

## DOCUMENT CONTROL DATA - R &amp; D

(Security classification of title, body of abstract and indexing annotation must be entered when the overall report is classified)

1. ORIGINATING ACTIVITY (Corporate author)		2a. REPORT SECURITY CLASSIFICATION	
Naval Research Laboratory Washington, D.C. 20390		Unclassified	
3. REPORT TITLE		2b. GROUP	
Thermal Blooming in Viscous and Thermally Conducting Fluids			
4. DESCRIPTIVE NOTES (Type of report and inclusive dates)			
This is an interim report on a continuing problem.			
5. AUTHOR(S) (First name, middle initial, last name)			
J. N. Hayes			
6. REPORT DATE	7a. TOTAL NO. OF PAGES	7b. NO. OF REFS	
August 21, 1972	22	2	
8a. CONTRACT OR GRANT NO.	9a. ORIGINATOR'S REPORT NUMBER(S)		
NRL Problem A-02-17.101	NRL Report 7383		
b. PROJECT NO.	9b. OTHER REPORT NO(S) (Any other numbers that may be assigned this report)		
RR 011-07-41-5152			
c.			
d.			
10. DISTRIBUTION STATEMENT			
Approved for public release; distribution unlimited.			
11. SUPPLEMENTARY NOTES		12. SPONSORING MILITARY ACTIVITY	
		Department of the Navy (Office of Naval Research) Arlington, Virginia 22217	
13. ABSTRACT			
<p>The problem of thermal blooming of a laser beam in viscous and thermally conducting fluids is treated by means of the model discussed in NRL Report 7213. (In that report blooming was treated for an ideal gas only.) It is shown that viscous effects are small, as should be expected. The longtime behavior of the density in the presence of thermal conductivity is discussed at length. The cases of beam diameters of 0.5 and 1.0 cm with Gaussian intensity profiles are treated. The effect of thermal conduction in these cases is small, but in the direction of reducing the thermal blooming. For smaller diameter beams, it is shown that conductive effects are more pronounced.</p>			

14. KEY WORDS	LINK A		LINK B		LINK C	
	ROLE	WT	ROLE	WT	ROLE	WT
Laser beams Propagation Thermal blooming Nonlinear optics Light transmission						

## CONTENTS

Abstract .....	
Problem Status .....	
Authorization .....	
I. INTRODUCTION .....	
II. DETERMINATION OF DENSITY CHANGES .....	
III. TRACING LIGHT RAYS .....	
REFERENCES .....	
APPENDIX A—Parameter Values .....	
APPENDIX B—Mathematical Considerations .....	
APPENDIX C—Basic Model Hypotheses .....	

## ABSTRACT

The problem of thermal blooming of a laser beam in viscous and thermally conducting fluids is treated by means of the model discussed in NRL Report 7213. (In that report blooming was treated for an ideal gas only.) It is shown that viscous effects are small, as should be expected. The longtime behavior of the density in the presence of thermal conductivity is discussed at length. The cases of diameters of 0.5 and 1.0 cm with beam Gaussian intensity profiles are treated. The effect of thermal conduction in these cases is small, but in the direction of reducing the thermal blooming. For smaller diameter beams, it is shown that conductive effects are more pronounced.

## PROBLEM STATUS

This is an interim report on a continuing problem.

## AUTHORIZATION

NRL Problem A0217.101  
Project RF 05-152-401

Manuscript submitted December 14, 1971.

# THERMAL BLOOMING IN VISCOUS AND CONDUCTING FLUIDS

## I. INTRODUCTION

In a previous report (1) the problem of thermal blooming of a laser beam was considered for the case of perfect gases. For that study, a model\* for the laser and the medium was defined, and within the framework of that model, light rays from the laser were traced and intensity profiles downbeam were determined, both as functions of time. The solution obtained was appropriate for those intervals of time up to the point where convection or wind effects, or both, became predominant. The calculation was deficient insofar as it applied only to gases and it ignored the effects of thermal conductivity and viscosity. Since these effects are much easier to study in experiments concerned with liquids, it would be desirable to extend the theory to include these fluids. The determination of these effects may prove necessary for a quantitative comparison between experiment and theory.

We may expect that viscous effects will not play too large a role in the blooming of a laser beam because the velocities involved are quite small. It will be seen that this expectation is justified by detailed calculations applied to a number of common fluids. The effects of thermal conductivity are another matter. Since a finite thermal conductivity for a medium acts as a heat sink in hotter regions of the medium and as a heat source for the cooler regions, the principal consequence expected in the problem at hand is a flattening of the density profile  $\rho_1$  compared to that obtained in Ref. 1, subsequently leading to a diminution of the blooming. Since thermal conduction depends upon temperature gradients, we must expect the effects of heat transport to depend upon the absolute dimensions of the laser beam, in contrast to the results of Ref. 1 where scaling was complete. Therefore we qualitatively expect that thermal conduction will be important in small beams and considerably less so in large beams (for a fixed power profile across the face of the laser). Since a change in the spatial mode of the laser alters the power profile, which in turn alters the temperature profile in the medium, the effects of thermal conduction will change from one mode to another. Generally speaking, heat transport will tend to diminish the differences between differing profiles and to diminish the blooming. An exception would be a beam with a flat power profile; in Ref. 1 we saw that this case led to no blooming. When thermal conduction is included, there exists an infinite temperature gradient in the medium and this would imply an infinite heat transport. This of course is quite unrealistic, but a very flat profile with smooth edges will have a large temperature gradient; thermal conduction will cause the temperature (and density) profile in the medium downbeam to spread, and thereby cause blooming.

Our qualitative expectations will be found justified by detailed calculations of these effects. The calculations are based on the same model as used in Ref. 1, with the aforementioned exceptions, and the notational conventions used there will be adopted here.

---

\*For a brief summary of the model as discussed in Ref. 1, see the present App. C.

## II. DETERMINATION OF DENSITY CHANGES

The fluid medium, which is now taken to be viscous, obeys the linearized equations of hydrodynamics (2) given by

$$\frac{\partial \rho_1(\mathbf{R}, t)}{\partial t} + \rho_0 \nabla \cdot \mathbf{v}_1(\mathbf{R}, t) = 0 \quad (1)$$

and

$$\rho_0 \frac{\partial \mathbf{v}_1(\mathbf{R}, t)}{\partial t} - \eta \nabla^2 \mathbf{v}_1(\mathbf{R}, t) - \left( \zeta + \frac{1}{3} \eta \right) \nabla \nabla \cdot \mathbf{v}_1(\mathbf{R}, t) = -\nabla p_1(\mathbf{R}, t) \quad (2)$$

where  $\eta$  and  $\zeta$  are the first and second viscosity coefficients,  $\rho_0$  is the initial ambient density,  $\mathbf{R}$  is the position vector to a point in the beam,  $t$  is time, and  $\mathbf{v}$  is velocity. The equation of energy transport reduces to

$$\frac{\partial T_1(\mathbf{R}, t)}{\partial t} = \frac{T_0 \beta B}{\rho_0^2 c_V} \frac{\partial \rho_1}{\partial t} + \frac{\kappa}{\rho_0 c_V} \nabla^2 T_1(\mathbf{R}, t) + \frac{q(\mathbf{R}, t)}{c_V} \quad (3)$$

where  $\beta$  is the volume expansivity,  $B$  is the isothermal bulk modulus,  $c_V$  is the specific heat at constant volume,  $\kappa$  is the coefficient of thermal conductivity, and  $q$  is the energy deposited per gram-sec by the laser beam. The quantities  $\rho_1$ ,  $\mathbf{v}_1$ ,  $p_1$ , and  $T_1$  are the deviations of density, velocity, pressure, and temperature of the fluid from their initial values  $\rho_0$ ,  $\mathbf{v}_0 = 0$ ,  $p_0$ ,  $T_0$ . The existence of an equation of state for the fluid gives a fourth relation

$$p_1 = \frac{B}{\rho_0} \rho_1 + b \beta T \quad (4)$$

between the dependent variables.

To get the density  $\rho_1$ , not all three vector components of the pressure gradient in Eq. (2) are needed. Taking the divergence of both sides of Eq. (2), we get

$$\frac{\partial}{\partial t} \rho_0 \nabla \cdot \mathbf{v}_1 - \left( \zeta + \frac{4}{3} \eta \right) \nabla^2 \nabla \cdot \mathbf{v}_1 = -\nabla^2 p_1. \quad (5)$$

We need only solve the scalar Eqs. (1), (3), (4), and (5) to get  $\rho_1$  in Eq. (5), thus reducing the equations to three in number. We take the Laplace transform of these equations with respect to the time variable, and the Fourier transforms with respect to the space variables. The resulting equations may then be solved algebraically to get the Laplace-Fourier transform  $\hat{\rho}_1(\mathbf{k}; s)$  of the density  $\rho_1$ . Omitting the detailed steps, we have as our result

$$\frac{\hat{\rho}_1(\mathbf{k}; s)}{\rho_0} = -\frac{3 \beta c^2}{2 c_p} \frac{1}{t_c} \frac{k^2 c^2 \hat{f}(\mathbf{k}; s)}{(s + \tau_\kappa k^2 c^2)(s^2 + k^2 c^2 + \tau_\eta k^2 c^2 s) - \frac{\gamma - 1}{\gamma} \tau_\kappa k^4 c^4} \quad (6)$$

where the function  $\hat{f}(\mathbf{k}; s)$  is related to the Laplace-Fourier transform  $\hat{q}(\mathbf{k}; s)$  of  $q(\mathbf{R}, t)$  by

$$\hat{q}(\mathbf{k};s) = \frac{3c^2}{2t_c} \hat{f}(\mathbf{k};s). \quad (7)$$

The constant  $t_c$  is the characteristic time constant  $t_c = 3\pi a^2 \rho_0 c^2 / 2\alpha W$ , as defined in Ref. 1,  $c$  is the velocity of sound in the medium and is given by  $c^2 = \gamma\beta/\rho_0$ ,  $\gamma$  is the ratio of specific heats  $c_p/c_V$ , and  $\tau_\kappa$  and  $\tau_\eta$  are new constants with the dimensions of time, defined by

$$\tau_\kappa \equiv \frac{\kappa}{\rho_0 c_V c^2} = \frac{\gamma\kappa}{\rho_0 c_p c^2} \quad (8a)$$

and

$$\tau_\eta \equiv \frac{\zeta + \frac{4}{3}\eta}{\rho_0 c^2}. \quad (8b)$$

For a perfect gas,  $\tau_\kappa = \tau_\eta = 0$  and  $3\beta c^2 / 2c_p = 1$ ; then Eq. (6) becomes the Laplace-Fourier transform  $\hat{\rho}_1(\mathbf{k};s)$  as derived in Ref. 1. Thus, for a liquid whose thermal conductivity and viscosity may be ignored, the only changes required of the theory in Ref. 1 is to multiply the density variation  $\rho_1$  given there by  $3\beta c^2 / 2c_p$ .

The effects of thermal conduction and viscosity are concentrated in the denominator of the right-hand side of Eq. (6); it is a cubic polynomial in  $s$  which we designate by  $F(s)$ . Multiplying out the terms,  $F(s)$  is given by

$$F(s) = s^3 + (\tau_\eta + \tau_\kappa)k^2 c^2 s^2 + (1 + \tau_\eta \tau_\kappa k^2 c^2)k^2 c^2 s + \tau_\kappa k^4 c^4 / \gamma. \quad (9)$$

The inversion of the Laplace transform is easy to accomplish when the zeroes of  $F(s)$  are known, in which case  $F(s)$  may be written as a product of three linear factors. Standard tables of Laplace transforms may then be used. Furthermore, the solutions for the roots of a cubic in terms of its coefficients are well known, so in principle we have no problem. However, the coefficients of  $F(s)$  are functions of the wave number  $k$ , or more precisely, of  $k^2$ . Hence the roots of the cubic will also be functions of  $k^2$ , and indeed they will be very complicated functions of  $k^2$ . Since we still have to invert the Fourier transform to obtain  $\rho_1(\mathbf{k},t)$ , one can foresee an analytical problem of considerable magnitude if one tries for an exact solution. Hence we seek to locate the roots only approximately, but with sufficient accuracy so as to leave us a result in which we have the confidence that it still represents the physical situations to a considerable degree of accuracy.

A reference to the tables in App. A is helpful at this point. There we list, among other things, the values of  $\tau_\eta$ ,  $\tau_\kappa$ , and  $c^2$  for some common substances which may easily be used in the thermal blooming experiments. (We note that the viscosity coefficients in some substances show a frequency dependence which, if included, would complicate the inversion of the Fourier transform. Since the viscous effects are expected to be small, choosing  $\zeta$  and  $\eta$  to be independent of wavenumber will simplify our discussion and give us a result that is representative of the real situation, even if it is not exact. Indeed, justification of this simplifying hypothesis will be a posteriori.) It is seen that  $\tau_\eta$  in all cases is of the order of  $10^{-12}$  to  $10^{-11}$  s, while  $\tau_\kappa \approx 10^{-13}$  to  $10^{-10}$  s. Hence  $\tau_\eta \tau_\kappa k^2 c^2$  will be small compared to unity if  $k^2 c^2$  remains sufficiently small. Now, if the radial dimension of the laser power profile is of the order of  $a$ , then the function  $f(\mathbf{k},s)$  in the numerator of Eq. (6) diminishes to zero rapidly when  $k$  becomes significantly larger than

$a^{-1}$ . Hence the behavior of the denominator  $F(s)$  for larger values of  $k$  is of no great consequence. Therefore,  $\tau_\eta \tau_\kappa k^2 c^2$  will be small compared to unity if  $\tau_\eta \tau_\kappa c^2/a^2$  is small compared to unity for significant values of  $k^2$ . For the fluids listed in the tables of App. A,  $\tau_\eta \tau_\kappa c^2/a^2 \approx 10^{-4} a^{-2}$ ; thus, the approximation improves as the radius of the beam increases and is excellent even for  $a \approx 1$  mm. Within the framework of this approximation, the polynomial  $F(s)$  now becomes

$$F(s) \approx s^3 + (\tau_\eta + \tau_\kappa)k^2 c^2 s^2 + k^2 c^2 s + \tau_\kappa k^4 c^4/\gamma. \quad (10)$$

To get the zeroes of  $F(s)$  as given in the above equation we could use Cardan's technique, but this proves to be very cumbersome. We shall want the roots only to order  $\tau$  anyway, so they will be obtained as follows. First we note that at  $\tau_\eta = \tau_\kappa = 0$ ,  $F(s)$  becomes  $s(s^2 + k^2 c^2)$ ; hence the roots in this case are at 0,  $+ikc$ , and  $-ikc$ . Now assuming nothing peculiar happens, the roots are generally analytic functions of the coefficients of the polynomial. Hence, the roots of Eq. (10) must be near 0 and  $\pm ikc$ . So, write  $F(s)$  in the form

$$F(s) = (s - s_1)(s^2 + \beta s + \gamma k^2 c^2).$$

The roots are then determined uniquely by multiplying the two factors of the last expression and comparing them with Eq. (10), and by requiring that, to the lowest order in  $\tau_\eta$  and  $\tau_\kappa$ , (a) the sum of the roots be  $(\tau_\eta + \tau_\kappa)k^2 c^2$ , (b) the sum of their products in pairs be  $k^2 c^2$ , and (c) their product be  $\tau_\kappa k^4 c^4/\gamma$ . The roots thus obtained are

$$s_1 = -k^2 c^2 \tau_\kappa / \gamma, \quad (11a)$$

$$s_2 = ikc - k^2 c^2 (\tau_\eta + (1 - \gamma^{-1})\tau_\kappa)/2, \quad (11b)$$

and

$$s_3 = -ikc - k^2 c^2 (\tau_\eta + (1 - \gamma^{-1})\tau_\kappa)/2. \quad (11c)$$

It remains to check if these roots are consistent with our assumptions. The ratio of the real part of  $s_2$  to its imaginary part has the magnitude  $kc(\tau_\eta + (1 - \gamma^{-1})\tau_\kappa)/2$ , which for significant values of  $k$  is smaller than, or of the order of,  $c(\tau_\eta + (1 - \gamma^{-1})\tau_\kappa)/2a$ . For a laser whose radius is of the order of 1 mm, this ratio is of the order of  $10^{-4}$ ; for larger beam diameters, the approximation improves.

Before the Laplace transform can be inverted, the function  $\hat{f}(\mathbf{k};s)$  has to be specified. Adopting the same model of energy deposition as in Ref. 1,  $f(\mathbf{R},t)$  is, in fact, independent of time so that  $\hat{f}(\mathbf{k};s)$  has the form  $\hat{f}(\mathbf{k})/s$ . Then  $\hat{\rho}_1(\mathbf{k};s)$  can be written as

$$\frac{\hat{\rho}_1(\mathbf{k};s)}{\rho_0} = -\frac{3\beta c^2}{2c_p} \frac{1}{t_c} \frac{k^2 c^2 \hat{f}(\mathbf{k})}{s(s - s_1)(s - s_2)(s - s_2^*)}. \quad (12)$$

In Eq. (12), we have incorporated the fact that  $s_3 = s_2^*$ ; also note that  $s_1$  is real, and that the denominator itself is real. Then

$$\begin{aligned} \frac{1}{sF(s)} &= \left( \frac{1}{s(s - s_1)(s - s_2)} - \frac{1}{s(s - s_1)(s - s_2^*)} \right) \frac{1}{s_2^* - s_2} \\ &= -(\text{Im } s_2)^{-1} \text{Im} \left( \frac{1}{s(s - s_1)(s - s_2)} \right). \end{aligned} \quad (13)$$

Inverting the Laplace transform and keeping terms to the order  $\tau_\kappa$  and  $\tau_\eta$  at most in the multiplicative factors, we get

$$\frac{\hat{\rho}_1(\mathbf{k}, t)}{\rho_0} = -\frac{3\beta c^2}{2c_p} \frac{1}{t_c} \hat{f}(\mathbf{k}) \left[ \frac{\gamma}{k^2 c^2 \tau_\kappa} (1 - e^{-(k^2 c^2 \tau_\kappa / \gamma)t}) - \frac{1}{kc} e^{-\frac{1}{2} k^2 c^2 (\tau_\eta + (1-\gamma^{-1})\tau_\kappa)t} \left( \sin kct - kc \left( \tau_\eta + \left( \frac{3}{2} - \frac{2}{\gamma} \right) \tau_\kappa \right) \cos kct \right) \right]. \quad (14)$$

The model in Ref. 1 assumed the laser beam to have an intensity profile which is rotationally symmetric about the beam axis. This assumption renders  $\hat{f}(\mathbf{k})$  to be a function of  $k^2$  rather than a function of  $\mathbf{k}$ . Then  $\hat{\rho}_1/\rho_0$  may be written as the product  $\hat{f}(k^2)$  and  $G(k^2)$ , where  $G(k^2)$  is the set of factors on the right-hand side of Eq. (14) that multiply  $\hat{f}$ . The density function in space and time may then be written

$$\frac{\hat{\rho}_1(\mathbf{R}, t)}{\rho_0} = \int_0^\infty dr' r' f\left(\frac{r'}{a}\right) \int_0^\infty dk k J_0(kr') J_0(kr) G(k^2) \quad (15)$$

where  $\hat{f}(r/a)$  is the reduced intensity profile defined in Ref. 1,  $r = (x^2 + y^2)^{1/2}$  is the distance of the point whose position vector is  $\mathbf{R}$  from the beam axis (the  $z$  axis), and  $J_0$  is the Bessel function of order zero.

From Eqs. (14) and (15) it is clear that the density variation  $\rho_1$  in the medium is composed of two additive parts, one which involves the effects of thermal conduction alone, and another which has both thermal and viscous effects intertwined. Further, the latter term involves a description of the propagation of sound as well. In Ref. 1 where viscous and thermal effects were ignored, we saw that this term went to zero rapidly when  $t$  became larger than the time required for sound to traverse the distance  $a$ . A similar result obtains here, with minor modification due to the presence of  $\tau_\eta$  and  $\tau_\kappa$ . It is readily seen that both terms tend to the results of Ref. 1 in the limit that  $\tau_\eta$  and  $\tau_\kappa$  vanish.

Write  $G(k^2) = G_L(k^2) + G_S(k^2)$ , where

$$G_L(k^2) = -\frac{3\beta c^2}{2c_p} \frac{1}{t_c} \frac{\gamma}{k^2 c^2 \tau_\kappa} (1 - e^{-(k^2 c^2 \tau_\kappa) / (\gamma)t}) \quad (16a)$$

and

$$G_S(k^2) = +\frac{3\beta c^2}{2c_p} \frac{1}{t_c} \frac{1}{kc} e^{-\frac{1}{2} k^2 c^2 (\tau_\eta + (1-\gamma^{-1})\tau_\kappa)t} \times \left( \sin kct - kc \left( \tau_\eta + \left( \frac{3}{2} - \frac{2}{\gamma} \right) \tau_\kappa \right) \cos kct \right) \quad (16b)$$

Correspondingly, we may write  $\rho_1(\mathbf{R}, t) = \rho_1(\mathbf{R}, t)_L + \rho_1(\mathbf{R}, t)_S$ . We study the behavior of  $\rho_{1S}$  for times  $t > a/c$ .

$$\rho_{1S}(\mathbf{R}, t) = -\frac{3\beta c^2}{2c_p} \frac{1}{ct_c} \int_0^\infty dr' r f\left(\frac{r'}{a}\right) \int_0^\infty dk J_0(kr') J_0(kr) e^{-\frac{1}{2} k^2 c^2 \tau t} \times (\sin kct - kc\tau' \cos kct). \quad (17)$$

In Eq. (17), we have substituted  $r'$  for  $\tau_\eta + (3/2) - (2/\gamma)\tau_\kappa$  and  $\tau$  for  $\tau_\eta + (1 - \gamma^{-1})\tau_\kappa$ . Indeed, for the liquids listed in the tables of App. A,  $\tau \approx r' \approx \tau_\eta$ , so that  $\rho_{1S}$  is dominated by viscous and sound propagation effects primarily. Arguing as we did in Ref. 1, the first of the two terms on the right-hand side of Eq. (17) vanishes by the Riemann-Lebesgue Lemma as  $t \rightarrow \infty$ ; this can be demonstrated to mean  $t \gg a/c$ . Next we look at the  $k$  integration in the second term; make a change in the variable of integration from  $k$  to  $x = kc\sqrt{t\tau}$ . The  $k$  integral then becomes

$$\frac{2r'}{ct\tau} \int_0^\infty dx x J_0\left(x \frac{r'}{c\sqrt{t\tau}}\right) \cos\left(x \sqrt{\frac{t}{\tau}}\right) e^{-x^2}.$$

This expression is bounded by

$$\frac{2r'}{ct\tau} \int_0^\infty dx e^{-x^2} x = \frac{r'}{ct\tau} \approx \frac{1}{ct}.$$

Therefore the second term on the right-hand side of Eq. (17) is smaller than

$$-\frac{3\beta c^2}{2c_p} \frac{1}{c^2 t_c t} \int_0^\infty dr' r' f\left(\frac{r'}{a}\right) = -\frac{3\beta c^2}{2c_p} \frac{a^2}{c^2 t_c t}.$$

This term will be of interest only if it is comparable to, or larger than, the first term  $\rho_{1L}$ . While we have not yet determined  $\rho_{1L}$  when thermal conductivity is present, we know from Ref. 1, and from our remarks on how to generalize those results to fluids, that  $\rho_{1L} \approx -(3/2)(\beta c^2/c_p)(t/t_c)$ ; hence the ratio of these terms is given approximately by  $a^2 t^2/c^2$ . For times of the order of 1 s, this quantity will be small. Hence we may ignore  $\rho_{1S}$  compared to  $\rho_{1L}$ .

The function  $\rho_{1L}$  is given by

$$\begin{aligned} \frac{\rho_{1L}}{\rho_0} = & -\frac{3\beta c^2}{2c_p} \frac{1}{t_c} \int_0^\infty dr' r' f\left(\frac{r'}{a}\right) \int_0^\infty dk k J_0(kr') J_0(kr) \\ & \times \frac{\gamma}{k^2 c^2 \tau_\kappa} (1 - e^{-(k^2 c^2 \tau_\kappa / \gamma)t}). \end{aligned}$$

Make the  $r'$  integration dimensionless by letting  $r' = a\lambda$  and  $k = x\sqrt{\gamma/c^2\tau t}$ ; then

$$\frac{\rho_{1L}}{\rho_0} = -\frac{3\beta c^2}{2c_p} \cdot \frac{t_k}{t_c} \int_0^\infty d\lambda \lambda f(\lambda) \int_0^\infty dx \frac{(1 - e^{-x^2})}{x} J_0\left(x\sqrt{\frac{t_k}{t}}\lambda\right) J_0\left(x\sqrt{\frac{t_k}{t}}\frac{r}{a}\right) \quad (18)$$

where  $t_k$  is a constant with the dimensions of time defined by

$$t_k = \frac{\gamma}{\tau_\kappa} \frac{a^2}{c^2} = \frac{\rho_0 c_p a^2}{\kappa}. \quad (19)$$

It will be seen that the value of  $t_\kappa$  is a measure of the importance of the effects of thermal conductivity.\* If  $\kappa$  is very small, then  $t_\kappa$  by Eq. (19) is very large. Now when thermal conductivity is very small, we would naturally expect that a long time would have to pass for its effects to manifest themselves. That  $t_\kappa$  is large for small  $\kappa$  indicates that  $t_\kappa$  is the appropriate time constant to associate with our qualitative description. Conversely, when  $\kappa$  is large,  $t_\kappa$  is small and, on intuitive grounds, we expect that conduction effects will show themselves in a short time in these circumstances. Further,  $t_\kappa$  depends upon the linear dimensions of the beam—the smaller the beam, the shorter the time constant. This again coincides with the notion that thermal conduction will make its effects pronounced over shorter dimensions more rapidly than over longer dimensions. For a laser of dimension  $a \approx 0.1$  cm and a medium with  $\rho_0 \approx 1$ , we have  $c_p \approx 1$ ,  $\kappa \approx 10^{-3}$ , and  $t_\kappa \approx 10$  s. For a laser three times as wide,  $t_\kappa \approx 10^2$  s. For air at standard temperatures and pressures and  $a \approx 0.1$  cm,  $t_\kappa \approx 0.25$  s. To determine conduction effects quantitatively, however, Eq. (18) must be evaluated.

In paragraphs 1 and 2 of App. B it is shown that for a laser with a Gaussian profile, for which  $f(x) = \exp(-x^2)$ , the integral may be computed exactly in the sense that it may be expressed in terms of known tabulated transcendental functions  $Ei(x)$ . For this case,

$$\frac{\rho_1(r,t)_L}{\rho_0} = -\frac{3\beta c^2}{2c_p} \frac{t_\kappa}{t_c} \frac{1}{4} \left[ Ei\left(\frac{r^2}{a^2 \left(1 + \frac{4t}{t_\kappa}\right)}\right) - Ei\left(\frac{r^2}{a^2}\right) \right] \quad (\text{Gaussian Profile}). \quad (20)$$

In the limit of zero thermal conductivity, i.e.,  $t_\kappa \rightarrow \infty$ , we should recover the results of Ref. 1. We obtain this limit, to the order  $(t/t_\kappa)^2$ , by expanding the first term in a Taylor-McLaurin series about  $r^2/a^2$ . Then

$$\frac{\rho_1(r,t)_L}{\rho_0} \approx -\frac{3\beta c^2}{2c_p} \left( \frac{t}{t_c} e^{-r^2/a^2} + \frac{4t^2}{t_\kappa t_c} \left(\frac{r}{a}\right)^6 e^{-r^2/a^2} \right) \quad (21)$$

for  $(t/t_\kappa) \ll 1$ . In the limit as  $t_\kappa \rightarrow \infty$ , we see, indeed, that the results of Ref. 1 are recovered, although the factor  $3\beta c^2/2c_p$  was unity there. However, from the evaluation of  $t_\kappa$  in the above paragraph for typical substances, we saw that  $t_\kappa$  varied from  $10^1$  to  $10^2$  s for typical laser dimensions and a liquid medium, and an even shorter time for air. The phenomenon of thermal blooming occurs over a time scale of the order of seconds. It is therefore apparent that the approximation given by Eq. (21) in many instances will not be applicable and that Eq. (20) must be used instead for determining the light rays in the medium and for the intensity profiles. The effects of thermal conduction in some experimentally realizable situations, therefore, cannot be ignored.

For the case of the general choice of  $f(\lambda)$ , it is shown in App. B, Sec. 3, that the density everywhere increases as  $\ln(t/4t_\kappa)$ , and that for  $r \gg a$  the spatial and temporal development of the density varies as  $Ei[(t_\kappa/4t)(r^2/a^2)]$ , regardless of the choice of  $f$ . For values of  $r \ll a$ , the spatial details are wiped out and a uniform growth given by  $Ei(t_\kappa/4t)$  takes place. Thus, thermal conductivity, given enough time, tends to obliterate the structural details of the initial beam profile. In any specific case, the time required is of the

\*In the Gaussian case, it would appear that  $t_\kappa/4$  is a more appropriate value for this constant.

order of  $t_\kappa/4$  for this to occur; if this quantity is significantly less than the time scales for phenomena that interrupt the blooming, then it appears reasonable that we may always take Eq. (20) as a good approximation to the behavior of the density. However, in applying this result to blooming, one must be extremely cautious because the blooming phenomenon depends primarily on the difference in densities between two points in space (see Ref. 1). Therefore, while structural differences in the densities due to differing selections of  $f(\lambda)$  may affect only slightly the absolute values of the densities  $\rho_1$ , these same structural details may well be important for blooming.

### III. TRACING LIGHT RAYS

The equations for the light rays derived in Ref. 1 require modification to be applicable to liquids as well as gases, and to include the effects of thermal conduction. The changes are slight, however. The Lorentz-Lorenz law is still valid:

$$\frac{n^2 - 1}{n^2 + 2} = N\rho \quad (22)$$

where  $n$  is the index of refraction of the field,  $\rho$  is the density, and  $N$  is the molecular refraction. If  $\rho_1$  represents a slight deviation of the density from  $\rho_0$ , then this gives rise to a slight change  $\delta n^2$  in the square of the index from its initial value  $n_0^2$ , which is easily shown to be

$$\delta n^2 = \frac{1}{3} N(n_0^2 + 2)^2 \rho_1. \quad (23)$$

The difference between liquids and gases is that the right-hand side of Eq. (22) may be taken to be very small for gases, but is generally of the order of unity for liquids. In Ref. 1, it was proved that the equation for the light rays could be cast into the form

$$\frac{dz}{dr} = \left[ \frac{n^2(r_0, 0; t) + \int_0^z dz' \frac{\partial n^2(r(z'), z'; t)}{\partial z'}}{n^2(r(z), z; t) - n^2(r_0, 0; t) - \int_0^z dz' \frac{\partial n^2(r(z'), z'; t)}{\partial z'}} \right]^{1/2} \quad (24)$$

Adopting the same model here as we did in Ref. 1,  $\rho_1$  is independent of  $z$ ; from the hypothesis and Eq. (23), it follows that  $\partial n^2/\partial z'$  vanishes and the differential equation for the light rays reduces to

$$\frac{dz}{dr} = \left[ \frac{n^2(r_0, 0; t)}{n^2(r(z), z; t) - n^2(r_0, 0; t)} \right]^{1/2} \quad (25)$$

The numerator changes with time, but since we are limiting ourselves only to small density changes, and thereby through Eq. (23) to small changes in  $n^2$ , the numerator may be replaced to a good approximation by  $n_0^2$ , the initial index squared. The principal source of changes in  $dz/dr$  come from the denominator in the right-hand side of Eq. (25).

Writing  $n^2(r, z, t) = n_0^2 + \delta n^2$ , and using Eq. (23), Eq. (25) may be integrated and cast into the form

$$\left[ \frac{1}{3} \frac{N(n_0^2 + 2)^2}{n_0^2} \right]^{1/2} z = \int_{r_0}^r \frac{dr'}{\sqrt{\rho_1(r',t) - \rho_1(r_0,t)}}. \quad (26)$$

For those liquids and gases for which the effects of thermal conductivity can be ignored, or equivalently, for very early times, we have seen that

$$\rho_1(r,t) = -\frac{3\beta c^2}{2c_p} \frac{t}{t_c} f\left(\frac{r}{a}\right). \quad (27)$$

Then Eqs. (26) and (27) combined can be written as

$$\left( \frac{1}{3} \frac{N(n_0^2 + 2)^2}{n_0^2} \frac{3\beta c^2}{2c_p} \frac{t}{t_c} \right)^{1/2} \frac{z}{a} = \int_{r_0/a}^{r/a} \frac{dx'}{\sqrt{f(x_0) - f(x')}} \quad (28)$$

where  $x_0 = r_0/a$ . Equation (28) is equivalent to the result of Ref. 1 for an ideal gas, but is now cast into a form that is applicable to a liquid as well. The results of Ref. 1 apply to Eq. (28) in all particulars, therefore, except for the definition of the reduced variable  $\zeta$  defined there. Here it becomes

$$\zeta = \left( \frac{1}{3} \frac{N(n_0^2 + 2)^2}{n_0^2} \frac{3\beta c^2}{2c_p} \frac{t}{t_c} \right)^{1/2} \frac{z}{a}. \quad (29)$$

Scaling is once again complete; with the appropriate choice of constants and beam intensity profile, the graphs given in Ref. 1 may be used to estimate the light rays and the intensity profiles for nonconducting liquids.

When thermal conduction is important, part of the scaling in space and time is lost, and the situation becomes more complicated. Because the Gaussian profile is probably one of the more important ones, and because it is the only profile for which tractable analytical results are available to use, we restrict our discussion below to this case. We introduce the scaled or reduced coordinates

$$x = \frac{r}{a}, \quad (30a)$$

$$x_0 = \frac{r_0}{a}, \quad (30b)$$

$$\tau = \frac{4t}{t_c}. \quad (30c)$$

The quantity  $\tau$  is a "reduced" time variable, which is not needed when conduction can be ignored. In addition, we define a "reduced" distance  $\zeta$  downbeam by

$$\zeta = \left[ \frac{1}{3} \frac{N(n_0^2 + 2)^2}{n_0^2} \frac{3\beta c^2}{2c_p} \frac{t_c}{4t_c} \right]^{1/2} \frac{z}{a}. \quad (31)$$

With Eqs. (30), (31), and (20), the equation for the light rays becomes

$$\zeta = \int_{x_0}^x \frac{dx'}{\left\{ (Ei(x_0^2) - Ei[x_0^2(1 + \tau)^{-1}]) - (Ei(x'^2) - Ei[x'^2(1 + \tau)^{-1}]) \right\}^{1/2}} \quad (32)$$

Note that the quantities  $\xi$  and  $\zeta$  differ from one another in that the factor  $t$  in  $\xi$  is replaced by  $t_K/4$  in  $\zeta$ ; observe also that time, through the reduced variable  $\tau$  appears in the denominator on the righthand side of Eq. (32) and cannot be factored out as it was in Eq. (28). For this reason, one has to compute a set of light rays for each value of the time variable  $\tau$ ; the light rays do not scale in the time variable as they did in Ref. 1, or as in the case of nonconducting liquids discussed above. In addition, scaling is lost in the downbeam intensity profiles. What must be done now is to select a distance downbeam and, using the intensity formula from Ref. 1, compute the modifications to the energy profile induced by the blooming; but this must be done separately for each downbeam distance. Thus, the numerical work increases considerably over what is needed for a non-conducting medium.

The effect of thermal conduction on the thermal blooming is shown in Figs. 1 and 2. The solid curves represent the blooming as calculated in Ref. 1 without thermal conduction effects taken into account, while the dashed curves show the blooming with thermal conduction included. The beam sizes, ranges, and other pertinent parameters are given in the figure captions. It is seen that the conduction effects cause only a small modification in the beam profile by the time that the linearization approximation is not valid any longer, or by the time that convection begins to set in. As expected, the effects are more pronounced for smaller beams. These same qualitative features will carry over to the case of a beam in a liquid.

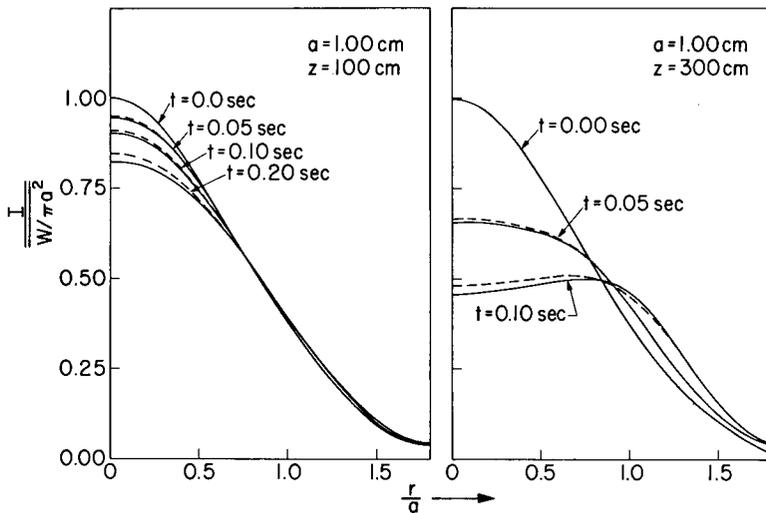


Fig. 1—Intensity profiles for a Gaussian beam. The solid curves show the bloomed beam without conduction effects taken into account, while the dashed curves include conduction. Specific parameter choices here are  $W = 100$  watts,  $\alpha = 10^{-3} \text{ cm}^{-1}$ ,  $\rho_0 = 10^{-3} \text{ gm/cm}^{-3}$ , and  $a = 1.00$  cm; all other parameters are those for air at STP. For these choices,  $t_c = 5.1$  s,  $t_{\text{conv}} = 0.33$  s,  $a/c = 0.3 \times 10^{-4}$  s, and  $t_K = 1$  s.

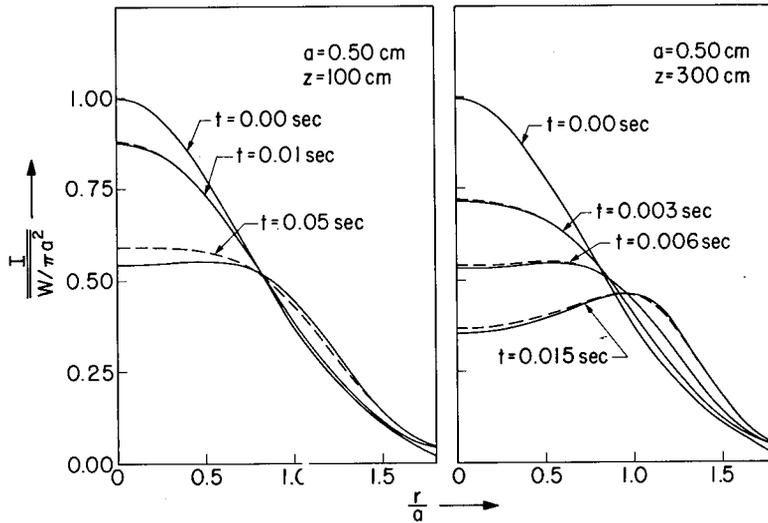


Fig. 2—Intensity profiles for a Gaussian beam. The solid curves show the bloomed beam without conduction effects taken into account, while the dashed curves include conduction. Here, the beam radius is  $a = 0.5$  cm, and the other parameters are as indicated on Fig. 4. For these choices,  $t_c = 1.0$  s,  $t_{conv} = 0.16$  s,  $a/c = 0.15 \times 10^{-4}$  s, and  $t_K = 0.25$  s.

#### REFERENCES

1. Hayes, J. N., "Thermal Blooming of Laser Beams in Gases," NRL Report 7213, February 11, 1971.
2. Landau, L. D., and Lifshitz, E.M., "Fluid Mechanics," Reading, Mass.: Addison-Wesley, 1959.

## APPENDIX A PARAMETER VALUES

The data shown in Tables A1 and A2 are adapted from the paper by Karim and Rosenhead\*, who list the quantities  $\mu$  and  $\mu'$  rather than our  $\zeta$  and  $\eta$ . These quantities are related by  $\mu = \eta$  and  $\mu' = \zeta - 2/3 \eta$ . The data are listed to indicate two things: first, that the bulk viscosity coefficient cannot be ignored, as it had been up to 1940. (Indeed it may dominate  $\eta$  by a substantial factor.) Second, these numbers indicate a frequency dependence. The numbers are not meant to be exact, in spite of their appearance; they should be taken as representative or indicative. Finally,  $\zeta$  and  $\eta$  in Table A1 have units of centipoise (cP).

The bulk viscosity coefficients  $\zeta$  have not been determined for as many substances as have the corresponding first viscosity coefficients  $\eta$ . To forestall the effects of convection in an experiment on blooming, one might choose a fluid that is much more viscous than those listed in Table A1. In such a case, it may prove to be important to obtain the value of  $\zeta$ . For such fluids, our treatment of the viscous terms in Sect. II of the present report is not correct.

Table A1  
Viscosity Coefficients for Common Substances

Substance	Freq (MHz)	$\eta$ (cP)	$\zeta/\eta$	$\zeta$ (cP)	$\zeta + 4/3 \eta$ (cP)
Water	5	1.00	3.06	3.06	4.40
Methyl Alcohol	5	0.60	1.96	1.18	1.98
Ethyl Alcohol	5	1.20	4.46	5.35	6.50
Acetone	5	0.30	3.76	1.13	1.43
Isopropyl Alcohol	5	2.20	5.66	12.45	15.25
Amyl Acetate	5	0.89	10.56	9.40	10.60
	2	0.89	10.26	9.35	10.55
Xylol(m)	5	0.62	11.66	7.25	8.05
	2	0.62	11.66	7.25	8.05
Ethylformate	5	0.40	15.66	6.26	6.78
	4	0.40	22.66	9.06	9.58
	3	0.40	31.66	12.66	13.16
	2	0.40	85.66	34.30	34.80
Chloroform	2	0.57	24.66	13.88	14.68
Carbon Tetrachloride	2	2.00	28.66	57.32	60.50
Benzene	2	0.65	107.66	70.00	70.90
Carbon Disulphide	2	0.37	200.00	74.25	74.80
Air	—	$1.82 \times 10^{-4}$	0.00	0.00	$2.43 \times 10^{-4}$

\*S. M. Karim and L. Rosenhead, "The Second Coefficient of Viscosity of Liquids and Gases," Rev. Mod. Phys., 24:108-116 (1952).

Table A2  
Parameter Values for Select Substances

Substance	Constants									
	$\rho$ $\left(\frac{\text{gm}}{\text{cm}^3}\right)$	$c_p$ $\left(\frac{\text{cal}}{\text{gm}\cdot^\circ\text{C}}\right)$	$\gamma$	$c^2$ $\left(10^{10}\frac{\text{cm}^2}{\text{sec}^2}\right)$	$k = B^{-1}$ $\left(10^{-12}\frac{\text{cm}^2}{\text{dyne}}\right)$	$\kappa$ $\left(10^{-3}\frac{\text{cal}}{^\circ\text{C}\cdot\text{cm}\cdot\text{sec}}\right)$	$\tau_\kappa$ $(10^{-13}\text{ sec})$	$\tau_\eta$ $(10^{-12}\text{ sec})$	$\tau_\eta\tau_\kappa$ $(10^{-25}\text{ sec}^2)$	$\frac{t_K^*}{4}$ $(\text{sec})$
Water	1.00	1.00	0.98	2.214	45	1.45	.642	2.01	1.290	1.72
Ethyl Alcohol	0.784	0.584	1.17	1.24	111	0.405	0.782	6.68	5.22	3.68
Acetone	0.783	0.514	1.40	1.378	126	0.380	0.959	1.325	1.272	2.65
Carbon Tetra-chloride	1.582	0.207	1.47	0.854	106	0.248	1.30	44.75	58.18	3.32
Benzene	0.872	0.410	1.42	1.685	95	0.343	0.808	48.3	39.12	2.60
Air	0.00118	0.240	1.40	0.1156	$10^6$	0.062	2790	188	$525 \times 10^3$	0.0108

\*For  $a = 10^{-1}$  cm.

**APPENDIX B**  
**MATHEMATICAL CONSIDERATIONS**

In this appendix we evaluate and estimate some integrals pertinent to the discussion of Sect. II of this report.

**Some Simplifications**

We make use, in the sequel, of the standard integral

and of the integral representation

$$\int_0^{\infty} dx x e^{-a^2 x^2} J_0(bx) = \frac{1}{2a^2} e^{-b^2/4a^2} \quad (\text{B1})$$

$$1 - e^{-x^2} = x^2 \int_0^1 ds e^{-sx^2}. \quad (\text{B2})$$

Also, we use the symbol  $Ei(x)$  to designate the exponential integral

$$Ei(x) = \int_x^{\infty} \frac{e^{-t}}{t} dt. \quad (\text{B3})$$

Using Eqs. (B3) and (B1) one can readily show

$$\int_0^{\infty} dx \frac{(1 - e^{-x^2})}{x} J_0(bx) = \frac{1}{2} Ei\left(\frac{b^2}{4}\right) \quad (\text{B4})$$

and

$$\int_0^{\infty} dx \frac{(1 - e^{-x^2})}{x} e^{-a^2 x^2} J_0(bx) = \frac{1}{2} \left[ Ei\left(\frac{b^2}{4(1+a^2)}\right) - Ei\left(\frac{b^2}{4a^2}\right) \right]. \quad (\text{B5})$$

In addition, we make a transformation of the kernel of Eq. (18) to obtain the integral representation

$$\begin{aligned} \int_0^{\infty} dx \frac{(1 - e^{-x^2})}{x} J_0(ax) J_0(bx) &= \frac{1}{2} \int_0^1 \frac{ds}{s} e^{-(a^2+b^2)/4s} I_0\left(\frac{ab}{4s}\right) \\ &= \frac{1}{2} \int_0^{\infty} \frac{ds}{s} e^{-(a^2+b^2/4)s} I_0\left(\frac{ab}{4}\right) \end{aligned} \quad (\text{B6})$$

which might prove useful.

To derive Eq. (B6), use Eq. (B2) and Eq. (11.601) of Wheelon\*. The second version of Eq. (B6) is obtained from the first by transforming the variable of integration from  $s$  to  $s^{-1}$ . With Eq. (B6) and one more simple change of the variable of integration, Eq. (18), the expression for the density change, may also be written as

$$\frac{\rho_1(r, t)}{\rho_0} = -\frac{3\beta c^2}{2c_p} \frac{t_K}{t_c} \times \frac{1}{2} \int_0^\infty d\lambda \lambda f(\lambda) \int_{t_K/4t}^\infty \frac{d\mu}{\mu} e^{-(\lambda^2 + (r^2/a^2))\mu} I_0\left(\frac{\lambda r}{a} \mu\right). \quad (\text{B7})$$

### The Gaussian Profile

For a laser whose intensity profile is Gaussian,  $f(\lambda) = \exp(-\lambda^2)$ . Then Eq. (18) becomes

$$\frac{\rho_1(r, t)}{\rho_0} = -\frac{3\beta c^2}{2c_p} \frac{t_K}{t} \times \int_0^\infty d\lambda \lambda e^{-\lambda^2} \int_0^\infty dx \frac{(1 - e^{-x^2})}{x} J_0\left(x \sqrt{\frac{t_K}{t}} \frac{r}{a}\right) J_0\left(x \sqrt{\frac{t_K}{t}} \lambda\right). \quad (\text{B8})$$

Interchange the orders of integration and use Eq. (B1) to do the integration over the variable  $\lambda$ . Then use Eq. (B5) to complete the remaining integral. This gives

$$\frac{\rho_1(r, t)}{\rho_0} = -\frac{3\beta c^2}{2c_p} \frac{t_K}{t_c} \frac{1}{4} \left[ Ei\left(\frac{r^2}{a^2} \left(1 + \frac{4t}{t_K}\right)^{-1}\right) - Ei\left(\frac{r^2}{a^2}\right) \right]. \quad (\text{B9})$$

### The General Profile

We consider  $f(\lambda)$  arbitrary and take the limit of very small thermal conductivity, or equivalently, very early times. In Eq. (18) make the transformation  $y = (t_K/t)x$ . Then

$$-\frac{2c_p}{3\beta c^2} \frac{\rho_1(r, t)_L}{\rho_0} = \frac{t_K}{t_c} \int_0^\infty d\lambda \lambda f(\lambda) \int_0^\infty dy \frac{(1 - e^{-y^2/(t_K/t)})}{y} J_0\left(y \frac{r}{a}\right) J_0(y\lambda).$$

For very large values of  $t_K/t$ , the exponential may be expanded in a powers series using just the first two terms; this approximates the first factor in the  $y$  integrand quite well for those values of  $y$  where the Bessel functions differ from zero by a substantial amount. Thus,

\*A. D. Wheelon, "Tables of Summable Series and Integrals Involving Bessel Functions," San Francisco: Holden-Day, 1968.

$$\begin{aligned}
-\frac{2c_p}{3\beta c^2} \frac{\rho_{1L}(r,t)}{\rho_0} &= \frac{t_K}{t} \int_0^\infty d\lambda \lambda f(\lambda) \frac{t}{t_K} \int_0^\infty dy y J_0\left(y \frac{r}{a}\right) J_0(y\lambda) \\
&= \frac{t}{t_c} \int_0^\infty d\lambda \lambda f(\lambda) \frac{\delta\left(\frac{r}{a} - \lambda\right)}{\lambda} \\
&= \frac{t}{t_c} f\left(\frac{r}{a}\right)
\end{aligned} \tag{B10}$$

which is the result of Ref. 1 where thermal conductivity was ignored.

To estimate the behavior of  $\rho_{1L}$  for long times or large conductivity, rewrite  $\rho_{1L}$  as

$$\begin{aligned}
\frac{\rho_{1L}}{\rho_0} &= -\frac{3\beta c^2}{2c_p} \frac{t_K}{t_c} \int_0^\infty dx \frac{(1-e^{-x^2})}{x} J_0\left(x \sqrt{\frac{t_K}{t}} \frac{r}{a}\right) \\
&\quad \times \int_0^\infty d\lambda \lambda f(\lambda) J_0\left(x \sqrt{\frac{t_K}{t}} \lambda\right).
\end{aligned}$$

Now if  $t$  is very large,  $x\lambda$  has to be very large in order that  $J_0(x\sqrt{t_K/t}\lambda)$  deviate significantly from unity. For  $r \gg a$ , however,  $J_0(x\sqrt{t_K/t}r/a)/x$  in the  $x$  integrand tends to zero for large  $x$ . Therefore, in the  $\lambda$  integrand, because  $f(\lambda)$  has finite width,  $\lambda_w \approx 1$  and  $J_0(xy\sqrt{t_K/t})$  may be replaced by unity. Since

$$\int_0^\infty d\lambda \lambda f(\lambda) = \frac{1}{2},$$

we get, using Eq. (B4),

$$\frac{\rho_{1L}}{\rho_0} \approx -\frac{3\beta c^2}{2c_p} \frac{t}{t_c} \frac{1}{4} Ei\left(\frac{t_K}{4t} \frac{r^2}{a^2}\right), \quad \text{for } r \gg a. \tag{B11}$$

To obtain the behavior of  $\rho_{1L}$  closer to the beam axis, i.e., for  $r \approx 0$ , we observe that the two  $J_0$ 's in the above argument may be interchanged. Specifically, put  $r = 0$ . Then

$$\begin{aligned}
\frac{\rho_{1L}(0,t)}{\rho_0} &= -\frac{3\beta c^2}{2c_p} \frac{t_K}{t_c} \int_0^\infty d\lambda \lambda f(\lambda) \frac{1}{2} Ei\left(\frac{t_K}{4t} \lambda^2\right) \\
&\approx -\frac{3\beta c^2}{2c_p} \frac{t_K}{t_c} \frac{1}{2} \int_0^\infty d\lambda \lambda f(\lambda) \left[-\gamma - \ln \lambda^2 - \ln\left(\frac{t_K}{4t}\right) + o\left(\frac{t_K}{4t}\right)\right] \\
&\approx -\frac{3}{2} \frac{\beta c^2}{c_p} \frac{t_K}{t_c} \left[-\frac{\gamma}{4} - \frac{\ln\left(\frac{t_K}{4t}\right)}{4} - \int_0^\infty d\lambda f(\lambda) \lambda \ln \lambda + o\left(\frac{t_K}{4t}\right)\right].
\end{aligned} \tag{B12}$$

Thus, the density at and near the beam axis decreases as  $\ln(4t/t_K)$  for  $t/t_K \gg 0$ .

These qualitative conclusions are easily verified in the special case of the Gaussian profile. For the general case, the kernel of our Fourier-Laplace inversion is much too complicated to say more. Each choice of  $f$  must be handled individually to make better approximations than the above, and it is unlikely that many choices of  $f$  will render the inversion reducible to the more common transcendental functions.

In spite of the precautions of the above paragraph, physical and statistical arguments may be combined to give weight to an interesting conjecture. First, we note that for a confined beam, i.e.,  $f(\lambda) \rightarrow 0$  rapidly as  $\lambda > 1$ , the heating and subsequent density changes for points  $r$ , such that  $r/a \gg 1$ , should not depend much on the profile of energy deposition within the beam but only on the rate. This led to Eq. (B11). Inside the beam, differences in energy deposition rates lead heat to flow into the cooler regions until a steady state is built up. This led to Eq. (B17). Said in another way, heat conduction is a statistical process and the final state in such processes never reflects the initial state. Hence, we conjecture that for *any* reasonably smooth choice of  $f(\lambda)$ , i.e., no infinite discontinuities, and one that satisfies the normalization condition

$$\int_0^\infty d\lambda \lambda f(\lambda) = \frac{1}{2}$$

as  $\tau$  becomes very small, then

$$\int_0^\infty d\lambda \lambda f(\lambda) \int_0^\infty dx \frac{(1 - e^{-x'})}{x} J_0(x\tau\lambda) J_0(x\tau\rho) \approx \frac{1}{4} \left[ Ei\left(\frac{\tau^2}{4} \rho^2\right) - Ei(\rho^2) \right] \quad (\text{B13})$$

$$+ O(\tau) + O(\rho) + O(\rho, \tau), \quad \text{for } \tau \approx 0.$$

This conjecture is consistent with the result of Eq. (B9), and with our special approximation. The integral is seen to be finite when  $\rho = 0$ . On the other hand,  $Ei[(\tau/4)\rho^2]$  is singular at  $\rho = 0$ , but with a logarithmic singularity  $Ei[(\tau^2/4)\rho^2] \approx \ln(\tau^2/4) + \ln\rho^2$ . Therefore the singularities of the two terms in the bracket on the right-hand side of Eq. (B13) cancel. The terms  $O(\tau)$  and  $O(\rho)$  will reflect the choice of  $f$ .

## APPENDIX C BASIC MODEL HYPOTHESES

A brief resumé of the hypotheses used in the model for laser propagation in an absorbing medium as discussed in NRL Report 7213 is presented here for convenience:

1. The beam was assumed to propagate through an initially homogeneous isotopic and quiescent medium.
2. Convection, viscosity, and thermal conduction were neglected.
3. The variation of intensity of the initial laser beam, as a function of distance, due to absorption was ignored.
4. Energy deposition by the laser beam into the medium was calculated using  $\alpha I_0$ , where  $I_0$  was the unperturbed beam. The model was therefore restricted to those times for which the calculated beam distortions were small.
5. The index of refraction was related to the density by the Lorentz-Lorenz law, and the model of energy deposition described above was used in conjunction with the linearized hydrodynamic equations to determine the subsequent behavior of the medium parameters, in particular, the density.
6. The optical changes in the beam were determined by the laws of geometric optics. This restricts the model to those times and distances where diffraction may be ignored and caustics do not form.

The present report does not neglect the viscosity and thermal conduction.