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A Flow-Diagram Technique for Time-Varying Coordinate Systems

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13. ABSTRACT

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Matrix						
Inertia						
Shipboard						
Gimbal						
Software						
Hardware						
Force						
Torque						
Position						
Velocity						
Acceleration						

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PROBLEM STATUS

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A FLOW-DIAGRAM TECHNIQUE FOR TIME-VARYING COORDINATE SYSTEMS

INTRODUCTION

A large class of problems in the design of antennas, weapon control systems, and navigation and guidance systems is inherently concerned with time-varying coordinate systems used to describe space directions or attitudes of one or more controlled members. A controlled member in this context is not limited to a mechanical structure with associated mass and inertia but may be regarded as a "fictitious" rigid body (without mass) represented by a coordinate system or reference frame. The latter interpretation may, in fact, be extended to any coordinate system and would be representative of one used to describe an inertialess "beam" produced by an electronic-steered phased-array antenna. The kinematics and kinetics of rigid bodies, well-treated topics of classical mechanics, have generally been more rigorously applied to problems of gyroscopic motion and inertial navigation systems than to problems of antenna design, especially as concerns drive-torque requirements.

Various topological configurations have been used to represent problems in network theory, feedback systems, computer programming, production scheduling, management, etc. One highly developed form is the so-called linear-signal flow graph (1), which may be used for a linear physical system capable of being expressed by a set of linear simultaneous equations. A space flow diagram has been given by Lange (2), but it is limited in its application to a simple symbolic representation of coordinate-system transformations given detailed treatment by conventional matrix methods. In dealing with rigid-body motion and moving platform problems, the desirability of a method for complete representation of problems involving time-varying coordinate systems has been evident. Such a method should permit direct formulation of a problem of this class into such a diagram without intervening mathematical manipulations. The resultant flow diagram should then contain all necessary details, providing complete visualization of the problem and permitting concentration at this level rather than repeatedly returning to the initial problem. The flow diagram should then be subject to simplifications and digital computer solution or hardware implementation.

It is the intention of this report to develop in an orderly manner a flow-diagram method for depicting time-varying coordinate systems, principally for problems of rotational motion involving angular position, velocity, acceleration, and angular momentum and its time rate of change as related to drive torque requirements. A tutorial approach to rigid-body problems in kinematics and kinetics is adopted for developing the flow-diagram method. Emphasis is placed on problem visualization, ease, and flexibility in constructing flow diagrams and in implementing both hardware and software solutions. While more typically employed matrix formulations may also be readily obtained from these flow diagrams, their use in this report is generally limited to developing the flow-diagram methods for time-varying coordinate systems. Hardware and software forms of resolvers are used as the principal tools for solution. The majority of examples are concerned with multigimbaled antenna design for moving platforms. These flow-diagram techniques, however, are equally applicable to other engineering problems involving time-varying coordinate systems.

LINEAR TRANSFORMATIONS

The n linear homogeneous equations

$$\begin{aligned} x_1' &= a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n \\ x_2' &= a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n \\ &\ldots\ldots\ldots\ldots\ldots\ldots \\ x_n' &= a_{n1}x_1 + a_{n2}x_2 + \ldots + a_{nn}x_n \end{aligned} \tag{1}$$

where det $(a_{ij}) \neq 0$, can be interpreted two ways:

1. As a coordinate transformation, Eq. (1) relates different descriptions of the same invariant position vector **x** of a point for two coordinate systems.* The coordinates of the point in either coordinate system are given by the components of the position vector in that system.

2. By operating on vector components, Eq. (1) maps one position vector into another, the vector components being expressed in the same system. With this interpretation, Eq. (1) defines a linear vector function.

For the admitted class of linear transformations, Eq. (1) applies equally as well to other vectors as to position vectors of a point, the components of these general vectors transforming in the same way as the coordinates $x_1, x_2, \ldots, x_n$. Similarly, both of the above interpretations may be applied to transformations of general vectors.

*Boldface type is used to represent invariant vectors or linear operators. Matrix representation of either, being an array of components, is not invariant and, therefore, is not boldface. It is important to distinguish between the invariant quantity and its component description as resulting confusion may otherwise lead to erroneous results.

In matrix form, Eq. (1) becomes

$$\begin{bmatrix} x_1' \\ x_2' \\ \cdots \\ x_n' \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \cdots \\ x_n \end{bmatrix} \tag{2}$$

or, symbolically,

$$x' = Ax.$$

For the coordinate transformation interpretation, the transformation matrix may also be used to relate descriptions of invariants other than vectors in two coordinate systems. A similar matrix representing a linear operator (a special case of a tensor of rank 2) is the other type of invariant considered here. For the linear vector function $y = Bx$, the same vector function in the primed system is given by $y' = B'x'$. Since $y' = Ay$ and $x' = Ax$, it is easily established that

$$B' = ABA^{-1}. \tag{3}$$

Equation (3) thereby relates the description of the invariant linear operator **B** in the primed and unprimed coordinate systems.*

The class of linear transformations considered in this report is limited to three-dimensional Euclidean vector spaces for which the transformation matrix is orthogonal. Both the coordinate transformation and the operation interpretations will be used to establish basic flow diagrams of successive coordinate systems and to relate vector and linear-operator invariants in these systems. The operation interpretation of a linear transformation will be used to express the rotation between coordinate systems and its time rate of change as invariants capable of expression in these systems.

*For the limited class of transformations considered in this report, no distinction is made between a linear operator, a dyadic, or an invariant tensor of the second rank. Component description of such invariants with regard to a selected coordinate system is generally given in matrix form.

ORTHONORMAL SYSTEMS

A transformation

$$\xi' = \begin{bmatrix} x' \\ y' \\ z' \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = A\xi \tag{4}$$

and the matrix A are orthogonal if the transpose of A is equal to its inverse. When x, y, z are the rectangular Cartesian coordinates of a point p, they constitute the components of a position vector $\overrightarrow{OP}$ for a mutually orthogonal basis **i**, **j**, **k**, fixed with respect to the common origin O. Likewise, x, y, z in Eq. (4) may be regarded as the components of a free vector. From an operation viewpoint, the transformation of Eq. (4) preserves vector magnitudes and angles between vectors in the mapping operation. Furthermore, $\det(A) = \pm 1$. For $\det(A) = 1$ (a proper rotation), an orthonormal vector triple (unitary orthogonal) maps into a similar set differing only by a rotation. For $\det(A) = -1$ (an improper rotation), the transform set is subject to a reflection in addition to a rotation so as to change from a right- to a left-handed set (or vice versa). From the coordinate transformation viewpoint, Eq. (4) provides a changing description of an invariant vector when the original orthonormal reference system is transformed to a new orthonormal system differing only by rotation of the base vectors for $\det(A) = 1$ and a rotation plus reflection for $\det(A) = -1$.

Since no advantage is seen for a change between right- and left-handed systems in a given problem, especially with techniques that can be applied conveniently to either sense, only proper rotations of the orthonormal base vectors are considered in subsequent developments. Moreover, depending on the manner of definition, improper rotations may entail mathematical difficulties in representing quantities which behave as vectors in three-dimensional geometry but more generally are represented by skew-symmetric tensors of rank 2. Vector cross products and angular velocity vectors are representative of this form of vector and are sometimes referred to as axial vectors.

A set of orthonormal base vectors having a common origin may be affixed to a rigid body (body-bound) so as to define the attitude of that rigid body with respect to a similar set. Such sets of vectors and the coordinate systems they define may be regarded simply as reference frames.* The angular motion of these rigid bodies and reference frames may be examined with respect to each other or with respect to some selected preferential reference frame, such as an assumed inertial frame. With a time-varying relation between two or more reference frames, the added dimension of time increases the three-dimensional problem to one of four dimensions. An analysis technique could be developed in a four-dimensional geometry, but for the assumed invariance of time, this problem is best treated as a time-varying three-dimensional geometry. In viewing invariant position vectors of objects and by

*The terms coordinate system and reference frame will be used interchangeably.

transforming coordinate systems for desired alignments of base vectors with respect to these position vectors, control gimbal angles may be determined. In examining time-varying relations between reference frames, such concepts as angular velocity and acceleration are evident.

Finite and Infinitesimal Rotations

In addition to relating descriptions of an invariant vector in different coordinate systems, it is frequently desired to express the rotation between associated reference frames. For the coordinate transformation $\xi' = A\xi$, A^{-1} is the unprimed system description of the rotation **R** of the primed with respect to the unprimed frame. As an operation **R**, maps a vector **a** in the unprimed frame into the corresponding vector **b** in the primed frame by $\mathbf{b} = \mathbf{R} \cdot \mathbf{a}$. The description of the invariant **R** in the primed system by Eq. (3) is $AA^{-1}A^{-1} = A^{-1}$ and, therefore, is the same in both systems. It must be kept in mind, however, that this is the rotation of the primed with respect to the unprimed frame. For the inverse, rotation $\mathbf{R}'$ of the unprimed with respect to the primed frame is described by A in either system.

For successive coordinate transformations $\xi_1 = A_1\xi_0, \xi_2 = A_2\xi_1, \ldots, \xi_n = A_n\xi_{n-1}$

$$\xi_n = (A_nA_{n-1} \ldots A_1)\xi_0. \tag{5}$$

The matrix representation of the rotation of the nth frame with respect to the initial frame is

$$(A_nA_{n-1} \ldots A_1)^{-1} = A_1^{-1}A_2^{-1} \ldots A_n^{-1}.$$

A finite rotation between reference frames or rigid bodies which they may represent cannot be expressed as a vector quantity susceptible to representation in either system. If this were so, the rotation corresponding to the result of two successive rotations would be independent of the order in which they occurred, a condition which is not satisfied. As has been shown, however, a finite rotation may be represented by an invariant linear operator, its description in a Cartesian coordinate system being given by an orthogonal matrix.

In contrast to the inability to represent finite rotations by vectors, the effect of an infinitesimal rotation may be expressed as a vector for three-dimensional quantities. One method of establishing this vector characterization is to examine the infinitesimal coordinate transformation $\xi' = (I + \Delta)\xi$, where I is the unit matrix (3). By showing Δ to be skew-symmetric, the change in a vector **u** mapped by the infinitesimal rotation may be expressed by an invariant form consisting of the vector cross product of a vector representing the infinitesimal rotation and **u**.

Rate of Change of a Vector

When two reference frames have a time varying relation, the matrix A of the coordinate transformation $\xi' = A\xi$ between systems fixed in the primed frame Σ' and the unprimed frame Σ, respectively, will necessarily have time-varying components.

By first examining the linear operator invariant **R**, representing the rotation of Σ' with respect to Σ, and having for its description the matrix A^{-1} in either system as previously noted, an alternative method is realized for establishing the vector character of an infinitesimal rotation or, in this case, the angular velocity of Σ' relative to Σ. Let the vector **f**, which is non-time-varying when observed from Σ, be mapped by **R** into the corresponding vector **g** in Σ'; therefore, $\mathbf{g} = \mathbf{R} \cdot \mathbf{f}$. The vector **g** will likewise appear to be non-time-varying when observed from Σ' but will be time-varying when observed from Σ.

Using the unprimed system matrix representation $g = A^{-1}f$ of the invariant relation $\mathbf{g} = \mathbf{R} \cdot \mathbf{f}$, $dg/dt = (dA^{-1}/dt)f$. Since $f = Ag$,

$$\frac{dg}{dt} = \left(\frac{dA^{-1}}{dt}A\right) g, \tag{6}$$

where dg/dt is the unprimed system (column) matrix representation of the time derivative of **g** when observed from Σ. By establishing skew-symmetry of the matrix given by $(dA^{-1}/dt)A$, an alternative expression for Eq. (6) is given by the vector cross product

$$\frac{d\mathbf{g}}{dt} = \boldsymbol{\omega} \times \mathbf{g}, \tag{7}$$

where $\boldsymbol{\omega}$ is defined as the angular-velocity vector representing the angular velocity of Σ' relative to Σ. The skew-symmetry of $(dA^{-1}/dt)A$ and its vector characterization are established in Appendix A.

For successive coordinate systems defined by the coordinate transformations $\xi_1 = A_1\,\xi_0$, $\xi_2 = A_2\,\xi_1, \ldots, \xi_n = A_n\,\xi_{n-1}$, and having a time-varying relation with respect to each other, it is shown in Appendix A that the angular velocity of the corresponding nth frame, Σ_n, with respect to the initial frame, Σ_0, is given by

$$\boldsymbol{\omega}_{0n} = \boldsymbol{\omega}_{01} + \boldsymbol{\omega}_{12} + \ldots + \boldsymbol{\omega}_{n-1,\,\mathrm{n}} = \sum_{k=1}^{n} \boldsymbol{\omega}_{k-1,\,k}, \tag{8}$$

where $\boldsymbol{\omega}_{k-1,\,k}$ is the angular velocity Σ_k with respect to Σ_{k-1}.

A vector which has a time-varying relation in Σ will necessarily have a different time-varying form when observed from Σ'.* Differentiation of a vector, therefore, must be qualified as to the frame from which it is being observed.

For the vector **u** having the components in the primed and unprimed system related by $u' = Au$

$$\frac{du'}{dt} = \frac{dA}{dt} u + A \frac{du}{dt}. \tag{9}$$

Multiplying both sides of Eq. (9) by A^{-1}

$$A^{-1} \frac{du'}{dt} = A^{-1} \frac{dA}{dt} u + \frac{du}{dt};$$

therefore,

$$\frac{du}{dt} = A^{-1} \frac{du'}{dt} - \left(A^{-1} \frac{dA}{dt}\right) u.$$

Since

$$A^{-1} A = I, \qquad A^{-1} \frac{dA}{dt} = -\frac{dA^{-1}}{dt} A$$

and

$$\frac{du}{dt} = A^{-1} \frac{du'}{dt} + \left(\frac{dA^{-1}}{dt} A\right) u. \tag{10}$$

The term $A^{-1}\, du'/dt$ is the unprimed system description of the time derivative of **u** observed from Σ'; du/dt is the unprimed system description of the time derivative of **u** when observed from Σ. The kth system description of the time derivative of a vector when observed from the kth frame is simply the time derivatives of its kth system components. This is a result of the apparent constancy (from the kth system) of the base vectors implicit to the kth system description. When the frame from which a vector is being observed differs from that for its description, differentiation in general is not so simply performed.

Using Eqs. (6) and (7), the invariant vector form of Eq. (10) is then

$$\frac{d\mathbf{u}}{dt} = \frac{d'\mathbf{u}}{dt} + \boldsymbol{\omega} \times \mathbf{u}, \tag{11}$$

*Note that in this case, the same (invariant) vector is being viewed in the two systems instead of the mapping operation used to derive Eqs. (6) and (7).

where $d'\mathbf{u}/dt$ is the time derivative of $\mathbf{u}$ when observed from Σ' and $\boldsymbol{\omega}$, as previously defined, is the angular velocity of the Σ' relative to Σ. If Σ is relabeled by the subscript a to be Σ_a and Σ' by the subscript b to be Σ_b, Eq. (11) may be rewritten as

$$\frac{d_a \mathbf{u}}{dt} = \frac{d_b \mathbf{u}}{dt} + \boldsymbol{\omega}_{ab} \times \mathbf{u}. \tag{12}$$

For this subscript notation, $\boldsymbol{\omega}_{ab} = -\boldsymbol{\omega}_{ba}$, where $\boldsymbol{\omega}_{ba}$ is the angular velocity of the Σ_a relative to Σ_b. In subsequent examples involving time differentiation of vectors, this subscript notation or a similar form* will be used to denote the systems from which the time behavior of these vectors is being observed.

Eulerian Angles

To specify an orientation of a rigid body or a body-bound coordinate system with respect to another coordinate system, a minimum of three parameters is required. Two parameters could specify the axis of rotation, and the third could specify the amount of rotation of one coordinate system about this axis to yield the other system. The Eulerian angles are a set of three parameters based on three successive rotations of a coordinate system performed about two or three of its axes (as transformed) in a preestablished order. Unfortunately, there is no general agreement what this order is except as applied to a specific type of problems, for example, the roll, pitch, and yaw of a ship. Based on the choice of axes and whether two or three axes are used as axes of rotation, there are twelve possible combinations of Eulerian angles. From Fig. 1, the rotation of the coordinate axes x, y about

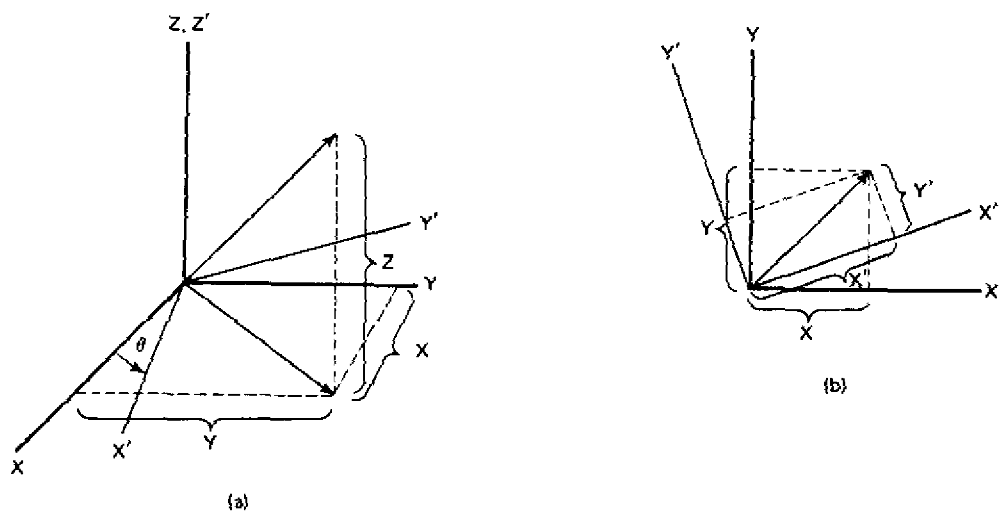


Fig. 1 – Planar coordinate transformation in the xy plane.

*Later the more convenient notation D_a will be used in lieu of d_a/dt.

the z axis yields, for a vector having initial components x, y, and z in the unprimed system, components x', y', and z', such that

$$\begin{aligned} x' &= x \cos\theta + y \sin\theta \\ y' &= -x \sin\theta + y \cos\theta \\ z' &= z. \end{aligned} \tag{13}$$

While the ccw sense for positive angles and the right-handed sense for the order x, y, z in Fig. 1 conform to frequent usage, no such restriction is placed on techniques developed in this report. The form of a general coordinate transformation from the unprimed to the primed system using one possible set of Eulerian angles is given by

$$\begin{bmatrix} x' \\ y' \\ z' \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\alpha & \sin\alpha \\ 0 & -\sin\alpha & \cos\alpha \end{bmatrix} \begin{bmatrix} \cos\beta & 0 & -\sin\beta \\ 0 & 1 & 0 \\ \sin\beta & 0 & \cos\beta \end{bmatrix} \begin{bmatrix} \cos\gamma & \sin\gamma & 0 \\ -\sin\gamma & \cos\gamma & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}.$$

GENERALIZED SENSE OF ANGLE AND ORDER OF AXES

For the planar rotation of Fig. 1, the matrix representation is

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}. \tag{14}$$

Since the inverse of an orthogonal matrix equals its transpose, the inverse transformation is

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x' \\ y' \end{bmatrix} \tag{15}$$

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \cos(-\theta) & \sin(-\theta) \\ -\sin(-\theta) & \cos(-\theta) \end{bmatrix} \begin{bmatrix} x' \\ y' \end{bmatrix} \tag{16}$$

when the transformation matrix is shown in the form of Eq. (14).

Positive Sense of Angle

Given the order of the two orthogonal axes in the plane, the positive sense of an angle shall always be taken as that direction obtained by moving the first ordered axis toward the second by the least effort for the same orientation.

Equation (14) rearranged for the order y, x and expressed in terms of $-\theta$ yields

$$\begin{bmatrix} y' \\ x' \end{bmatrix} = \begin{bmatrix} \cos(-\theta) & \sin(-\theta) \\ -\sin(-\theta) & \cos(-\theta) \end{bmatrix} \begin{bmatrix} y \\ x \end{bmatrix}. \tag{17}$$

The angle of rotation is now $-\theta$, corresponding to the fact that the angle is now taken opposite to that direction resulting from moving the y axis toward the x axis. Equation (17) yields results identical to Eq. (14), demonstrating the utility of the above definition.

Rule for Planar Coordinate Transformations

To simply write the coordinate transformation equations for a planar rotation the following rule may be applied:

$$\begin{aligned} a_2 &= a_1 \cos\gamma + b_1 \sin\gamma \\ b_2 &= -a_1 \sin\gamma + b_1 \cos\gamma, \end{aligned} \tag{18}$$

where a,b is the chosen order and γ is signed according to the previous definition for sign of an angle; that is, if the rotation from the a_1,b_1 axes to the a_2,b_2 axes is in the same direction as moving the a axis toward the b axis, then γ is positive; otherwise, it is negative. Using complex rotation, an equivalent form is

$$(a_2 + jb_2) = (a_1 + jb_1)e^{-j\gamma}.$$

Equation (18) relates the coordinates or vector components of an invariant vector in two different coordinate systems differing by an angle γ in the ab plane, this angle being signed according to the rule for positive sense of angle. By specifying certain combinations of three of the five parameters a_1, b_1, γ, a_2, and b_2, the remaining two admit the real solution (additional constraints may be required in some cases to insure unique solutions).

The inverse of Eq. (18) constitutes the problem of given a_2, b_2, and γ, find a_1 and b_1. The solution of this problem, however, yields the same rules as given by Eq. (18) for planar coordinate transformations when the rule for positive sense of angle is applied. This variation, being a restatement of Eq. (18) is, therefore, not treated as a separate case.

Two other forms based on a different set of specified parameters are derived from Eq. (18) for use in later problems. Given a_1 and b_1, both not zero, and the constraints that a_2 is positive and $b_2 = 0$, the angle of rotation γ from the a_1, b_1 to the a_2, b_2 axes and the nonzero output a_2 may be determined. In terms of Eq. (18), this problem is of a recursive nature, requiring control of γ to achieve the constrained output. However, analytical solution is direct and given by

$$a_2 = \sqrt{a_1{}^2 + b_1{}^2}$$
$$\gamma = \tan^{-1}(b_1/a_1). \tag{19}$$

The quadrant of γ is uniquely determined from the signs of a_1 and b_1. For *ab* being the chosen order, γ is signed according to the rule for positive sense of angle. Alternately, Eq. (19) may be expressed by the complex form

$$a_2 = \text{modulus}\,(a_1 + jb_1)$$
$$\gamma = \text{argument}\,(a_1 + jb_1).$$

Another possible set of input parameters for Eq. (18) required in the solution of later problems is a_1, b_2 and γ ($\gamma \neq \pm 90$ degrees). Again, finding a_2 and b_1 in terms of Eq. (18) requires control of b_1 to achieve the constrained output b_2. Similarly, the analytical solution of Eq. (18) for this specified set yields

$$a_2 = \frac{a_1 + b_2 \sin\gamma}{\cos\gamma}.$$
$$b_1 = \frac{b_2 + a_1 \sin\gamma}{\cos\gamma}. \tag{20}$$

Other variations on Eq. (18) are possible but not necessary if care is exercised in the choice of ordering. Table 1 summarizes the three forms for coordinate transformations in the *ab* plane.

Table 1
Transformations in the *ab* Plane

CASE	INPUT	OUTPUT	SOLUTION
I	a_1, b_1, γ	a_2, b_2	$a_2 = a_1 \cos\gamma + b_1 \sin\gamma$ $b_2 = -a_1 \sin\gamma + b_1 \cos\gamma$
II	a_1, b_1 $b_2 = 0$ $a_2 +$	a_2, γ	$a_2 = \sqrt{a_1^2 + b_1^2}$ $\gamma = \tan^{-1} \frac{b_1}{a_1}$
III	a_1, b_2, γ	a_2, b_1	$a_2 = \frac{a_1 + b_2 \sin\gamma}{\cos\gamma}$ $b_1 = \frac{b_2 + a_1 \sin\gamma}{\cos\gamma}$

FLOW DIAGRAMS

The development of the flow-diagram techniques of this report is directed to kinematic and kinetic problems involving rigid-body motion and other problems dealing with the time-varying behavior of coordinate systems. After a problem has been formulated in terms of a flow diagram, it may then be directly converted to software form for solution on a digital computer or implemented by electromechanical components for hardware solution. Since emphasis is placed on ease in problem visualization, digital computer solution, in general, is not concerned with the most efficient form for a particular problem. For many problems, loss in overall efficiency for solution on a high-speed digital computer, even for a large number of computation, will normally be minimal.

Nodes

Nodes, identified by an assigned number, are used to represent initial, intermediate, and final coordinate systems or reference frames, each capable of describing the same invariant quantity. In the flow diagrams of this report, invariants are generally vectors representing position, velocity, acceleration, momentum, angular velocity, angular acceleration, angular momentum, and torque. While the concept on an invariant to represent rotation between coordinate systems has been exploited to derive the vector characterization of angular velocity and the relation for time differentiation of a vector when observed from different reference frames, its use would be redundant to these flow diagrams since the associated coordinate transformations inherently describe these rotational invariants. An exception to the general use of vectors is the inertia tensor which appears in flow diagrams depicting kinetic relations. This inertia tensor is described by a matrix for the system in which it operates on a vector.

Branches

The presence of a vector invariant is represented by a directed line or branch entering or leaving a node. At the point of entrance or exit, its x, y, and z component description is with respect to the coordinate system represented by the node. One or more directed lines entering a node, possibly modified by an adjacent minus sign for negation, represents a vector summation. The value of the resultant vector is applied to all directed lines leaving this node. In Fig. 2 the column matrices p and q are describing the invariant vectors summed at the kth mode. For each vector, the order of component description is always xyz.

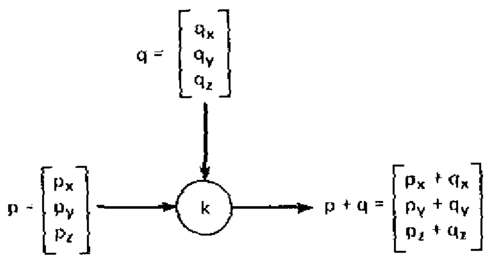


Fig. 2 – Summation of vectors at a node.

Coordinate Transformation

Description change of a vector invariant, the result of a coordinate transformation, is indicated by supplemental information on both sides of the branch connecting two nodes, these nodes representing the predecessor and successor coordinate systems. The coordinate transformation is limited to the xy, yz, or zx planes, the particular plane of rotation being indicated by the coordinate pair (in either order) on one side of the directed line and the angle of rotation between the coordinate systems by the number or symbol on the other side. The order of the coordinate pair is taken to be in the direction of the branch. This order and the angle are signed in accordance with the rule for positive sense of angle.

In Fig. 3, the vector **r** given by the column matrix r in the $(k-1)$th coordinate system is given a new description r' in the kth system, the change between the kth and the $(k-1)$th systems being the result of rotation in the xy plane for an angle θ*. The x axis is the first ordered; the y axis, the second ordered. The rotation by which the $(k-1)$th coordinate system becomes the kth system is in the direction requiring least action to align the x axis with the y axis. Since the coordinate transformation is in the xy plane, the components corresponding to x and y undergo the change in accordance with Eq. (18); the z component is unmodified. Therefore,

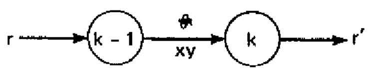


Fig. 3 – Planar coordinate transformation in the xy plane.

$$\begin{aligned} r_x' &= r_x \cos\theta + r_y \sin\theta \\ r_y' &= -r_y \sin\theta + r_y \cos\theta \\ r_z' &= r_z \end{aligned} \quad \text{or} \quad \begin{bmatrix} r_x' \\ r_y' \\ r_z' \end{bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} r_x \\ r_y \\ r_z \end{bmatrix} . \qquad (21)$$

Cross-Product Operation

As defined here, the vector cross-product operation, which is frequently used in the flow diagrams to relate the time derivatives of vectors when observed from different coordinate systems, has meaning only when the components of the two input vectors are expressed in same system†. This operation is indicated by a box containing the cross-product operation symbol x. Since this operation is not commutative with respect to the two input vectors, the convention used here is that the order is, as it appears on the flow diagram, left to right.

Figure 4, a typical stage in the simultaneous determination of angular velocity and acceleration, is representative of a flow diagram in which a vector cross product is used.

*While the xy plane with the axes so ordered has been selected for this and some following examples, such representation is equally applicable to other axis pairs for either order.

†The cross-product operation is an operation between invariant vectors. Cross-product operation between two column matrices, therefore, can be construed to have meaning only when they provide a component description of two vectors with reference to the same coordinate system.

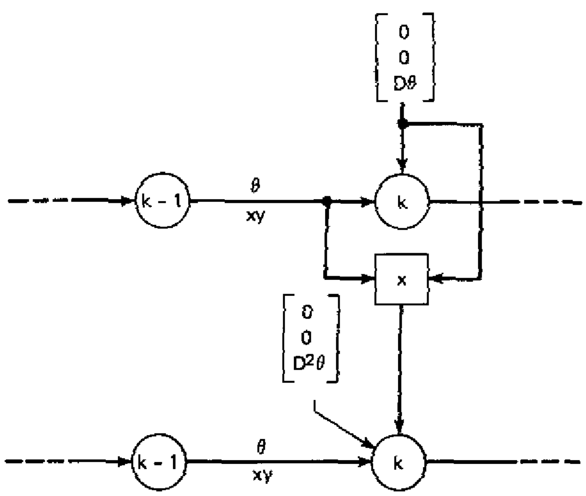


Fig. 4 – Example of vector cross product in a flow diagram.

As discussed at length in Appendix A, the vector cross product $\mathbf{v} = \boldsymbol{\omega} \times \mathbf{u}$ is defined in this report by

$$\begin{bmatrix} v_x \\ v_y \\ v_z \end{bmatrix} = \begin{bmatrix} \omega_y u_z - \omega_z u_y \\ \omega_z u_x - \omega_x u_z \\ \omega_x u_y - \omega_y u_x \end{bmatrix} \qquad (22)$$

irrespective of whether a left- or right-hand system is chosen so that a uniform set of rules is obtained. Since reflection in a coordinate transformation is not permitted under the rules employed in this report, this offers no difficulty except to recognize a left-hand sense to this cross product when a left-hand system has been chosen. As angular velocity will also have a left-hand sense for such a choice, **v** will be defined in the same sense of **u**.

Other Flow-Diagram Operations

Other flow-diagram operations, like the cross-product operation, operate on vectors (branches). The presence of these operations is noted by the operation box containing the appropriate symbol or quantity and having one or two input vectors, depending on the operation. These operations, which do not appear as frequently as the vector cross-product operation, include multiplication of a vector by a scalar, scalar (inner) product of two vectors and inner product of a tensor and a (input) vector. In the latter operation, the order is always taken with the tensor first unless specially qualified. In the system (node) in which it appears, this is given by the product of the square-matrix component description of the tensor and the column-matrix component description of the vector. Output of the scalar-product operation, denoted by a "dot," being a scalar, must be distinguished from a branch, which represents a vector.

For the operations of multiplication by a scalar or inner product with a tensor, it would appear that a more useful flow-diagram form would be a simple modifier along the branch to indicate its "transmittance." The operation box is used, however, to provide a clear perspective of paths.

Paths

A path in a flow diagram is taken to be associated only with the entrance of vectors, their summing with other vectors at a node, the transmittal from a node, and their changing description as the flow diagram proceeds from node to node. A path starts with a branch having no source node, proceeds only along branches having continuity of direction but excluding all "operations," and terminates in a branch having no destination mode. Loops are not permitted.

Equivalent Relations

By applying Eq. (18), each group of Table 2 may be seen to be equivalent.

The inverted flow diagrams in case I of Table 2 corresponds to an inverse transformation. If the direction of the branch between nodes, the angle of rotation between coordinate systems, and the order of the axis pair defining the plane of rotation are taken as three defining parameters of the transformations of case I, inversion of any two results in an equivalent form. Since the defined order of the axis pair is inverted when the direction of the branch is inverted, inversion of the direction of flow may be simply performed by redirecting the branch.

Path inversion in a flow diagram is performed only for one path at a time. With several branches entering a node and one branch leaving, inversion of one of the paths through that node requires the noninverted branches entering that node to be negated. With one branch entering a node and several leaving, inversion of one path is readily performed. However, with more than one branch entering and more than one branch leaving a node, a dummy node (having the same number) must be introduced prior to inversion of any path by the above procedures. This is necessary since the sum of all vectors entering a node is transmitted equally on all branches leaving the node. An alternative form for a dummy node is a simple tie-point to a branch at a point adjacent to the node representing the desired system. Equivalent forms of two representative cases involving path inversion are shown in Table 3. In case II, a dummy node (or equivalent branch tie-point) is created to separate multiple branch inputs and outputs.

Constraints

Representative cases of two constraints encountered in subsequent problems, which take place only within the same transformation, are given in Table 4 (case III being a more representative situation than case II). Solution for these constraints is given by Eqs. (19) and (20).

The ordering shown in Table 4 corresponds to the required ordering for use of Eqs. (19) and (20) in solutions for these transformations. If the original ordering had been opposite, use of relations for case I of Table 2 would be required for proper ordering prior to solution by these equations.

Table 2
Equivalent Flow Diagrams

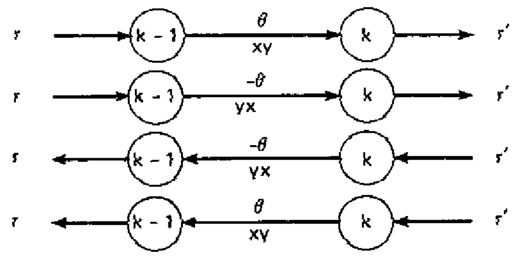
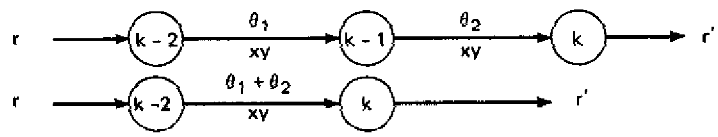
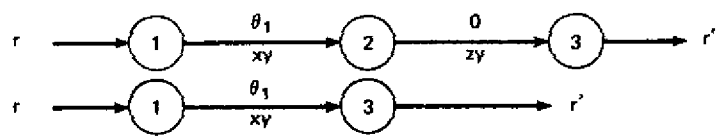
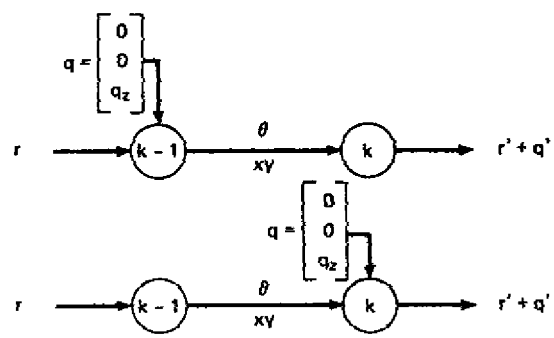
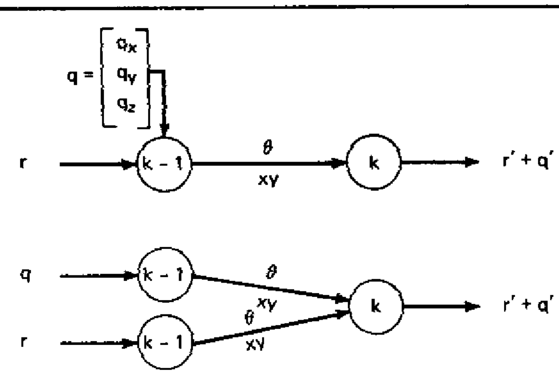
CASE	EQUIVALENT GROUP
I	
II	
III	
IV	
V	

Table 3
Representative Flow-Diagram Inversions

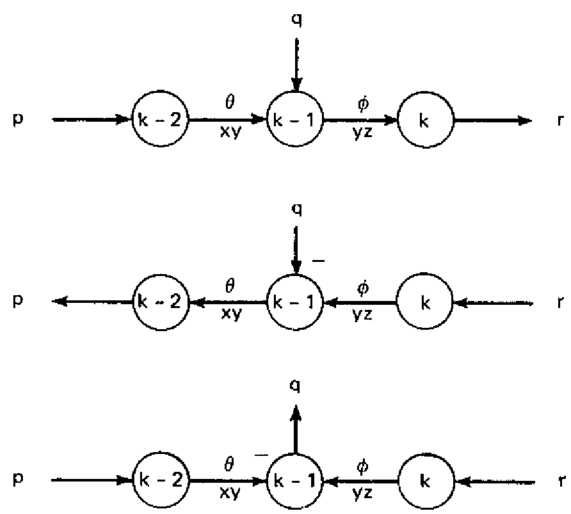
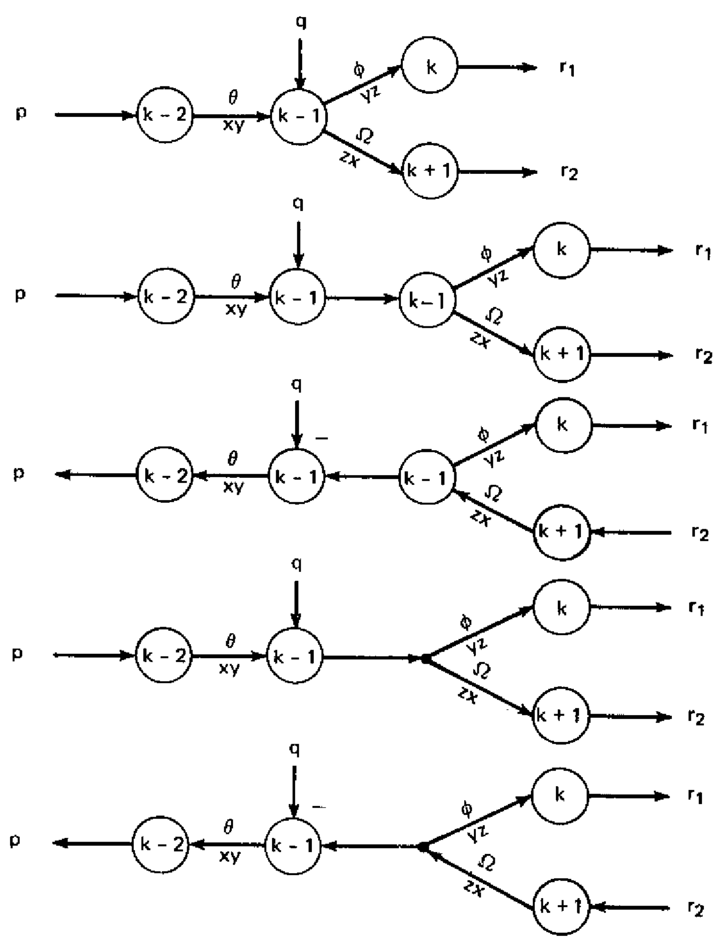
CASE	EQUIVALENT GROUP
I	
II	

Table 4
Constrained Transformations

CASE	CONSTRAINT	SOLUTION
I	$\begin{bmatrix} r_x \\ r_y \\ r_z \end{bmatrix} \rightarrow (k-1) \xrightarrow[xy]{\theta} (k) \rightarrow \begin{bmatrix} r_x' \\ 0 \\ r_z' \end{bmatrix}$	$r_x' = \sqrt{r_x^2 + r_y^2}$ $\theta = \tan^{-1} \frac{r_y}{r_k}$
II	$\begin{bmatrix} \omega_x \\ \omega_y \\ \omega_z \end{bmatrix} \rightarrow (k-1) \xrightarrow[xy]{\theta} (k) \rightarrow \begin{bmatrix} \omega_x' \\ \omega_y' \\ \omega_z' \end{bmatrix}$	$\omega_x' = \dfrac{\omega_y' + \omega_x \sin\theta}{\cos\theta}$ $\omega_y = \dfrac{\omega_x + \omega_y' \sin\theta}{\cos\theta}$
III	$\begin{bmatrix} 0 \\ q_y \\ 0 \end{bmatrix} \downarrow (k-1)$ $\begin{bmatrix} p_x \\ p_y \\ p_z \end{bmatrix} \rightarrow (k-1) \xrightarrow[xy]{\theta} (k) \rightarrow \begin{bmatrix} \omega_x' \\ \omega_y' \\ \omega_z' \end{bmatrix}$	$\omega_x' = \dfrac{p_x + \omega_y' \sin\theta}{\cos\theta}$ $q_y = \dfrac{\omega_y' + p_x \sin\theta}{\cos\theta} - p_y$

Rules for Flow-Diagram Symbols

A summary of the rules for flow-diagram symbols is given by:

Rule 1 – The number assigned to a node uniquely defines the coordinate system associated with that node. A branch tie-point is, in effect, a dummy node and is identified with the system corresponding to the number of the nearest node.

Rule 2 – Vectors "travel" along branches only in the directions of arrows and, at the point of entrance or exit, have their description in the system represented by a node. Branches may (a) be between two nodes, having both a source and a destination node; (b) have no source node for vector input; or (c) have no destination node for vector output.

Rule 3 – The component description of a vector "traveling" along a branch between two nodes undergoes a planar coordinate transformation with respect to parameters provided on that branch. When both the order of the coordinate pair and the sign of the angle are changed, the result is unchanged.

Rule 4 – The value of the vector at any node is the sum of all vectors (negated when modified by a minus sign) entering that node and is transmitted on all branches leaving that node.

Rule 5 – The cross product between two vectors is represented by convergence of two branches (without transformations) on an operation box, the output being directed without transformation to another node. The source nodes (or equivalent source nodes) for the two input vectors and the destination node (or equivalent destination node) for the outputs must all be of the same number. The order of the vector cross-product operation as it appears on the flow diagram is taken to be left to right. Other operations, which appear less frequently, include multiplication of an input vector (branch) by a scalar, scalar product of two input vectors and inner product of a tensor and a vector.

Rule 6 – A path starts with a branch having no source node, proceeds along any branch having continuity of direction but excluding all operations, and terminates in a branch having no destination node. Loops are not permitted.

Rule 7 – Inversion is performed for one path at a time. Multiple inputs and multiple outputs to and from the same node must be separated by a dummy node (or branch tie-point) having the same number prior to a path inversion through that node.

Rule 8 – Inverting a path through a node having multiple inputs requires that noninverted branch inputs be negated.

Rule 9 – Inverting a branch in which the component description of a vector undergoes a planar coordinate transformation is simply performed by redirecting the branch.

RESOLVER IMPLEMENTATION

The two methods used in this report for the solution of flow diagrams are (a) hardware implementation by electromechanical resolvers and (b) software forms suitable for solutions on a digital computer. One commonly employed form for representing an electromechanical resolver is shown in Fig. 5a. By redrawing Fig. 5a in the manner of Figs. 5b and 5c for stator and rotor excitation, respectively, the symbolic form more closely approaches its use in solution of planar coordinate transformations.

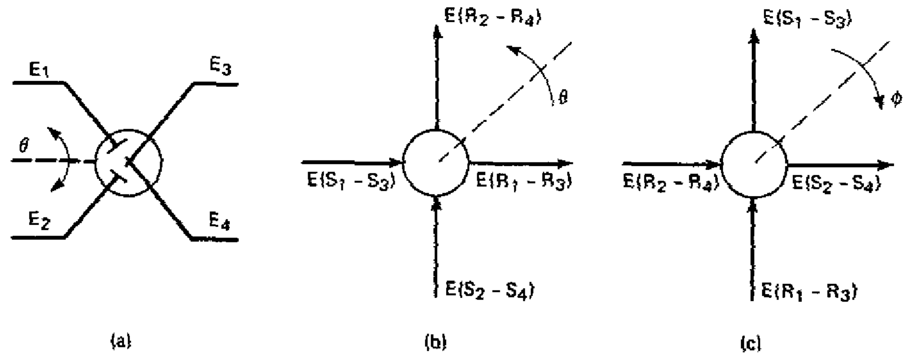


Fig. 5 – Electromechanical resolver representations.

In Figs. 5b and 5c, $E(S_1 - S_3)$ is the voltage across the stator winding, having terminals S_1 and S_3; $E(R_1 - R_3)$ is the voltage across the rotor winding, having terminals R_1 and R_3. The quantities $E(S_2 - S_4)$ and $E(R_2 - R_4)$ are similarly defined. The relations between stator and rotor windings for Figs. 5b and 5c are

$$\begin{aligned} E(R_1 - R_3) &= K[E(S_1 - S_3)\cos\theta + E(S_2 - S_4)\sin\theta] \\ E(R_2 - R_4) &= K[-E(S_1 - S_3)\sin\theta + E(S_2 - S_4)\cos\theta] \end{aligned} \tag{23a}$$

$$\begin{aligned} E(S_1 - S_3) &= K[E(R_1 - R_3)\cos\varphi + E(R_2 - R_4)\sin\varphi] \\ E(S_2 - S_4) &= K[-E(R_1 - R_3)\sin\varphi + E(R_2 - R_4)\cos\varphi]. \end{aligned} \tag{23b}$$

The transformation ratio K will be assumed to be unity. The indicated direction of mechanical rotation corresponds to the direction viewed when looking toward the shaft end.

While there are a number of alternative methods by which a planar coordinate transformation may be implemented with a resolver (depending on choice of inputs and outputs, sign reversals, and sign reversal and adding or subtracting multiples of 90 degrees to the angle), the limited few considered here are sufficient to implement the required coordinate transformations.

A further simplification of the graphic representation of a resolver is shown in Fig. 6. Through Eqs. (18) and (23), it is readily shown that Fig. 6a or Fig. 6b is independent of whether the stator or rotor is excited. Either may be directly related to Fig. 5b or Fig. 5c to determine the appropriate terminals for hardware implementation. For each of the four possible cases, the direction of shaft rotation as viewed from the shaft end of the resolver is given by Fig. 6a or Fig. 6b. The forms of Fig. 6 further correspond graphically to the concept of axis ordering and the rule for positive sense of angle. It is, therefore, a simple matter to go directly from flow diagrams to this representation of hardware resolvers.

Constrained Forms

Prior to implementing a flow diagram involving constrained forms for solution by hardware resolvers, the flow diagram must be modified, if necessary, to insure that the constrained variables are ordered to correspond to case II or III of Table 1. The equivalent forms for the noninverted portion of case I of Table 2 are used to provide the required ordering.

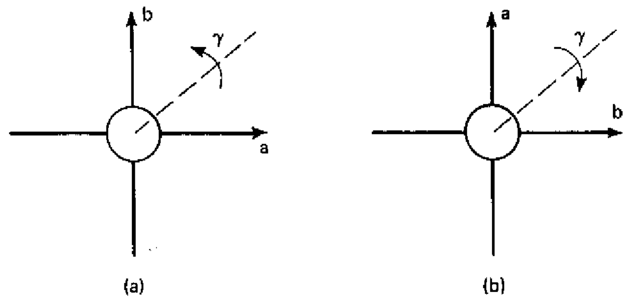


Fig. 6 – Simplified electromechanical resolver representation.

Corresponding to the constrained form for case II of Table 1 and case I of Table 4, either form of Fig. 7 provides solution. The feedback nature is evident in the requirement for a closed-loop servo to control the shaft angle.

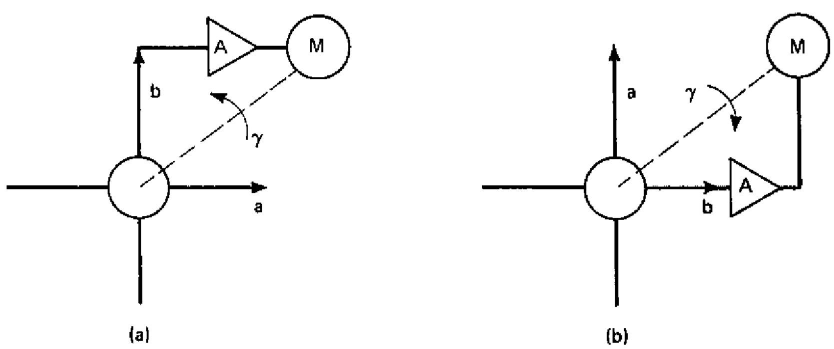


Fig. 7 – Control of angle for constrained second ordered output.

The second ordered output b is constrained to be zero and the first ordered output a to be positive by the closed-loop servo simply represented by the high-gain amplifier A and the motor M. In addition to being the angle from the original to the new axis, γ is also the direction in which the resolver input shaft must move for positive input to the amplifier.

Loop Gain

A fundamental problem in any feedback network (or its computer equivalent) is stability. Variation of open-loop gain, in general, is detrimental to maintaining the desired level of stability. In large structures, limitations of structural resonances frequently result in conditional stable servos to meet required performance and, therefore, require close control of loop gain. From Eq. (18), the total differential of b_2 for constant a_1 and b_1 is

$$db_2 = -a_2\, d\gamma. \tag{24}$$

In the region of servo null, loop gain for the constraint of Fig. 7 is proportional to the first ordered output a_2. To maintain constant loop gain, the gain of other portions of the loop must be modified by $1/a_2$. An agc amplifier controlled in gain by a_2 provides one method of maintaining constant loop gain. In solving gimbal angles for two-axis systems, a_2 is generally constant for the inner gimbal stage and related to the inner gimbal angle for the outer gimbal stage. By using a nonlinear device to provide a correction proportional to the secant of the inner gimbal angle (except near ± 90 degrees), loop gain in the constrained transformation for the outer gimbal angle may be maintained relatively constant over a wide range of values for a_2. As a_2 becomes very small, however, the solution represented by Fig. 7 becomes indeterminate.

The class of problems involving the other constrained form, that of case II of Table 1 and generally encountered as case II of Table 4 in the determination of angular rates and accelerations, is not generally concerned with hardware solution in this report. Figure 8, however, shows two equivalent forms using a comparison of the second ordered output b_2 with its constrained value c for closed-loop control of the unknown second ordered input b_1 and a given first ordered input a_1.

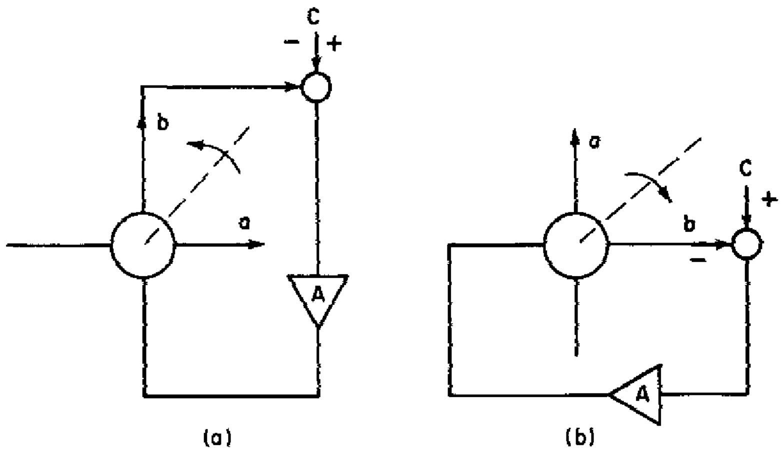


Fig. 8 – Control of second ordered input for constrained second ordered output.

Since a_1 and γ are constants, the total differential of b_2, again using Eq. (18), is

$$db_2 = \cos \gamma \, db_1 . \tag{25}$$

Loop gain, therefore, may be maintained constant (within limits) by controlling a secant potentiometer with the angle γ. Indeterminancy results for $\gamma = \pm 90$ degrees, as is the case for any other method of solution.

Other types of constraints which involve feedback over more than one stage are sometimes encountered. Each of these cases is generally analyzed on an individual basis. The basic hardware form for these is readily derived, but loop gain problems may require further analysis. Subsequent hardware implementations of flow diagrams in this report deal only with the basic form without regard to loop gain and other factors within the feedback loop which may contribute to stability problems.

FORTRAN IMPLEMENTATION

Direct Transformations

The software form chosen here for computer solution is based on a version of FORTRAN IV. However, equivalent forms could be developed for other languages, including efficient assembly languages that might be required in real-time computer control of systems involving time-varying coordinate systems. For convenience, the form developed in the chosen language should reflect the concept of ordering of axis pairs and the rule for positive sense of angle. With the limitations imposed on the number of alternative forms for resolver implementation, the FORTRAN equivalent of these resolvers is best adapted to a subroutine subprogram in which the parameters may serve the functions of both input and output.*

The subroutine for the basic nonconstrained planar transformations given by Eq. (18) (case I of Table 1) is denoted by RESOLV1. The most obvious form for its implementation is

```
SUBROUTINE RESOLV1 (A,B,GAMMA)
TEMP = A *COSF (GAMMA) +B *SINF (GAMMA)
B = -A *SINF (GAMMA +B *COSF (GAMMA)
A = TEMP
RETURN
END.
```

The flow diagram of Fig. 3 is then implemented in a FORTRAN computer program by the call

```
CALL RESOLV1 (RX,RY,THETA)
```

or by

```
CALL RESOLV1 (RY,RX,-THETA).
```

The ordering of the parameters, therefore, conforms to the concept of ordering for a planar transformation and the rule for positive sense of angle.

*While it is also possible to define a similar set of subroutines based on matrix-vector notations and rules, a simpler translation from flow diagram to software form is possible using the resolver definitions. Alternately, the subroutines defined may be simply viewed as products of transformation matrices and vectors in the manner of (21).

An alternate form having faster object language execution is obtained by effectively computing the sine and cosine functions only once in a subroutine.

```
SUBROUTINE RESOLV1 (A,B,GAMMA)
TEMP1 = COSF (GAMMA)
TEMP2 = SINF (GAMMA)
TEMP3 = A *TEMP1 +B *TEMP2
B = -A *TEMP2 +B *TEMP1
A = TEMP3
RETURN
END.
```

In a program involving a large number of computations where one or more of these nonconstrained planar transformations is each associated with a constant angle, it may be desirable to "construct" a separate resolver for each transformation or to use a table of angles and computed cosines and sines for one resolver such that the computation of sine and cosine is necessitated only when the value of that angle changes. Such an approach would be especially attractive in a real-time control system. For many general programs, however, the loss in overall efficiency, when using a single resolver that requires computations of sine and cosine each time a call is made, is minimal in comparison to overhead for other computation functions.

A form of the basic resolver, retaining the concept of ordering and the rule for positive sense of angle in the subroutine call but using a table of cosines and sines for different values of the same angle variable, is given by the subroutine

```
SUBROUTINE RESOLV1 (A,B,SIGN,TABLE,I)
TEMP = A *TABLE (2*I) +B *SIGN *TABLE (2*I+1)
B = -A *SIGN *TABLE (2*I+1) +B *TABLE (2*I)
A = TEMP
RETURN
END.
```

In the call, ±1.0 would be substituted for SIGN and the angle variable used to define the table of cosines and sines would be subscripted for the particular angle by the number (or integer expression) substituted for I.

Constrained Forms

For the constrained planar transformation given by Eq. (19) (case II of Table 1), the FORTRAN subroutine is designated RESOLV2. One possible form for the subroutine subprogram is

```
SUBROUTINE RESOLV2 (A,B,GAMMA)
GAMMA = ATAN2 (B,A)
A = SQRTF (A**2+B**2)
B = 0.
RETURN
END.
```

In the computation of angles corresponding to the formal parameter GAMMA by the function subprogram ATAN (B,A), quadrant information is supplied by the individual signs of B and A. Since the formal parameter GAMMA is an output, the actual parameter must not be an expression, for example, −AZIMUTH.

In translating a flow diagram involving a constraint to a software form, it is necessary that proper ordering of coordinates be observed in the subroutine call. The flow diagram of Fig. 9a, in which the x and z components of a vector **u** are to be constrained in the new coordinate system for a positive x component and a zero z component, is first converted to Fig. 9b (mentally or otherwise) and then solved by

```
CALL RESOLV2 (UX,UZ,TEMP)
ALPHA = -TEMP.
```

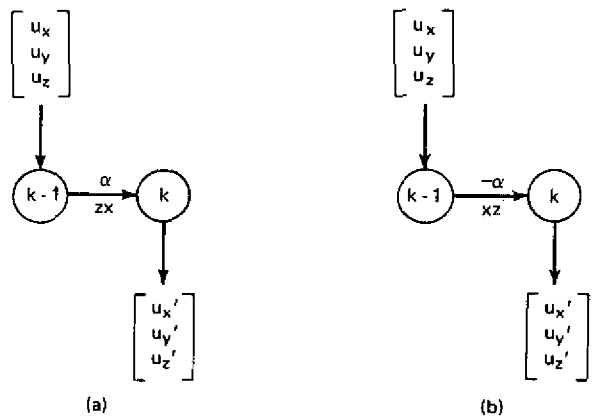


Fig. 9 – Flow diagram conversion for constrained form prior to solution.

The other constrained form, corresponding to Eq. (20) (case III of Table 1), has a solution given by the FORTRAN subroutine RESOLV3. Again, as for the prior constrained form, it is necessary to insure that ordering corresponds to Eq. (20). The subroutine subprogram is

```
SUBROUTINE RESOLV3 (A,B,GAMMA)
TEMP1 = COSF (GAMMA)
TEMP2 = SINF (GAMMA)
TEMP3 = (A+B*TEMP2)/TEMP1
B = (B+A*TEMP2)/TEMP1
A = TEMP3
RETURN
END.
```

Unique to this constrained resolver is the situation that vector components of the subroutine input or output are not in the same coordinate system but are paired as a_1, b_2 and a_2, b_1, respectively, in accordance with Eq. (20).

MULTIPLE PLANAR TRANSFORMATIONS

Through use of the tools developed thus far, a variety of problems in time-varying coordinate systems may be put in flow-diagram form and the solution then implemented in hardware or software form. While a set of three Eulerian angles is sufficient to specify orientation of a rigid body or a coordinate system with respect to another coordinate system, multiple planar transformations employed in the solution of problems generally involve in excess of three planar rotations. The natural subdivision of a problem into successive planar rotations is related to the manner in which directions in space and attitudes of rigid bodies are specified and to the arrangement of multigimbaled devices necessary to achieve control of the line of sight or the attitude of a rigid body.

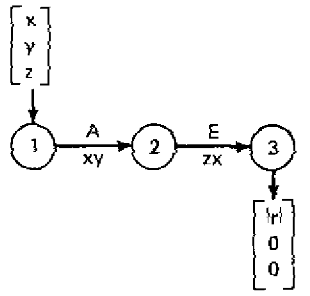


Fig. 10 – Conversion of Cartesian coordinate to azimuth, elevation, and range.

Concepts of spherical coordinates and more generalized curvilinear coordinates are of great importance in general classes of coordinate transformations and the study of invariants under these more general classes. Common usage of spherical coordinates in the type of problems considered here is generally limited to alternative descriptions of position vectors and the space direction which they define. Inasmuch as this usage of spherical coordinates is equivalent to two planar coordinate transformations performed by successive planar rotations of coordinate axes so as to align one of the original axes with the position vector, spherical coordinates are not given special status in this report. In Fig. 10, a position vector **r**, having a direction specified by familiar definitions of azimuth and elevation, is equivalent to two planar rotations of a right-handed xyz coordinate system, the original x axis in the direction of north, the z axis in a direction opposite to zenith pointing, and the xy plane a horizontal plane, such that the resultant x axis becomes aligned with the invariant position vector.

Two constrained transformations, each of the type shown for case I of Table 4, are required to align the x axis with the vector **r**. Although a right-handed system was chosen and the x axis was aligned with the position vectors, a variety of combinations, including those involving left-hand systems, may be used to achieve a similar result. Using the relationships shown in Fig. 9, software solution is given by

```
.
.
CALL RESOLV2 (X,Y,A)
CALL RESOLV2 (X,Z,TEMP)
E = -TEMP
RANGE = X
.
.
```

Based on the choice of Fig. 6a or 6b for a stage, several forms are possible for hardware solution. One form, shown in Fig. 11, may be directly related to Fig. 5b or 5c for terminal connections when stator or rotor excitation is selected, respectively.

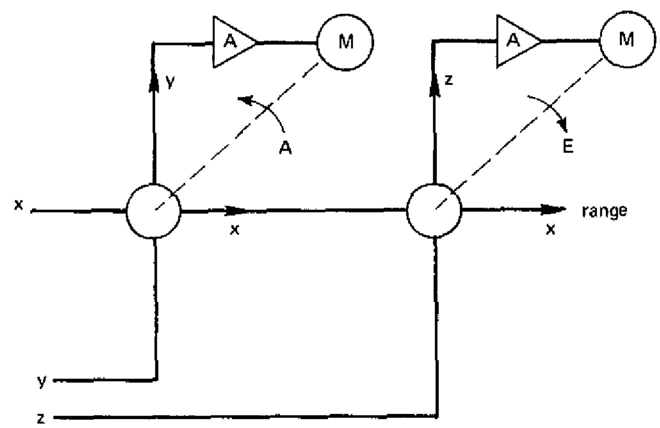


Fig. 11 – Hardware implementation for conversion of Cartesian coordinates (of Fig. 10) to azimuth, elevation, and range.

Inversion of Fig. 10b which yields conversion from azimuth, elevation, and range to Cartesian coordinates, requires two nonconstrained transformations shown in Fig. 12.

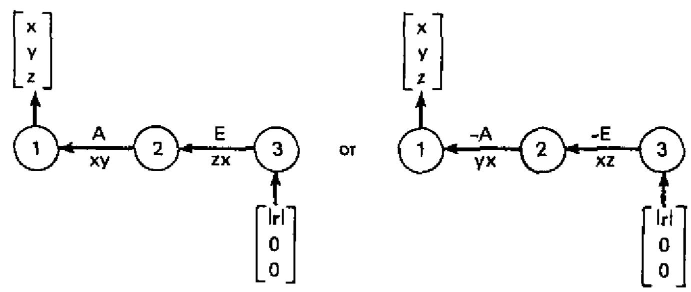


Fig. 12 – Conversion of azimuth, elevation, and range to Cartesian coordinates.

Software solution is

.		.
.		.
X = RANGE		X = RANGE
Y = 0.		Y = 0.
Z = 0.	or	Z = 0.
CALL RESOLV1 (X,Z,E)		CALL RESOLV1 (Z,X,−E)
CALL RESOLV1 (Y,X,A)		CALL RESOLV1 (X,Y,−A)
.		.
.		.

GIMBALING

To provide the degrees of freedom necessary for the control of a rigid body, gimbals are generally employed. Such a rigid body may be a platform on which other equipment (with other gimbaling) may be mounted or may itself be the final controlled element, such as an antenna. In Fig. 13, representative two-, three-, and four-gimbal systems are shown. When adjacent gimbal axes are orthogonal, the controlled member is related to the reference frame for which the outer gimbal axis is stationary by a succession of planar transformations, one for each degree of freedom.

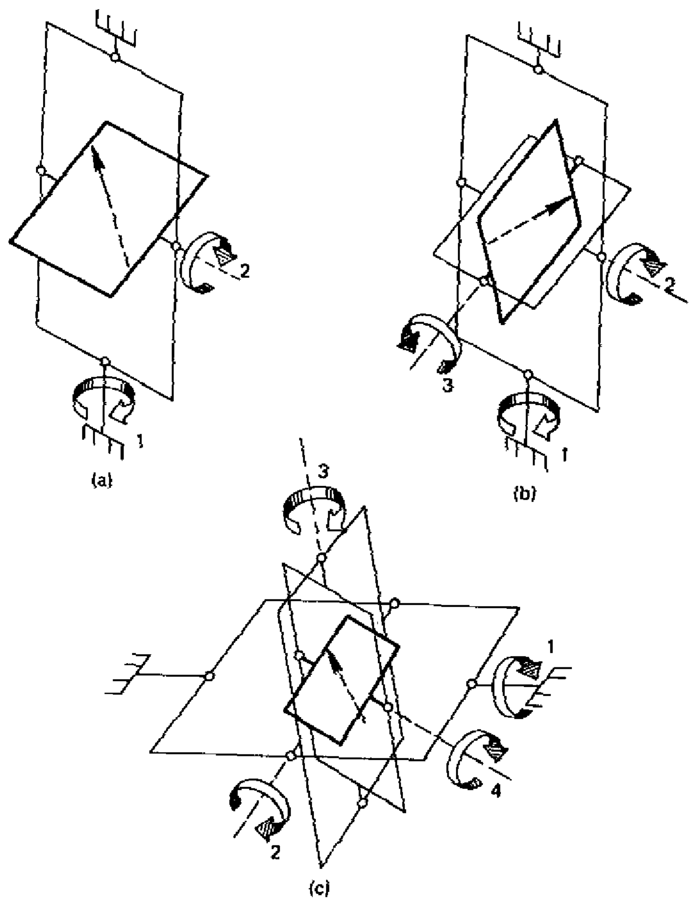


Fig. 13 – Examples of gimbaling.

In certain special designs, such as found in the so-called differential antenna pedestal, adjacent gimbal axes may not be orthogonal. In this event, additional planar transformations are required to orient coordinate system axes with gimbal axes. A similar problem occurs when only a specified direction of the controlled member is to be maintained and this direction is not orthogonal to (or aligned with) the axis of the innermost gimbal.

Since directions in space, attitudes of moving platforms, and satellite orbits, etc., are generally defined by angles equivalent to rotation about adjacent axes that are orthogonal, planar transformations are generally in direct correspondence to these angles. By regarding the controlled member of a gimbaled system as a rigid body and relating the changing description of selected invariant vectors in properly ordered successive coordinate systems, directions in space and angular position, velocity, and acceleration of gimbals may be determined.

For control objectives of orienting the controlled member with respect to a given reference frame, either for control of a specific direction of this member (as for an antenna tracking line) or the attitude of the entire controlled member, a coordinate system defining this orientation is chosen both for convenience and for minimizing the number of planar transformations. Even within these restrictions, a number of possible alternatives exist. While selection of these coordinate systems may depend on individual preference, choice and order of angles defining the successive planar transformations are generally much more restrictive in the sense that, within certain classes of problems, definitions are standardized. The heading, pitch, and roll of a ship are examples of ordered angles defining the attitude with respect to local earth-referenced stable space. Definitions and symbols for these and a number of other related angle quantities for shipboard problems are given by Navy standards (4). Another instance of established definitions of ordered angles is the orbital parameters of argument of perigee, inclination, and right ascension of ascending node, which define the orientation of the Keplerian ellipse with respect to the celestial sphere.

Windup and Unwind

If the successive planar transformations from the initial coordinate system defining the desired orientation of the controlled member to the system to which the outermost gimbal axis is attached is regarded as a "windup" procedure, then an obvious method of gimbaling to achieve the desired objective is to "unwind" back to the original system. In Fig. 14, the original system is chosen to be right handed with the x axis aligned with a desired direction and the xz plane a vertical plane with the z axis downward. The pointing direction is specified by an ordered set of angles, in this case the azimuth A and elevation E. The equivalent relations of case II of Table 2 are used in the step-by-step unwind procedure, resulting in the output vector being the same as the input for any choice of input vector.

The trivial result is that a gimbal arrangement to unwind a direction specified by azimuth and elevation is simply implemented by an azimuth-elevation gimbal arrangement. Node 3 of Fig. 14 is the system to which the outer gimbal axis is attached. The azimuth axis is, therefore, the outer gimbal. In unwinding, it is seen that since the original system and the final system are the same, the attitude as well as the desired pointing direction is retained.

A more involved windup and unwinding results when the true bearing By (equivalent to azimuth) and the true elevation E specify a direction in local earth-referenced stable space and the gimbaled device is mounted on a moving platform having heading Cqo, pitch Eio, and roll Zdo*. In the unwinding process of Fig. 15, By and Cqo are combined in a single

*Symbols and definitions of angles here are in accordance with Ref. 4. Positive sense is clockwise for (own ship) heading, top view; bow down for pitch; and clockwise for roll, front view. The defined order corresponds to their use in Fig. 15.

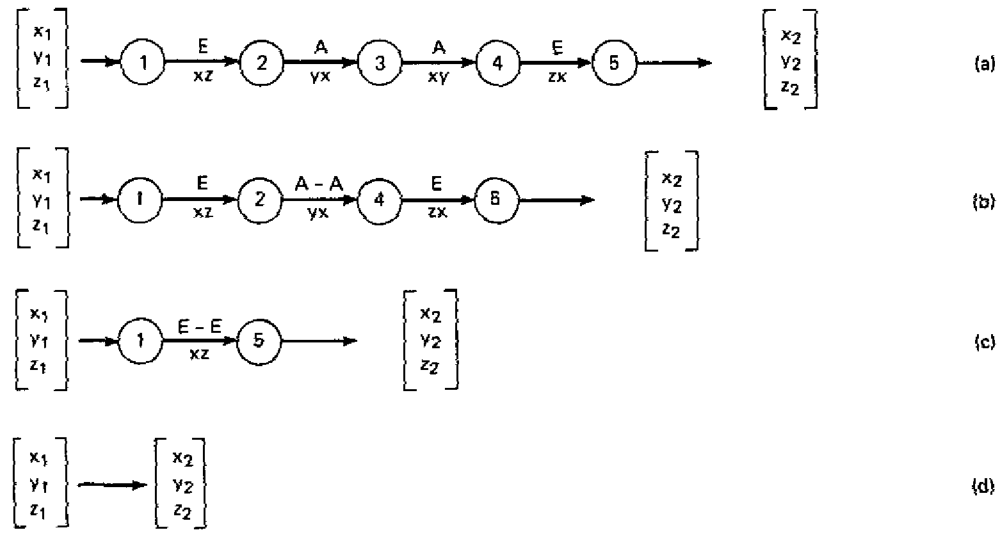


Fig. 14 – Successive steps in wind-unwind for azimuth and elevation.

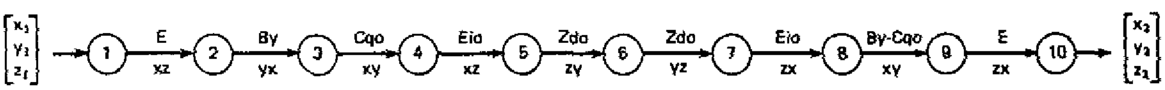


Fig. 15 – Windup and unwind of bearing, elevation, and platform position.

gimbal of an arrangement which has its outer axis attached to the system of node 6. Again, the inferred attitude as well as the desired direction is preserved. The choice of the initial coordinate system is the same as that for Fig. 14.

In this case a four-axis gimbal arrangement is required. When translation (and hence parallax) is absent, this procedure may be carried out for any number of motions and corresponding planar transformations. Complex design, mechanical problems, and costs of multigimbal devices, however, precludes this direct approach in many instances.

Other Director Methods

As previously noted for a general case, a minimum of two degrees of freedom is necessary to specify a direction and a minimum of three degrees of freedom to specify a desired attitude. Two-degree-of-freedom systems are subject to two singular directions, corresponding to either direction of the outer gimbal axis, for which the outer gimbal angle becomes undefined and its required angular rate becomes unbounded. In some instances a two-gimbaled device will be used to provide a more optimum (for a particular problem) choice of the singular directions, even when the desired pointing direction may be specified in such a way that the simple windup and unwind for the two degrees of freedom cannot be used. This situation is typical of the X-Y antenna-mount* use when azimuth A and elevation E are pointing-direction specifications.

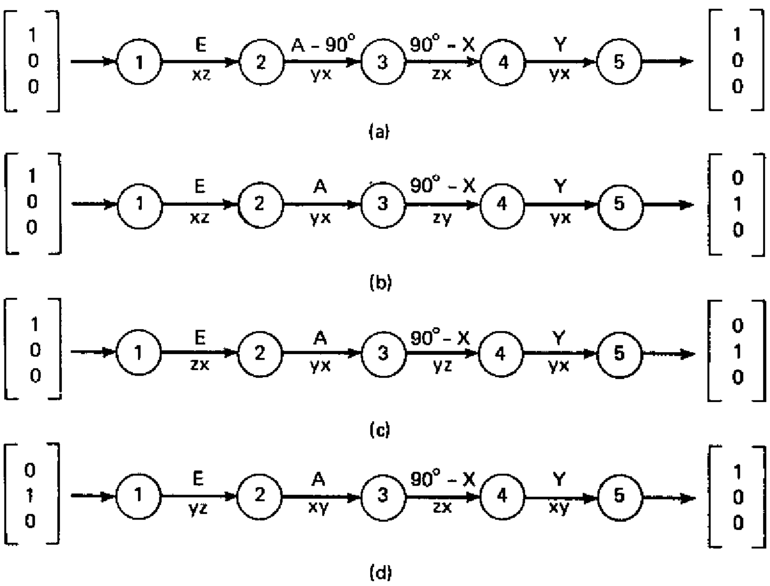


Fig. 16 – Directing X-Y gimbal arrangement from azimuth-elevation input.

Figure 16 shows four of a number of possible flow diagrams for directing an X-Y antenna from inputs of azimuth and elevation. The X (antenna) axis is north aligned; positive X angles are eastward and positive Y angles are northward; zero X and Y angles correspond to zenith pointing. Figures 16a and 16b use right-handed coordinate systems, and Figs. 16c and 16d use left-handed systems. While initial and final orientations of coordinate axes permit a large number of possible choices for solution of this problem, it is seen that the choice and order of angles (except possibly modified by integer multiples of 90 degrees) does not. Since a windup and an unwind procedure was not used and since there are only two degrees of freedom, the attitude of the controlled member is, in general, not the same as defined or implied by the initial coordinate system. The rotation of the controlled member about the controlled direction (tracking line) is known as cross roll.

If in Fig. 15 the transformation between nodes 8 and 9 is made in the zx plane and that between nodes 9 and 10 in the xy plane, a three-axis arrangement is sufficient, since the transformations between nodes 7 and 9 may be performed by a single gimbal. A partial unwind of the original windup (node 4 through node 8 of Fig. 15) is then effected, resulting in a simple three-axis director system. The remaining transformations (node 8 through node 10), which cannot be inherently directly performed by gimbal geometry, are tantamount to conversion from azimuth and elevation to X-Y commands. The X axis in this case is parallel to the pitch axis.

*X and Y are the antenna axes and, therefore, do not necessarily bear a particular relation to the x and y axes of a chosen coordinate system.

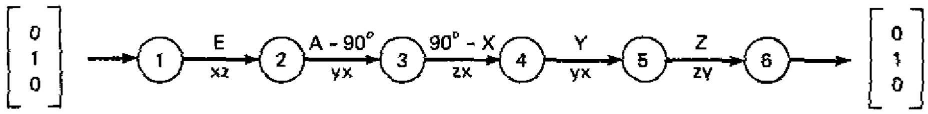


Fig. 17 – Determination of cross roll of linear polarization for *X-Y* antenna.

In Fig. 17, the flow diagram of Fig. 16a is modified to compute the change Z in the angle of linear polarization from that for $Y = 0$. This is performed by examining the vector having the initial description (0,1,0)* after X and Y have been determined. Z is positive for counterclockwise when viewing from the front. The computer solution for Fig. 16a and Fig. 17 is given by

```
 •
 •
 X = 1.
 Y = 0.
 Z = 0.
 CALL RESOLV1 (X,Z,E)
 CALL RESOLV1 (Y,X,A -PI/2)
 CALL RESOLV2 (X,Z,TEMP)
 XAXIS = PI/2 +TEMP
 CALL RESOLV2 (X,Y,TEMP)
 YAXIS = -TEMP
 •
 •
 X = 0.
 Y = 1.
 Z = 0.
→CALL RESOLV1 (X,Z,E)
 CALL RESOLV1 (Y,X,A -PI/2)
 CALL RESOLV1 (Z,X,PI/2 - XAXIS)
 CALL RESOLV1 (Y,X,YAXIS)
 CALL RESOLV2 (Y,Z,TEMP)
 ZROLL = -TEMP
 •
 •
```

The first resolver of Fig. 17, indicated by the arrow in the FORTRAN program above, may, of course, be omitted since x and z inputs are both zero (see case IV of Table 2).

*In the text of this report, (x,y,z) is used to represent the column matrix description of a vector, such that $(x,y,z) \triangleq \begin{bmatrix} x \\ y \\ z \end{bmatrix}$.

The vector (0,0,1) also could have been used as input and output to determine cross roll. In this event, the last two statements in the above program would also require minor modification. The first resolver, however, cannot be deleted in this case. Judicious choice of vectors, therefore, frequently results in computational savings.

Electromechanical analog solution is readily obtained through substitution of either form of Fig. 6 for each connecting branch and then relating to Fig. 5b or 5c for stator or rotor excitation, respectively.

The inverse of the problem is simply a matter of inverting the path of the flow diagram according to the rules for the flow diagrams and then providing either software or hardware solution.

Elevation Over Train

A typical two-degree-of-freedom configuration on a moving platform is given by a shipboard-mounted elevation-over-train antenna. The outer gimbal axis (train) of this arrangement is normal to the deck. In Fig. 18, the windup from the initially chosen coordinate system to the system to which the train axis is attached is the same as for Fig. 15. The initial coordinate system, having the x axis directed along the line of sight, the xz plane vertical, the z axis downward, and xyz a right-handed set, is one of a number of possible choices for solution of this problem.

Software solution of Fig. 18 for the train and elevation angles Bd' and Ed', respectively, is given by

```
•
•
X = 1.
Y = 0.
Z = 0.
CALL RESOLV1 (X,Z,E)
CALL RESOLV1 (Y,X,BY–CQO)
CALL RESOLV1 (X,Z,EIO)
CALL RESOLV1 (Z,Y,ZDO)
CALL RESOLV2 (X,Y,BD)
CALL RESOLV2 (X,Z,TEMP)
ED = –TEMP
•
•
```

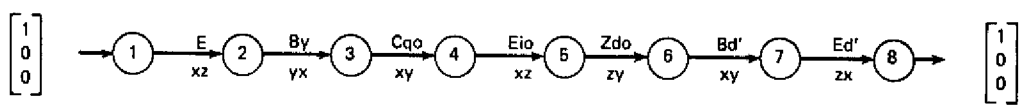


Fig. 18 – Elevation-over-train configuration on a moving platform.

The use of two degrees of freedom result in cross roll of the controlled member about the antenna tracking line as *Eio* and *Zdo* change. The amount of this cross roll may be determined in a manner similar to that of Fig. 17 after *Bd′* and *Ed′* have been determined.

Cross Roll Over Elevation Over Train

Figure 19 may be interpreted as a flow diagram for determining the cross roll resulting when an elevation-over-train antenna is used on a moving platform. It also may be regarded as the flow diagram of one form of a three-axis arrangement, the added degree of freedom being the result of an inner cross-roll gimbal.

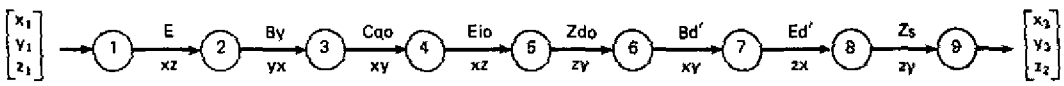


Fig. 19 – Cross-roll-over-elevation-over-train configuration on a moving platform.

The use of three degrees of freedom enables control of the attitude of the controlled member as well as the tracking line. To have an attitude defined by the initial coordinate system, the final system, which is bound to the controlled member, must be the same as the initial system. Any input vector, therefore, will have the same components after its description has been transformed from the initial system by successive planar rotations of coordinate axes to the final system. Train and elevation angles, *Bd′* and *Ed′*, respectively, are first determined by selecting a vector normal to the plane in which the last planar rotation takes place, since its component description is not altered by this last transformation. This vector has the initial description (1,0,0) and is parallel to the tracking line. Software solution for *Bd′* and *Ed′* is the same as that used in determining gimbal angles for Fig. 18. The cross-roll angle *Zs* is then determined by choosing a convenient vector which is orthogonal to the one used in the first path.* The choice of the vector having the initial description (0,1,0) permits elimination of the first resolver in the computation of *Zs* in the following manner:

```
.
.
X = 0.
Y = 1.
Z = 0.
CALL RESOLV1 (Y,Z,BY -CQO)
CALL RESOLV1 (X,Z,EIO)
CALL RESOLV1 (Z,Y,ZDO)
CALL RESOLV1 (X,Y,BD)
CALL RESOLV1 (Z,X,ED)
```

*When attitude control is the objective and the final coordinate system, therefore, must be constrained by gimbal angles to be the same as the initial system, selection of any two noncollinear vectors are, in principle, sufficient to determine the gimbal angles. For convenience and computational efficiency, these are selected in accordance with the above.

```
CALL RESOLV2 (Y,X,TEMP)
ZS = -TEMP
•
•
```

Zs is defined here to be positive for a clockwise angle when viewed from the front. A similar choice is made for the traverse and cross-level angles of subsequent examples. For opposite definitions, it is necessary only to invert the corresponding coordinate pair on the flow diagram or change the sign of the angle.

Traverse (Cross Elevation) Over Elevation Over Train

Another form of a three-axis arrangement, the traverse (or cross-elevation)-over-elevation-over-train antenna, has a flow diagram shown in Fig. 20. This is obtained from Fig. 19 by performing the last transformation in a different plane, requiring the innermost gimbal to be cross elevation. When used for attitude control, train and elevation angles Bd' and Ed', respectively, are first determined by selecting a vector normal to the plane in which the last

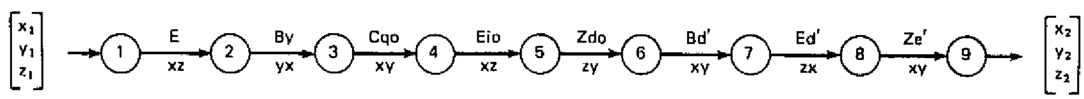


Fig. 20 – Traverse (cross-elevation)-over-elevation-over-train configuration on a moving platform.

planar rotation of coordinate axis occurs. In contrast to the first vector used in connection with Fig. 19, this vector, having initial description (0,0,1), is not along the tracking line. The vector (0,1,0) is a convenient choice for determining the traverse angle Ze' following solution for Bd' and Ed'.* As before, this convenient choice permits elimination of the first resolver in this portion. Software solution of gimbal angles is given by

```
•
•
X = 0.
Y = 0.
Z = 1.
CALL RESOLV1 (X,Z,E)
CALL RESOLV1 (Y,X,BY -CQO)
CALL RESOLV1 (X,Z,EIO)
CALL RESOLV1 (Z,Y,ZDO)
CALL RESOLV2 (X,Y,BD)
CALL RESOLV2 (Z,X,ED)
```

*Strictly speaking, Bd' and Ed' as used here do not conform to definition in Ref. 4 when used in regard to the line of sight. Moreover, Ze' has not been defined in Ref. 4.

```
.
.
X = 0.
Y = 1.
Z = 0.
CALL RESOLV1 (Y,X,BY -CQO)
CALL RESOLV1 (X,Z,EIO)
CALL RESOLV1 (Z,Y,ZDO)
CALL RESOLV1 (X,Y,BD)
CALL RESOLV1 (Z,X,ED)
CALL RESOLV2 (Y,X,TEMP)
ZE = -TEMP
.
.
```

A typical strategem in using this configuration is not attitude control but reduction of the high angular rates and accelerations associated with a two-axis antenna as the singular pointing directions are approached. One method is to control the train gimbal so as to minimize excursions of the traverse gimbal, the maximum values of which are generally limited to moderate amounts by mechanical restraints. Cross roll of the controlled member about the tracking line is an inevitable result of this mode of control. Software solution of Ed' and Ze' in this case presumes knowledge of Bd', requiring only a single "pass" with the vector having an initial description (1,0,0).

Elevation Over Cross Level Over Train

While several other three-axis arrangements with adjacent axes orthogonal are possible, the only other one for which use of a flow diagram is discussed here is the elevation-over-cross-level-over-train configuration.* The gimbal configuration shown in Fig. 13b is representative of this and the preceding arrangement. Mechanically, however, the two are considerably different. As shown in the flow diagram of Fig. 21, the elevation and train gimbals are separated by a gimbal for cross-level control of the elevation axis. The vector having the initial description (0,1,0), being normal to the plane of the last transformation, is used to determine the train and cross-level angles Bd and Zd, respectively. Since the final and initial systems are the same for attitude control and since the vector (0,1,0) is horizontal in the initial system, the elevation axis is maintained horizontal. Either of the vectors having initial description (1,0,0) or (0,0,1) suffice to determine the elevation angle Ed.

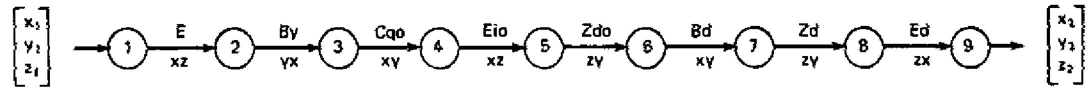


Fig. 21 – Elevation-over-cross-level-over-train configuration on a moving platform.

*In Appendix B, a three-axis arrangement having a pair of nonorthogonal adjacent axes is treated.

Software implementation of Fig. 21 is

```
.
.
X = 0.
Y = 1.
Z = 0.
CALL RESOLV1 (Y,X,BY -CO)
CALL RESOLV1 (X,Z,EIO)
CALL RESOLV1 (Z,Y,ZDO)
CALL RESOLV2 (Y,X,TEMP)
BD = -TEMP
CALL RESOLV2 (Y,Z,TEMP)
ZD = -TEMP
.
.
X = 1.
Y = 0.
Z = 0.
CALL RESOLV1 (X,Z,E)
CALL RESOLV1 (Y,X,BY -CO)
CALL RESOLV1 (X,Z,EIO)
CALL RESOLV1 (Z,Y,ZDO)
CALL RESOLV1 (X,Y,BD)
CALL RESOLV1 (Z,Y,ZD)
CALL RESOLV2 (X,Z,TEMP)
ED = -TEMP
.
.
```

The obvious form of hardware implementation requires two resolver chains, one for each of the two vectors which establish attitude of the controlled member. By using dual resolvers, eight such units and three servos are required for direct correspondence to Fig. 21. The first and last resolver may be a single type, these two resolvers being required only to implement the second "pass" on Fig. 21. By combining *By* and *Cqo* with a mechanical differential, one unit may be deleted. An alternative method is to use a single resolver chain but time-share its use for each of the two vectors which establish the attitude of the controlled member. This implementation, which necessarily involves a form of sampled-data servos, is shown in Fig. 22.

In attempting to simplify hardware implementation, a vertical reference is frequently mounted on moving portions of the antenna corresponding to either node 7 or 8 of Fig. 21. Alternately, the antenna cross-level gimbal may be used as the outer gimbal of such a vertical reference, the vertical reference inner gimbal and vertical sensing elements (accelerometers) being attached to nodes 8 or 9. Without going into specific details, it is sufficient to state

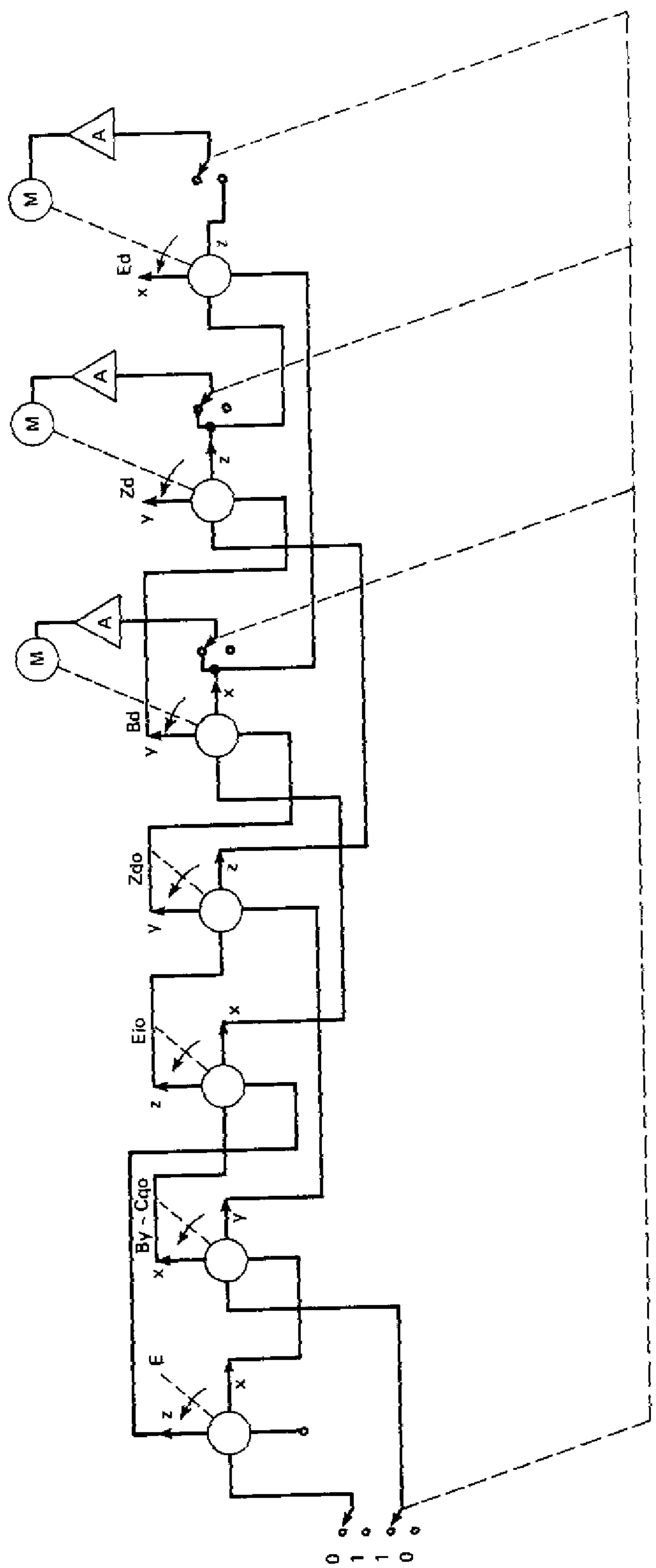


Fig. 22 – Hardware director for elevation-over-cross-level-over-train configuration on a moving platform.

that a horizontal plane is established by controlling the cross-level axis so as to keep the elevation axis level and by providing a horizon reference for the elevation axis. With respect to this horizontal plane, the elevation axis is positioned in accordance with true elevation. Unfortunately, the relative bearing $B = By - Cqo$ only approximates the train angle required. The difference between this train angle and the relative bearing may be determined by the first "pass" on the flow diagram of Fig. 21.

Determination of true bearing and true elevation for each of the three-axis configurations is simply a matter of inverting the path according to the flow-diagram rules and solution by a single "pass" since only two angles are unknown. The vector (1,0,0) for input and output in Figs. 19, 20, and 21 is required for this determination.

TRANSLATION

In the preceding discussion, only angular displacement was dealt with. In some problems, linear as well as angular displacement is required to determine pointing angles or gimbal angles for directing a controlled member. A typical situation is the linear displacement between the coordinate system of an observer on the earth's surface and an earth-centered coordinate system used to define satellite orbits and the position of celestial objects.

The orbital parameters of argument of perigee ω, inclination i, and right ascension of ascending node Ω are Eulerian angles defining the orientation of the Keplerian ellipse (of semimajor axis a) with respect to the celestial sphere. Reference is made to the rotating earth by the coordinate transformation for the sidereal time angle ST and the Greenwich hour angle GHA of the vernal equinox. The input vector to the flow diagrams of Fig. 23 is the position vector from the earth's center to the satellite. Translation to the earth's surface, shown by the added vector summed at node 7, results in the output vector (the vector from the observer to the satellite) not being the same as the input vector. Therefore, the input and output coordinate systems cannot be the same because they are oriented for alignment of the input and output vectors with one of the corresponding coordinate system axes.

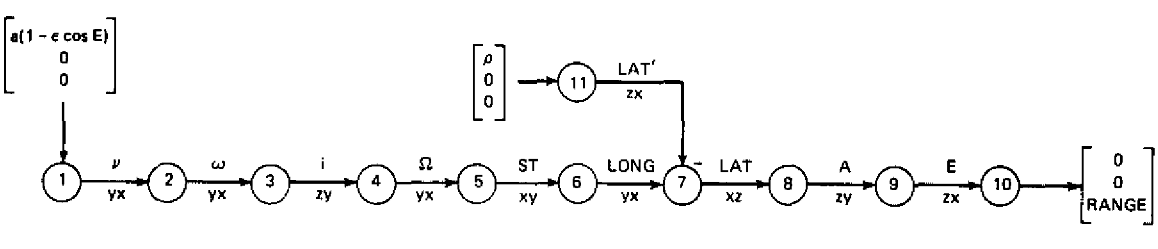


Fig. 23 – Determination of azimuth and elevation pointing angles of a satellite (ellipsoid earth).

The true anomaly ν is related to eccentric anomaly E by

$$\sin \nu = \frac{\sqrt{1-\epsilon^2}\,\sin E}{1-\epsilon\cos E}, \quad \cos \nu = \frac{\cos E - \epsilon}{1-\epsilon\cos E}.$$

E is determined by solving Kepler's equation $M = E - \epsilon \sin E$, typically by an iteration method for known values of mean anomaly M and eccentricity ϵ. The radius vector to the observer on the earth's surface, ρ, may be expressed as a function of the geodetic latitude LAT for the assumed ellipsoidal earth. Similarly, the geocentric latitude LAT′ is also a function of LAT.

Examination of Fig. 23 reveals the possibility of combining the branches between nodes 1 and 3 and between nodes 4 and 7. If a spherical earth is assumed, LAT = LAT′ and $\rho = 1$ for measurement in units of earth radii. Using the equivalence of case V of Table 2, a simpler point for translation is then node 8. The flow diagram for combined branches and assumption of a spherical earth is shown in Fig. 24. By using inputs to node 2 for Fig. 24, direct determination of ν is avoided.

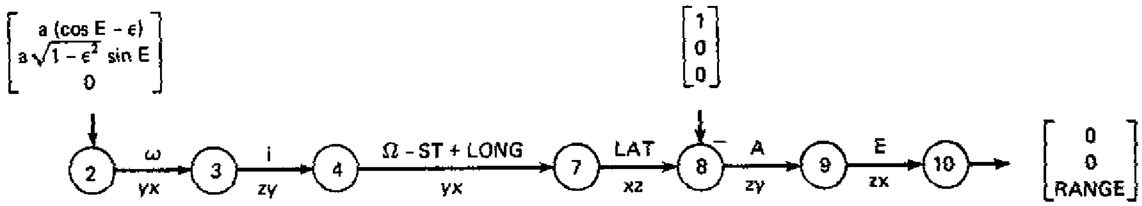


Fig. 24 -- Determination of azimuth and elevation pointing angles of a satellite (spherical earth).

Software solution for a spherical earth might then be of the form

```
.
.
X = A* (COSF(EANOM)-ECCEN)
Y = A* SQRTF (1.0-ECCEN**2) *SINF (EANOM)
Z = 0.
CALL RESOLV1 (Y,X,ARGPER)
CALL RESOLV1 (Z,Y,SINCL)
CALL RESOLV1 (Y,X,RA-SIDTIM+SLONG)
CALL RESOLV1 (X,Z,SLAT)
X = X-1.
CALL RESOLV2 (Z,Y,AZ)
CALL RESOLV2 (Z,X,ELEV)
RANGE = Z.
.
.
```

Similar forms may be used to derive subsatellite data from the orbital parameters or for solution of pointing angles when subsatellite data are given. These flow diagrams may be expanded to include the attitude of moving platforms and required angles of multigimbaled arrangements.

An important consequence of the linear displacement between two coordinate systems is that the attitude of a controlled member cannot, in general, be the same as the attitude implied or defined by the original choice of coordinate system. Similarly, the simple windup and unwind procedures for directing a controlled member cannot be used.

ANGULAR VELOCITY FLOW DIAGRAMS

The concept of angular velocity of one reference frame with respect to another has been exploited in dealing with the time rate of change of a vector. The vector characterization of angular velocity is demonstrated in Appendix A, even though finite rotations do not possess similar characterizations. For the coordinate transformation $\xi' = A\xi$, which relates the unprimed system description of position or other invariant vectors to their description in the primed system, the unprimed system description of the angular velocity of Σ' relative to Σ is given by Eq. (A8). This equation repeated here is

$$\omega = \begin{bmatrix} \mathbf{a}_2 \cdot \dot{\mathbf{a}}_3 \\ \mathbf{a}_3 \cdot \dot{\mathbf{a}}_1 \\ \mathbf{a}_1 \cdot \dot{\mathbf{a}}_2 \end{bmatrix} = \begin{bmatrix} \omega_x \\ \omega_y \\ \omega_z \end{bmatrix}, \tag{A8}$$

where the components of $\mathbf{a}_1$, $\mathbf{a}_2$, $\mathbf{a}_3$ and $\dot{\mathbf{a}}_1$, $\dot{\mathbf{a}}_2$, $\dot{\mathbf{a}}_3$ are given by the column vectors of the matrix A and their time derivatives. When the coordinate transformation is in the xy, yz, or zx planes, simple relations evolve.

In Table 5, the $(k-1)$th system description of angular velocity $\boldsymbol{\omega}_{k-1,k}$ of Σ_k relative to Σ_{k-1} is computed to Eq. (A8) for the corresponding transformation matrix. Since the angular velocity vectors are invariants capable of being described in the $(k-1)$th system or any system having known relation to the $(k-1)$th system, a flow diagram may be constructed expressing the property.

In Fig. 25a, a flow diagram is shown for case I of Table 5 and assumed subsequent planar transformations, the output being the angular velocity of Σ_k relative to Σ_{k-1}, expressed in the $(k+2)$th system.

Since the nonzero component of angular velocity for each case of Table 5 corresponds to the coordinate axis not in the plane of transformation, the $(k-1)$th and kth system descriptions of the angular velocity are the same. The equivalence of Fig. 25a and Fig. 25b is illustrated by case IV of Table 2.

The angular velocity of Σ_{k+2} relative to Σ_{k-1} in correspondence to Eq. (8) is given by

$$\boldsymbol{\omega}_{k-1,k+2} = \boldsymbol{\omega}_{k-1,k} + \boldsymbol{\omega}_{k,k+1} + \boldsymbol{\omega}_{k+1,k+2}\,.$$

Table 5
Basic Planar Transformations

CASE	FLOW DIAGRAM	TRANSFORMATION MATRIX	ANGULAR VELOCITY $\omega_{k-1,k}$
I	$\xi_{k-1} \to (k-1) \xrightarrow[xy]{\alpha} (k) \to \xi_k$ or $\xi_{k-1} \to (k-1) \xrightarrow[yx]{-\alpha} (k) \to \xi_k$	$\begin{bmatrix} \cos\alpha & \sin\alpha & 0 \\ -\sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix}$	$\begin{bmatrix} 0 \\ 0 \\ \frac{d\alpha}{dt} \end{bmatrix}$
II	$\xi_{k-1} \to (k-1) \xrightarrow[yz]{\beta} (k) \to \xi_k$ or $\xi_{k-1} \to (k-1) \xrightarrow[zy]{-\beta} (k) \to \xi_k$	$\begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\beta & \sin\beta \\ 0 & -\sin\beta & \cos\beta \end{bmatrix}$	$\begin{bmatrix} \frac{d\beta}{dt} \\ 0 \\ 0 \end{bmatrix}$
III	$\xi_{k-1} \to (k-1) \xrightarrow[zx]{\gamma} (k) \to \xi_k$ or $\xi_{k-1} \to (k-1) \xrightarrow[xz]{-\gamma} (k) \to \xi_k$	$\begin{bmatrix} \cos\gamma & 0 & -\sin\gamma \\ 0 & 1 & 0 \\ \sin\gamma & 0 & \cos\gamma \end{bmatrix}$	$\begin{bmatrix} 0 \\ \frac{d\gamma}{dt} \\ 0 \end{bmatrix}$

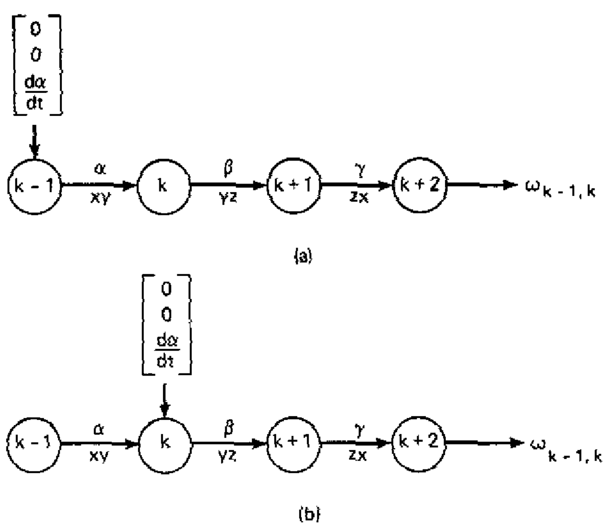


Fig. 25 – Angular velocity $\omega_{k-1,k}$.

It is now apparent that $\omega_{k-1,k+2}$, expressed in the $(k+2)$th system, is given by the flow diagram of Fig. 26a or Fig. 26b.

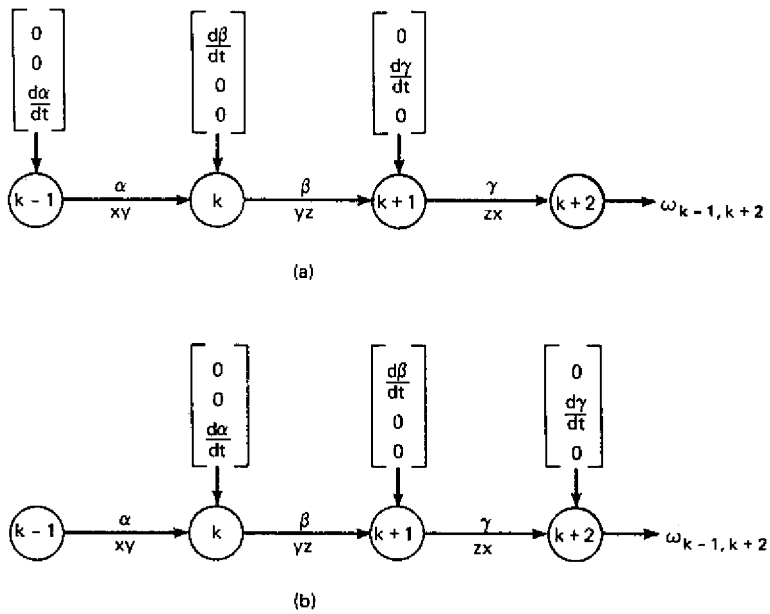


Fig. 26 – Angular velocity $\omega_{k-1,k-2}$.

An angular velocity flow diagram is, therefore, derived by deleting "sourceless" inputs for the basic portion of the original flow diagram associated with position vector relations and then summing at each node the angular velocities between adjacent coordinate systems in the manner of Fig. 26a or 26b. The method of Fig. 26b, however, is preferred and is used in subsequent flow diagrams. When the order of transformation is inverted from the xy, yz, or zx used above, the negative of the time derivative of the angle of rotation is the nonzero component of angular velocity between adjacent systems (see Fig. 27).

If the original basic system of Fig. 26 is inverted and the angular velocity relationships are applied, the output as seen in Fig. 27 is $\boldsymbol{\omega}_{k+2,k-1}$, the angular velocity of Σ_{k-1} relative to Σ_{k+2}.

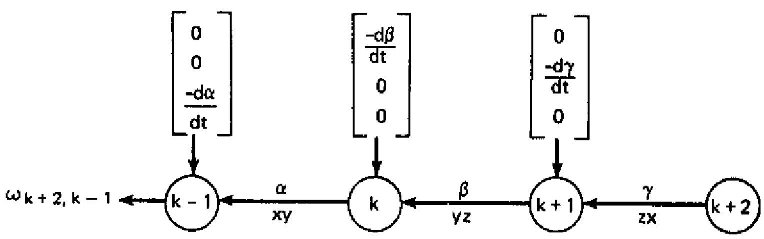


Fig. 27 – Angular velocity $\omega_{k+2,k-1}$.

Procedures for Angular Velocity Flow Diagrams

The following procedures may be summarized for drawing angular velocity flow diagrams:

1. Determine all angle displacements between successive coordinate systems by establishing the flow diagram defining geometric relations and transforming one or more position vectors in the manner previously shown for problems of angular displacements.

2. Modify the above flow diagram, if necessary, so that a path exists in the direction from a selected reference node to the node whose angular velocity relative to the reference node is desired.

3. Delete all branches having no source or no destination node to derive the basic flow diagram depicting geometric relationship of successive coordinate systems.

4. At either successor node of each branch between the reference node and the node whose angular velocity is referenced, input a vector having zero components in the plane of rotation for that branch and set the third component equal to the time derivative of the angle displacement for an order of the coordinate pair of xy, yz, or zx. For an inverted order of any of the above coordinate pairs, set the third component equal to the negative of the time derivative.

5. Provide an output—a branch having no destination node—from the node whose angular velocity is to be determined or any subsequent node in the previously established direction for expression of the angular velocity in one of the corresponding subsequent systems.

6. If expression of the angular velocity is desired in any other system not obtainable in step 5, it may be inputed at the previous output node for the basic diagram of step 3, appropriately modified for the direction of flow, and output then taken from the desired node.

7. As a result of the manner in which angular velocity is defined in Appendix A, the above rules apply equally for right- and left-hand coordinate systems. The angular velocity vector for left-hand systems, however, will have a left-hand sense corresponding to a left-hand thumb rule.

Elevation Over Train

The flow diagram used to determine the angular velocity of Σ_8 relative to Σ_3 for the configuration of Fig. 18 is shown in Fig. 28. Differentiation with respect to time is indicated by $D \triangleq d/dt$. Bd' and Ed' in Fig. 28 are necessarily first determined by solving the flow diagram of Fig. 18.*

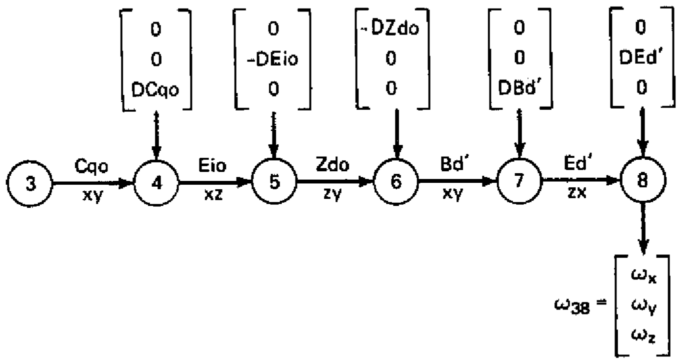


Fig. 28 – Angular velocity for elevation over train: controlled member relative to earth.

If the true elevation E and true bearing By of Fig. 18 are constant or considered to be constant, then $\boldsymbol{\omega}_{38}$ for Fig. 28 has the components $\omega_y = \omega_z = 0$ and ω_x (the cross-roll rate) for the tracking line to be directed in the constant earth-referenced direction. That this is true aside from an intuitive approach may be established by considering a vector $\mathbf{u}$ body-bound to the final system (node 8) and directed along the line of sight and tracking line (the x axis for nodes 3 and 8). The time derivatives of $\mathbf{u}$, when observed from Σ_3 or Σ_8, are equal since $\mathbf{u}$ is of constant direction in these systems. From Eq. (12), $D_3\mathbf{u} = D_8\mathbf{u} + \boldsymbol{\omega}_{38} \times \mathbf{u}$. Since $D_3\mathbf{u} = D_8\mathbf{u}$, then $\boldsymbol{\omega}_{38} \times \mathbf{u} = 0$. The general solution to this is $\boldsymbol{\omega}_{38} = \omega_x\, \mathbf{i}_3 = \omega_x\, \mathbf{i}_8$, where $\mathbf{i}_3$ and $\mathbf{i}_8$ are unit vectors along the x axis of nodes 3 and 8 (the line of sight). Hence, in the system of node 8, $\omega_{38} = (\omega_x, 0, 0)$†.

At this point it is assumed that all quantities of Fig. 28 are known except DBd', Ded', and ω_x. These quantities, however, are constrained for $\omega_y = \omega_z = 0$. Software solution is given by

```
•
•
X = 0.
Y = 0.
Z = DCQO
CALL RESOLV1 (X,Z,EIO)
Y = Y–DEIO
CALL RESOLV1 (Z,Y,ZDO)
```

*In general, a hierachy in solving problems involving angular motion is of the order: angular position, angular velocity, and angular acceleration.

†When dealing with problems involving translation, such as would be involved in determining angular velocity of the controlled member relative to a system-defining position of a satellite (earth-centered and having one axis directed toward the satellite), a more complicated relation exists.

```
X = X–DZDO
CALL RESOLV1 (X,Y,BD)
TEMP = 0.
CALL RESOLV3 (X,TEMP,–ED)
DBD = TEMP–Z
DED = –Y
DXROLL = X
•
•
```

The use of the subroutine RESOLV3 is similar to that shown in case III of Table 4. It is to be noted that reordering of the coordinate pair *zx* to *xz* was required to satisfy Eq. (20) and, hence, RESOLV3. From Fig. 28, it is evident that the train axis provides the cross-roll motion. Irrespective of a changing *E* and *By*, the solution above may always be interpreted as providing the train and elevation rates necessary to overcome platform motion.

If the earth-referenced motion of the tracking line is specified by *DE* and *DBy*, a similar analysis of the angular velocity Σ_8 relative to Σ_1 will provide the required train and elevation rates, which include the earth-referenced tracking problem as well as that for platform motion.

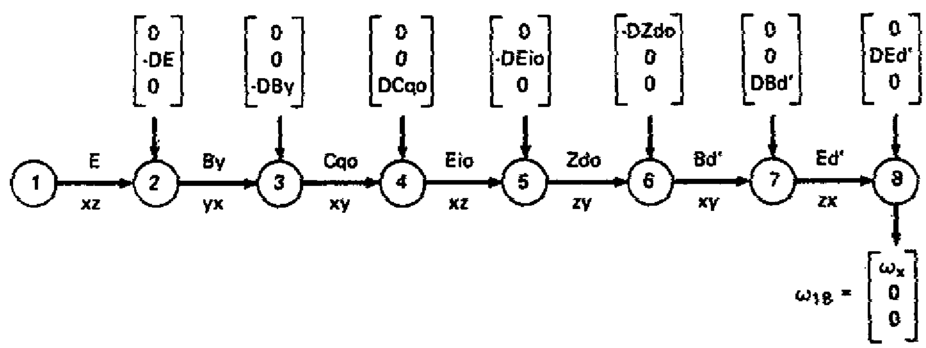


Fig. 29 – Angular velocity for elevation over train: controlled member relative to the system for director designation.

In the more general case of Fig. 29, the requirement for directing the tracking line in a direction prescribed by *E* and *By* is that $\omega_{18} = (\omega_x,0,0)$ in the system of node 8. When transformations and angular rates for true bearing and heading are combined, software solution for *DBd'* and *DEd'* is of the form

```
•
•
X = 0.
Y = –DE
Z = 0.
CALL RESOLV1 (Y,X,BY–CQO)
```

```
Z = Z-DBY+DCQO
CALL RESOLV1 (X,Z,EIO)
Y = Y-DEIO
CALL RESOLV1 (Z,Y,ZDO)
X = X-DZDO
CALL RESOLV1 (X,Y,BD)
TEMP = 0.
CALL RESOLV3 (X,TEMP,-ED)
DBD = TEMP-Z
DED = -Y
DXROLL = X
.
.
```

Elevation Over Cross Level Over Train

When gimbal angles of a three-degree-of-freedom arrangement are controlled so as to preserve attitude of the controlled member with respect to a given reference system, the angular velocity of the controlled member with respect to this reference system is zero. Disregarding intuitive validation of this statement, proof is similar to that for tracking-line control of an elevation-over-train configuration except that two noncollinear vectors body-bound to the controlled member are examined. For the elevation-over-cross-level-over-train configuration, the flow diagram used to determine angular velocity of the controlled member with respect to the earth-referenced attitude (defined by E and By) is shown in Fig. 30.

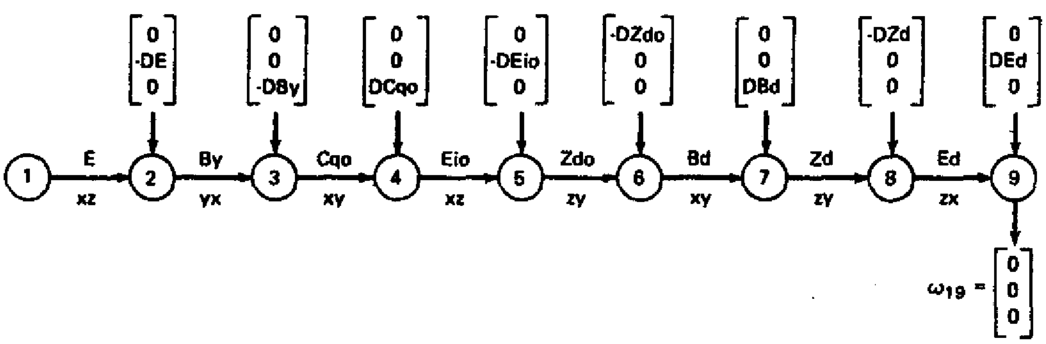


Fig. 30 – Angular velocity for elevation over cross level over train: controlled member relative to the system for director designation.

All values of Fig. 30 are assumed to be known except for DBd, DZd, and DEd. Since at the output of node 8, the components corresponding to the coordinate pair of the last planar transformation must both be zero for zero output velocity, the computer solution is of the form

```
      .
      .
      X = 0.
      Y = -DE.
      Z = 0.
      CALL RESOLV1 (Y,X,BY-CQO)
      Z = Z-DBY+CQO
      CALL RESOLV1 (X,Z,EIO)
      Y = Y-DEIO
      CALL RESOLV1 (Z,Y,ZDO)
      X = X-DZDO
      CALL RESOLV1 (X,Y,BD)
      TEMP = 0.
      CALL RESOLV3 (Y,TEMP,-ZD)
      DBD = TEMP-Z
      DZD = X
      DED = -Y
      .
      .
```

From Fig. 30, it is seen that the cross-level axes counteract the cross roll that would be introduced by train motion and geometric relations. This cross-roll elimination is, of course, with regard to the referenced system, in this case corresponding to node 1. For a changing true elevation and true bearing, there will be a cross-roll component of the controlled member relative to earth (represented by node 3). In considering platform motion only, this cross-roll component is eliminated.

Traverse Over Elevation Over Train

If the traverse-over-elevation-over-train configuration is used to stabilize the attitude of the controlled member, then the angular velocity of the controlled member relative to the initial system is zero and software solution would proceed in a manner similar to that for the previous configuration. When this configuration is used to minimize high angular rates and acceleration rather than maintain a prescribed attitude of the controlled member, cross roll results as indicated in the flow diagram of Fig. 31.

For this manner of control, DBd' and other values of Fig. 31, except DEd', DZe', and ω_x, are assumed to be known. Software solution is straightforward and similar (except for coordinate pairs) to that for the elevation-over-train problem. Cross roll in this case is the result of incomplete compensation between DEd' and DBd'.

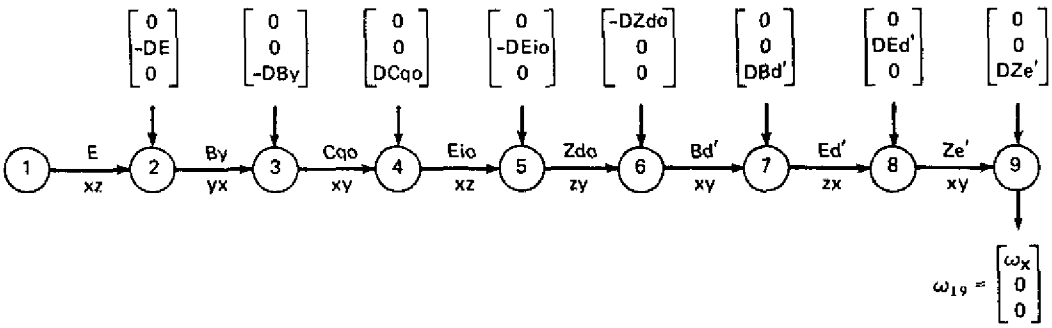


Fig. 31 – Angular velocity for traverse over elevation over train: controlled member relative to the system for director designation.

ANGULAR ACCELERATION FLOW DIAGRAMS

The flow diagrams presented thus far perform two basic functions. First, these diagrams relate component descriptions of the same invariant vector for different coordinate systems from which the vector is being observed. In certain instances, such a coordinate system defines the attitude of a rigid body as well as being a reference frame. Second, these diagrams depict vector relations in problems that involve more than one vector input, previous examples being translation and angular velocity.

The angular velocity of one reference frame with respect to another is shown in Appendix A to possess vector character for three-dimensional space geometry. It has been further shown that time differentiation of a vector must be qualified as to the reference frame from which this vector is being observed. The time derivative of angular velocity, being a vector quantity must, therefore, be similarly qualified.*

The angular acceleration of one reference frame with respect to another as observed from a third frame is the time derivative of the angular velocity vector of the first frame relative to the second frame when this angular velocity vector is observed from the third frame. The resulting angular acceleration, like angular velocity, is an invariant capable of component description in the referenced system or any other system having known relation to the reference system.

For convenience, the operation of taking a time derivative of a vector $\mathbf{u}$ when reviewed from Σ_k is denoted by $D_k \mathbf{u} \triangleq d_k \mathbf{u}/dt$. Accordingly, the angular acceleration of frame Σ_r relative to Σ_p as observed from Σ_q is $D_q\ \boldsymbol{\omega}_{pr}$. When $q = p$,

$$D_p\ \boldsymbol{\omega}_{pr} = D_r\ \boldsymbol{\omega}_{pr} + \boldsymbol{\omega}_{pr} \times \boldsymbol{\omega}_{pr} = D_r\ \boldsymbol{\omega}_{pr}, \tag{26a}$$

*Angular velocity, by virtue of the manner in which it was defined and not being the time derivative of a vector, does not require similar qualification.

since the cross product of a vector with itself vanishes. By using the notation ${}_q(\boldsymbol{\alpha}_{pr})$ for $D_q\ \boldsymbol{\omega}_{pr}$,

$$_p(\boldsymbol{\alpha}_{pr}) = {}_r(\boldsymbol{\alpha}_{pr}). \tag{26b}$$

The angular acceleration of one frame relative to another, therefore, is the same when observed from either of the two frames.

For Σ_p and Σ_r separated by a number of intermediate frames, from Eq. (8)

$$\boldsymbol{\omega}_{pr} = \sum_{q=p+1}^{r} \boldsymbol{\omega}_{q-1,\,q}.$$

Hence, observed from Σ_p, the angular acceleration of Σ_p relative to Σ_r is

$$_p(\boldsymbol{\alpha}_{pr}) = D_p\ \boldsymbol{\omega}_{pr} = D_p \sum_{q=p+1}^{r} \boldsymbol{\omega}_{q-1,\,q} = \sum_{q=p+1}^{r} D_p\ \boldsymbol{\omega}_{q-1,\,q}. \tag{27}$$

Using Eq. (12)

$$_p(\boldsymbol{\alpha}_{pr}) = \sum_{q=p+1}^{r} (D_q\ \boldsymbol{\omega}_{q-1,q} + \boldsymbol{\omega}_{pq} \times \boldsymbol{\omega}_{q-1,q}). \tag{28}$$

From Eq. (26) and since $\boldsymbol{\omega}_{pq} = \boldsymbol{\omega}_{p,q-1} + \boldsymbol{\omega}_{q-1,q}$, other forms for ${}_p(\boldsymbol{\alpha}_{pr})$ are

$$_p(\boldsymbol{\alpha}_{pr}) = \sum_{q=p+1}^{r} (D_{q-1}\ \boldsymbol{\omega}_{q-1,q} + \boldsymbol{\omega}_{pq} \times \boldsymbol{\omega}_{q-1,q}) \tag{29}$$

$$_p(\boldsymbol{\alpha}_{pr}) = \sum_{q=p+1}^{r} (D_q\ \boldsymbol{\omega}_{q-1,q} + \boldsymbol{\omega}_{p,q-1} \times \boldsymbol{\omega}_{q-1,q}) \tag{30}$$

$$_p(\boldsymbol{\alpha}_{pr}) = \sum_{q=p+1}^{r} (D_{q-1}\ \boldsymbol{\omega}_{q-1,q} + \boldsymbol{\omega}_{p,q-1} \times \boldsymbol{\omega}_{q-1,q}). \tag{31}$$

Since

$$\boldsymbol{\omega}_{p,q-1} \times \boldsymbol{\omega}_{q-1,q} = \boldsymbol{\omega}_{p,q-1} \times (\boldsymbol{\omega}_{p,q-1} + \boldsymbol{\omega}_{q-1,q}) = \boldsymbol{\omega}_{p,q-1} \times \boldsymbol{\omega}_{pq},$$

$$_p(\boldsymbol{\alpha}_{pq}) = \sum_{q=p+1}^{r} (D_q\ \boldsymbol{\omega}_{q-1,q} + \boldsymbol{\omega}_{p,q-1} \times \boldsymbol{\omega}_{pq}) \tag{32}$$

and

$$_p(\boldsymbol{\alpha}_{pq}) = \sum_{q=p+1}^{r} (D_{q-1}\, \boldsymbol{\omega}_{q-1,q} + \boldsymbol{\omega}_{p,q-1} \times \boldsymbol{\omega}_{pq}). \tag{33}$$

The term to the right of the summation sign of Eq. (27) may be regarded as relative angular accelerations between successive frames as observed from either Σ_p or Σ_r (since $D_p\, \boldsymbol{\omega}_{pr} = D_r\, \boldsymbol{\omega}_{pr}$). The term $_p(\boldsymbol{\alpha}_{pr})$ may then be expressed by

$$_p(\boldsymbol{\alpha}_{pr}) = {}_r(\boldsymbol{\alpha}_{pr}) = \sum_{q=p+1}^{r} {}_p(\boldsymbol{\alpha}_{q-1,q}) \tag{34a}$$

in a manner similar to that for angular velocity in Eq. (8). When the frame of observation is different from either of the two frames for which relative angular acceleration is to be determined,

$$D_m\, \boldsymbol{\omega}_{pr} = -D_m\, \boldsymbol{\omega}_{mp} + D_m\, \boldsymbol{\omega}_{mr}$$

or

$$_m(\boldsymbol{\alpha}_{pr}) = -{}_m(\boldsymbol{\alpha}_{mp}) + {}_m(\boldsymbol{\alpha}_{mr}). \tag{34b}$$

Hence

$$_m(\boldsymbol{\alpha}_{pr}) = -\sum_{q=m+1}^{p} {}_m(\boldsymbol{\alpha}_{q-1,q}) + \sum_{q=m+1}^{r} {}_m(\boldsymbol{\alpha}_{q-1,q}). \tag{34c}$$

If in system $q-1$, $\boldsymbol{\omega}_{q-1,q}$ is described by the column matrix $\omega_{q-1,q}$, $D_{q-1}\,\boldsymbol{\omega}_{q-1,q}$ (or $D_q\,\boldsymbol{\omega}_{q-1,q}$) is then given by $D\omega_{q-1,q}$ in the same system. Moreover, since $\boldsymbol{\omega}_{q-1,q}$ is described for the basic planar transformations by the same column matrix (see Fig. 25) in both systems $q-1$ and q, $D\omega_{q-1,q}$ also provides the q description of $D_{q-1}\,\boldsymbol{\omega}_{q-1,q}$ (or $D_q\,\boldsymbol{\omega}_{q-1,q}$). Using consistent notation, a flow diagram for the angular acceleration of Σ_p relative to Σ_r (together with that for angular velocity) may then be composed of a chain of stages for frames between Σ_p and Σ_r. The most direct form for an angular acceleration flow diagram is based on Σ_p being the frame of observation. Eight possible forms of a typical stage are shown in Figs. 32a through 32f, corresponding to Eqs. (28) through (33), respectively. For frames of observation other than ones for which angular acceleration is being determined, a combination flow diagram based on Eq. (34c) is used.

For any form used, caution must be used in insuring that inputs to the cross-product operation box are with reference to the same system, since vector representation in the flow diagram deals with component description and not the invariant forms. Subsequent flow diagrams dealing with angular acceleration generally employ the form of Fig. 32c, although any of the eight forms shown are equally applicable.

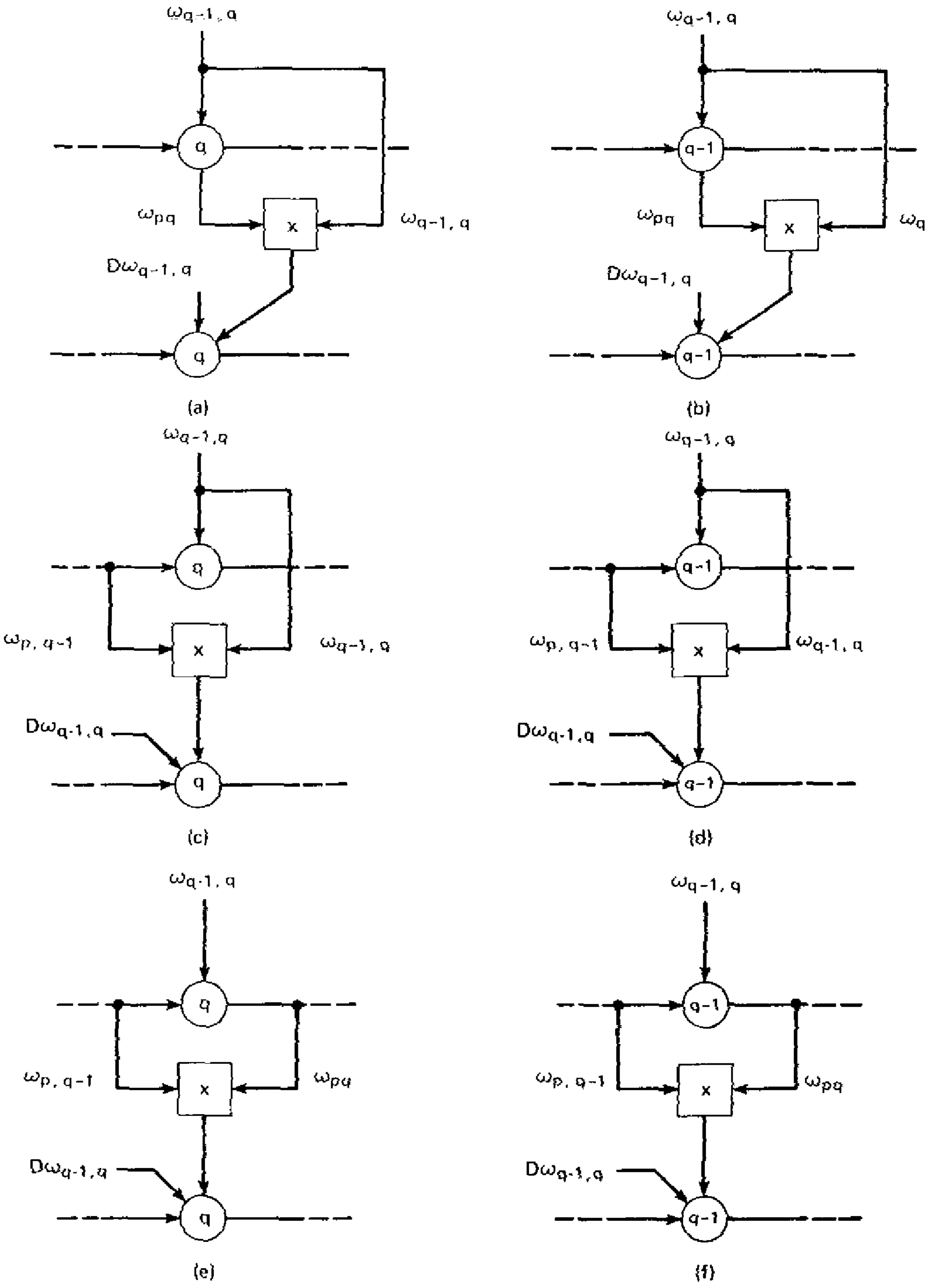


Fig. 32 – Possible forms in a typical stage for angular velocity-acceleration.

In Fig. 33, several examples of flow diagrams are given for angular acceleration. In Figs. 33b and 33c, the frame of observation differs from the frames for which angular acceleration is determined. The first cross-product term in Figs. 33a and 33b and that for each portion of Fig. 33c are, of course, zero, since one of the input vectors in each case is zero. These cross products are drawn in Fig. 33, however, to demonstrate the general applicability of the form of Fig. 32d. In Fig. 33c, the flow diagram is drawn to show the output described in system 3.

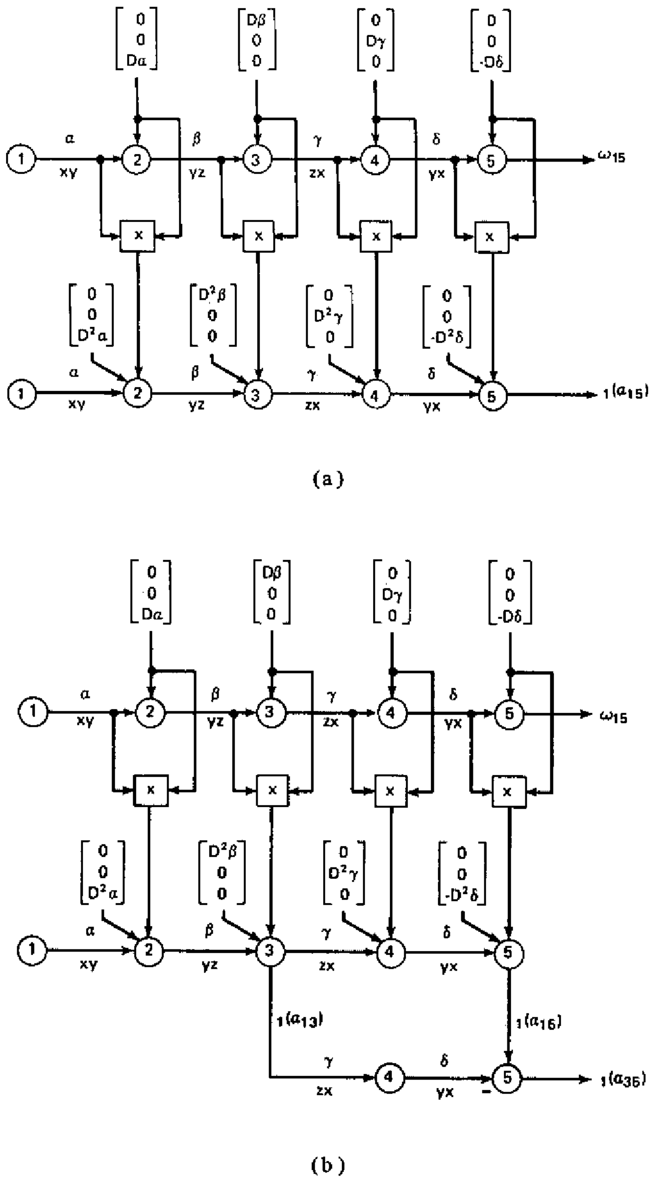
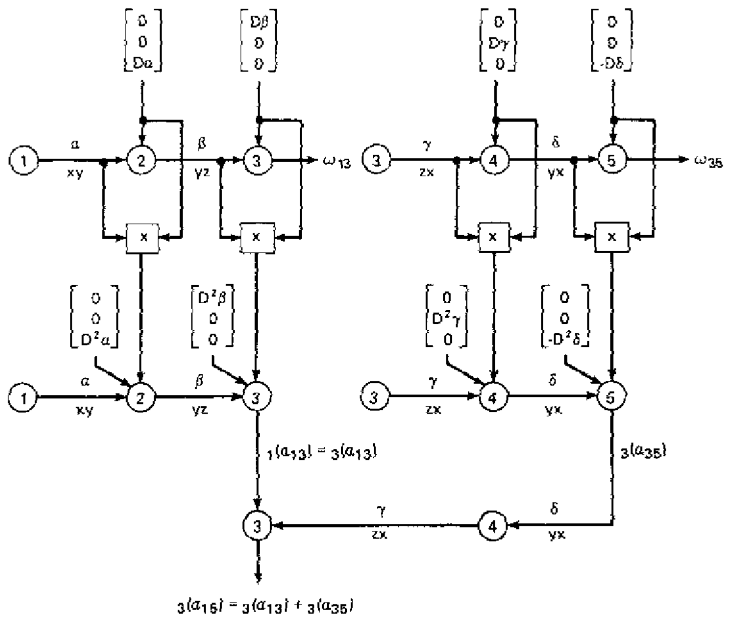


Fig. 33 – Examples of angular velocity-acceleration flow diagrams.



(c)

Fig. 33 – Example of angular velocity-acceleration flow diagram.

Procedures for Angular Acceleration-Velocity Flow Diagrams

The following procedures may be used to construct a dual flow diagram for angular acceleration and velocity of one frame represented by a node relative to the frame represented by a reference node and, in the case of angular acceleration, as observed from either of these frames. For more general frames of observation, a combination of flow diagrams based on these procedures is required to implement Eq. (34c).

1. Initially construct two flow diagrams, each in accordance with the previously given procedures for angular velocity except that one diagram is for angular acceleration. In the latter, second-order time derivatives of the angle displacement are used instead of the first order.

2. Provide on the angular velocity diagram a branch from a tie point (or dummy node) on either side of each node or from the node itself and input this to the left side of a cross-product operation box. The other input to this operation box is then taken from a tie point on the branch for the sourceless angular velocity input to that node. Alternately, the points on the input and output sides of the node may be used to provide left- and right-side inputs, respectively, to the cross-product operation box. The output from the cross-product box is inputed to the node in the angular acceleration diagram having the same number.

3. The output angular velocity and acceleration taken from the desired node of each of the two portions of the dual flow diagram are for that node relative to the reference node and expressed in the system of the output node.* The angular acceleration is further qualified as being observed from the frames corresponding to the reference or output nodes. If expression of either angular velocity or acceleration is desired in other systems, the procedure previously given for angular velocity is applicable.

4. In a manner similar to that for angular velocity, the angular acceleration vector will have a left-hand sense when left-hand coordinate systems are employed. This is a consequence of the definitions of the angular velocity vector (Appendix A) and the cross-product operation (Eq. (22)).

Elevation Over Train

Through reasoning similar to that for angular velocity, it may be established that the vector giving angular acceleration of the controlled member of the two-axis elevation-over-train configuration, relative to a frame (and as observed from this frame) prescribing the space direction by true elevation E and true bearing By, must be collinear with this direction. In the flow diagram of Fig. 34, which is based on the same choice of coordinate systems as Fig. 18, this angular acceleration vector is $D_1\ \boldsymbol{\omega}_{18}$. Since $\boldsymbol{\omega}_{18}$ is described by $(\omega_x,0,0)$ in system 8 for the tracking line to be directed in accordance with E and By (see Fig. 29), $D_8\ \boldsymbol{\omega}_{18}$ in the same system is described by $(D\omega_x,0,0)$. From Eq. (26a), $D_1\ \boldsymbol{\omega}_{18} = D_8\ \boldsymbol{\omega}_{18}$ and, therefore, is collinear with the controlled direction.

Software implementation of the angular velocity portion of Fig. 34 is basically the same as that for Fig. 29. Solution of angular acceleration is performed in a similar manner, except for the addition of a cross-product term at each node. Both may be solved concurrently or the angular velocity portion may be treated first, storing necessary vectors for later solution of angular acceleration. Again By and Cqo may be combined, either in a modified flow diagram or mentally, prior to writing a software program.

A new subroutine performing the cross product of two vectors and adding the result to a third is defined by

```
SUBROUTINE XADD (U,VX,VY,VZ,W)
DIMENSION U(3), W(3)
W(1) = W(1)+U(2)*VZ-U(3)*VY
W(2) = W(2)+U(3)*VX-U(1)*VZ
W(3) = W(3)+U(1)*VY-U(2)*VX
RETURN
END.
```

*In this case, the components of output angular velocity and acceleration for this node are simply related in that each component of angular acceleration is the time derivative of the corresponding component of angular velocity. This is a consequence of Eq. (26a) and is not such a trivial result as it might initially seem.

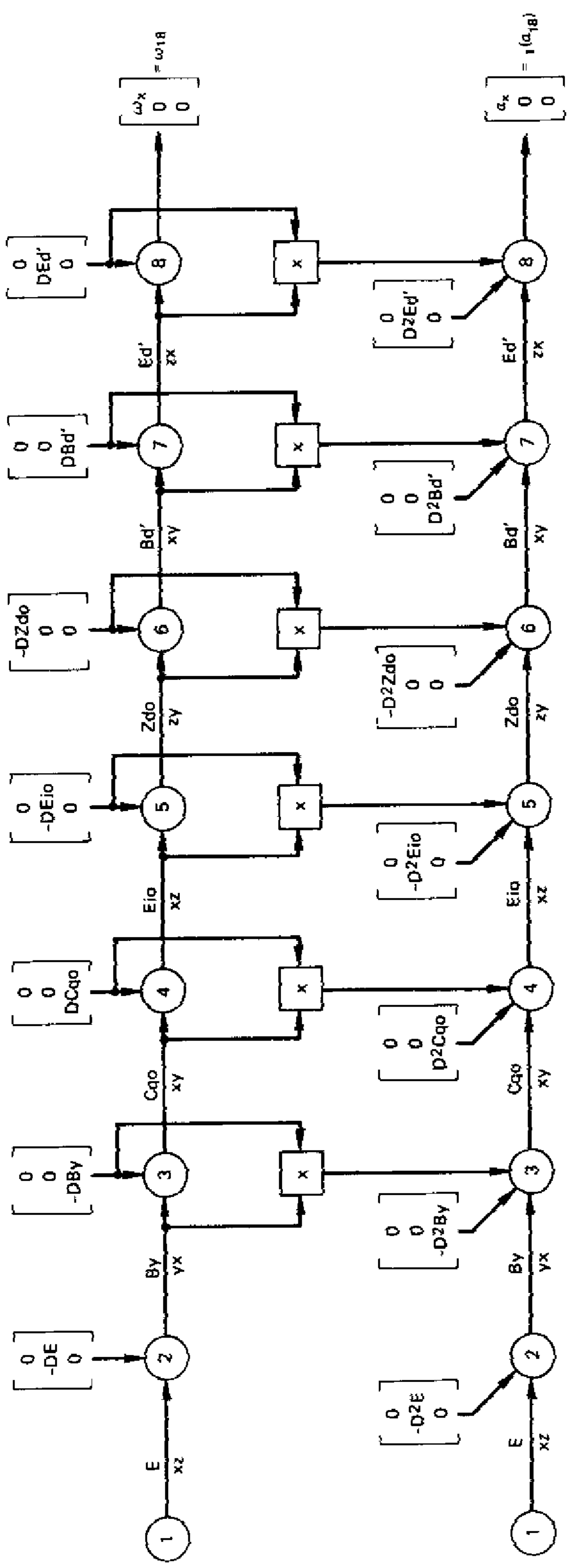


Fig. 34 -- Angular velocity-acceleration for elevation over train: controlled member relative to the system for director designation.

With Bd' and Ed' having been previously determined by solving Fig. 18, concurrent solution of both angular velocity and acceleration may take the form shown below. The angular acceleration portion of the software program is indented for clarity. The dashed lines show how the program would be modified if the form of Fig. 32a had been used instead of Fig. 32c. The vectors A, V, and W necessarily require proper dimensioning and equivalence to AX, AY, AZ for A, etc.

```
        .
        .
        VX = 0.
        VY = -DE
        VZ = 0.
             AX = 0.
             AY = D2E
             AZ = 0.
   ----->CALL RESOLV1 (VY,VX,BY-CQO)
  |          CALL RESOLV1 (AY,AX,BY-CQO)
  |          AZ = AZ-D2BY+D2CQO
  |          CALL XADD (V,0.,0.,-DBY+DCQO,A)
   ------VZ = VZ-DBY+DCQO
   ----->CALL RESOLV1 (VX,VZ,EIO)
  |          CALL RESOLV1 (AX,AZ,EIO)
  |          AY = AY-D2EIO
  |          CALL XADD (V,0.,-DEIO,0.,A)
   ------VY = VY-DEIO
   ----->CALL RESOLV1 (VZ,VY,ZDO)
  |          CALL RESOLV1 (AZ,AY,ZDO)
  |          AX = AX-D2ZDO
  |          CALL XADD (V,-DZDO,0.,0.,A)
   ------VX = VX-DZDO
        CALL RESOLV1 (VX,VY,BD)
             WX = VX
             WY = VY
             WZ = VZ
        TEMP = 0.
        CALL RESOLV3 (VX,TEMP,-ED)
        DBD = TEMP-VZ
        DED = -VY
        DXROLL = VX
             CALL RESOLV1 (AX,AY,BD)
             CALL XADD (W,0.,0.,DBD,A)
             TEMP = 0.
             CALL RESOLV3 (AX,TEMP,-ED)
             D2BD = TEMP-AZ
             D2ED = -AY
             D2XROLL = AX
        .
        .
```

Similar dual flow diagrams may be constructed for the three-axis configurations (and the assumed modes of control) of Figs. 30 and 31. When attitude control of the controlled member is the objective of the control mode for a three-axis design, the angular acceleration of the controlled member relative to the frame defining the desired attitude is zero when observed from either of these two frames. Again, the forms of software implementation are basically the same for the angular velocity and acceleration portions, except a cross-product term is added to each node of the latter. Figure 35 is the angular velocity-acceleration flow diagram for a traverse-over-elevation-over-train configuration, the mode of control being to minimize high angular rates and accelerations.

The preceding methods for determining angular velocity and acceleration of gimbal angles deal with direct computation. It is certainly possible (and often done) to perform only direct computation of gimbal angular position and then compute the angular velocity and acceleration for each gimbal through numerical differentiation techniques. This, however, necessitates computation of a number of values to determine angular acceleration. For a number of successively tabulated values with time as argument, this becomes a fixed overhead. When isolated or nonsuccessive values are desired, the excess sometimes becomes burdensome. Furthermore, care must be exercised in regions near a singularity to insure that numerical differentiation yields accurate results. The flow-diagram technique enables direct visualization of the factors involved from which accurate, direct computation of isolated time values can be performed for a variety of assumed conditions. This allows convenient, separate adjustment of angular position, velocity, and acceleration for yaw, pitch, and roll, a useful capability since these are far more independent than simple sinusoidal approximation would suggest. In many problems, such as the determination of forces and moment, the direct technique is a necessary part of a flow-diagram procedure.

Efficiency when using direct computation may be improved by the previously shown method of storing sine and cosine values of pertinent angles and using the more generalized definition of the RESOLV1 subroutine. If the problem is of a nature that allows use of numerical differentiation, the flow-diagram procedure may be used up to the intermediate level where such differentiation is employed.

ANGULAR MOMENTUM AND ITS TIME RATE OF CHANGE

Up to now no reference frame or coordinate system has been given preferential treatment. Quite obviously a coordinate system used to establish a desired space direction or attitude which may be rapidly changing does not, in general, define a reference frame in which Newton's Second Law of Motion is valid. In dealing with the concept of angular momentum of a rigid body, however, we are generally concerned with the total moment of inertially referenced linear momentum about a point.

Presuming the existence of an inertial frame of reference for which Newton's Second Law of Motion is valid, the angular momentum H of a rigid body about a point O of this body, fixed with respect to the inertial frame, is the integral

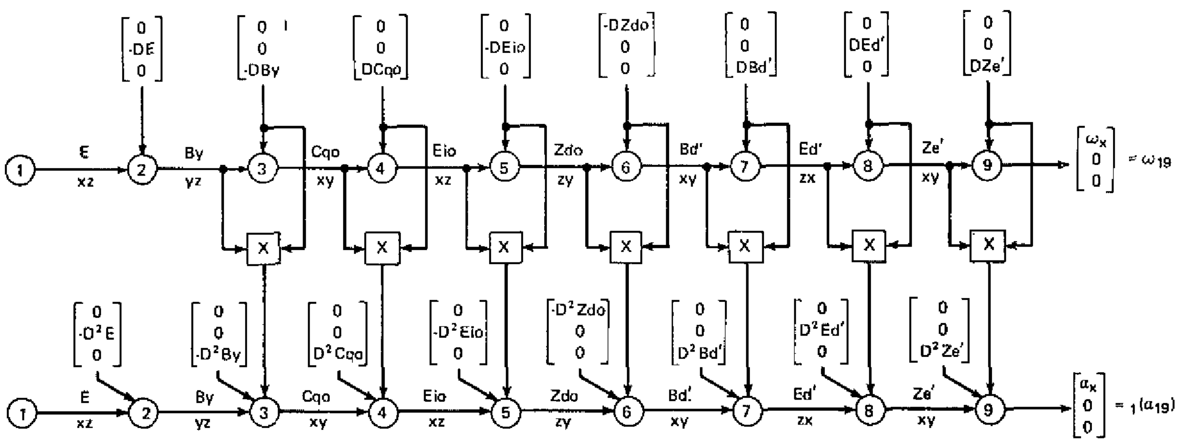


Fig. 35 – Angular velocity-acceleration for traverse over elevation over train: controlled member relative to the system for director designation.

$$\mathbf{H} = \int_B \mathbf{r} \times \mathbf{v}\, dm, \tag{35}$$

where $\mathbf{r}$ is the position vector (referenced to an inertial frame) to each elemental mass dm and $\mathbf{v}$ is the inertially observed and referenced velocity of dm. In the usual engineering parlance, where quantities about an axis* rather than a point are normally considered, the angular momentum about an axis passing through O parallel to the unit vector $\mathbf{e}$ is $\mathbf{H} \cdot \mathbf{e}$. By taking the time derivative of $\mathbf{H}$ observed from an inertial system,

$$D_i\mathbf{H} = D_i \int_B \mathbf{r} \times \mathbf{v}\, dm = \int_B \left(D_i\mathbf{r} \times \mathbf{v} + \mathbf{r} \times D_i\mathbf{v}\right) dm.$$

Since $\mathbf{v} = D_i\mathbf{r}$, $D_i\mathbf{r} \times \mathbf{v} = \mathbf{0}$. Replacing $D_i\mathbf{v}$ by $\mathbf{a}$ for linear acceleration†,

$$D_i\mathbf{H} = \int_B \mathbf{r} \times \mathbf{a}\, dm = \Gamma\,. \tag{36}$$

By Newton's Second Law of Motion, the term $\mathbf{a}\, dm$ is equal to the combined effect of external forces and forces of constraint (rigid-body constraint) acting on the elemental mass. In integrating over the full rigid body only the external forces are effective, so that Γ is the externally applied torque or moment of force about O. The torque about an axis through O, parallel to the unit vector $\mathbf{e}$, is $D_i\mathbf{H} \cdot \mathbf{e}$.

Since any motion of this rigid body must be the result of a rotation about an axis through O (which is fixed in this inertial frame),

$$\mathbf{H} = \int_B \mathbf{r} \times \mathbf{v}\, dm = \int_B \mathbf{r} \times (\boldsymbol{\omega}_{iB} \times \mathbf{r})\, dm. \tag{37}$$

*Quantities about an axis are generally expressed as scalars whereas those about a point are higher rank invariants, such as the vector angular momentum, torque, and the inertia tensor.

†Both reference and observation qualification is required for linear acceleration as well as angular acceleration, although a point reference is used in the former in contrast to the frame reference in the latter. Since the frame of observation is generally an assumed inertial frame and the point of reference is frequently fixed in this frame, special notation is avoided in the case of linear acceleration. In instances where this is deviated from, as in the discussion on Corolis acceleration, specific qualification is noted.

Expanding the vector triple product,

$$\mathbf{H} = \int_B \left[\boldsymbol{\omega}_{iB}\, \mathbf{r}^2 - \mathbf{r}(\mathbf{r} \cdot \boldsymbol{\omega}_{iB})\right] dm, \tag{38}$$

where $\mathbf{r}^2$ is $\mathbf{r} \cdot \mathbf{r}$. By defining the inertia tensor with respect to O to be

$$\mathbf{J} = \mathbf{I} \int_B \mathbf{r}^2\, dm - \int_B \mathbf{r}\mathbf{r}\, dm, \tag{39}$$

where $\mathbf{I}$ is the idem tensor (described by the unit matrix in any system),

$$\mathbf{H} = \mathbf{J} \cdot \boldsymbol{\omega}_{iB}. \tag{40}$$

It is seen that angular momentum of a rigid body about a point of that body fixed with respect to an inertial frame is a linear function of the angular velocity vector. Component description of the inertia tensor is frequently given with respect to a body-bound system by the matrix

$$J = \begin{bmatrix} J_{xx} & J_{xy} & J_{xz} \\ J_{yx} & J_{yy} & J_{yz} \\ J_{zx} & J_{zy} & J_{zz} \end{bmatrix}, \tag{41}$$

because this representation is unchanging regardless of motion of the rigid body.

From the symmetry of Eq. (39), the matrix (component) description given by Eq. (41) is also symmetrical. Inasmuch as J is symmetric, a set of three mutually perpendicular axes (principal axes) may be found which diagonalizes J. The angular momentum vector, in general, is not parallel to the angular velocity vector except when the latter is directed along one of the principal axes. The moment of inertia about an axis is given by $\mathbf{e} \cdot \mathbf{J} \cdot \mathbf{e}$, where $\mathbf{e}$ is a unit vector defining that axis. Using Eq. (39), this moment of inertia is

$$J_{ee} = \mathbf{e} \cdot \mathbf{J} \cdot \mathbf{e} = \int_B \left[\mathbf{r}^2 - (\mathbf{r} \cdot \mathbf{e})^2\right] dm = \int_B p^2\, dm, \tag{42}$$

where p is the perpendicular distance from this axis to the elemental mass dm. This conforms to familiar engineering definitions.

J_{xx}, J_{yy}, J_{zz} of Eq. (41) are the moment of inertia about the x, y, and z axes, respectively, and are given by

$$J_{xx} = \mathbf{i} \cdot \mathbf{J} \cdot \mathbf{i} = \int_B (r^2 - x^2)\, dm,$$

$$J_{yy} = \mathbf{j} \cdot \mathbf{J} \cdot \mathbf{j} = \int_B (r^2 - y^2)\, dm,$$

$$J_{zz} = \mathbf{k} \cdot \mathbf{J} \cdot \mathbf{k} = \int_B (r^2 - z^2)\, dm.$$

$J_{xy} = J_{yx}$, $J_{yz} = J_{zy}$ and $J_{zx} = J_{xz}$ are products of inertia for the corresponding axes and are given by

$$J_{xy} = -\mathbf{i} \cdot \mathbf{J} \cdot \mathbf{j} = \int_B xy\, dm$$

$$J_{yz} = -\mathbf{j} \cdot \mathbf{J} \cdot \mathbf{k} = \int_B yz\, dm$$

$$J_{zx} = -\mathbf{k} \cdot \mathbf{J} \cdot \mathbf{i} = \int_B zx\, dm.$$

In general, as the rigid body moves about the point O fixed to an inertial reference frame, **J**, when observed from this frame, is also a function of time. The matrix description of **J** in the inertial frame will, therefore, have time-varying coefficients. Transformed to a body-bound reference frame, the coefficients will be constant (so long as the body remains rigid). As observed from this body-bound frame, **J** is reckoned to be non-time-varying.* A preferential coordinate system, of course, is given by the three principal axes for which the matrix is diagonalized. The external torque about the inertially fixed point O of Eqs. (36) and (12) is

$$\Gamma = D_i\, \mathbf{H} = D_i(\mathbf{J} \cdot \boldsymbol{\omega}_{iB}) = D_B\, (\mathbf{J} \cdot \boldsymbol{\omega}_{iB}) + \boldsymbol{\omega}_{iB} \times (\mathbf{J} \cdot \boldsymbol{\omega}_{iB}). \tag{43}$$

A frequent situation encountered in gimbaled structures is that it is desired to know the torque about a point on a body which does not necessarily coincide with the center of mass and which further has motion of its own with respect to an assumed inertial frame†. For example, in a single-axis antenna on a moving platform, motion of the "point" of gimbaling results when the platform is subject to roll, pitch, and yaw angular motions and sway, surge, and heave translational motions. The effect of gravity may be included as a constant added to heave acceleration.

In Fig. 36, the point P, fixed to the rigid body B having a total mass M, is given by the position vector $\mathbf{r}_p$ from O, which is fixed with respect to the assumed inertial frame. Position vectors **r** and **r**′ of any point Q of B are with reference to O and P, respectively. The moment of force or torque about P is‡

$$\begin{aligned}\Gamma_P &= \int_B \mathbf{r}' \times D_i\mathbf{v}\, dm \\ &= \int_B \mathbf{r}' \times D_i\, (\mathbf{v}_P + \mathbf{v}')\, dm \\ &= \left(\int_B \mathbf{r}'\, dm\right) \times D_i\, \mathbf{v}_P + D_i \int_B \mathbf{r}' \times \mathbf{v}'\, dm,\end{aligned}$$

*By defining cross-product operations between a tensor of the second rank and vector, an expression in the manner of Eq. (12) may be used to relate the time derivative of **J** as observed from either of two frames. This relation from Milne (5) is $D_a\mathbf{J} = D_b\mathbf{J} + \boldsymbol{\omega}_{ab} \times \mathbf{J} - \mathbf{J} \times \boldsymbol{\omega}_{ab}$.

†The assumed inertial frame takes one of two possibilities in this report. The first, which is generally adequate for the determination of antenna drive torques, assumes the earth to establish such a frame. For problems of inertial navigation and related components (which may be used for geometric stabilization of antennas affixed to a moving base), a reference frame having its origin at the center of the earth, but not rotating with respect to the "fixed" stars, is used. A further refinement could treat such a frame as being fixed at the center of the sun.

‡This is also given by the rate of change of angular momentum about a fixed point with which the movi point P coincides momentarily.

since $\mathbf{v}'$, the velocity of Q relative to P as observed from the inertial frame, is given by $D_i\,\mathbf{r}'$ and $D_i\,\mathbf{r}' \times \mathbf{v}' = 0$. Since $\mathbf{r}'_G$, the position vector from P to the center of mass G, is given by

$$\mathbf{r}'_G = \frac{\int_B \mathbf{r}'\,dm}{M}$$

and by letting $\mathbf{a}_P = D_i\,\mathbf{v}_P$,

$$\Gamma_P = \mathbf{r}'_G \times M\,\mathbf{a}_P + D_i \int_B \mathbf{r}' \times \mathbf{v}'\,dm. \tag{44}$$

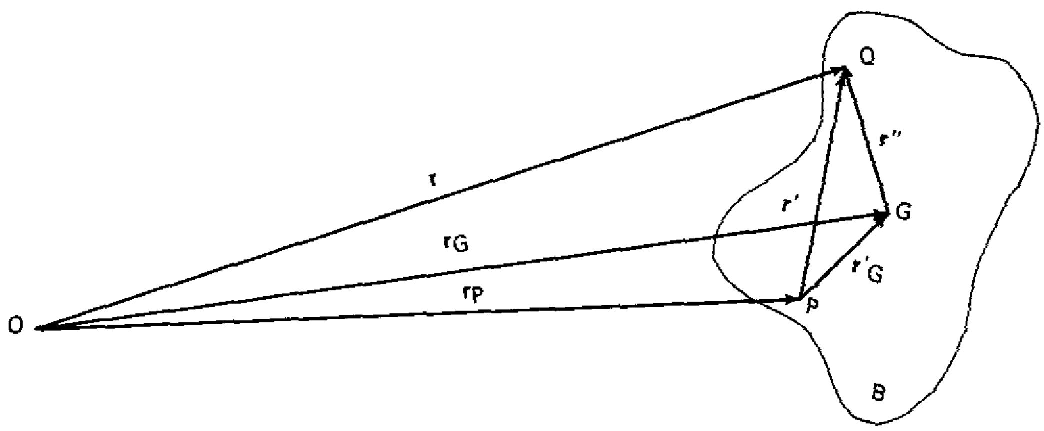


Fig. 36 – Rotation about a moving point.

In effect, Eq. (44) shows that the torque about P is composed of (a) the torque about P due to the force for accelerating (relative to an inertial frame) the entire mass concentrated at the center of mass with an acceleration equal to that of P and (b) the time derivative of angular momentum about P of the motion relative to P. If either P is located at the center of mass or if P is moving uniformly with respect to an inertial frame, the first term of Eq. (44) vanishes. The component of the torque vector along the axis of gimbaling is the load torque a drive would have to provide.* The other two orthogonal components are torques which must be sustained by the gimbal bearings in addition to other loads resulting from forces of constraint.

An alternate expression for Eq. (44), obtained by defining $\mathbf{H}'_P = \int_B \mathbf{r}' \times \mathbf{v}'\,dm$, is

$$\Gamma_P = \mathbf{r}'_G \times M\mathbf{a}_P + D_i\mathbf{H}'_P, \tag{45}$$

*The effects of gearing and motor inertia are not included and must be separately dealt with as their motions are generally different from that of the gimbaled body.

where $\mathbf{H}'_P$ is the angular momentum about P of the motion relative to P. Since motion for $\mathbf{H}'_P$ is relative to P, $\mathbf{v}' = \boldsymbol{\omega}_{iB} \times \mathbf{r}'$. Although Eq. (40) is concerned with angular momentum of a rigid body about a fixed point, it may be generalized about any point for motion relative to that point. Therefore,

$$\Gamma_P = \mathbf{r}'_G \times M\mathbf{a}_P + D_i\,(\mathbf{J}_P \cdot \boldsymbol{\omega}_{iB}). \tag{46}$$

An alternate form may be derived to relate M_P to the angular momentum about G of the motion relative to G. From Fig. 36, by substituting $\mathbf{r}'_G + \mathbf{r}''$ for $\mathbf{r}'$ and $D_i\,(\mathbf{r}'_G + \mathbf{r}'')$ for $\mathbf{v}'$ and since $\int_B \mathbf{r}''\,dm = 0$ (as the origin of $\mathbf{r}''$ is the center of mass),

$$\int_B \mathbf{r}' \times \mathbf{v}'\,dm = \mathbf{r}'_G \times M\mathbf{v}'_G + \int_B \mathbf{r}'' \times \mathbf{v}''\,dm.$$

Since $\mathbf{r}_G = \mathbf{r}_P + \mathbf{r}'_G$ and $D_i\mathbf{r}_G = \mathbf{v}_G = \mathbf{v}_P + \mathbf{v}'_G$, Eq. (44) becomes

$$\Gamma_P = \mathbf{r}'_G \times M\mathbf{a}_G + D_i \int_B \mathbf{r}'' \times \mathbf{v}''\,dm. \tag{47}$$

$$\Gamma_P = \mathbf{r}'_G \times M\mathbf{a}_G + D_i\mathbf{H}''_G. \tag{48}$$

Since motion for $\mathbf{H}''_G$ is relative to G,

$$\Gamma_P = \mathbf{r}'_G \times M\mathbf{a}_G + D_i\,(\mathbf{J}_G \cdot \boldsymbol{\omega}_{iB}).^* \tag{49}$$

An alternate statement with regard to the torque about the point P in Fig. 36 is, therefore, that this torque is composed of (a) the torque about P due to the force for accelerating (relative to an inertial frame) the entire mass concentrated at the center of mass with an acceleration equal to that of the center of mass G and (b) the time derivative of angular momentum of G of the motion relative to G.

If either P is located at the center of mass of if the center of mass is moving uniformly with respect to an inertial frame, the first term of Eq. (47) vanishes. While the first condition is the same as that for Eq. (44), the second condition is quite different for the two expressions.

Using Eq. (12) to modify Eq. (46), the torque about the moving point P is

$$\Gamma_P = \mathbf{r}'_G \times M\mathbf{a}_P + D_B(\mathbf{J}_P \cdot \boldsymbol{\omega}_{iB}) + \boldsymbol{\omega}_{iB} \times (\mathbf{J}_P \cdot \boldsymbol{\omega}_{iB}). \tag{50}$$

*This may also be obtained by substituting in Eq. (46) the relation between an inertia tensor about a point P and that about its center of mass, G: $\mathbf{J}_P = M[\mathbf{r}'^2_G\mathbf{I} - \mathbf{r}'_G\mathbf{r}'_G] + \mathbf{J}_G$.

Observed from the inertial frame, $\mathbf{J}_P$ varies with time. However, observed from the moving reference frame Σ_B, represented by a body-bound coordinate system, $\mathbf{J}_P$ is a constant tensor. Consequently,

$$\Gamma_P = \mathbf{r}'_G \times M\mathbf{a}_P + \mathbf{J}_P \cdot D_B \boldsymbol{\omega}_{iB} + \boldsymbol{\omega}_{iB} \times (\mathbf{J}_P \cdot \boldsymbol{\omega}_{iB}). \tag{51}$$

Since

$$D_B \boldsymbol{\omega}_{iB} = D_i \boldsymbol{\omega}_{iB} = {}_i(\boldsymbol{\alpha}_{iB}),$$

$$\Gamma_P = \mathbf{r}'_G \times M\mathbf{a}_P + \mathbf{J}_P \cdot {}_i(\boldsymbol{\alpha}_{iB}) + \boldsymbol{\omega}_{iB} \times (\mathbf{J}_P \cdot \boldsymbol{\omega}_{iB}). \tag{52}$$

A similar relationship, developed from Eq. (49), yields

$$\Gamma_P = \mathbf{r}'_G \times M\mathbf{a}_G + \mathbf{J}_G \cdot {}_i(\boldsymbol{\alpha}_{iB}) + \boldsymbol{\omega}_{iB} \times (\mathbf{J}_G \cdot \boldsymbol{\omega}_{iB}). \tag{53}$$

Euler's Equations of Motion

If P moves uniformly with respect to an inertial frame or if P is the center of mass, the first term of Eq. (51) vanishes. The resulting equation

$$\Gamma_P = \mathbf{J}_P \cdot D_B \boldsymbol{\omega}_{iB} + \boldsymbol{\omega}_{iB} \times (\mathbf{J}_P \cdot \boldsymbol{\omega}_{iB}) \tag{54}$$

is the vector form for Euler's equations of motion. Figure 37 is a flow diagram representation of Eq. (54), using the relation $D_B \boldsymbol{\omega}_{iB} = D_i \boldsymbol{\omega}_{iB} = {}_i(\boldsymbol{\alpha}_{iB})$. The inputs $\boldsymbol{\omega}_{iB}$ and ${}_i(\boldsymbol{\alpha}_{iB})$ may be regarded as that which would be determined by a velocity and acceleration flow diagram using any convenient set of Eulerian angles describing the attitude of Σ_B with respect to an inertial frame.

The flow diagram operation given by the box labeled J_P, denoting the matrix description of the inertia tensor $\mathbf{J}_P$ in the system used to define Σ_B, is taken to be the matrix product $J_P u$, where the column matrix u corresponds to the branch input to this operation. Figure 37 shows two such products corresponding to their appearance in Eq. (52).

When the choice of the body-bound system in B is the triad of principal axes,

$$J_P = \begin{bmatrix} A & 0 & 0 \\ 0 & B & 0 \\ 0 & 0 & C \end{bmatrix}.$$

The matrix form of Euler's Equations is then

$$\begin{bmatrix} \Gamma_x \\ \\ \Gamma_y \\ \\ \Gamma_z \end{bmatrix} = \begin{bmatrix} A\dfrac{d\omega_x}{dt} - (B-C)\omega_y\omega_z \\ \\ B\dfrac{d\omega_y}{dt} - (C-A)\omega_z\omega_x \\ \\ C\dfrac{d\omega_z}{dt} - (A-B)\omega_x\omega_y \end{bmatrix}. \qquad (55)$$

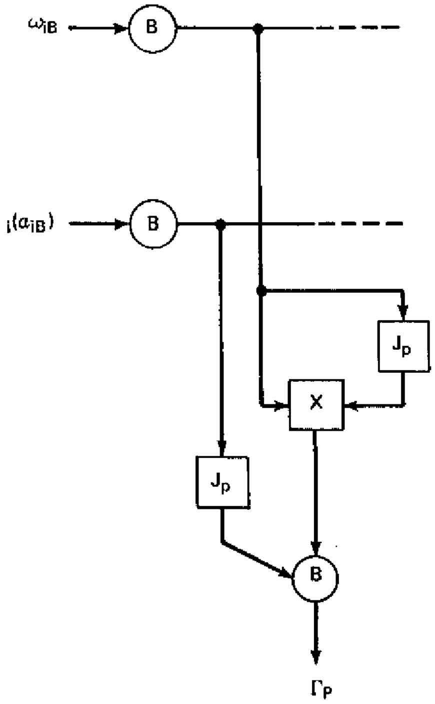


Fig. 37 – Moment about a uniformly moving point, single body.

While these differential equations (and more complicated forms for coupled rigid bodies in the study of multigimbaled structures) may be derived from flow diagrams depicting kinetic relations of these systems, the point of view largely taken in this report is that kinematic relations are known and that torque requirements and bearing loads are to be determined. Further, the many possible variations in the design of multigimbaled structures make it difficult to generalize procedures for constructing flow diagrams. For example, axes may be intersecting or nonintersecting, the distribution of mass about an axis or a point may be balanced or unbalanced, and motors may be rotary or linear, geared or direct-drive. Furthermore, this report is generally not concerned with the kinetic behavior of the moving platform. The motion of the moving platform is taken to be a part of input specifications or measured by inertial sensors or by relating its position to that of stable references. Rather than attempt to formulate general rules, several representative examples are considered.

Single-Axis Gimbaling

When the rigid body B of Fig. 36 is the controlled member of a single-axis gimbaled structure and P is taken to be on this axis at the interface between controlled and supporting members, the component of Γ_P along this axis is the torque which must be provided for this single degree of freedom. The component of torque normal to this axis represents an overturning moment which must be sustained by the bearing interface.

Since two systems of forces (represented by line vectors) are equivalent if they have the same vector sum and same total moment about any point, the system of forces acting on the controlled member is equivalent to a single force acting at P and a couple acting independent of P. The force acting at P is the vector sum of the system of forces and, therefore, has a value independent of P. The couple, however, which acts independent of P, is equal to the total moment of the system of forces about P and, therefore, has a value dependent on P.

Since the sum of external forces acting on B is equal to the sum of the products of elemental masses and their inertial accelerations, the equivalent force acting at P is

$$\mathbf{F}_P = \int_B D_i^2\, \mathbf{r}\, dm = D_i^2 \int_B \mathbf{r}\, dm = D_i^2 (M\mathbf{r}_G)$$

or the familiar relation,

$$\mathbf{F}_P = M\mathbf{a}_G . \tag{56}$$

Equations (52) and (53) give alternate formulations for Γ_P, the couple in this equivalent system of forces. When P is taken to be a point on the gimbal axis, $\mathbf{F}_P$ makes no contribution to the component of torque about this axis but would contribute to bearing overturning moments unless P is taken to be in the bearing plane. The components of $\mathbf{F}_P$ along this axis and normal to it, however, constitute thrust and radial bearing loads, respectively.

Although not considered here, forces other than constraining and inertial forces may be included, one example being the system of forces corresponding to wind pressure acting on elemental surface areas. By summing these forces and computing the total moment about P, contributions are made to both $\mathbf{F}_P$ and Γ_P.

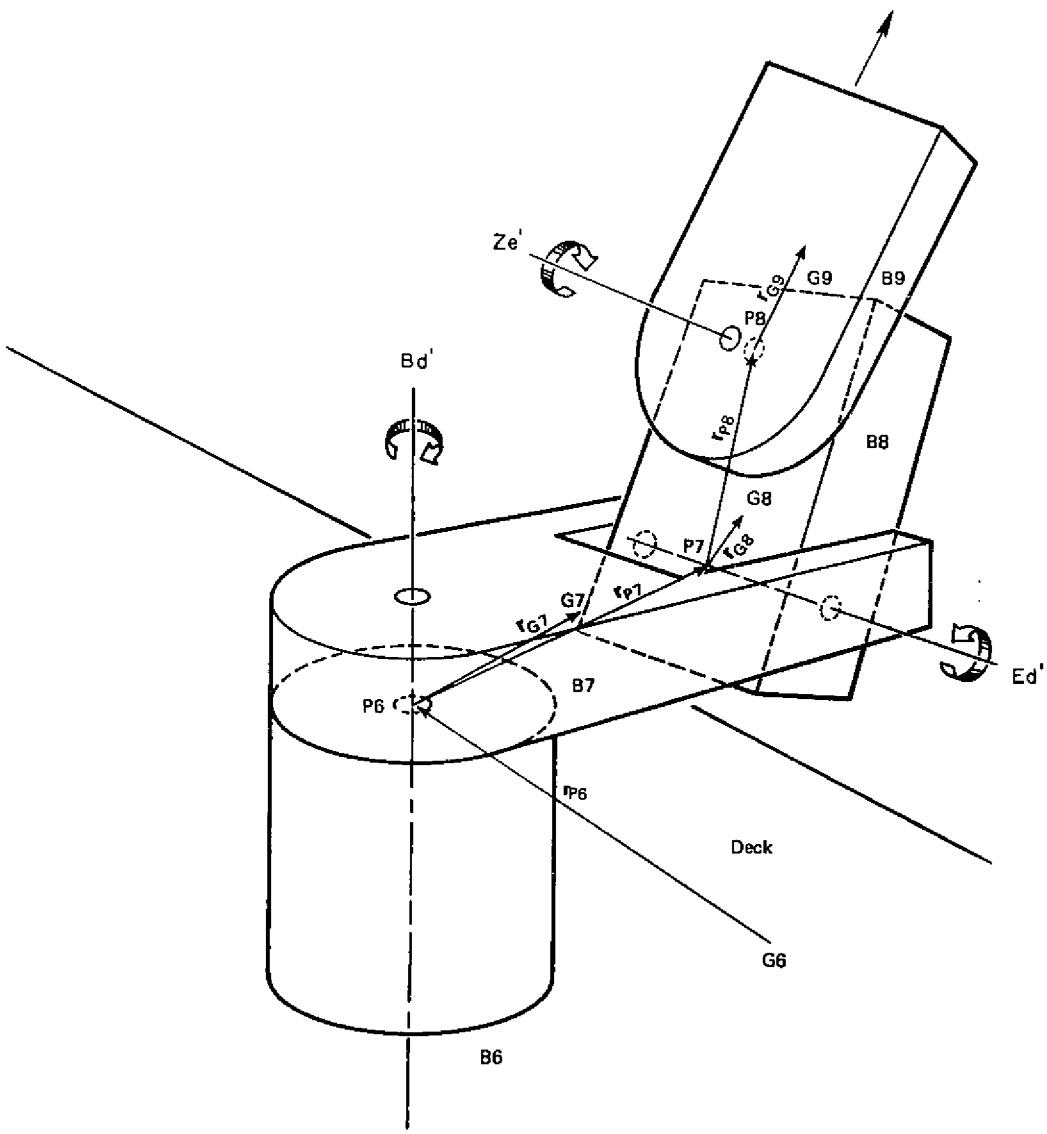
Multiaxis Gimbaling

When gimbaling is used to provide more than one rotational degree of freedom for a controlled member, an analysis of forces and moments necessary to produce the required motion is best performed by separate examination of each rigid member of the structure and the forces of action and reaction at the interfaces between it and adjacent members. In Fig. 38a, an unlikely design of a traverse-over-elevation-over-train configuration emphasizes a general nature in the distribution of masses and location of axes. The members of this structure are denoted as $B6$, $B7$, $B8$, and $B9$. The corresponding reference frames are Σ_6, Σ_7, Σ_8, and Σ_9, respectively. No distinction is made in Fig. 38a between the portion of the structure attached to the moving platform and the moving platform itself, this grouping being labeled $B6$. The points $P6$, $P7$, and $P8$ are on the axes between adjacent members and are, therefore, fixed with respect to the corresponding adjacent members. The point $P8$, for example, is fixed with respect to $B8$ and $B9$. These points, which will be referred to as "points of gimbaling," are taken to be symmetrically located with respect to the bearing interfaces. These points are used to reference an equivalent system of forces (consisting of a force acting at the point and a couple) due to the action of an adjacent member.

Point $G6$ of Fig. 38a, the center of mass of the moving platform (including the rigidly attached portion of the structure), is used to define the three translational and three rotational degrees of freedom of this platform. Position vectors $\mathbf{r}_{P6}$, $\mathbf{r}_{P7}$, and $\mathbf{r}_{P8}$ define $P6$, $P7$, and $P8$ with respect to $G6$, $P6$, and $P7$, respectively. Position vectors $\mathbf{r}_{G7}$, $\mathbf{r}_{G8}$, and $\mathbf{r}_{G9}$ define $G7$, $G8$, and $G9$, the center of masses of $B7$, $B8$, and $B9$ with respect to $P6$, $P7$, and $P8$, respectively. The most natural choices of coordinate systems for describing these vectors are ones body-bound to the member having the same number, their resulting descriptions being unchanging irrespective of gimbal motions. In general, inertial acceleration of both the P's and the G's must be determined to evaluate the equivalent system of forces for each point of gimbaling.

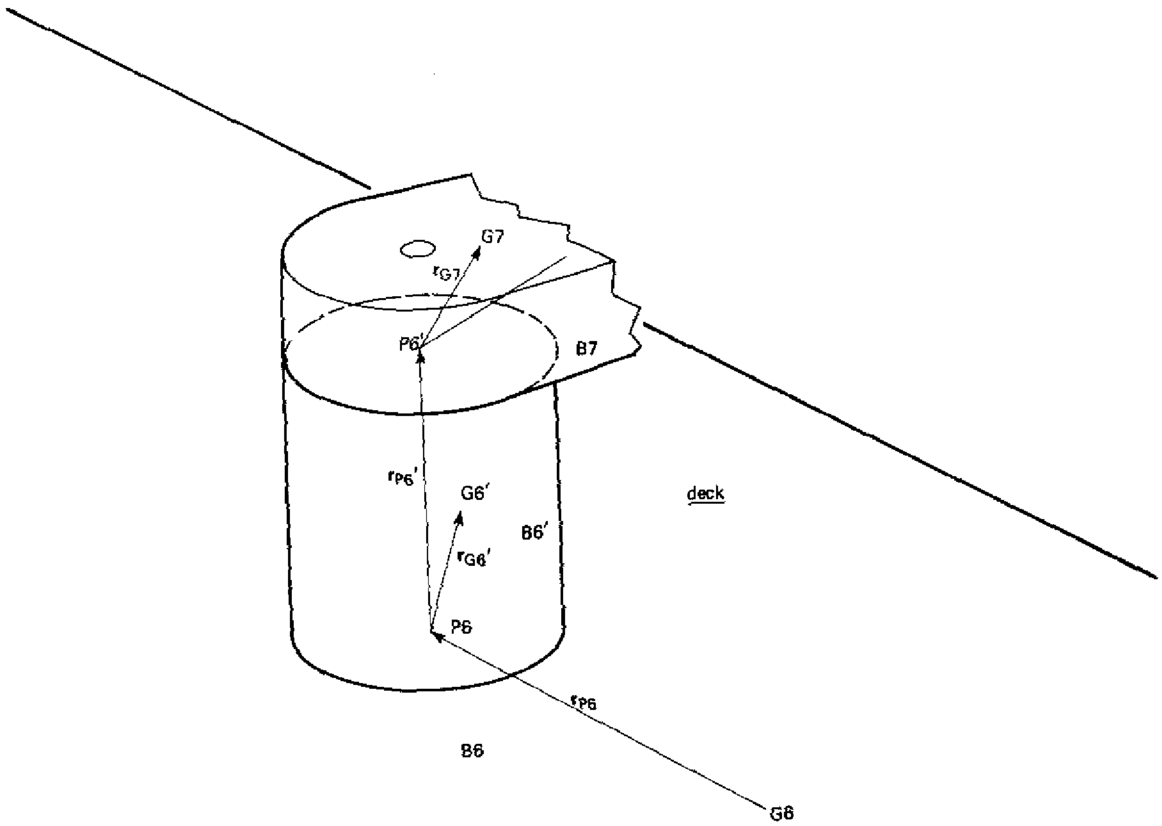
The inertial motion of $P6$ is the sum of the motion of $G6$ and $P6$ relative to $G6$ as observed from an inertial frame. Since $\mathbf{r}_{P6}$ is a constant vector when observed from Σ_6, $D_6 \mathbf{r}_{P6} = \mathbf{0}$. Therefore,

$$D_i \mathbf{r}_{P6} = \boldsymbol{\omega}_{i6} \times \mathbf{r}_{P6}$$



(a)

Fig. 38 – Traverse-over-elevation-over-train structure.



(b)

Fig. 38 – Traverse-over-elevation-over-train structure.

and

$$D_i^2\ \mathbf{r}_{P6} = {}_i(\boldsymbol{\alpha}_{i6}) \times \mathbf{r}_{P6} + \boldsymbol{\omega}_{i6} \times (\boldsymbol{\omega}_{i6} \times \mathbf{r}_{P6}). \tag{57}$$

The inertial acceleration of $P6$ is given by

$$\mathbf{a}_{P6} = \mathbf{a}_{G6} + D_i^2\ \mathbf{r}_{P6}$$

or

$$\mathbf{a}_{P6} = \mathbf{a}_{G6} + {}_i(\boldsymbol{\alpha}_{i6}) \times \mathbf{r}_{P6} + \boldsymbol{\omega}_{i6} \times (\boldsymbol{\omega}_{i6} \times \mathbf{r}_{P6}). \tag{58}$$

Acceleration of the other P's and G's are determined in a similar manner. For example,

$$\mathbf{a}_{G8} = \mathbf{a}_{G6} + D_i^2\ (\mathbf{r}_{P6} + \mathbf{r}_{P7} + \mathbf{r}_{G8})$$

or

$$a_{G8} = a_{G6} + {}_i(\boldsymbol{\alpha}_{i6}) \times r_{P6} + \boldsymbol{\omega}_{i6} \times (\boldsymbol{\omega}_{i6} \times r_{P6})$$

$$+ {}_i(\boldsymbol{\alpha}_{i7}) \times r_{P7} + \boldsymbol{\omega}_{i7} \times (\boldsymbol{\omega}_{i7} \times r_{P7})$$

$$+ {}_i(\boldsymbol{\alpha}_{i8}) \times r_{G8} + \boldsymbol{\omega}_{i8} \times (\boldsymbol{\omega}_{i8} \times r_{G8}). \quad (59)$$

In Fig. 39, translation components of platform motion are defined with respect to its center of mass. These components are further qualified in that the coordinate system chosen follows the transformation for heading.* Using the typical choice of right-handed axes and the z-axis downward, the x, y, and z components of a_{G6} for this choice of coordinate system are surge, sway, and heave accelerations, respectively. In determining forces and moments, the effect of gravity may be treated as an equivalent acceleration summed with the z component of a_{G6} and, thereby, is appropriately included in the calculation of the a_P's and a_G's. Flow-diagram representation of both the acceleration of the points of gimbaling and the centers of mass is included in Fig. 39. Brief examination of the acceleration portion of Fig. 39 shows the pattern in which a flow diagram is drawn and how it implements Eqs. (58) and (59) and similarly derived equations for the other points.

Node 3, corresponding to the earth frame, is assumed to be an inertial frame in Fig. 39. In many cases of determining forces and moments acting on structures, this is acceptable. Where this choice of inertial frame is not sufficiently accurate as may be the case for inertial platforms, etc., Fig. 39 may be revised to reference all motions to a nonrotating frame having the earth's center as a fixed point. In this event, a minimum of two additional planar transformations is required, the first taking into account the earth's rotation by inclusion of local sidereal time and the second using latitude to provide a local reference. The earth-observed acceleration of $G6$ given by surge, sway, and heave components may then be related to the inertial acceleration of this point by expressions given later in this report in the discussion of Coriolis acceleration.

In Fig. 39, the operation of simple multiplication of a vector by a scalar is shown, in this case the mass of a member. The respective rows for force and moment are inverted in the direction to that for the other rows, a result of the chaining effect on forces and moments from the inner gimbal to the outer.

The output from each node of the force and moment rows corresponds to the force and moment of an equivalent system of forces comprised of a force and a couple acting on a member, the force acting at the point of outermost gimbaling for that member. The reaction on the next outer member is, of course, the negative of this equivalent system. The force and moment of the equivalent system includes not only the inertial effects embodied in Eqs. (52) and (56) but also the effects of reaction of the next inner member at the point of its

*The translational components of platform motion necessarily depend on the point of reference and choice of coordinate system. For the latter, the coordinate system may well have been selected to be that preceding the transformation for the yaw component of heading.

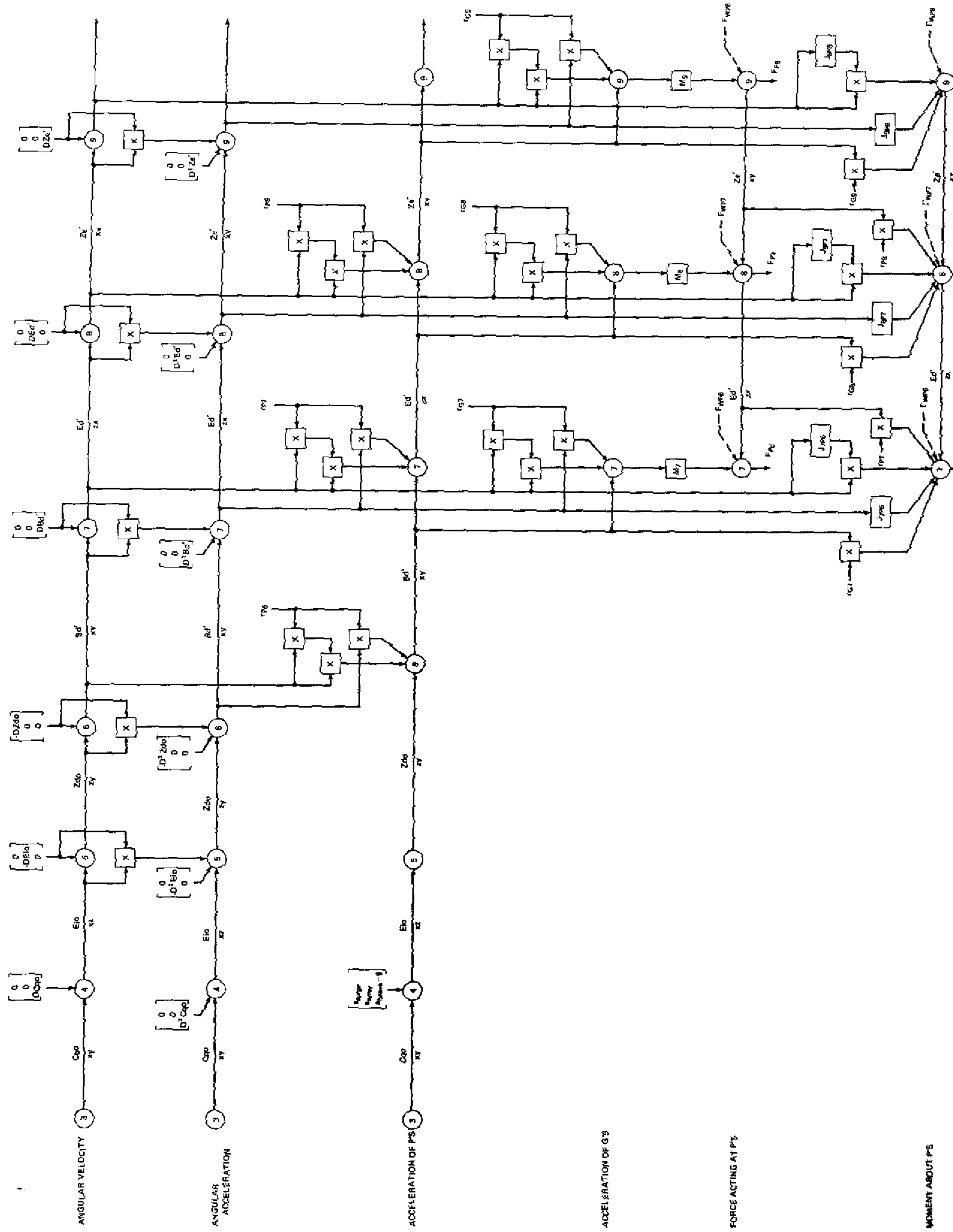


Fig. 39 – Force and moment relations for traverse-over-elevation-over-train structure.

gimbaling. The components of force and moment take on the same meaning as discussed for the single-axis case. A further breakdown could be made for the two-bearing mounting of $B8$ on $B7$ by providing two systems, one acting at each bearing interface in lieu of that referenced to $P7$. The proportioning would necessarily involve a knowledge of the design of these bearings and assumptions on their ability to share the load.

The dashed line inputs to the nodes of the force and moment rows of Fig. 39 represent an equivalent system of forces due to wind-pressure forces acting on each member.

If the position vector definitions of Fig. 38a are modified in the manner of Fig. 38b to separate the moving platform and the portion of the multigimbaled structure rigidly attached to it, a modified version of Fig. 39 may be constructed. The corresponding modifications of the flow diagram are shown in Fig. 40. From this, the forces and overturning and twisting moments may be determined for the interface between the structure and the moving platform.

If Eq. (53) had been used instead of Eq. (52) in implementing the flow diagram, a modification to Fig. 39 would be required. This modification, a typical stage shown in Fig. 41, requires a simple change in a branch for each node column and relabeling of the inertia matrices to be with respect to the centers of mass. However, no particular advantage is seen for its use.

Although the flow diagram of Fig. 39 may initially appear to be complex, repetitious patterns are seen to result throughout. The methods used for computer solution of previous problems may be readily extended to compute forces and moments in Fig. 39. An added routine useful in this solution is the product of a square matrix and a column matrix. When certain simplifying assumptions can be made or if mass distribution is such that some or all of the P's coincide with the G's, a much simpler appearing flow diagram results since the corresponding $\mathbf{r}_G$'s and $\mathbf{r}_P$'s are $\mathbf{0}$.

CORIOLIS ACCELERATION

Inasmuch as any discussion of time-varying coordinate systems generally includes the relation between acceleration of a point object when observed from two different reference frames, the utility of the flow-diagram approach is examined in this context.

Given two frames Σ_i and Σ_j in Fig. 42 with their relative motion having both translational and rotational components, points O and A are origins fixed in Σ_i and Σ_j, respectively.

From Fig. 42,

$$\mathbf{r} = \mathbf{r}_A + \mathbf{r}'.$$

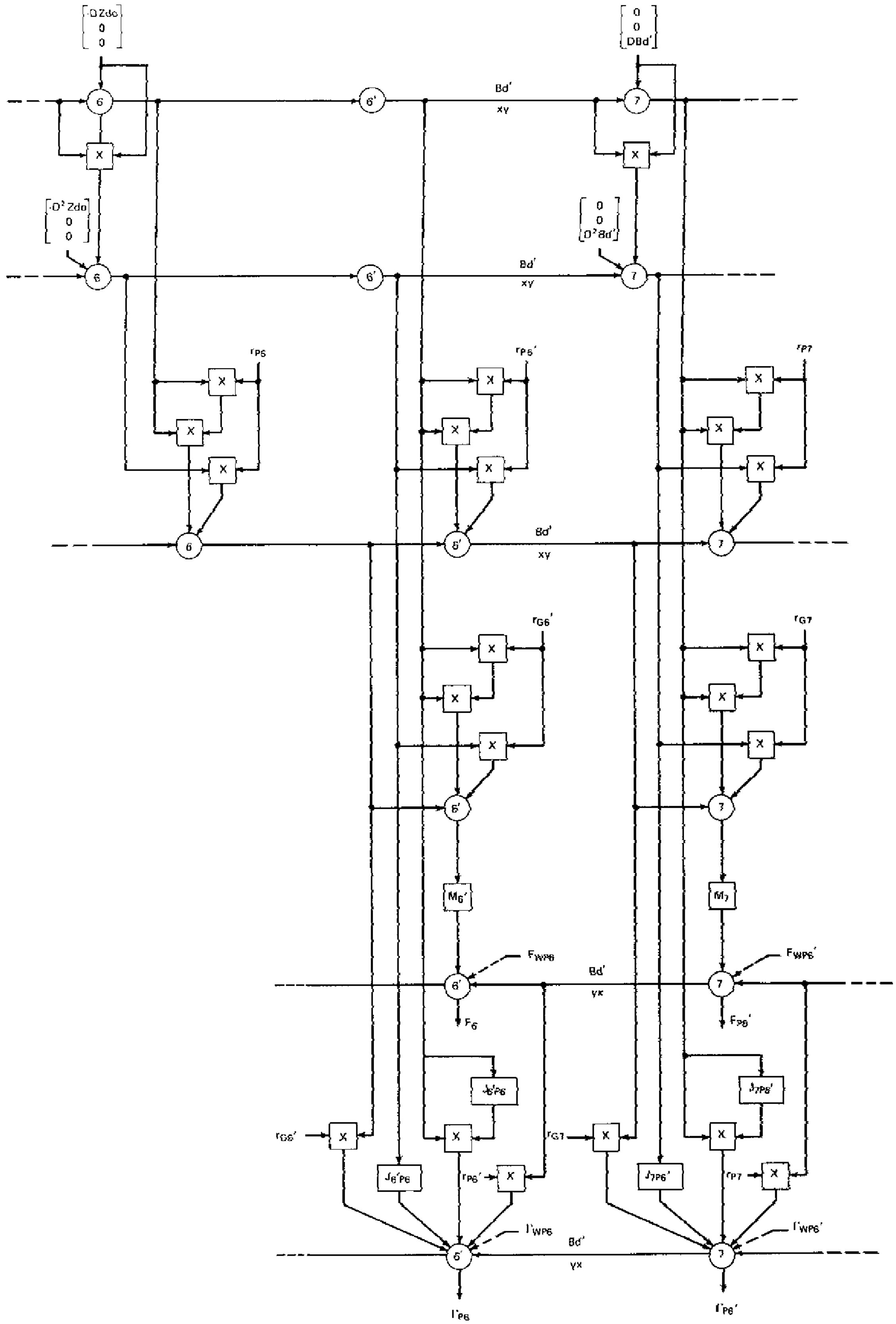


Fig. 40 – Modification for force and moment relations of the base-deck interface.

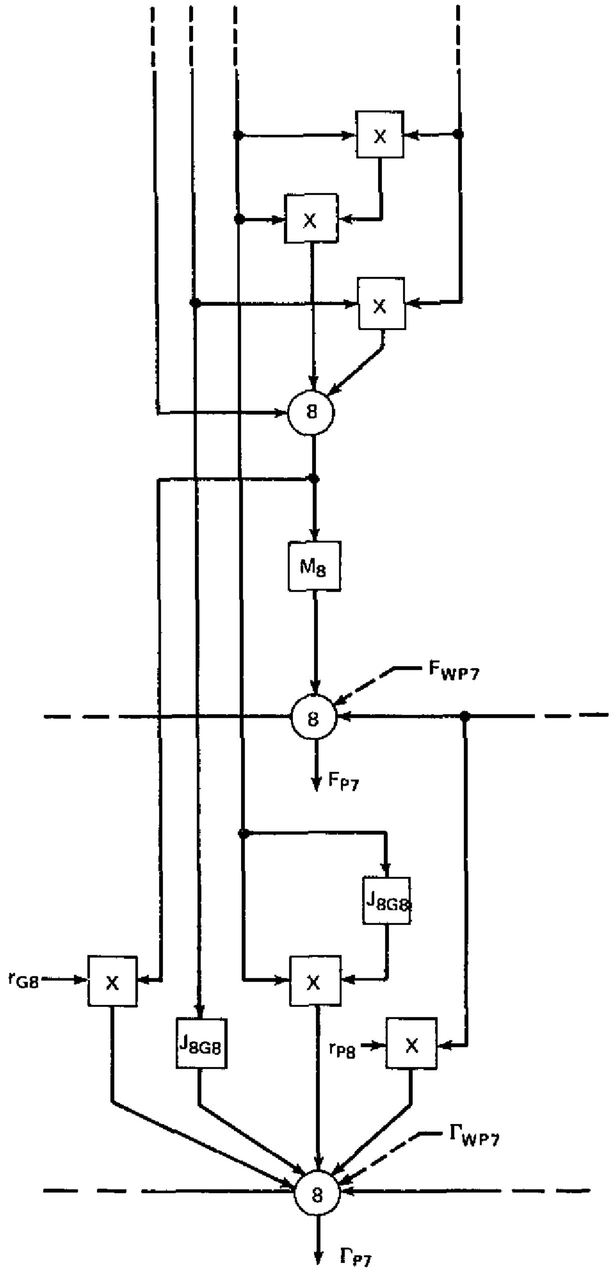


Fig. 41 – Modification for Center-of-mass reference inertia.

Using the notation of this report for qualifying time differentiation of a vector when observed from different reference frames,

$$D_i\mathbf{r} = D_i\mathbf{r}_A + D_i\mathbf{r}'$$

and

$$D_i\mathbf{r} = D_i\mathbf{r}_A + D_j\mathbf{r}' + \boldsymbol{\omega}_{ij} \times \mathbf{r}'. \tag{60}$$

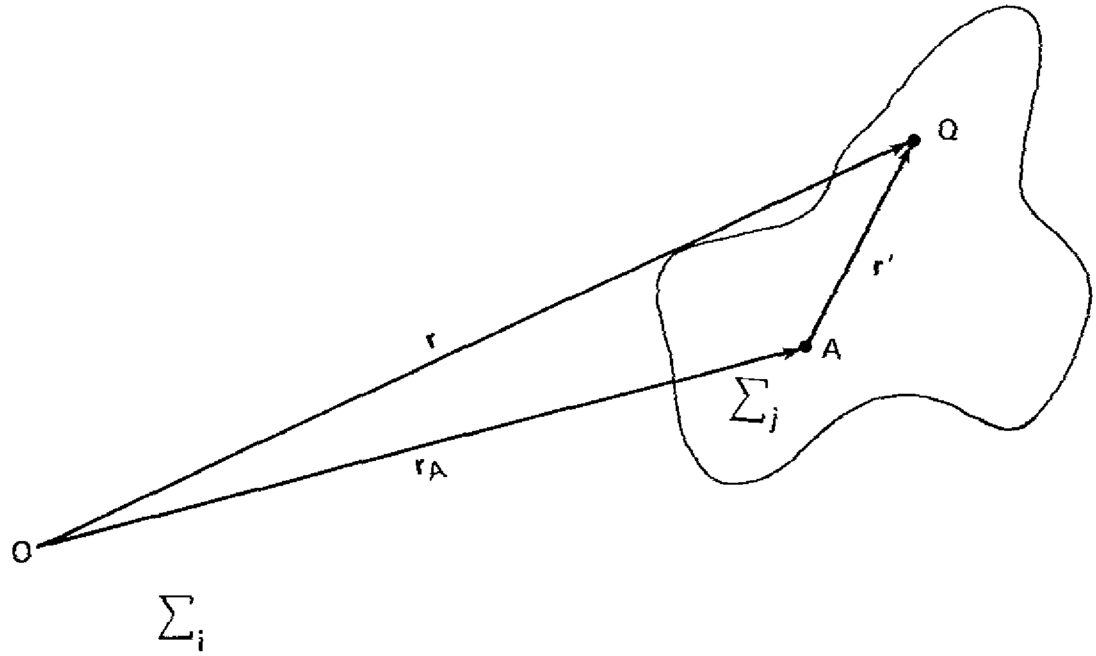


Fig. 42 – Frames of observation and geometry for Coriolis Acceleration.

Continuing,

$$\begin{aligned} \mathbf{a}_Q &= D_i^2\mathbf{r} = D_i^2\mathbf{r}_A + D_i(D_j\mathbf{r}') + D_i(\boldsymbol{\omega}_{ij} \times \mathbf{r}') \\ &= \mathbf{a}_A + D_j^2\mathbf{r}' + \boldsymbol{\omega}_{ij} \times D_j\mathbf{r}' + (D_i\boldsymbol{\omega}_{ij}) \times \mathbf{r}' \\ &\quad + \boldsymbol{\omega}_{ij} \times (D_j\mathbf{r}' + \boldsymbol{\omega}_{ij} \times \mathbf{r}'). \end{aligned}$$

Using ${}_i(\boldsymbol{\alpha}_{ij}) = D_i\boldsymbol{\omega}_{ij}$ and letting $D_j\mathbf{r}' = \mathbf{v}'$ and $D_j^2\mathbf{r}' = \mathbf{a}'$, the velocity and acceleration of Q relative to O as observed from Σ_j,

$$\mathbf{a}_Q = \mathbf{a}_A + {}_i(\boldsymbol{\alpha}_{ij}) \times \mathbf{r}' + \boldsymbol{\omega}_{ij} \times (\boldsymbol{\omega}_{ij} \times \mathbf{r}') + 2\boldsymbol{\omega}_{ij} \times \mathbf{v}' + \mathbf{a}'. \tag{61}$$

In Eq. (61), $2\boldsymbol{\omega}_{ij} \times \mathbf{v}'$ is the Coriolis acceleration term. Flow-diagram representation of Eqs. (60) and (61) is simple and direct. Previous flow-diagram forms for angular velocity and acceleration (Fig. 33) are used to obtain $\boldsymbol{\omega}_{ij}$ and ${}_i(\boldsymbol{\alpha}_{ij})$ when the rotational component of motion between Σ_i and Σ_j is known. With the relative velocity $\mathbf{v}'$ and acceleration $\mathbf{a}'$ of Q measured by an observer in Σ_j, the corresponding velocity and acceleration as would be observed from Σ_i is readily described in Σ_i by using nodes for summing and the cross-operation box for vector cross products. The position vector of Q with respect to the origin A in Σ_j and the acceleration of A, of course, must be known quantities and described in Σ_j. The resultant Σ_i observed velocity and acceleration may be described in Σ_i by further extension of the flow diagram.

In the acceleration determinations of Eqs. (58) and (59), an apparently different approach was taken. Equation (61), however, could have been applied in these instances, althrough perhaps not as conveniently as use of Eqs. (58) and (59). In any event, it is not difficult to show the equivalence of these forms.

DOPPLER

In dealing with frequency and phase tracking circuits in satellite communication receivers and modem design, it is necessary to know doppler shift in frequency and its time rate of change that results from relative motion between a satellite and a receiver. Since rf components are generally designed to keep a constant path length between an antenna feed and receiver irrespective of platform motion, this may be rephrased to be the relative motion between a satellite and an antenna feed whose tracking line is constantly directed toward the satellite. Without regard to relativistic effects and when relative motion is much less than the speed of light, doppler shift in frequency Δf is given by

$$\Delta f = f_R - f_T \approx -\frac{\nu}{c} f_T,$$

where f_R and f_T are the received and transmitted frequencies, c is the velocity of light, and ν is the range-rate or time rate of change of the magnitude of the position vector from the antenna feed to the satellite.

Denoting the position vector from the antenna feed to the satellite by $\mathbf{r}$, the range rate is $D(\mathbf{r} \cdot \mathbf{r})^{1/2}$. Since $\mathbf{r} \cdot \mathbf{r}$ is a scalar invariant, no qualification is necessary as to the frame from which this invariant is being observed. However, upon expanding $D(r \cdot r)^{1/2}$, differentiation of a vector is involved for

$$D(\mathbf{r} \cdot \mathbf{r})^{1/2} = 1/2(\mathbf{r} \cdot \mathbf{r})^{-1/2} (D_\alpha \mathbf{r} \cdot \mathbf{r} + \mathbf{r} \cdot D_\alpha \mathbf{r}) = (\mathbf{r} \cdot \mathbf{r})^{-1/2} (\mathbf{r} \cdot D_\alpha \mathbf{r}). \tag{62}$$

Since vector differentiation must be qualified as to the frame of observation and since no such qualification was required for $D(\mathbf{r} \cdot \mathbf{r})^{-1/2}$, it must follow that Σ_α may be any frame. This can be further demonstrated since

$$\mathbf{r} \cdot D_\alpha \mathbf{r} = \mathbf{r} \cdot (D_\beta \mathbf{r} + \boldsymbol{\omega}_{\alpha\beta} \times \mathbf{r}) = \mathbf{r} \cdot D_\beta \mathbf{r}.$$

On differentiating Eq. (62),

$$D^2(\mathbf{r} \cdot \mathbf{r})^{1/2} = -(\mathbf{r} \cdot \mathbf{r})^{-3/2} (\mathbf{r} \cdot D_\beta \mathbf{r}) (\mathbf{r} \cdot D_\alpha \mathbf{r})$$
$$+ (\mathbf{r} \cdot \mathbf{r})^{-1/2} (D_\beta \mathbf{r} \cdot D_\alpha \mathbf{r} + \mathbf{r} \cdot D_\beta D_\alpha \mathbf{r}), \tag{63}$$

for Σ_α and Σ_β any frames, not necessarily the same. By substituting $D_\alpha \mathbf{r} = D_\gamma \mathbf{r} + \boldsymbol{\omega}_{\alpha\gamma} \times \mathbf{r}$ or $D_\beta \mathbf{r} = D_\gamma \mathbf{r} + \boldsymbol{\omega}_{\beta\gamma} \times \mathbf{r}$ and $D_\beta D_\alpha \mathbf{r} = D_\gamma D_\alpha \mathbf{r} + \boldsymbol{\omega}_{\beta\gamma} \times D_\alpha \mathbf{r}$ in Eq. (63), the same form results except now in β and γ or α and γ, respectively, demonstrating the arbitrary choice of frames for α and β.

Letting $R = (\mathbf{r} \cdot \mathbf{r})^{1/2} = |\mathbf{r}|$, Eqs. (62) and (63) become

$$DR = \frac{\mathbf{r}}{R} \cdot D_\alpha \mathbf{r} = \mathbf{e} \cdot D_\alpha \mathbf{r} \tag{64}$$

and

$$D^2R = -\frac{1}{R}(\mathbf{e} \cdot D_\beta \mathbf{r})(\mathbf{e} \cdot D_\alpha \mathbf{r}) + \frac{1}{R}(D_\beta \mathbf{r} \cdot D_\alpha \mathbf{r}) + \mathbf{e} \cdot D_\beta D_\alpha \mathbf{r}, \tag{65}$$

respectively, where $\mathbf{e} = \mathbf{r}/R$ is a unit vector parallel to $\mathbf{r}$. In particular, if $\beta = \alpha$

$$D^2R = -\frac{1}{R}(\mathbf{e} \cdot D_\alpha \mathbf{r})^2 + \frac{1}{R}(D_\alpha \mathbf{r} \cdot D_\alpha \mathbf{r}) + \mathbf{e} \cdot D_\alpha{}^2 \mathbf{r}. \tag{66}$$

When the choice of frame α is given by the controlled member, in which $\mathbf{e}$ is observed to be constant and the apparent change in $\mathbf{r}$ is, therefore, only in magnitude, $(\mathbf{e} \cdot D_\alpha \mathbf{r})^2 = D_\alpha \mathbf{r} \cdot D_\alpha \mathbf{r}$. The expected result is that Eq. (66) becomes

$$D^2R = \mathbf{e} \cdot D_\alpha{}^2 \mathbf{r}, \text{ and } \Sigma_\alpha \text{ is the controlled member.} \tag{67}$$

Equations (64) and (65) reveal a variety of possible flow diagrams. For D^2R, the simplest form results when $\alpha = \beta$ in Eq. (65) and Σ_α is given by the controlled member, resulting in Eq. (67). When dealing with doppler alone, the preference of frame is much more arbitrary since Eq. (64) is not subject to greater simplification by such a choice. In this case, any frame involved in the basic flow diagram would be equally convenient. The discussion concerning motion for the problem of Figs. 38 and 39 is largely applicable except for the choice of frame. Though not shown in Fig. 39, a row corresponding to velocities of P's would necessarily be constructed in a flow diagram for doppler.

CONCLUSION

Like most flow-diagram representations, the immediate virtue of the approach of this report is the ability to depict complex problems. With limited practice, one is able to directly formulate problems involving time-varying coordinate systems into the flow-diagram language and then to perform certain manipulations to simplify or restate the problem. For some classes of problems, for example in determining forces and moments when motion is known, the flow diagram admits of simple computer modeling and solution. For most problems in time-varying coordinate systems, the flow-diagram approach provides an effective method of bookkeeping and of visualizing various contributions to the quantity being examined. This alone justifies the flow-diagram approach. Unfortunately, no magic is performed. A problem having an inherently difficult solution when using more conventional approaches will still be difficult to solve, even though the flow diagram itself may appear to be straightforward.

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Appendix A

SKEW-SYMMETRY OF $(dA^{-1}/dt)A$ AND ITS VECTOR CHARACTERIZATION

For an orthogonal coordinate transformation given by the matrix equation

$$\begin{bmatrix} x' \\ y' \\ z' \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} \tag{A1}$$

or symbolically

$$\xi' = A\xi, \tag{A2}$$

the set of vectors whose components consist of the rows or columns of A constitute a unitary orthogonal (orthonormal) set.

Let vectors having the columns of A as components be defined by

$$\begin{aligned} \mathbf{a}_1 &= a_{11}\ \mathbf{i} + a_{21}\ \mathbf{j} + a_{31}\ \mathbf{k} \\ \mathbf{a}_2 &= a_{12}\ \mathbf{i} + a_{22}\ \mathbf{j} + a_{32}\ \mathbf{k} \\ \mathbf{a}_3 &= a_{13}\ \mathbf{i} + a_{23}\ \mathbf{j} + a_{33}\ \mathbf{k}, \end{aligned} \tag{A3}$$

where **i**, **j**, and **k** are a unitary orthogonal set. No restriction is placed on the right- or left-handedness of this set at this time.

$$\mathbf{a}_\nu \cdot \mathbf{a}_\mu = \delta_{\nu\mu}, \tag{A4}$$

where the Kronecker delta $\delta_{\nu\mu} = 1$, when $\nu = \mu$ and $\delta_{\nu\mu} = 0$, when $\nu \neq \mu$.

Since A is orthogonal

$$A^{-1} = A^T = \begin{bmatrix} a_{11} & a_{21} & a_{31} \\ a_{12} & a_{22} & a_{32} \\ a_{13} & a_{23} & a_{33} \end{bmatrix}$$

and

$$\frac{dA^{-1}}{dt} = \begin{bmatrix} \dot{a}_{11} & \dot{a}_{21} & \dot{a}_{31} \\ \dot{a}_{12} & \dot{a}_{22} & \dot{a}_{32} \\ \dot{a}_{13} & \dot{a}_{23} & \dot{a}_{33} \end{bmatrix},$$

where

$$\dot{a}_{a\nu} = \frac{da_{a\nu}}{dt}.$$

Since matrix multiplication is given by the array of scalar products of the row vectors of the left matrix with the column vectors of the right matrix, the result is

$$\frac{dA^{-1}}{dt} A = \begin{bmatrix} \dot{\mathbf{a}}_1 \cdot \mathbf{a}_1 & \dot{\mathbf{a}}_1 \cdot \mathbf{a}_2 & \dot{\mathbf{a}}_1 \cdot \mathbf{a}_3 \\ \dot{\mathbf{a}}_2 \cdot \mathbf{a}_1 & \dot{\mathbf{a}}_2 \cdot \mathbf{a}_2 & \dot{\mathbf{a}}_2 \cdot \mathbf{a}_3 \\ \dot{\mathbf{a}}_3 \cdot \mathbf{a}_1 & \dot{\mathbf{a}}_3 \cdot \mathbf{a}_2 & \dot{\mathbf{a}}_3 \cdot \mathbf{a}_3 \end{bmatrix}, \tag{A5}$$

where

$$\dot{\mathbf{a}}_\nu = \dot{a}_{1\nu}\,\mathbf{i} + \dot{a}_{2\nu}\,\mathbf{j} + \dot{a}_{3\nu}\,\mathbf{k}.$$

From Eq. (A4), $\dot{\mathbf{a}}_\nu \cdot \mathbf{a}_\mu = -\mathbf{a}_\nu \cdot \dot{\mathbf{a}}_\mu$, establishing the skew symmetry of $(dA^{-1}/dt)A$. In array form

$$\frac{dA^{-1}}{dt} A = \begin{bmatrix} 0 & -\mathbf{a}_1 \cdot \dot{\mathbf{a}}_2 & \mathbf{a}_3 \cdot \dot{\mathbf{a}}_1 \\ \mathbf{a}_1 \cdot \dot{\mathbf{a}}_2 & 0 & -\mathbf{a}_2 \cdot \dot{\mathbf{a}}_3 \\ -\mathbf{a}_3 \cdot \dot{\mathbf{a}}_1 & \mathbf{a}_2 \cdot \dot{\mathbf{a}}_3 & 0 \end{bmatrix}. \tag{A6}$$

Since $A^{-1} A = I$,

$$\frac{dA^{-1}}{dt} A = -A^{-1} \frac{dA}{dt}. \tag{A7}$$

By defining

$$\omega = \begin{bmatrix} \mathbf{a}_2 \cdot \dot{\mathbf{a}}_3 \\ \mathbf{a}_3 \cdot \dot{\mathbf{a}}_1 \\ \mathbf{a}_1 \cdot \dot{\mathbf{a}}_2 \end{bmatrix} = \begin{bmatrix} \omega_x \\ \omega_y \\ \omega_z \end{bmatrix}, \tag{A8}$$

Eq. (6) becomes

$$\frac{dg}{dt} = \left(\frac{dA^{-1}}{dt} A\right) g = \begin{bmatrix} \omega_y g_z - \omega_z g_y \\ \omega_z g_x - \omega_x g_z \\ \omega_x g_y - \omega_y g_x \end{bmatrix}. \tag{A9}$$

With $\mathbf{g} = g_x\mathbf{i} + g_y\mathbf{j} + g_z\mathbf{k}$ and $\omega = \omega_x\mathbf{i} + \omega_y\mathbf{j} + \omega_z\mathbf{k}$,

$$\frac{d\mathbf{g}}{dt} = \boldsymbol{\omega} \times \mathbf{g} = \begin{bmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \omega_x & \omega_y & \omega_z \\ g_x & g_y & g_z \end{bmatrix} \begin{aligned} &(\omega_y g_z - \omega_z g_y)\,\mathbf{i} \\ &= + (\omega_z g_x - \omega_x g_z)\,\mathbf{j} \\ &+ (\omega_x g_y - \omega_y g_x)\,\mathbf{k}, \end{aligned} \tag{A10}$$

the frame of observation being the unprimed system. The vectors **i**, **j**, and **k** are the orthonormal base vectors of the unprimed system (and assumed at this time to be right-handed). The equivalence of Eqs. (A9) and (A10) is readily seen. By using Eq. (A8), the components of the angular velocity vector in the unprimed system may then be determined from the column vectors of the transformation matrix A and their time derivatives.

When a left-hand system is chosen, use of Eq. (A8) to define the components of $\boldsymbol{\omega}$ results in a left-hand sense. The use of the right-handed expansion for the cross product for the left-hand system, however, still yields the correct orientation for $d\mathbf{g}/dt$.

A linear-operator invariant represented by a skew-symmetric matrix in one coordinate system is also represented by a skew-symmetric matrix in any other coordinate system related to the first by an orthogonal transformation. Using Eq. (3) and an orthogonal matrix A, the transformed skew-symmetric matrix Ω is given by*

$$\Omega' = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{bmatrix} \begin{bmatrix} a_{11} & a_{21} & a_{31} \\ a_{12} & a_{22} & a_{32} \\ a_{13} & a_{23} & a_{33} \end{bmatrix}. \tag{A11}$$

By expanding Eq. (A11) and using the property of an orthogonal matrix that each element is its own cofactor, it may be shown that Ω' is also skew-symmetric, having the form

$$\Omega' = \begin{bmatrix} 0 & -\omega_3' & \omega_2' \\ \omega_3' & 0 & -\omega_1' \\ -\omega_2' & \omega_1' & 0 \end{bmatrix},$$

where

$$\begin{bmatrix} \omega_1' \\ \omega_2' \\ \omega_3' \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{bmatrix} \quad \text{or } \omega' = A\omega. \tag{A12}$$

*In this case no relation is assumed between the matrices Ω and A, and the results of Eqs. (A11) and (A12) are valid for any skew-symmetric matrix Ω and orthogonal matrix A.

This demonstrates that $\boldsymbol{\omega}$ transforms like a vector, a required property for the vector characterization of $(dA^{-1}/dt)A$.

In applying the vector mapping relationships used to develop Eqs. (6) and (7) to multiple successive coordinate systems defined by $\xi_1 = A_1\xi_0$, $\xi_2 = A_2\xi_1$, ..., $\xi_n = A_n\xi_{n-1}$,

$$g_0 = (A_n A_{n-1} \ldots A_1)^{-1} f_0$$

$$g_0 = (A_1^{-1}A_2^{-1} \ldots A_n^{-1}) f_0. \tag{A13}$$

Differentiating Eq. (A13) with respect to time (f_0 is constant) and replacing f_0 by $(A_n A_{n-1} \ldots A_1)\, g_0$,

$$\frac{dg_0}{dt} = \left[\frac{dA_1^{-1}}{dt}A_1 + A_1^{-1}\left(\frac{dA_2^{-1}}{dt}A_2\right) A_1 + \ldots\right.$$

$$\left. + A_1^{-1}A_2^{-1} \ldots A_{n-1}^{-1}\left(\frac{dA_n^{-1}}{dt}A_n\right) A_{n-1} \ldots A_2A_1\right] g_0, \tag{A14}$$

where dg_0/dt is the 0th system description rate of change of g when observed from Σ_0. Since $A_1^{-1}A_2^{-1} \ldots A_{k-1}^{-1} = (A_{k-1} \ldots A_2A_1)^{-1}$, the kth term dg_0/dt is

$$\left[(A_{k-1} \ldots A_2A_1)^{-1}\left(\frac{dA_k^{-1}}{dt}A_k\right) (A_{k-1} \ldots A_1A_2)\right] g_0.$$

From Eq. (3), the linear-operator invariant, given by

$$(A_{k-1} \ldots A_2A_1)^{-1}\left(\frac{dA_k^{-1}}{dt}A_k\right) (A_{k-1} \ldots A_1A_2)$$

in the 0th system, has the $(k-1)$th system description

$$\frac{dA_k^{-1}}{dt}A_k.$$

The kth term of dg/dt is then $\boldsymbol{\omega}_{k-1,k} \times$ g, where $\boldsymbol{\omega}_{k-1,k}$ is the angular velocity of Σ_k with respect to Σ_{k-1}. From Eq. (A14) and since $dg/dt = \boldsymbol{\omega}_{0n} \times$ g,

$$\boldsymbol{\omega}_{0n} = \boldsymbol{\omega}_{01} + \boldsymbol{\omega}_{12} + \ldots + \boldsymbol{\omega}_{n-1,n} = \sum_{k=1}^{n} \boldsymbol{\omega}_{k-1,k}. \tag{A15}$$

Appendix B

CONFIGURATIONS WITH NONORTHOGONAL ADJACENT AXES

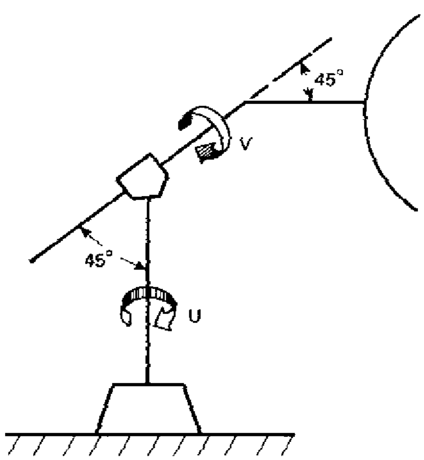

Fig. B1 – Differential mount.

Geometric relations for the two-axis configuration of Fig. B1 include not only nonorthogonal axes but also a tracking line which is nonorthogonal to the upper axis of rotation (the axis of the equivalent innermost gimbal). This arrangement, sometimes referred to as a differential mount, provides hemispherical coverage in relation to the platform on which it is mounted. Continuously available earth-referenced coverage, however, would be less than hemispherical when this platform is subject to roll and pitch motions. The effect of these nonorthogonal relations may be observed in the flow diagrams of Fig. B2.

Given elevation E and azimuth A, the computer solution for U and V is complicated by the presence of these nonorthogonal relations. In particular, the subroutine RESOLV2 cannot be simply employed when computating U. However, the analytic solution of Fig. B2a is not difficult. Starting with the input on the left of Fig. B2, the vector at node 2 is

$$\begin{bmatrix} x_2 \\ y_2 \\ z_2 \end{bmatrix} = \begin{bmatrix} \cos E \\ 0 \\ -\sin E \end{bmatrix}. \tag{B1}$$

By inverting the flow diagram and going from right to left, the vector at node 2 is

$$\begin{bmatrix} x_2 \\ y_2 \\ z_2 \end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix} -\sin V \sin (U-A) + \dfrac{1}{\sqrt{2}} + \dfrac{1}{\sqrt{2}}\cos V \ \ \cos (U-A) \\ \sin V \cos (U-A) + \dfrac{1}{\sqrt{2}} + \dfrac{1}{\sqrt{2}}\cos V \ \ \sin (U-A) \\ -\dfrac{1}{\sqrt{2}} + \dfrac{1}{\sqrt{2}}\cos V \end{bmatrix}. \tag{B2}$$

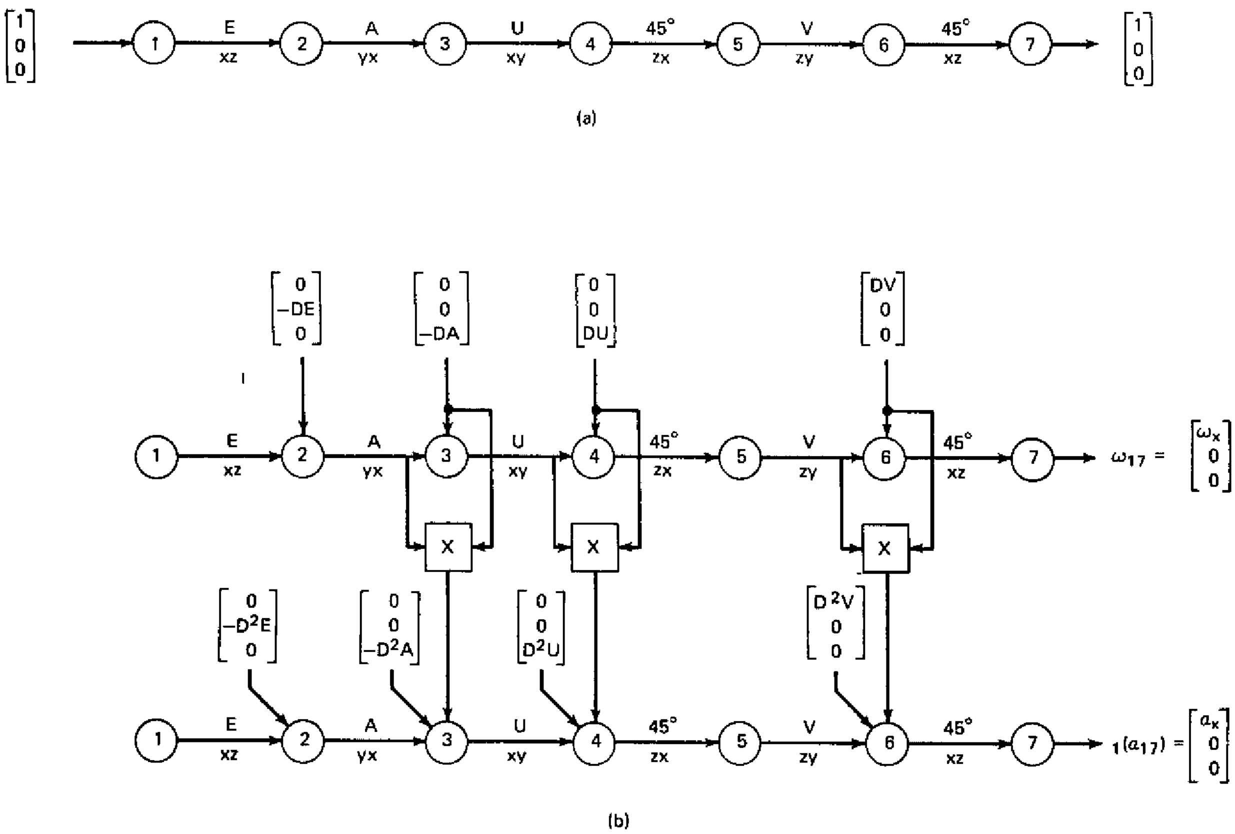

Fig. B2 – Position and velocity-acceleration relations for the differential mount.

Equating the y components of the vectors of Eqs. (B1) and (B2)*,

$$A = U + \tan^{-1} \frac{\sqrt{2} \sin V}{1 + \cos V} . \tag{B3}$$

By equating the z components in a similar manner,

$$E = \sin^{-1} \left[\frac{1}{2} (1 - \cos V) \right] . \tag{B4}$$

Given A and E or U and V as inputs, the other pair may be determined through use of Eqs. (B3) and (B4).

*By defining the subroutines RESOLV2 and RESOLV3, previous analysis of unknown quantities in a path has been performed in an apparent forward direction. A more general method is to determine the vector output of a computationally convenient intermediate node in this path and the vector output at the same node when the path is inverted. By equating the components of these vectors, pertinent unknown quantities may be determined. Similarly, it is also possible to determine unknown angular rates and accelerations about an axis by writing the transformation equations in matrix form and obtaining the first and second derivatives of the vector at the intermediate node, using rules of matrix differentiation. By equating the differentiated vector at the intermediate node obtained from the forward and inverted paths, pertinent unknown quantities are determined.

In Fig. B2b, angular velocity and acceleration relations are shown for the differential mount when affixed to a stationary horizontal platform. Since the solution for DU and DV is also complicated by the presence of the nonorthogonal relations, similar analysis is required in dealing with Fig. B2a.

By adding the third axis shown in Fig. B3, the resulting configuration is suitable for use on a moving platform. Furthermore, attitude control may be obtained in a manner similar to the control of the elevation-over-cross-level-over-train configuration.

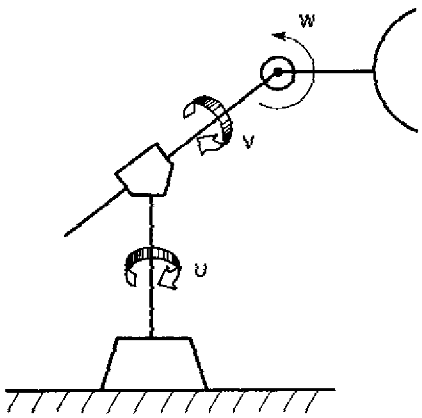

Fig. B3 – Three-axis configuration obtained by modifying the two-axis differential mount.

Previous arguments on location of a vertical reference for the elevation-over-cross-level-over-train configuration are also applicable here, except that the control method using an off-mount vertical reference is more complicated by virtue of the nonorthogonality between the lower two axes.

The first of two vectors used to compute U, V, and W is chosen orthogonal to the plane of the last transformation. The solution for U and V in this case must be dealt with in a manner similar to the proceeding two-axis case. Having determined U and V, selection of a convenient vector orthogonal to the first results in a simple solution for W. Although not shown, the velocity-acceleration flow diagram for the configuration of Fig. B3 is straightforward. Its solution, however, is also complicated by the nonorthogonal relation of the U and V axes.

$$\begin{bmatrix}x_1\\y_1\\z_1\end{bmatrix} \longrightarrow (1) \xrightarrow[xz]{E} (2) \xrightarrow[yx]{By} (3) \xrightarrow[xy]{Cqo} (4) \xrightarrow[xz]{Eio} (5) \xrightarrow[zy]{Zdo} (6) \xrightarrow[xy]{U} (7) \xrightarrow[zx]{45^\circ} (8) \xrightarrow[zy]{V} (9) \xrightarrow[zx]{W-45^\circ} (10) \longrightarrow \begin{bmatrix}x_{10}\\y_{10}\\z_{10}\end{bmatrix}$$

Fig. B4 – Director for three-axis modification of differential mount.

Appendix C

SHIPBOARD MULTIPLE TWO-AXIS INSTALLATION

A desirable capability of a military shipboard satellite communication system is the ability to provide continuous communications irrespective of the ship's course and attitude. To achieve this in a single narrowbeam antenna would require a high location free from blockage and at least a three-axis pedestal design, assuming that sufficiently low-elevation coverage is required so as to preclude possible use of an *X-Y* antenna. Limitations in available ship real estate and avoidance of special maintenance problems associated with a relatively inaccessible location require a multiple antenna installation to meet this objective. With judicious choice of locations, two antennas should provide the necessary coverage.*

Within probable ship-motion extremes, it is generally possible to take advantage of inherent redundancy afforded by a daul installation so that only two-axis antenna mounts are required. For full earth-referenced hemispherical coverage without the possibility of unbounded rates and accelerations, however, the outer-gimbal axes of the two antennas must be directed in different directions. The typical use of a normal-to-the-deck outer-gimbal axis for both antennas is, therefore, not possible since this use would contradict this requirement.

The outer-gimbal axis, which will be called the primary axis, is generally inclined to the deck and different for each of the two antennas. The inner-gimbal axis, designated the secondary axis, is taken to be orthogonal to the primary axis. Deck orientation of the primary axis of each antenna requires a minimum of two numbers for its description. If two Eulerian angles providing this description are chosen to be the same as that giving the tracking line of a deck-mounted *X-Y* antenna with the *X* axis parallel to its own ship center line, an apparent reduction in complexity would be possible since this angle could be combined in a single planar transformation with the roll angle used to define ship attitude. However, since a shipboard dual antenna installation would typically be fore-aft or athwartships with the primary axes oppositely inclined at train angles of 0 and 180 degrees or −90 and 90 degrees, respectively, primary-axis designation by deck train and elevation angles would result in an equally simple flow diagram and would generally be more amenable to standard measuring techniques. The corresponding transformations in a plane parallel to the deck for these train angles would be trivial and could be performed without a resolver. The choice for sense of directoin being arbitrary, Fig. C1 uses the downward (into the deck) direction to define orientation of the primary axis.

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*In some instances, it may also be desired to avoid frequent switching between antennas as well as providing full coverage. Computer determination of shipboard coverage including the antenna switching problem is treated in detail in the work of T. Schmeckpeper and C. A. Bass (NRL Report in Preparation).

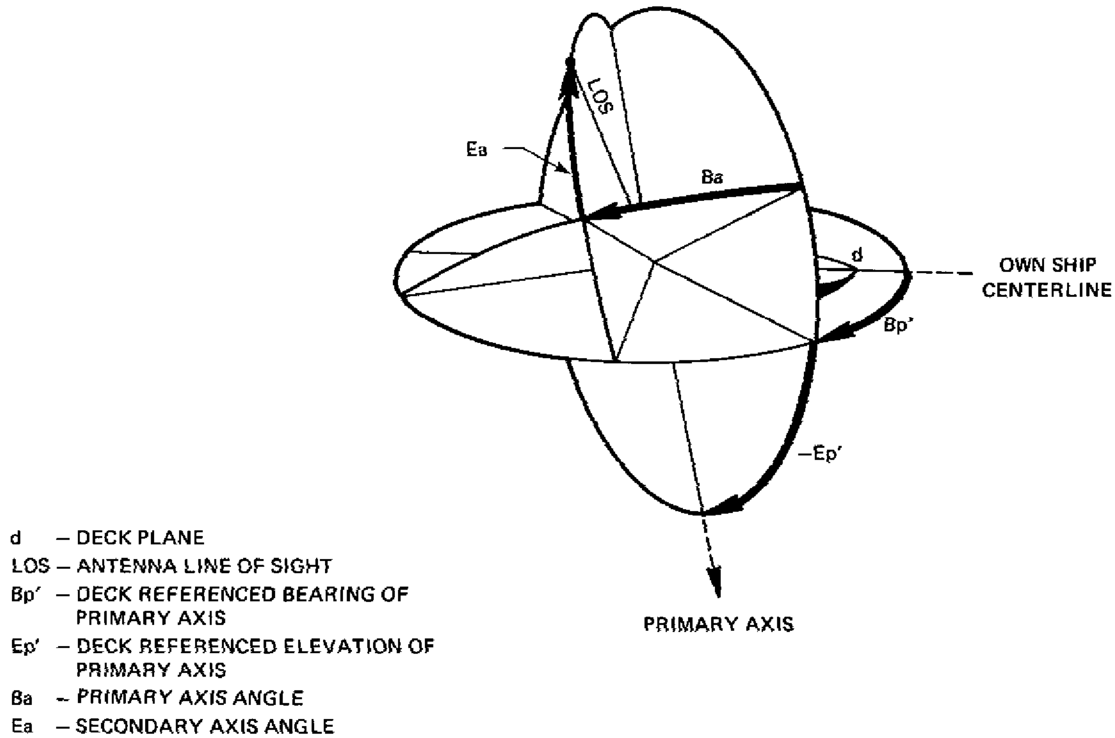

Fig. C1 – Deck-referenced geometry for a shipboard-mounted two-axis antenna having general orientation.

Starting with the right-hand system frequently used in this report for defining an earth-referenced space direction*, the flow-diagram used to compute B_a and E_a is given by Fig. C2a. Separate transformations are shown for true bearing and ship heading, although they may be combined. According to the selected sense for the primary axis, Ep' is generally negative. If $Ep' = -90$ degrees, the antenna is the typical train-elevation configuration; if $Ep' = 0$ degree, a deck-mounted X-Y configuration results. In Fig. C2b, the angular velocity-acceleration flow diagram, Bp' and Ep' are both treated as constant angles.

A position flow diagram for a dual antenna installation may be drawn in the manner of Fig. C3a. This represents the problem of directing two such antennas from a single source. Figures C3b and C3c show modes corresponding to followup on the director and slave of one antenna from the other.

*The x axis is directed along the line of sight, the z axis downward, and the y axis chosen for a right-hand system.

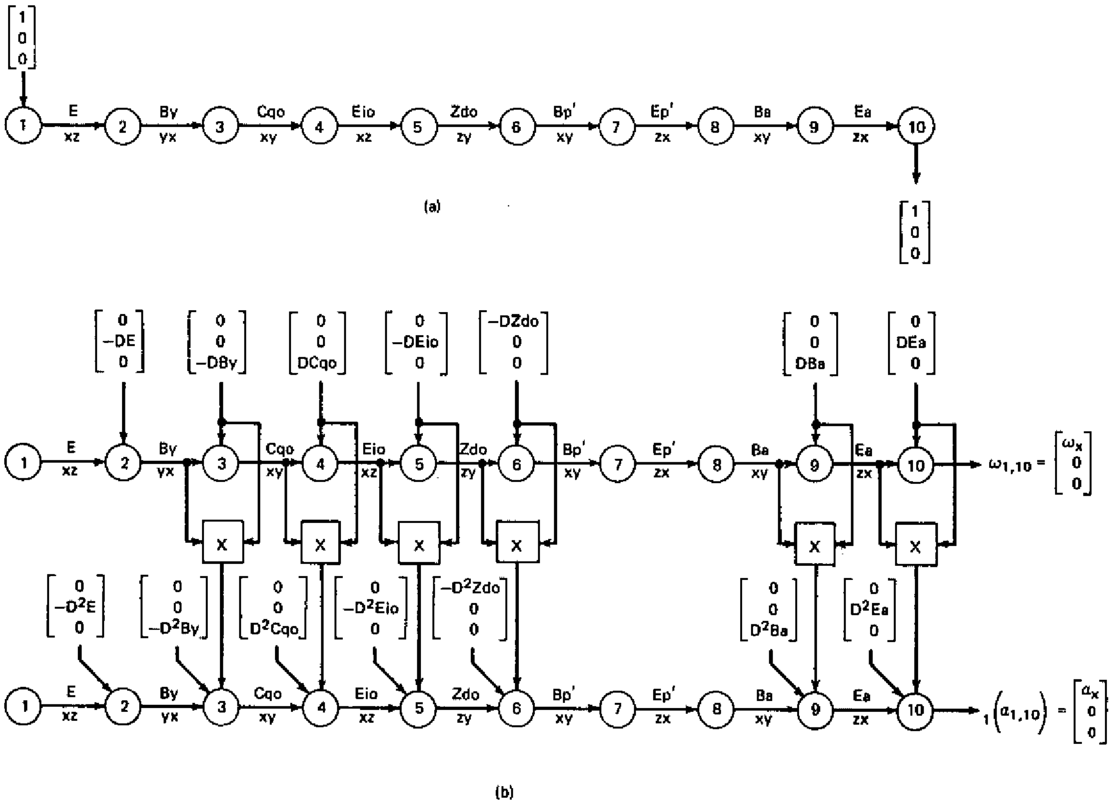

Fig. C2 – Single shipboard two-axis system having general orientation.

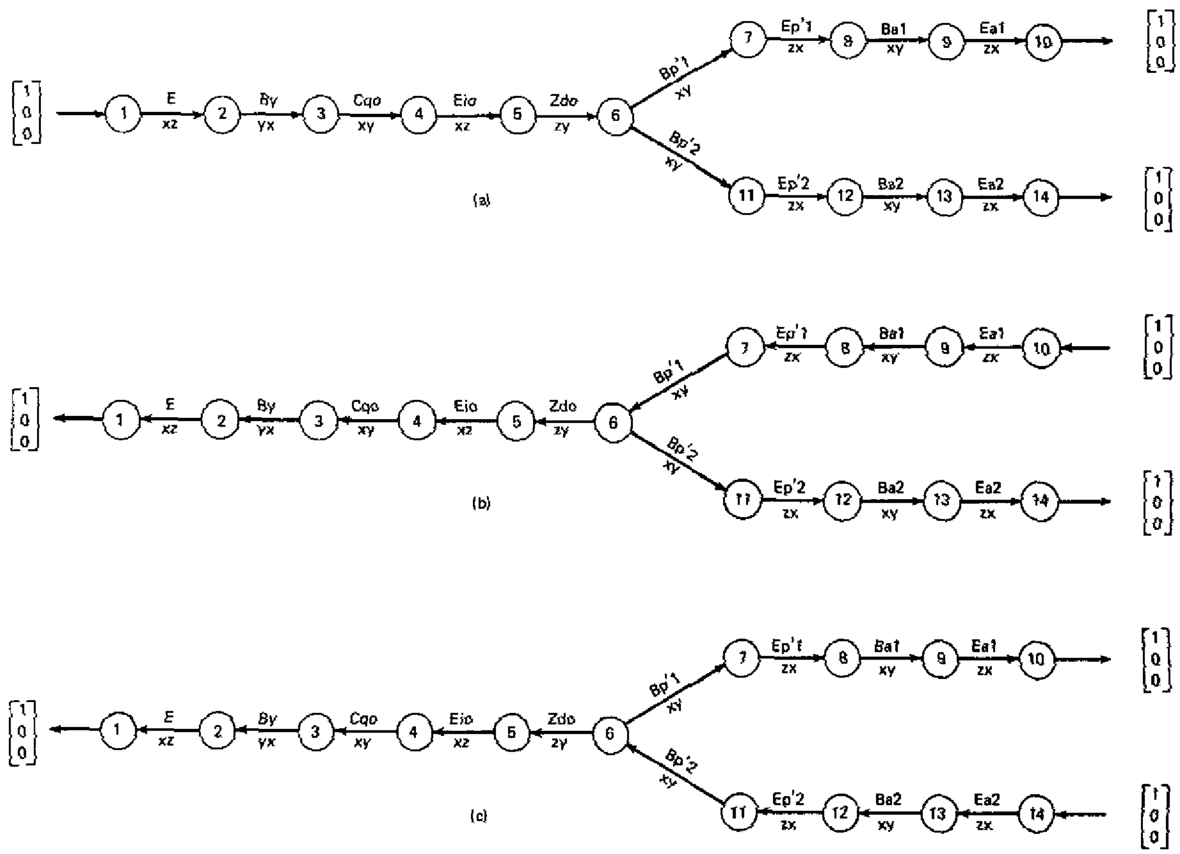

Fig. C3 – Dual shipboard two-axis system.

Appendix D

GEOMETRIC STABILIZATION

Stabilization, or more properly geometric stabilization, is taken here to be the action of isolating the tracking line or attitude of the controlled member from platform motion. In the broad sense, this may include either velocity or position measurement with respect to some preferred reference frame, such as earth or an inertial frame. Stabilization of the tracking line of a shipboard-mounted elevation-over-train antenna is then inherently accomplished by implementing a director system corresponding to Fig. D1.* Isolation of the tracking line is performed by the portion of the director system between node 3 and node 8, while directing the tracking line relative to earth corresponds to that portion between node 1 and node 3. A similar overall objective is achieved by an automatic tracking system which uses the positional reference afforded by the angle of arrival of rf energy. However, common usage of the term stabilization and the desire to optimize tracking-loop bandwidth for minimum tracking error independent of requirements to overcome platform motion generally precludes this interpretation. Stabilization of the attitude of the controlled member requires a configuration with a minimum of three axes. A director system having attitude control as the objective also provides stabilization under the above interpretation but with the added refinement of attitude control. The limited discussion of this appendix, however, is confined to the use of flow diagrams to analyze techniques for stabilizing the tracking line of an elevation-over-train configuration.

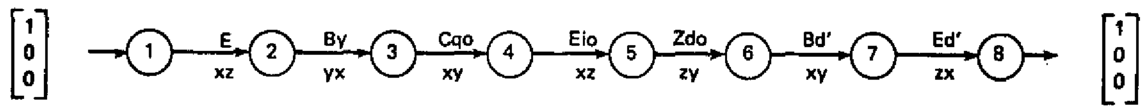

Fig. D1 – Train-elevation director.

If position control is used to obtain isolation of the controlled member from platform motion, velocity control is inherent. When stabilization is achieved by velocity control, an absolute positional reference is not obtained. This may be supplied by a director (Fig. D1, for example) or by an automatic tracking system. Either of these two positional references performs the bulk of stabilization for lower frequency components of platform motion, especially offset and drift, while velocity control provides stabilization for the higher frequency components.

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*The flow diagrams of this appendix use coordinate systems based on the controlled-member system having the x axis directed along the tracking line, the z axis in a downward direction, and the y axis chosen for a right-hand system.

When heading, pitch, and roll are available, their rates, obtained through use of appropriate hardware or software, may be used to calculate angular velocity of the moving platform relative to earth. Starting with node 3 (earth frame), the velocity flow diagram of Fig. D2 may be solved in the manner of that for Fig. 28. The rates *DBd'* and *DEd'*, after having been computed, may then be used for "feed-forward" stabilization through input to tachometer servo loops designed and calibrated for this purpose.

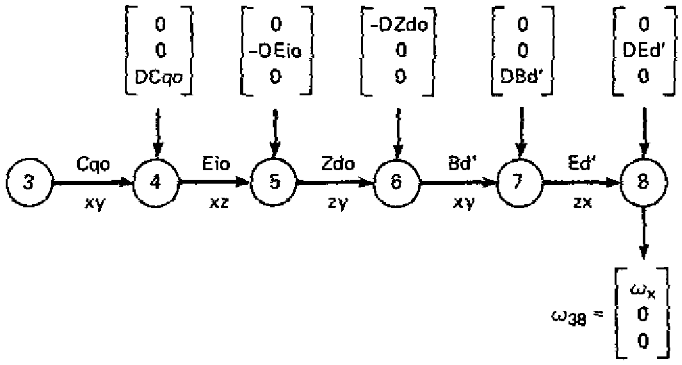

Fig. D2 – Flow diagram for forward stabilization using heading, pitch, and roll rates.

In lieu of the availability of heading, pitch, and roll and to reduce computing complexity, the angular velocity of Σ_3 relative to an inertial frame may be obtained through use of three single-degree-of-freedom rate gyros. Figure D3a shows pertinent transformations and contributions to inertially referenced angular velocities of the moving platform and the controlled member when the celestial sphere is taken to be an inertial frame. With the center of the celestial sphere as the origin of the coordinate system of node i, the x axis is directed toward the vernal equinox, the z axis toward the south celestial pole, and the y axis chosen for a right-hand system. The transformations between node i and node 3 involve local hour angle *LHA* and latitude *LAT*. Earth's sidereal rate $-D(LHA)$ accounts for the motion of the earth relative to the inertial frame. Translational quantities are of no concern here as only parallel space directions are being considered.

The difference between the correction established by an inertial reference and that by an earth reference is generally not important if a director system or automatic tracking is used for positional reference. In the absence of a position reference, motion of the tracking line would experience an apparent drift due to Earth's rotation as well as that resulting from imperfections of inertial-rate measurements and control. A triad of rate gyros on the moving platform and respectively aligned with the x, y, and z axes of node 6 of Fig. D3b provide the components of $\boldsymbol{\omega}_{i6}$ in this system, where i denotes an inertial frame. Following their calculation, *DBd'* and *DEd'* are used in the previously discussed forward method of isolating the tracking line from platform motion.

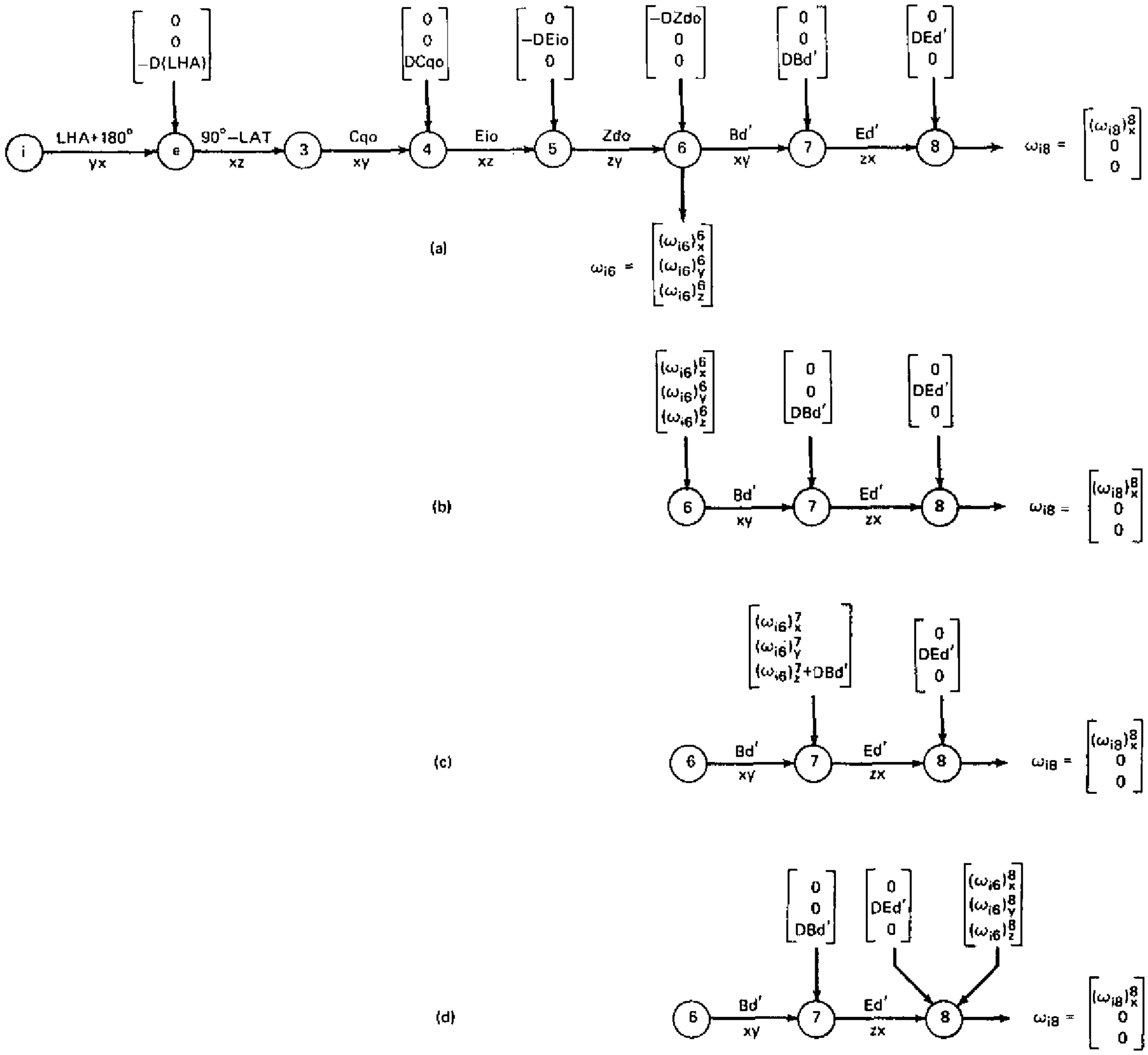

Fig. D3 – Flow diagram forms for analysis of velocity-controlled stabilization.

Figure D3c is obtained by describing $\boldsymbol{\omega}_{i6}$ in node 7 rather than node 6. The notation $(\omega_{i6})_x^6$ is used to denote the x component of $\boldsymbol{\omega}_{i6}$ when described in system 6*. Since $(\omega_{i6})_z^7 = (\omega_{i6})_z^6$ cannot be measured by a rate gyro on Σ_7 by virtue of the added z component DBd', this term is still most conveniently measured by a rate gyro on Σ_6. The other two rate gyros, however, are moved to Σ_7 to measure $(\omega_{i6})_x^7$ and $(\omega_{i6})_y^7$, respectively. The motion of Σ_7 relative to Σ_6 will have no effect on the output of these two gyros as $\boldsymbol{\omega}_{67}$ has only the z component DBd' when described in node 7.

*This notation is not really necessary in a flow diagram, since the system of description for a vector is inherent to that diagram.

Solution of Fig. D3c is straightforward and simple. The output vector

$$\begin{bmatrix} (\omega_{i8})_x^8 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} -[DBd' + (\omega_{i6})_z^7]\sin Ed' + (\omega_{i6})_x^7 \cos Ed' \\ DEd' + (\omega_{i6})_y^7 \\ [DBd' + (\omega_{i6})_z^7]\cos Ed' + (\omega_{i6})_x^7 \sin Ed' \end{bmatrix}$$

may be solved for DBd', DEd', and the cross-roll rate $(\omega_{i8})_x^8$. Alternately, by inverting the order for the transformation between nodes 7 and 8 and using Eq. (20) with $a_1 = (\omega_{i6})_x^7$, $b_2 = 0$, and $\gamma = -Ed'$,

$$DBd' + (\omega_{i6})_z^7 = \frac{(\omega_{i6})_x^7 \sin(-Ed')}{\cos(-Ed')}$$

or

$$DBd' = -[(\omega_{i6})_z^7 + (\omega_{i6})_x^7 \tan Ed'] \tag{D1}$$

and

$$(\omega_{i8})_x^8 = \frac{(\omega_{i6})_x^7}{\cos Ed'} \tag{D2}$$

Direct observation of Fig. D3c yields

$$DEd' = -(\omega_{i6})_y^7. \tag{D3}$$

By describing $\boldsymbol{\omega}_{i6}$ in the system of node 8, the flow diagram obtained in Fig. D3d is in a form suitable to analyze a feedback method for stabilization. If the effect of platform motion and angular velocity of the controlled member relative to the platform are considered separately, the resulting expression is

$$\boldsymbol{\omega}_{i8} = \boldsymbol{\omega}_{i6} + \boldsymbol{\omega}_{68}.$$

A feedback scheme drawn with the geometry in the system of node 8 has points which may be conveniently related to gimbal rates DBd' and DEd'. The servos shown in Fig. D4 implement this method by using two rate (or integrating-rate) gyros mounted on the controlled member to measure $(\omega_{i8})_y^8$ and $(\omega_{i8})_z^8$. The commands to the loops will be nonzero if the tracking line is required to move relative to inertial space. If the cross-roll rate $(\omega_{i8})_x^8$ is desired, a third gyro on the controlled member is required, its input axis being aligned along the tracking line for the simplest form.

Since $\boldsymbol{\omega}_{i7} + \boldsymbol{\omega}_{78} = \boldsymbol{\omega}_{i8}$ and $\boldsymbol{\omega}_{78}$ has only a y component when described in node 8, an alternate schemes is to use two gyros on Σ_7 to measure $(\omega_{i7})_x^7$ and $(\omega_{i7})_z^7$ and then transform these via a resolver to give $(\omega_{i7})_z^8 = (\omega_{i8})_z^8$ and $(\omega_{i7})_x^8 = (\omega_{i8})_x^8$. A third gyro, this being mounted on Σ_8, is used as before to measure $(\omega_{i8})_y^8$.

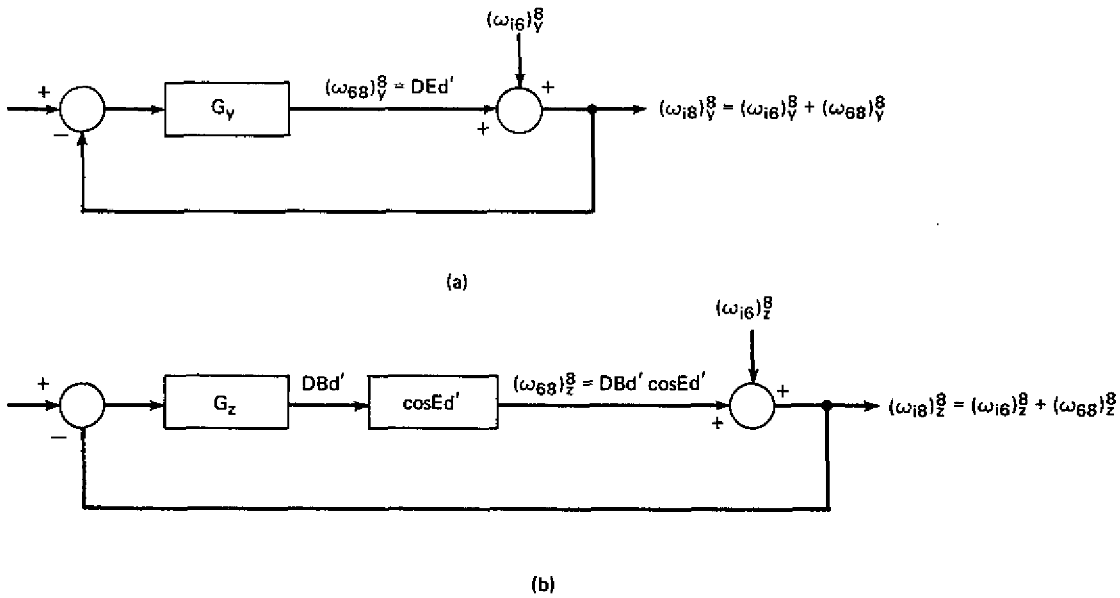

Fig. D4 – Feedback method for stabilization

G_y and G_z are high gain terms containing time constants either of deliberate design or inherent to the hardware being employed. If the gyros are of the integrating-rate type and connected in the integrating-mode, G_y and G_z will each have an integrating term. In addition, G_z may have the term sec Ed' (except near $Ed' = \pm 90$ degrees) to compensate for the cos Ed' term in Fig. D4b which results from the geometric relations.

Depending on configuration and availability of position and velocity sensors and their mounting, a wide variety of schemes is available for geometric stabilization. Use of flow diagrams allows the designer to quickly visualize the geometry involved in these methods.