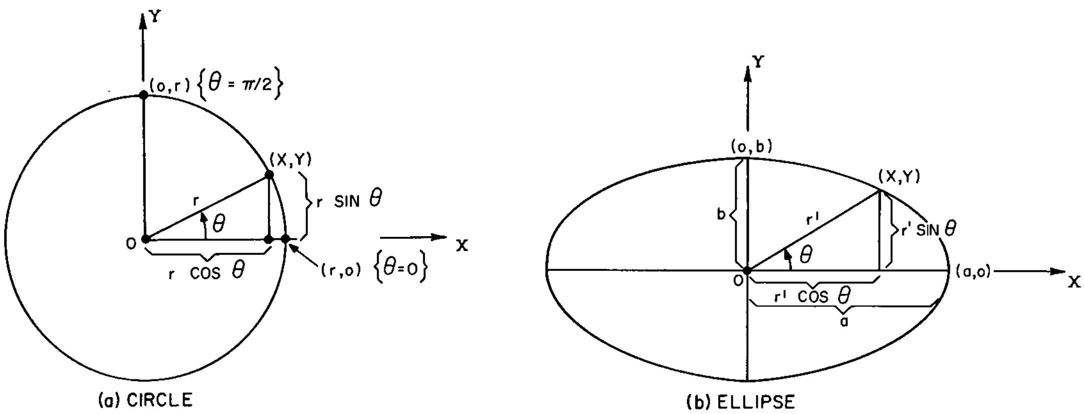


Fig. 3 - Angle of rotation θ used in computation of perimeter points

For an ellipse of intersection,

$$x = r' \cos \theta$$

$$y = r' \sin \theta, \tag{86}$$

where

$$r' = \sqrt{\frac{b^2}{1 - e^2 \cos \theta}}, \tag{87}$$

and the semimajor axis b is computed from Eq. (81), the eccentricity e from Eq. (82), and the polar angle θ is illustrated in Fig. 3b.

When the discrete points for the ellipse are computed in this fashion, the possible rotation of the ellipse has not been taken into account. Therefore if $\phi \neq 0$, transformation is needed. The rotational transformation is defined as follows:

$$x_\phi = x \cos \bar{\phi} + y \sin \bar{\phi},$$

$$y_\phi = -x \sin \bar{\phi} + y \cos \bar{\phi}, \tag{88}$$

where

(x, y) are the values of the point before applying the rotation,

(x_ϕ, y_ϕ) are the values of the point after applying the rotation, and $\bar{\phi} = -\phi$.

Translation of the set of discrete points for both the circle and ellipse intersection is accomplished by

$$x \leftarrow x + c_x ,$$

$$y \leftarrow y + c_y . \tag{89}$$

After the necessary translation and rotation the plot of the circle or ellipse will be properly oriented with other intersections in the HOPE plane. (There is, of course, no rotation needed for the case of a circle of intersection.) This completes the equation development for the situation wherein the HOPE plane intersects a cylinder which is considered to be infinite in extent, so that the curve of intersection is either a circle or a complete ellipse.

INTERSECTION BETWEEN HOPE PLANE AND BOUNDED CYLINDER

So far the development has created a cylinder coordinate system--twelve possible ways--, identified the curve of intersection between a cylinder and the HOPE plane--circle or ellipse--, and computed for plotting purposes a discrete set of points lying on the curve's perimeter. For these considerations the cylinder has been mathematically infinite in extent, so that any elliptical intersection is a complete ellipse.

Consider now the situation where the cylinder is finite in length, made so by two parallel planes, ℓ units apart. These planes are called individually the base plane and the top plane, but jointly referred to as the bounding planes, or if considering a general case a single plane is called a bounding plane. Thus, for example, it can be said that both bounding planes are perpendicular to the axis of the cylinder. The finite cylinder is in effect then a right circular cylinder, the only type (other than infinite) being considered in this report, thereby ruling out such surfaces as a tabulated cylinder.

An ellipse of intersection can be truncated by either, or both, bounding planes. A truncation line is defined as part of the line of intersection between the HOPE plane and a bounding plane. A line of truncation is parallel to the minor axis of the ellipse and lies within the ellipse, except for its two end points that lie on the perimeter like the discrete plotting points. A line of truncation is illustrated in Fig. 4.

For computing purposes it is necessary to have a test algorithm for the existence or nonexistence of a line of truncation when an ellipse of intersection is present. This is done in two steps. First, the existence of all or part of the ellipse *between* the planes is decided upon, and second, having confirmed that they exist, we compute the end point values of the truncation line or lines as needed.

The Ellipse and the Bounding Planes

In considering the question of whether or not at least a part of the ellipse lies between the bounding planes it is convenient to deal with one angle and two distances. The angle θ is the angle between the HOPE plane and a bounding plane. Excluding 0°, which is the special case of a circle when the HOPE plane is parallel to the bounding planes, there is *no* angular restriction. The plane of the ellipse can have *any* inclination to the bounding planes and still lie between them. It can also lie outside them. This is only the start of the test. It merely says that if θ exists and is not 0°, the possibility of the ellipse lying between the bounding planes exists because the ellipse exists somewhere in the infinite cylinder. The existence of an ellipse is confirmed.

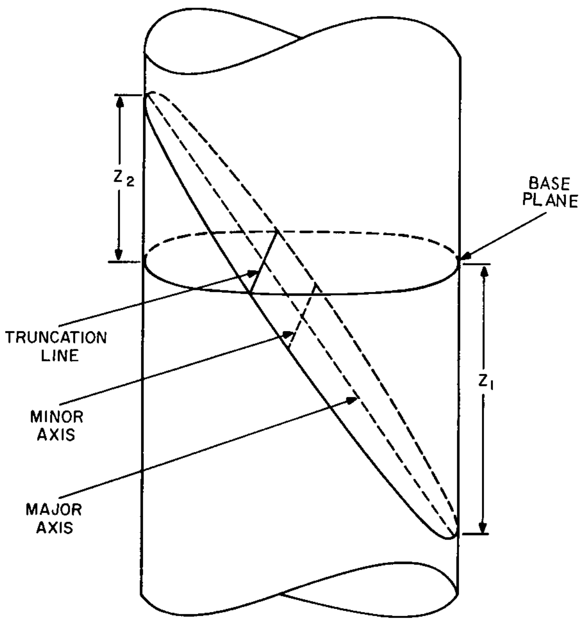


Figure 4 - Truncation line parallel to minor axis, and the major axis endpoint distances from the base plane, z_1 and z_2

The equation for θ is

$$\theta = \arccos\,(A\,u_x + B\,u_y + C\,u_z)\,. \tag{90}$$

In this equation A, B, C are the direction cosines of the HOPE plane and u_x, u_y, u_z are the direction cosines of the bounding planes (as well as being the axis vector components).

The two distances involved are the distances of the ends of the ellipse's major axis from the base plane. In the *cylinder's* coordinate system let these two distances be z_1 and z_2, corresponding to the positive and negative ends, respectively, of the major axis when the ellipse is considered centralized about the origin. One of six possible situations can then exist:

1. $z_1 < 0$ and $z_2 < 0$ — ellipse lies *below* the finite cylinder, or
2. $z_1 > \ell > 0$ and $z_2 > \ell > 0$ — ellipse lies *above* the finite cylinder, or
3. $0 < z_1 < \ell$ and $0 < z_2 < \ell$ — ellipse lies entirely *within* the finite cylinder without truncation, or
4. (a) $z_1 < 0$ and $0 < z_2 < \ell$ or (b) $z_2 < 0$ and $0 < z_1 < \ell$ — *only* the *base plane* is involved in ellipse truncation, or

5. $\left\{\begin{array}{l}\text{(a) } 0 < z_1 < \ell \text{ and } z_2 > \ell \\ \text{or} \\ \text{(b) } 0 < z_2 < \ell \text{ and } z_1 > \ell\end{array}\right\}$ *only* the *top plane* is involved in the ellipse truncation, or

6. $\left\{\begin{array}{l}\text{(a) } z_1 < 0 \text{ and } z_2 > \ell \\ \text{or} \\ \text{(b) } z_2 < 0 \text{ and } z_1 > \ell\end{array}\right\}$ *both* the *base plane* and the *top plane* are involved in ellipse truncation.

Figure 4 illustrates a 4(a) condition for z_1 and z_2.

The first test then involves the angle between the HOPE plane and the base plane and the distances from the endpoints of the major axis to the base plane and determines exactly if truncation is involved once, or twice, or not at all.

Truncation Line Computation

The second test is considered only if the first test is true, that at least a part of the ellipse exists between the bounding planes. If the line of intersection between the HOPE plane and a bounding plane intersects the ellipse in two points, then a truncation line exists. The second test culminates, specifically, in seeking two unequal roots to a quadratic equation. The base plane situation will be considered in detail, and the similarities for the top plane merely touched on.

The base plane equation is

$$u_x x + u_y y + u_z z = u_x x_c + u_y y_c + u_z z_c . \tag{91}$$

The HOPE plane equation is

$$Ax + By + Cz = D. \tag{92}$$

Now since both of these equations are in the XYZ coordinate system, the intersection between them is also in this system and is

$$(A - u_x) x + (B - u_y) y + (C - u_z) z = [D - (u_x x_c + u_y y_c + u_z z_c)] . \tag{93}$$

Let

$$P = A - u_x, \quad Q = B - u_y, \quad R = C - u_z ,$$
$$S = [D - (u_x x_c + u_y y_c + u_z z_c)] . \tag{94}$$

Then

$$Px + Qy + Rz = S. \tag{95}$$

Now the conversion equations from the XYZ system to the HOPE plane coordinate system are

$$
\begin{aligned}
x &= t_{11} x' + t_{12} y' + t_{13} z' + x_0 ,\\
y &= t_{21} x' + t_{22} y' + t_{23} z' + y_0 ,\\
z &= t_{31} x' + t_{32} y' + t_{33} z' + z_0 ,
\end{aligned} \tag{96}
$$

where (x_0, y_0, z_0) is the origin of the HOPE system at (DA, DB, DC). Substituting (96) into (95) gives

$$(Pt_{11} + Qt_{21} + Rt_{31})\, x' + (Pt_{12} + Qt_{22} + Rt_{32})\, y' + (Pt_{13} + Qt_{23} + Rt_{33})\, z' = S - (x_0 P + y_0 Q + z_0 R) . \tag{97}$$

Let

$$
\begin{aligned}
P' &= (P\, t_{11} + Q\, t_{21} + R\, t_{31}) ,\\
Q' &= (P\, t_{12} + Q\, t_{22} + R\, t_{32}) ,\\
R' &= (P\, t_{13} + Q\, t_{23} + R\, t_{33}) ,\\
S' &= S - (x_0 P + y_0 Q + z_0 R) .
\end{aligned} \tag{98}
$$

Then

$$P' x' + Q' y' + R' z' = S' . \tag{99}$$

Equation (99) is the line of intersection between the HOPE plane and the base plane expressed in HOPE plane coordinates. Setting $z' = 0$ and solving for x',

$$x' = \left(\frac{-Q'}{P'}\right) y' + \frac{S'}{P'} ,$$

or

$$x' = m y' + b , \tag{100}$$

where

$$m = \left(\frac{-Q'}{P'}\right) , \qquad b = \frac{S'}{P'} . \tag{101}$$

Now the ellipse can be defined in the HOPE system as

$$e_{11} x'^2 + 2e_{12} x' y' + e_{22} y'^2 + 2e_{13} x' + 2e_{23} y' + e_{33} = 0 \tag{102}$$

Substituting (100) for x' in the ellipse Eq. (102) and gathering terms gives a quadratic in y',

$$a' y'^2 + 2b' y + c' = 0 , \tag{103}$$

where

$$
\begin{aligned}
a' &= e_{11} m^2 + 2e_{12} m + e_{22} ,\\
b' &= e_{11} m b + e_{12} b + e_{13} m + e_{23} ,\\
c' &= e_{11} b^2 + e_{33} + 2e_{13} b .
\end{aligned} \tag{104}
$$

If the two roots of Eq. (103) are y_1' and y_2', then

$$x_1' = m y_1' + b$$

and (105)

$$x_2' = m y_2' + b,$$

and the two points on the circumference of the ellipse defining the truncation line in the base plane are (x_1', y_1') and (x_2', y_2').

The situation for the top plane is similar. In this case the bounding plane equation is

$$u_x x + u_y y + u_z z = u_x t_x + u_y t_y + u_z t_z, \tag{106}$$

where t_x, t_y, and t_z are defined as

$$\begin{aligned} t_x &= x_c + u_x \times \ell, \\ t_y &= y_c + u_y \times \ell, \\ t_z &= z_c + u_z \times \ell, \end{aligned} \tag{107}$$

resulting in two points defining the truncation line for that plane.

The second test, then, is determining if two real and unequal roots for y' exist in Eq. (102), first for the base plane and then for the top plane; and, if successful, the test results in the computation of the HOPE plane coordinates (x', y') of the endpoints of the truncation line on the perimeter of the ellipse.

It then remains merely to define the proper subset of the discrete points which along with the truncation line define the bounded intersection. An $x'y'$ view of a truncated ellipse is shown in Fig. 6, and a perspective view in Fig. 5. The algorithm that selects the subset of points must take several facts into account. The most important of these are

1. The center of the ellipse does not necessarily coincide with the origin of the HOPE system.

2. The truncation line, while parallel to the minor axis, can lie on either side of the center of the ellipse.

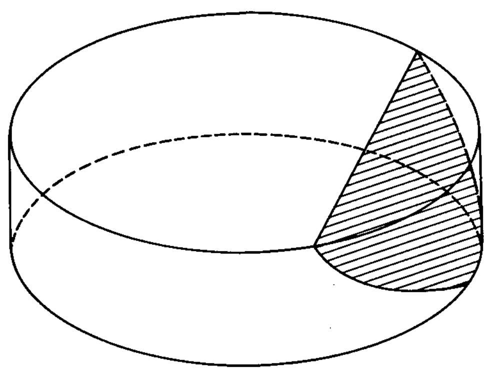


Fig. 5 - Perspective view of an ellipse of intersection truncated by a top bounding plane

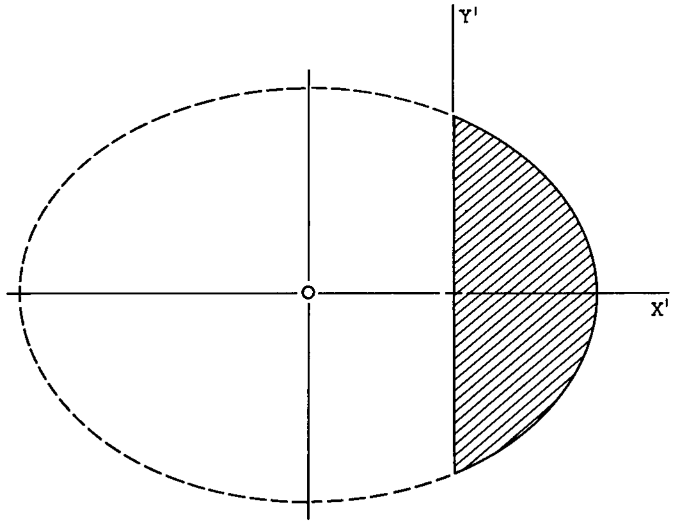


Fig. 6 - HOPE plane view of an ellipse truncated by a top bounding plane

3. Depending on which bounding plane is being considered, different criteria exist for accepting or rejecting points as the scan of the discrete set of points is made.

4. The relative locations of the starting point of the discrete set of points and the points (x_1', y_1'), (x_2', y_2').

This concludes the discussion relevant to the truncation of an elliptical intersection resulting from the bounding planes of a finite cylinder.

HOPE PLANE PARALLEL TO CYLINDER AXIS

The case of the HOPE plane intersecting both an infinite and a finite cylinder has been presented. Remaining to be considered is a special case, the situation wherein the HOPE plane is parallel to a bounded cylinder axis but close enough to intersect the cylinder. Under these circumstances the intersection becomes a single line or a pair of parallel lines. Such solutions are termed degenerate.

It is rewarding in this situation to consider the vector diagram of Fig. 7 because it provides a spatial picture. With the origin at 0 and the cylinder axis point at (x_c, y_c, z_c), vector $\mathbf{a}$ is defined between these two points. The axis of a *finite* cylinder is defined in space by the vector $\boldsymbol{\ell}$, and vector $\mathbf{b}$ joins the origin with the tip of $\boldsymbol{\ell}$. These vectors will be used as the discussion continues.

Distance from Cylinder Axis Point to HOPE Plane

The perperdicular distance from the cylinder axis point to the HOPE plane is involved in the derivation of the equations in this section. This distance is identified by d_0 (see Fig. 7) and is found by

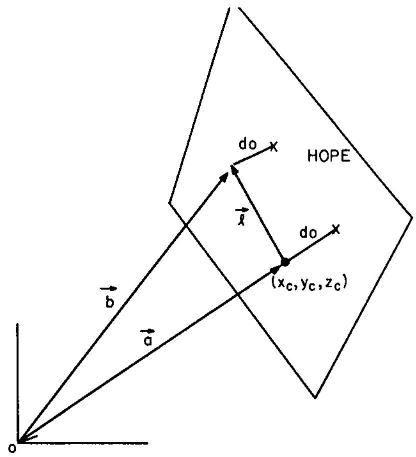


Fig. 7 - Showing d_0, the perpedicular distance from cylinder axis ℓ to the HOPE plane, and the vector **a** and **b** from the origin 0 to the base and tip of ℓ, respectively.

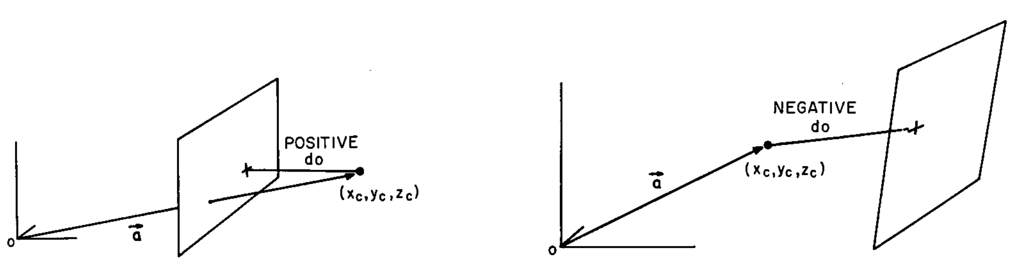
$$d_0 = \frac{A\,x_c + B\,y_c + C\,z_c - D}{(A^2 + B^2 + C^2)^{1/2}}, \tag{108}$$

where

> A, B, C, D are the HOPE plane constants, and (x_c, y_c, z_c) are the coordinates of the cylinder axis point.

The sign of the denominator's square root in Eq. (108) is chosen opposite to the sign of D. For example, if D is positive the negative square root is used. This establishes the convention in that if d_0 is *positive*, the HOPE plane lies *between* the origin and the cylinder axis point; and that if d_0 is *negative*, the HOPE plane lies *beyond* the cylinder axis point. This sign convention for d_0 is shown in Fig. 8a and 8b.

A line connecting the low point, (see below) with the high point, (see below) is parallel to the axis of the cylinder partly defined by the axis point (x_c, y_c, z_c) and the axis vector ℓ (missing and not needed in this context is the radius r). The same line connecting the low point with the high point lies in the HOPE plane since both points lie in the HOPE plane. Therefore, it follows that the low point and the high point can be used to define the termination points for the line, or lines, of intersection obtained in the degenerate solutions for finite cylinders. The actual equation of the line of intersection is obtained from an analysis of the "e" set of coefficients obtained in the manner of the third section of this report. Once the analysis shows that a degenerate form is present, then it is safe to assume that the low point(s) and the high point(s) are on the line(s). These degenerate



(a) (b)

Fig. 8 - Sign convention on d_0

line forms will be discussed next. First to be considered are the horizontal and vertical single line and parallel line forms, and finally the situation when the lines are not orthogonally oriented with respect to the HOPE plane axes.

The Low Point and the High Point

Since d_0 is perpendicular to the HOPE plane, its direction cosines are A, B, and C. Its axis-oriented components are d_0 A, d_0 B, d_0 C. A vector $\mathbf{d}_0$ can therefore be defined from the axis point. The tip of $\mathbf{d}_0$ lies in the HOPE plane and is called the low point. In general terms, for XYZ coordinates, the low point is identified by the subscript "lo," thus (x_{lo}, y_{lo}, z_{lo}). These low points coordinates are computed as follows:

$$
\begin{aligned}
x_{lo} &= x_c - d_0 A, \\
y_{lo} &= y_c - d_0 B, \\
z_{lo} &= z_c - d_0 C.
\end{aligned}
\tag{109}
$$

In HOPE plane coordinates the low point is (x'_{lo}, y'_{lo}) and is obtained by passing (x_{lo}, y_{lo}, z_{lo}) through a conversion matrix.

There is a corresponding point in the top bounding plane called the high point, identified by the subscript "hi," viz., (x_{hi}, y_{hi}, z_{hi}), and is located by erecting $\boldsymbol{\ell}$ at the low point. The components of the high point are therefore,

$$
\begin{aligned}
x_{hi} &= (x_c + \ell u_x) - d_0 A, \\
y_{hi} &= (y_c + \ell u_y) - d_0 B, \\
z_{hi} &= (z_c + \ell u_z) - d_0 C,
\end{aligned}
\tag{110}
$$

where ℓ is the length of $\boldsymbol{\ell}$ and (u_x, u_y, u_z) are the direction cosines of $\boldsymbol{\ell}$. In HOPE plane coordinates the high point is identified as (x'_{hi}, y'_{hi}).

Single Lines

When invariants w_2 and w_3 (see Eqs. (69) and (70) are both zero, the curve of intersection between the HOPE plane and the cylinder is *not* a circle or an ellipse, but some degenerate form. Of interest are those forms resulting in a single (tangent) line or a pair of parallel lines. An additional invariant is needed for the description of the degenerate cases. This one, w_4, is defined as

$$
w_4 = \begin{vmatrix} e_{22} & e_{23} \\ e_{32} & e_{33} \end{vmatrix} + \begin{vmatrix} e_{11} & e_{13} \\ e_{31} & e_{33} \end{vmatrix}
\tag{111}
$$

Consider first the simple situation of

$$
x'^2 = 0,
\tag{112}
$$

whence it develops that $e_{11} = 1$ (all other coefficients = 0) and $w_2 = 0$, $w_3 = 0$, $w_4 = 0$. Figure 9 shows the line extending along the HOPE Y′ axis from $(0, y'_{lo})$ to $(0, y'_{hi})$. In this figure the normal from the origin to the HOPE plane is D units in length.

If the form of the equation is

$$(x' - h)^2 = 0, \tag{113}$$

or in expanded form,

$$(x'^2 - 2hx + h^2); \tag{114}$$

then

$$e_{11} = 1, \quad e_{13} = -h = e_{31}, \quad e_{33} = h^2, \quad \text{(all others = 0)}.$$

The line extends from $(-e_{13}, y'_{lo})$ to $(-e_{13}, y'_{hi})$. Computation will show that it is still true that

$$w_2 = 0, \quad w_3 = 0, \quad w_4 = 0.$$

A comparable situation exists when

$$y'^2 = 0, \tag{115}$$

in which case,

the coefficients are

$$e_{22} = 1, \quad \text{(all others = 0)},$$

the invariants are

$$w_2 = 0, \quad w_3 = 0, \quad w_4 = 0, \text{ and}$$

the HOPE plane contains the line

from $(x'_{lo}, 0)$ to $(x'_{hi}, 0)$.

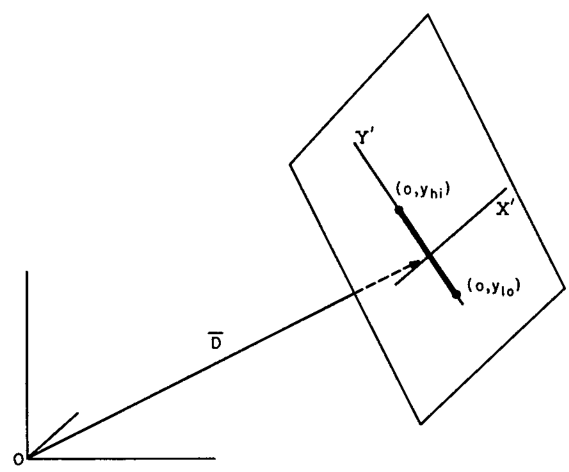


Fig. 9. $x'^2 = 0$

Also, when

$$(y' - k)^2 = 0 \tag{116}$$

or

$$y'^2 - 2ky' + k^2 = 0, \tag{117}$$

then

the coefficients are

$$e_{22} = 1, \quad e_{23} = -k = e_{32}, \quad e_{33} = k^2, \quad (\text{all others} = 0),$$

the invariants are

$$w_2 = 0, \quad w_3 = 0, \quad w_4 = 0,$$

and the HOPE plane contains the line

from $(x'_{1o}, -e_{23})$ to $(x'_{hi}\ -e_{23})$.

Parallel Lines

If the equation for the intersection is

$$x'^2 = m^2, \tag{118}$$

then

$$e_{11} = 1, \quad e_{33} = -m^2, \quad \text{all others} = 0,$$

and it follows that

$$w_2 = 0, \quad w_3 = 0, \quad w_4 < 0.$$

Figure 10 shows two parallel lines of intersection for Eq. 118; one extends

from $(+\sqrt{|e_{33}|},\ y'_{1o})$ to $(+\sqrt{|e_{33}|},\ y'_{hi})$, and the other

from $(-\sqrt{|e_{33}|},\ y'_{1o})$ to $(-\sqrt{|e_{33}|},\ y'_{hi})$.

If the form of the intersection equation is

$$(x' - h)^2 = m^2, \tag{119}$$

then

$$e_{11} = 1, \quad e_{13} = -h = e_{31}, \quad e_{33} = (h^2 - m^2), \quad (\text{all others} = 0),$$

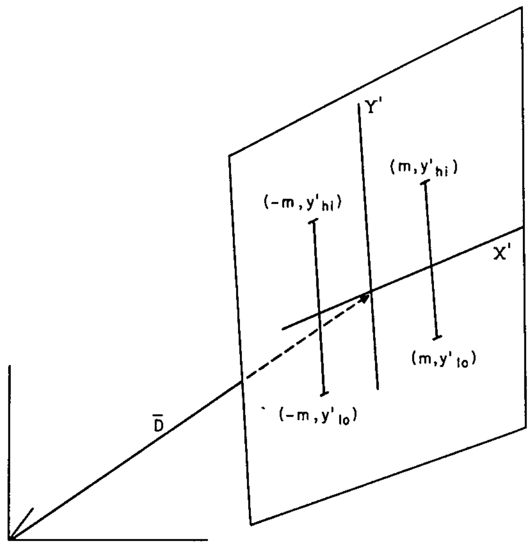


Fig. 10. $x'^2 = m^2$

and

$$w_2 = 0, \quad w_3 = 0, \quad w_4 < 0.$$

The two roots for x' are

$$x_1' = +\sqrt{|e_{13}|^2 - e_{33}} + (-e_{13}) = m + h \tag{120}$$

$$x_2' = -\sqrt{|e_{13}|^2 - e_{33}} + (-e_{13}) = -m + h. \tag{121}$$

The two parallel lines extend

from (x_1', y_{1o}') to (x_1', y_{hi}') and

from (x_2', y_{1o}') to (x_2', y_{hi}').

A perspective view of this situation is shown in Fig. 11.

Considering y' variable. When

$$y'^2 = n^2 \tag{122}$$

the coefficients are

$$e_{22} = 1, \quad e_{33} = -n^2, \quad (\text{all others} = 0),$$

which causes the invariants to be

$$w_2 = 0, \quad w_3 = 0, \quad w_4 < 0,$$

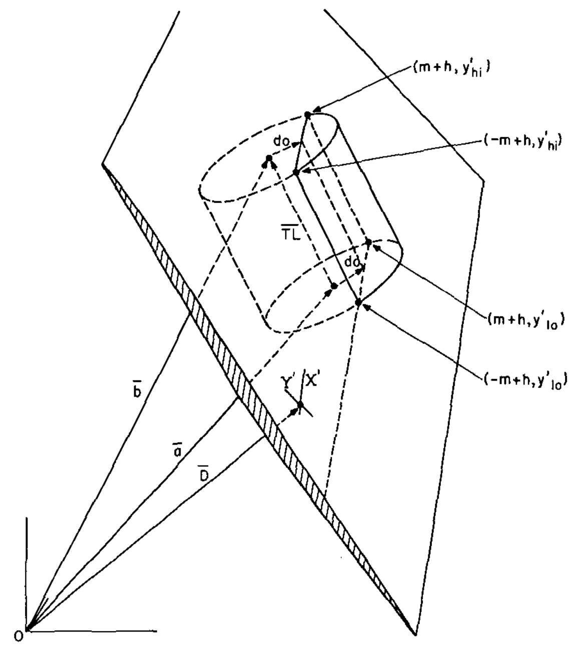


Fig. 11 - Perspective view of the $(x' - h)^2 = m^2$ solution with vectors **a** and **b** from the origin to the ends of the cylinder axis, **TL** and **D**, normal vector to the HOPE plane

and the two lines to extend

from $(x'_{lo}, +\sqrt{|e_{33}|})$ to $(x'_{hi}, +\sqrt{|e_{33}|})$ and

from $(x'_{lo}, -\sqrt{|e_{33}|})$ to $(x'_{hi}, -\sqrt{|e_{33}|})$

Finally, if

$$(y' - k)^2 = n^2 \tag{123}$$

or

$$y'^2 - 2ky' + (k^2 - n^2) = 0, \tag{124}$$

then the coefficients are

$$e_{22} = 1, \quad e_{23} = -k = e_{32}, \quad e_{33} = (k^2 - n^2), \quad \text{(all others = 0)},$$

the invariants are

$$w_2 = 0, \quad w_3 = 0, \quad w_4 < 0,$$

the y' roots are

$$y_1' = +\left(\sqrt{|e_{23}|^2 - e_{33}}\right) + (-e_{23}) = n + k \tag{125}$$

$$y_2' = -\left(\sqrt{|e_{23}|^2 - e_{33}}\right) + (-e_{23}) = n + k, \tag{126}$$

and the two parallel lines extend

from (x'_{lo}, y'_1) to (x'_{hi}, y'_1) and

from (x'_{lo}, y'_2) to (x'_{hi}, y'_2).

The nine degenerate solutions are summarized in Table 2 which lists the equations, the coefficient values, and/or the endpoint coordinates of the lines for the horizontal and vertical line cases, as well as the generalized rotated case, which is discussed next.

Slopes Other Than 0° or 90°

Line intersection solutions (either single or parallel) that are neither horizontal or vertical are possible when the HOPE plane is parallel to the cylinder's axis. The X′ Y′ axes are not forced to be orthogonally oriented to these line solutions, and therefore these so-called rotated solutions are possible. An XY plane cross section through Cyl 10 of Table 1 is an example of such an intersection.

If $w_2 = 0$ and $w_3 = 0$, a degenerate solution is present. In addition, if the coefficient of the $x'y'$ term (e_{12}) is nonzero, then the line(s) has (have) a slope other than 0° or 90°. If $w_4 < 0$ the solution is a single line running from the high point to the low point. If $w_4 = 0$ the solution is a pair of parallel lines which have a slope equal to the slope of the high-low point line, and whose y' intercepts can be determined by setting all x' terms in the intersection equation to zero and solving for y'. The resulting quadratic in y' is

$$e_{22}\, y'^2 + 2\, e_{23}\, y' + e_{33} = 0. \tag{127}$$

Let the two roots be b_1 and b_2; then the two parallel lines are

$$y' = (\tan \phi_a)\, x' + b_1 \tag{128}$$

and

$$y' = (\tan \phi_a)\, x' + b_2, \tag{129}$$

where

$$\phi_a = \tan^{-1}\left(\frac{y_{hi} - y_{lo}}{x_{hi} - x_{lo}}\right). \tag{130}$$

To plot the rectangle the end-plane intersection lines are needed. The slope of these lines is $\tan \phi_b$,

Table 2
Degenerate Line Forms of Plane-Cylinder Intersections

Equation	Coefficients									w_4	Line Coordinates
	e_{11}	e_{12}	e_{13}	e_{21}	e_{22}	e_{23}	e_{31}	e_{32}	e_{33}		
SINGLE LINE											
1) $x'^2 = 0$	1	0	0	0	0	0	0	0	0	0	$(0, y'_{lo})$ to $(0, y'_{hi})$
2) $(x'-h)^2=0$ $x^2-2hx'+h^2=0$	1	0	$-h$	0	0	0	$-h$	0	h^2	0	$(-e_{13}, y'_{lo})$ to $(-e_{13}, y'_{hi})$
3) $y'^2 = 0$	0	0	0	0	1	0	0	0	0	0	$(x'_{lo}, 0)$ to $(x'_{hi}, 0)$
4) $(y'-k)^2$ $y'^2 2ky'+k^2=0$	0	0	0	0	1	$-k$	0	$-k$	k^2	0	$(x'_{lo}, -e_{23})$ to $(x'_{hi}, -e_{23})$
DOUBLE LINE											
5) $x'^2 = m^2$	1	0	0	0	0	0	0	0	$-m^2$	<0	$\left(+\sqrt{\lvert e_{33}\rvert}, y'_{lo}\right)$ to $\left(+\sqrt{\lvert e_{33}\rvert}, y'_{hi}\right)$ $\left(-\sqrt{\lvert e_{33}\rvert}, y'_{lo}\right)$ to $\left(-\sqrt{\lvert e_{33}\rvert}, y'_{hi}\right)$
6) $(x'-h)^2 = m^2$	1	0	$-h$	0	0	0	$-h$	0	h^2-m^2	<0	$x'_1 = +\sqrt{\lvert e_{13}\rvert^2 - e_{33}} + (-e_{13}) = m + h$ $x'_2 = -\sqrt{\lvert e_{13}\rvert^2 - e_{33}} + (-e_{13}) = -m + h$ $(x'_1; y'_{lo})$ to (x'_1, y'_{hi}) (x'_2, y'_{lo}) to (x'_2, y'_{hi})
7) $y'^2 = n^2$	0	0	0	0	1	0	0	0	$-n^2$	<0	$\left(x'_{lo}, +\sqrt{\lvert e_{33}\rvert}\right)$ to $\left(x'_{hi}, +\sqrt{\lvert e_{33}\rvert}\right)$ $\left(x'_{lo}, -\sqrt{\lvert e_{33}\rvert}\right)$ to $\left(x'_{hi}, -\sqrt{\lvert e_{33}\rvert}\right)$
8) $(y-k)^2=n^2$	0	0	0	0	1	$-k$	0	$-k$	k^2-n^2	<0	$y'_1 = +\sqrt{\lvert e_{23}\rvert^2 - e_{33}} + (-e_{23}) = n + k$ $y'_2 = -\sqrt{\lvert e_{23}\rvert^2 - e_{33}} + (-e_{23}) = -n + k$ (x'_{lo}, y'_1) to (x'_{hi}, y'_1) (x'_{lo}, y'_2) to (x'_{hi}, y'_2)
9) General Rotated Non-orthogonal All terms possible	*	0	*	0	*	*	*	*	*	≤0	Line equations $y' = (\tan \phi_a)\, x' + b_1$ $y' = (\tan \phi_a)\, x' + b_2$ where, b_1, b_2 are roots of $e_{22}\, y'^2 + 2\; e_{23}\, y' + e_{33} = 0$ $\phi_a = \tan^{-1} (y_{hi} - y_{lo})/(x_{hi} - x_{lo})$

NOTE: All nine forms also have invariants $w_2 = 0$ and $w_3 = 0$.
*Any value

where

$$\phi_b = \phi_a + \frac{\pi}{2}. \tag{131}$$

Let the y' intercepts of the end-plane intersection lines be b_3 and b_4; then using the high point and the low point,

$$b_3 = y_{hi} - \tan\phi_b \times x_{hi} \tag{132}$$

and

$$b_4 = y_{lo} - \tan\phi_b \times x_{lo}. \tag{133}$$

If we have

$$y' = x' \tan\phi_b + b_3 \tag{134}$$

and

$$y' = x' \tan\phi_b + b_4 \tag{135}$$

as equations of these lines, the coordinates of the corners of the rectangle for plotting purposes then work out to be

$$x_1 = (b_3 - b_1)/\text{den},\ \ y_1 = x_1 \tan\phi_a + b_1, \tag{136}$$

$$x_2 = (b_3 - b_2)/\text{den},\ \ y_2 = x_2 \tan\phi_a + b_2, \tag{137}$$

$$x_3 = (b_4 - b_2)/\text{den},\ \ y_3 = x_3 \tan\phi_a + b_2, \tag{138}$$

$$x_4 = (b_4 - b_1)/\text{den},\ \ y_4 = x_4 \tan\phi_a + b_1, \tag{139}$$

where

$$\text{den} = \tan\phi_a - \tan\phi_b$$

Figure (12) shows a general configuration for the terms used to define the rotated case.

This concludes the consideration of the degenerate line solutions for cylinder intersections when the cross-sectional plane is parallel to the cylinder's axis. The sphere intersections are next considered.

INTERSECTION BETWEEN HOPE PLANE AND SPHERE

Sphere Defined in XYZ System

If a sphere of radius r has an orthogonal coordinate system (X_s, Y_s, Z_s) whose origin is at the center of the sphere, its equation can be expressed as

$$x_s^2 + y_s^2 + z_s^2 = r^2. \tag{140}$$

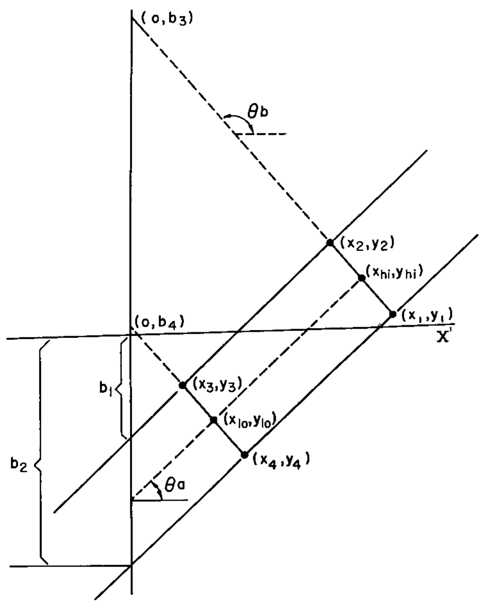


Fig. 12 - General configuration for rotated line cases showing a rectangle of intersection

Now if with respect the XYZ system the sphere's system has an

X_s axis with direction cosines t_{11}, t_{21}, t_{31}, a

Y_s axis with direction cosines t_{12}, t_{22}, t_{32}; and a

Z_s axis with direction cosines t_{13}, t_{32}, t_{33};

then, the conversion equations which allow for both translation and rotation are

$$x_s = t_{11} \cdot x + t_{21} \cdot y + t_{31} \cdot z - K_4, \tag{141}$$

$$y_s = t_{12} \cdot x + t_{22} \cdot y + t_{32} \cdot z - K_5, \tag{142}$$

$$z_s = t_{13} \cdot x + t_{23} \cdot y + t_{33} \cdot z - K_6, \tag{143}$$

where

$$K_4 = t_{11} \cdot x_{sp} + t_{21} \cdot y_{sp} + t_{31} \cdot z_{sp}, \tag{144}$$

$$K_5 = t_{12} \cdot x_{sp} + t_{22} \cdot y_{sp} + t_{32} \cdot z_{sp}, \tag{145}$$

$$K_6 = t_{13} \cdot x_{sp} + t_{23} \cdot y_{sp} + t_{33} \cdot z_{sp}, \tag{146}$$

with (x_{sp}, y_{sp}, z_{sp}) being the center of the sphere in XYZ coordinates. This point is part of the APT canonical form for a sphere along with the radius r. It should be noted that K_4, K_5, and K_6 are constants.

To express the sphere in the XYZ system it is necessary to square Eqs. (141), (142) and (143), substitute them into Eq. (140) and regroup the resulting terms. A convenient regrouping because of its symmetry is the following:

$$
\begin{aligned}
&(a_{11} \cdot x + a_{12} \cdot y + a_{13} \cdot z + a_{14})\, x \\
&+ (a_{21} \cdot x + a_{22} \cdot y + a_{23} \cdot z + a_{24})\, y \\
&+ (a_{31} \cdot x + a_{32} \cdot y + a_{33} \cdot z) + a_{34})\, z \\
&+ (a_{41} \cdot x + a_{42} \cdot y + a_{43} \cdot z + a_{44}) = 0. \qquad (147)
\end{aligned}
$$

This "a" series of coefficients is computed as follows:

$$a_{11} = t_{11}^2 + t_{12}^2 + t_{13}^2 \qquad (148)$$

$$a_{12} = t_{11} \cdot t_{21} + t_{12} \cdot t_{22} + t_{13} \cdot t_{23} \qquad (149)$$

$$a_{13} = t_{11} \cdot t_{31} + t_{12} \cdot t_{32} + t_{13} \cdot t_{33} \qquad (150)$$

$$a_{14} = -(K_4 \cdot t_{11} + K_5 \cdot t_{12} + K_6 \cdot t_{13}) \qquad (151)$$

$$a_{21} = a_{12} \qquad (152)$$

$$a_{22} = t_{21}^2 + t_{22}^2 + t_{23}^2 \qquad (153)$$

$$a_{23} = t_{21} \cdot t_{31} + t_{22} \cdot t_{32} + t_{23} \cdot t_{33} \qquad (154)$$

$$a_{24} = -(K_4 \cdot t_{21} + K_5 \cdot t_{22} + K_6 \cdot t_{23}) \qquad (155)$$

$$a_{31} = a_{13} \qquad (156)$$

$$a_{32} = a_{23} \qquad (157)$$

$$a_{33} = t_{31}^2 + t_{32}^2 + t_{33}^2 \qquad (158)$$

$$a_{34} = -(K_4 \cdot t_{31} + K_5 \cdot t_{32} + K_6 \cdot t_{33}) \qquad (159)$$

$$a_{41} = a_{14} \qquad (160)$$

$$a_{42} = a_{24} \qquad (161)$$

$$a_{43} = a_{34} \qquad (162)$$

$$a_{44} = K_4^2 + K_5^2 + K_6^2 - r^2. \qquad (163)$$

The HOPE plane in the XYZ stem is expressed as

$$A x + B y + C z - D = 0. \qquad (164)$$

The simultaneous solution of the HOPE plane and the sphere equation to get the intersection is accomplished by adding the first-degree coefficients and the constant terms. This operation can be expressed in the form of replacement statements as follows:

$$a_{14} \leftarrow a_{14} + A/2 \qquad (165)$$

$$a_{24} \leftarrow a_{24} + B/2 \qquad (166)$$

$$a_{34} \leftarrow a_{34} + C/2 \qquad (167)$$

$$a_{44} \leftarrow a_{44} - D. \qquad (168)$$

The remainder of the equation development for the sphere is identical to that already presented for the cylinder on p. 9.

Simplification

In practice, because of spherical symmetry, the direction cosines for the sphere's coordinate system are always chosen to be

$$\begin{bmatrix} t_{11} & t_{21} & t_{31} \\ t_{12} & t_{22} & t_{32} \\ t_{13} & t_{23} & t_{33} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}. \tag{169}$$

As a result, the following equation simplifications can be made.

From Eqs. (144) to (146):

$$K_4 = x_{sp}$$

$$K_5 = y_{sp}$$

$$K_6 = z_{sp}.$$

From Eqs. (141) to (143):

$$x_s = x - x_{sp}$$

$$y_s = y - y_{sp}$$

$$z_s = z - z_{sp}.$$

From Eqs. (148) to (151):

$$a_{11} = 1$$

$$a_{12} = 0$$

$$a_{13} = 0$$

$$a_{14} = -x_{sp}.$$

From Eqs. (153) to (155):

$$a_{22} = 1$$

$$a_{23} = 0$$

$$a_{24} = -y_{sp}.$$

From Eqs. (158) and (159):

$$a_{33} = 1$$

$$a_{34} = -z_{sp}.$$

From Eq. (163):

$$a_{44} = x_{sp}^2 + y_{sp}^2 + z_{sp}^2 - r^2 .$$

From Eqs. (165) to (168):

$$a_{14} \leftarrow -x_{sp} + A/2$$

$$a_{24} \leftarrow -y_{sp} + B/2$$

$$a_{34} \leftarrow -z_{sp} + C/2$$

$$a_{44} \leftarrow x_{sp}^2 + y_{sp}^2 + z_{sp}^2 - r^2 - D .$$

Or in more understandable form the sphere equation in the XYZ system is

$$(x - x_{sp})\,x + (y - y_{sp})\,y + (z - z_{sp})\,z + \left[-(x_{sp})\,x - (y_{sp})\,y - (z_{sp})\,z + x_{sp}^2 + y_{sp}^2 + z_{sp}^2 - r^2\right] = 0 \tag{170}$$

or

$$(x^2 - (2x_{sp}\,x) + (y^2 - 2y_{sp}\,y) + (z^2 - 2z_{sp}\,z) + \left(x_{sp}^2 + y_{sp}^2 + z_{sp}^2 - r^2\right) = 0 . \tag{171}$$

The curve of intersection in the XYZ system is

$$(x^2 - (2x_{sp} + A)\,x) + (y^2 - (2y_{sp} + B)\,y) + (z^2 - (2z_{sp} + C)\,z) + x_{sp}^2 + y_{sp}^2 + z_{sp}^2 - r^2 - D = 0 . \tag{172}$$

Example

If a sphere of radius 1, centered at ($x_{sp} = 0$, $y_{sp} = 0$, $z_{sp} = 0$), is intersected by the plane $x = 0$ (i. e., $A = 1$, $B = 0$, $C = 0$, $D = 0$), then the curve of intersection in the XYZ system, after coefficient and constant substitution, is

$$x^2 - x + y^2 + z^2 - 1 = 0 . \tag{173}$$

Since x does equal 0, the intersection is a circle in the yz plane;

$$y^2 + z^2 = 1 . \tag{174}$$

Test on Perpendicular Distance

A valuable quantity to know when dealing with sphere intersections is the perpendicular distance from the center to the HOPE plane. This distance, called d_{sp}, is computed as follows:

$$d_{sp} = \left| \frac{A\,x_{sp} + B\,y_{sp} + C\,z_{sp} - D}{(A^2 + B^2 + C^2)^{1/2}} \right| . \tag{175}$$

It is wise to make a test on the size of d_{sp} before proceeding to make computations on the sphere's intersection.

If $0 \leq d_{sp} < r$, a circle of intersection is present.

If $d_{sp} = r$, a point of intersection is present at $(x_{sp} + rA,\ y_{sp} + rB,\ z_{sp} + rC)$ in XYZ coordinates.

If $d_{sp} > r$, no intersection is present.

This concludes the presentation of the computational forms for cross sections of cylinders and spheres. Some illustrative examples of actual plots obtained by means of a computer program are next presented.

SOME PLOTTING EXAMPLES

A rather involved computer program, called CROSEC MOD 2.0, has been written as an extension of APT's Section I to implement a cross-sectional plotting capability through cylinders and spheres. This program which is described in Ref. 2, incorporates the previously written program for plotting cross sections through planes (Ref. 1) and considerable additional material. The output is in the form of CalComp or Gerber plots about a center, or origin, that is the tip of the normal vector from the XYZ origin to the cross-sectional (HOPE) plane. The HOPE plane origin is represented on the plots by a +, its axes are X′ and Y′, and the unit distance scale is given in each figure.

PARTNO TESTING, a simple APT program presented initially in Ref. 1 (Figs. 3 and 6, and Appendix C), is the subject of the first eight illustrations in this section. Two definitions have been added to the part program since it was introduced initially; these are a cylinder and a sphere:

CYL = CYLINDER/CANON, 3, 1, 0, 0, 0, 1, 1,

SPH = SPHERE/3, 1, 1, 1.

A listing for the augmented PARTNO TESTING is reproduced in Table 3, and a perspective view of the part is presented in Fig. 13. Eight planes form a step which has a rounded end formed by a cylinder. The sphere is inscribed inside the cylinder, resting on the XY plane. The base point for the cylinder is $(x = 3,\ y = 1,\ z = 0)$, and the center of the sphere is at $(x = 3,\ y = 1,\ z = 1)$. Both the sphere and the cylinder have a radius of 1. The cylinder is oriented vertically.

First a HOPE plane is passed in the XY plane $(z = 0)$. The resulting plot is shown in Fig. 14. The cylinder gives a circle of intersection, and the sphere a tangent point. The plane intersections result in two adjacent 2-in. squares. The circle is inscribed in the square on the right, which is formed by planes 4, 6, 7, 8. The next HOPE plane selected was $z = 0.5$, Fig. 15, parallel to the one just described and one-half unit above it. The one difference in the plot is that now the intersection with the sphere is no longer a tangent point but rather a circle of radius 0.87 which is concentric with the cylinder's circle of intersection.

Figure 16 shows the intersection with the XZ $(y = 0)$ plane. This plane is tangent to both the cylinder and the sphere. The cylinder's intersection is a vertical line segment five units long, from $(x' = 3,\ y' = 0)$ to $(x' = 3,\ y' = 5)$. The sphere gives a tangent point at $(x' = 3,\ y' = 1)$. The plane intersections are labeled. A similar situation for the cylinder and sphere intersections exists in Fig. 17 for the cross section plane $x = 4$. The cylinder intersection is again a line, and the sphere intersection a point.

The cross-sectional plane that created the plot for Fig. 18 is inclinded at 45° to both the X and Z axes while being parallel to the Y axis. The X intercept point is

Table 3
Augmented PARTNO TESTING

```
PARTNO   TESTING
      NOPOST
      CLPRNT
      SYN/P,POINT,C,CIRCLE,PL,PLANE
  P1      =P/0,0,0
  P2      =P/2,0,0
  P3      =P/2,2,0
  P4      =P/0,2,0
  P5      =P/0,0,2
  SETPT   =P/-5,-5,0
  SETPT2  =P/6,-1, 3
  PL1     =PL/P1,P2,P3
  PL2     =PL/1,0,0,0
  PL3     =PL/P5,PARLEL,PL1
  PL4     =PL/PARLEL,PL2,XLARGE,2
  PL5     =PL/PARLEL,PL1,ZLARGE,1
  PL6     =PL/0,1,0,0
  PL7     =PL/PARLEL,PL6,YLARGE,2
  PL8     =PL/PARLEL,PL2,XLARGE,4
  CYL     =CYLNDR/CANON,3,1,0,0,0,1,1
SPH=SPHERE/3,1,1,1
HOXYMIN=POINT/-5,-5,-5
HOXYMAX=POINT/10,10,10
PP1=POINT/1,0,0
PP2=POINT/0,1,0
PP3=POINT/0,0,1
HOPE1=PLANE/ 0,0,1,0
HOPE2=PLANE/0,0,1,0,5
HOPE3=PLANE/0,1,0,0,
HOPE4=PLANE/1,0,0,4
HOPE5=PLANE/0,7071068,0,0,7071068,2,8284271
HOPE6=PLANE/0,981,0,0,196,3.4335
HOPEN=PLANE/PP1,PP2,PP3
     FINI
```

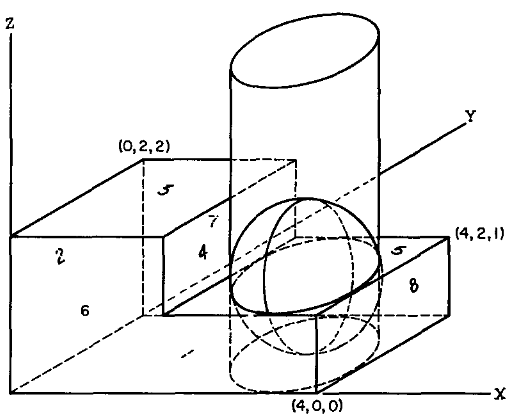


Fig. 13 - Augmented PARTNO TESTING

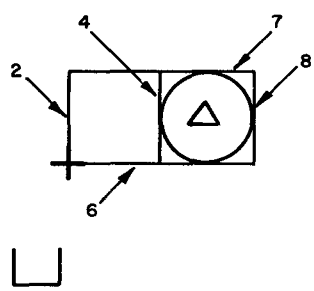


Fig. 14. z = 0

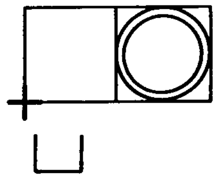


Fig. 15. $z = 0.5$

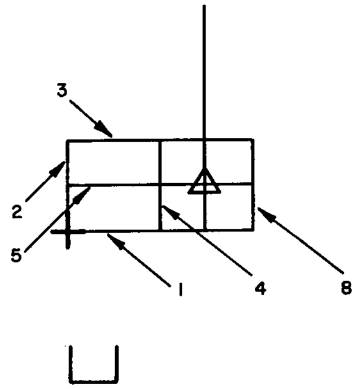


Fig. 16. $y = 0$

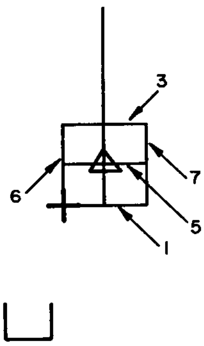


Fig. 17. $x = 4$

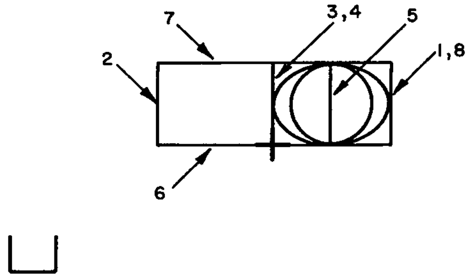


Fig. 18. $x \cos 45° + z \cos 45° = 2.828$

($x = 4$, $y = 0$, $z = 0$) and the Z intercept point is ($x = 0$, $y = 0$, $z = 4$). The plane intersections are identified and it is seen that the intersection for planes 3 and 4 are coincident. Planes 1 and 8 also have a coincident intersection. The cylinder's intersection is an ellipse bounded by planes 4, 6, 7, and 8. The semimajor axis is $\sqrt{2}$. The sphere's intersection is a circle of radius 1 because the HOPE plane passes through its center.

The plot of Fig. 19 is an interesting constrast with Fig. 18 because of the increased size of the ellipse and in the fact that the four planes to which it is tangent (4, 6, 7, 8) form the boundary of the external rectangle. The HOPE plane that produced this plot is again parallel to the Y axis, only this time inclined at 78.7° to X and 11.3° to Z. It has an X intercept point of ($x = 3.5$, $y = 0$, $z = 0$) and passes through the sphere's center. The intersections for planes 3 and 4 are not coincident; neither are those for planes 1 and 8. The intersection with plane 2 is not shown in Fig. 19 because it was beyond the bounds set by the HOXYMIN and HOXYMAX cards, namely.

HOXYMIN = POINT/-10, -10, -10,

HOXYMAN = POINT/10, 10, 10.

That is, plane 2 which is the YZ plane ($x = 0$) contains the Z axis. The Z axis intercept point in this case ($x = 0$, $y = 0$, $z = 17.5$) is beyond the plotting limits and was not plotted.

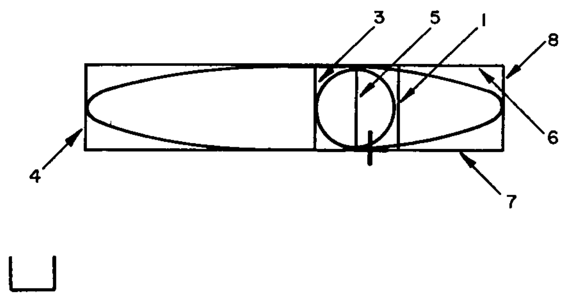


Fig. 19. x cos 78.7° + z cos 11.3° = 3.4335

The last plot related to PARTNO TESTING is Fig. 20. It is for the cross-sectional plane x + y + z = 1. This plane and its cross section with the eight planes is discussed in some detail in Ref. 1, pp. 7-10. This HOPE plane does not intersection the sphere, but does give an ellipse of intersection with the cylinder in addition to the previously discussed series of plane intersections. This ellipse is tangent to the lines of intersection from planes 4, 6, 7, and 8, with its center at HOPE dimension ($x' = 3.27$, $y' = 2.83$). It semimajor axis is 1.73, semiminor axis is 1.00, the eccentricity is 0.8165, and it is tilted at 60° with respect to the horizontal. This amount of detail is given to illustrate the output of the program and to emphasize the fact that the length of the semimajor axis of an ellipse of intersection with a cylinder is always equal to the radius of the cylinder being intersected.

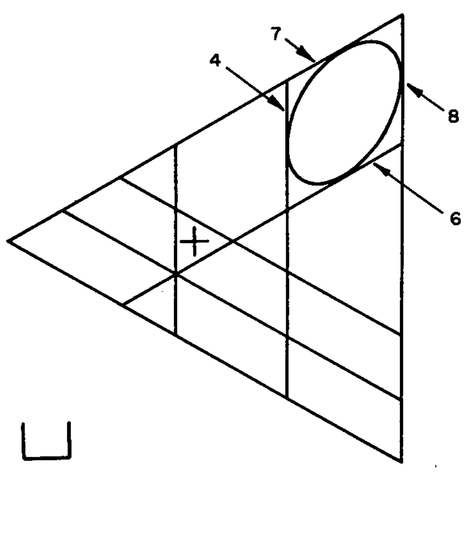


Fig. 20. x + y + z = 1

The next example of representative plots is given in Fig. 21, where the twelve sample cylinders of Table 1 are sliced by four cross-sectional planes to give a total of 48 plots. All twelve cylinders have a radius of 1. The point and vector sketches of Table 1 are helpful in getting a feeling for the space orientation of these twelve sample cylinders. The first six cylinders have *both* their axis points on, and their axis vectors along, one of the major axes, X, Y, or Z. Cylinders 7, 8, and 9 are characterized by having the axis point on one of the major axes and one vector component nonzero. The last three cylinders, 10, 11, and 12, have their axis points removed from a major axis and one nonzero vector component.

The cross-sectional planes, left to right, are the three major (principal) planes (XY, XZ, YZ) and the equianglar plane $x + y + z = 1$. Lettering the planes left to right as a, b, c, d, along with the number of the cylinder, allows a two-dimensional referencing to the plots, such as 5d, an example of a truncated ellipse, and 12c, which is an example of a single tangent line.

The a, b, and c columns of plots are readily understood because they represent the traditional views of mechanical drawing, namely front, plan, and end. Referring to the axes of a plot as X' and Y', for column a, $X' = X$ axis, $Y' = Y$ axis. For column b, $X' = X$ axis and $Y' = Z$ axis. For column c, $X' = X$ axis, $Y' = Z$ axis. The plotting axes for column d are assigned in the usual manner for CROSEC. The HOPE plane origin is on the tip of the normal vector to the plane, and the X' axis runs from the origin toward the X-axis intercept with the HOPE plane. For the plane $x + y + z = 1$, the origin is at (1/3, 1/3, 1/3) and the X axis intercept at (1, 0, 0). The X' axis starts at (1/3, 1/3, 1/3) and goes through (1, 0, 0), while the Y' axis has direction cosines $(0, \sqrt{1/2}, \sqrt{1/2})$, begins at (1/3, 1/3, 1/3), and is parallel to the line from (0, 0, 1) to (0, 1, 0). See Fig. 22 and Ref. 1, pp. 4-6. A given HOPE plane axes' orientations do not change in a given column series of 12.

Only one example of a truncated ellipse (5d) has been included in this set of 48 plots. The others have been suppressed for the sake of clarity.

CYL 12 breaks the expected pattern established in CYL 7 through CYL 11 by having a set of direction cosines for its vector identical to those of CYL 11 rather than CYL 9. The base point for CYL 12 is in the XZ plane at (1, 0, 1) and the vector is *parallel* to the YZ plane but inclined at an angle. This accounts for the single, sloping line of the plot of 12c.

There is a common scale for all the plots. The rectangles are all five units long. This arises from the arbitrary property that all cylinders are considered to have a length of five units in CROSEC.

To test his understanding of the cylinder types, let the reader answer this question. Of the twelve defined classes of cylinders, what class would a cylinder be whose point was at (5, 0, 0), whose vector had the cosines (cos 45°, 0, cos 45°) and any nonzero radius?

Would it be a class 7 with the point being on the X axis?

No, it would not because the u_y vector component is zero.

The answer from Table 1 is type 12. It is not on the Z axis and its $u_z \neq 0$.

CYLINDER/CANON,			HOPE PLANES			
POINT	VECTOR	RADIUS	(a) Z = 0	(b) Y = 0	(c) X = 0	(d) X+Y+Z=1
CYL 1 = 0, 0, 0,	1,0,0,	1				
CYL 2 = 0, 0, 0,	-1,0, 0,	1				
CYL 3 = 0, 0, 0,	0, 1, 0,	1				
CYL4 = 0, 0, 0,	0,-1, 0,	1				
CYL5 = 0,0, 0,	0, 0, 1,	1				
CYL6 = 0, 0, 0,	0,0,-1,	1				
CYL7 = 0, 0, 0,	COS 60° COS 30° 0,	1				
CYL8 = 0, 1, 0,	0, COS 60° COS 30°	1				
CYL9 = 0,0,1,	COS 60° 0, COS 30°	1				
CYL10 = 1,1,0,	COS 60° COS 30° 0,	1				
CYL11 = 0, 1, 1,	0, COS 60° COS 30°	1				
CYL12 = 1, 0, 1,	0, COS60° COS 30°	1				

1 UNIT = □

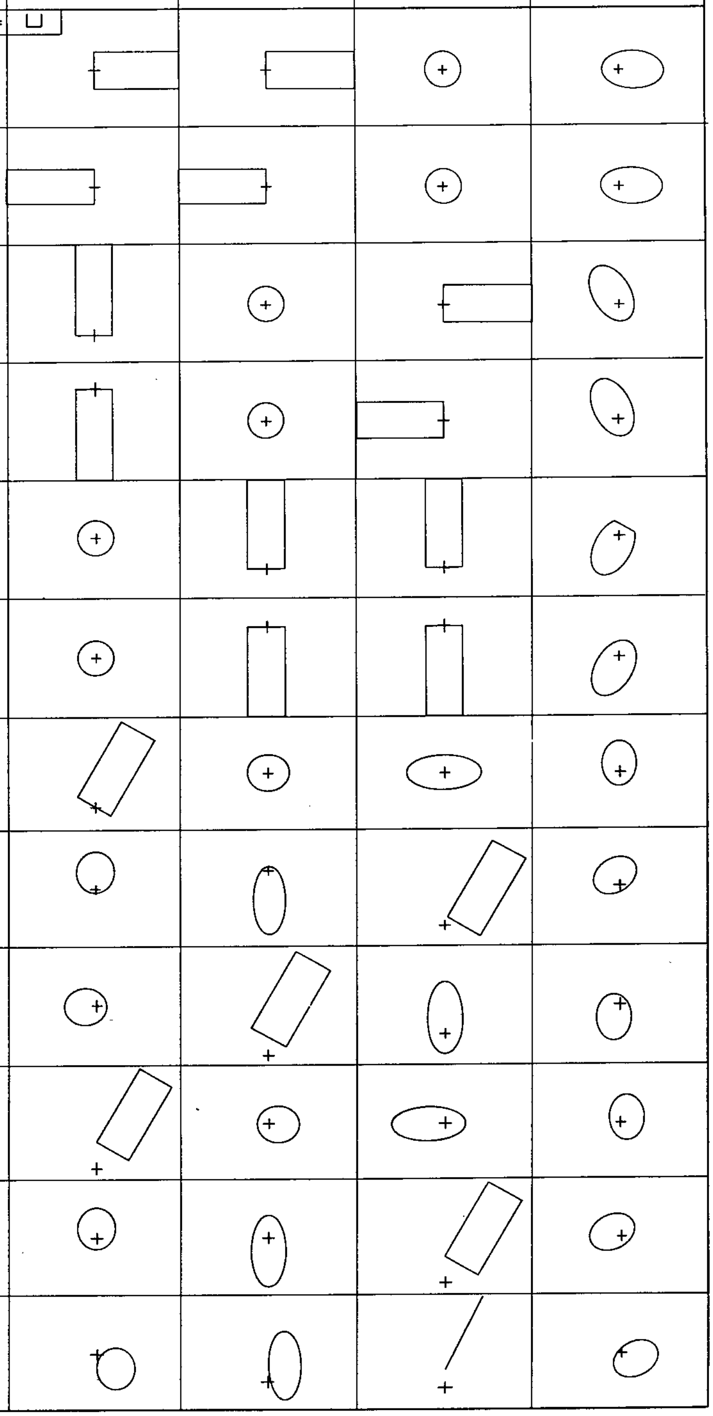


Fig. 21 - Cross-sections of the 12 single cylinders; z = 0, y = 0, x = 0, x + y + z = 1

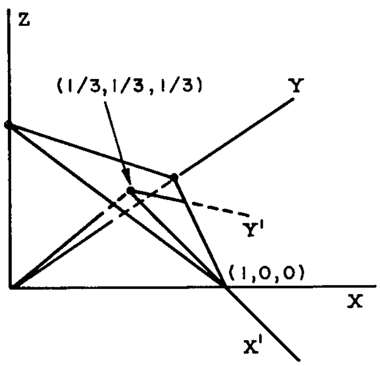


Fig. 22 - Emphasizing the location of the origin and direction of the X′ and Y′ axes for the HOPE plane, $x + y + z = 1$

CONCLUSION AND RECOMMENDATION

This report of computational forms and its implementing computer program CROSEC MOD 2.0 clearly demonstrate the feasibility of plotting the forms obtained from cross-sectional cuts through cylinders and spheres.

The tested method of local coordinate systems in the surfaces and on the HOPE plane could obviously be extended to include APT-defined ellipses and parabolas in the plane as well as three-dimensional cones and a generalized quadric surface.

The availability of cross-sectional forms for the *finite* cylinder leads naturally into the bounded geometry of the APT IV generation. The cylinder definition should be extended to include length. The considerations for plane intersections should be reexamined in the light of the APT IV geometry in hopes of eliminating unwanted internal lines by means of the bounded surface definition.

Finally, the feasibility of a data structure should be seriously considered as a further natural extension of this work.

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REFERENCES

1. Thompson, K.P., "Cross-Sectional Plots of Plane Intersections: An Adaptation of the APT System," NRL Report 7025, Jan. 27, 1970

2. Thompson, K.P., "CROSEC, A Fortran IV - APT Program to give Orthographic, Section and Definable Perspective Views of a Planar-Curved Surface," NRL Report 7228

3. Korn, G.A., and Korn, T.M., "Mathematical Handbook for Scientists and Engineers," New York: McGraw-Hill, 1961

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13. ABSTRACT

This report is the second in a series devoted to the development of a cross-sectional plotting capability for the numerical control APT system in use at NRL. The report covers the development of equations in a computational form for the plotting of intersections between a plane and a cylinder, and a plane and a sphere. Initially, a local coordinate system for a cylinder is established in one of twelve possible ways. Subsequently, the intersection of the cylinder with a plane is considered in the general (infinite) case, in a bounded situation, and in the degenerate condition when the cross-sectional plane is parallel to the axis of the cylinder. The development for the sphere concludes the equation presentation and is obviously less complicated than the cylinder case because of the symmetry, but similar to the extent that a circular intersection and one degenerate case, a tangent point, can be obtained. Computer-generated plots illustrating the types of intersections developed in the report are included. These plots were made using a program that was written to implement the equations developed in this report. The computer program is described in detail in "CROSEC, A Fortran IV - APT Program to give Orthographic, Section and Definable Perspective Views of a Planar-Curved Surface," NRL Report 7228.

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Geometry						
Plane intersections						
Computer-aided design						
Cross-sectional plotting						
Graphics						
Computer graphics						
Numerical control						
Cylinder sections						