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L. Bruce Palmer

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CONTENTS

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THE FOURIER TRANSFORM AND SOME OF ITS APPLICATIONS

INTRODUCTION

In April 1965 J. W. Cooley and J. W. Tukey published an article titled "An Algorithm for the Machine Calculation of Complex Fourier Series" (1). Since then, this algorithm and its various improved versions have become known as the fast Fourier transform (FFT). This algorithm provides for the rapid calculation of the discrete finite Fourier transform (DFT) of N data points and the inverse transform (IDFT) by digital computers. Before the discovery of this algorithm, the length of time required by a digital computer to calculate these transforms inhibited the use of Fourier methods in the computer analysis in such areas as digital filtering and spectral analysis. The computer time required to calculate these transforms by means of the algorithm is minimized when the number N of data points to be transformed is of the form 2^k, where k is a positive integer. Therefore, most existing computer programs require than N be such a power of 2, although the algorithm only requires that N be a composite number (1).

It is not the purpose of this paper to dwell on the FFT algorithm itself, since it is well documented. Rather, the present purpose is to investigate the DFT and the IDFT which the algorithm evaluates. In particular, the first section is concerned with defining the DFT and developing some notation for later use. The second section presents the DFT transform method for evaluating convolutions. In the third section this method is extended for use in digital filtering and calculation of lagged products. The use of the DFT to estimate amplitude spectra is also discussed here. Attention will be paid along the way to hazards encountered when attempting to use a computer to apply the FFT algorithm in each of these areas. The final section is devoted to the relative merits of the transform methods when implemented using the FFT algorithm for computers. Computation times are compared for both the transform method and the direct evaluation of defining formulae in the cases of digital filtering and lagged products.

THE DISCRETE FINITE FOURIER TRANSFORM (DFT)

Let W be the principal N^{th} root of unity, i.e.,

$$W = \exp(2\pi i/N) \tag{1}$$

where $i = \sqrt{-1}$. If k is an integer, then

$$W^k = \exp(2\pi ik/N) \tag{2}$$

and

$$W^{kN} = \exp(2\pi ik) = \cos 2\pi k + i \sin 2\pi k = 1. \tag{3}$$

The periodicity expressed by Eq. 3 is used repeatedly in calculations with W, e.g.,

$$W^{-(N-j)k} = W^{-Nk}W^{jk} = W^{jk}. \tag{4}$$

Consider a sequence X of N complex numbers and let

$$X = \{X_0, X_1, \ldots, X_{N-1}\}. \tag{5}$$

X can be thought of as a point in complex N-space, since complex N-space is a normed N-dimensional linear vector space C^N of N-tuples of complex numbers. Thus, for any basis for C^N, say $b_0, b_1, \ldots, b_{N-1}$, X can be written as the linear combination

$$X = \sum_{k=0}^{N-1} A_k b_k \tag{6}$$

where the A_k are complex. Also, this representation is unique, i.e., the coefficients A_k are unique for a given basis. Let

$$b_k = (1, W^k, W^{2k}, \ldots, W^{(N-1)k}),\ k = 0, 1, \ldots, N-1. \tag{7}$$

It is a simple calculation (2) to verify that

$$b_m \cdot b_n = \sum_{j=0}^{N-1} W^{jm} W^{-jn} = \begin{cases} 0, & m \neq n \\ N, & m = n \end{cases} \tag{8}$$

where $b_m \cdot b_n$ is the usual N-tuple dot product in the space C^N. Thus the N vectors $b_0, b_1, \ldots, b_{N-1}$ are mutually orthogonal. Hence, from a well-known theorem of finite dimensional vector spaces (3), it follows that the vectors $b_0, b_1, \ldots, b_{N-1}$ form a basis for C^N.

Substituting Eq. 7 into Eq. 6, one finds the coordinates X_k of X to be given by

$$X_k = \sum_{j=0}^{N-1} A_j W^{jk},\ k = 0, 1, 2, \ldots, N-1. \tag{9}$$

By multiplying both sides by W^{-km} for $m = 0, 1, \ldots, N-1$ and summing over k, we get

$$\begin{aligned} \sum_{k=0}^{N-1} X_k W^{-km} &= \sum_{j=0}^{N-1} A_j \sum_{k=0}^{N-1} W^{jk} W^{-km} \\ &= \sum_{j=1}^{N-1} A_j \sum_{k=0}^{N-1} W^{(j-m)k} \\ &= \sum_{j=0}^{N-1} A_j (b_j \cdot b_m) \\ &= A_m N,\ m = 0, 1, \ldots, N-1. \end{aligned} \tag{10}$$

Thus, the coefficients A_k, which are unique for the basis $b_0, b_1, \ldots, b_{N-1}$, are given by

$$A_m = \frac{1}{N} \sum_{k=0}^{N-1} X_k W^{-km}, \quad m = 0, 1, \ldots, N-1. \tag{11}$$

Substituting j for m in Eq. 11 yields

$$A_j = \frac{1}{N} \sum_{k=0}^{N-1} X_k W^{-jk}, \quad j = 0, 1, \ldots, N-1. \tag{12}$$

The sequence $A = \{A_0, A_1, \ldots, A_{N-1}\}$, given by (12), is defined to be the discrete finite Fourier transform (DFT) of X. Vice versa, the sequence X given in Eq. 9 is defined to be the inverse discrete finite Fourier transform (IDFT) of A. The preceding definitions of the DFT and IDFT are somewhat arbitrary and by no means standard. Often the $1/N$ is associated with the IDFT rather than the DFT (4,5). Which transform is to be called the "inverse" is also rather arbitrary and is often avoided by referring to (9) and (12) as the transform pair (2). However, the definitions chosen here are consistent with those of J. W. Cooley (6).

Given a sequence $Y = \{Y_0, Y_1, \ldots, Y_{N-1}\}$ of complex numbers, the usual FFT computer program calculates

$$B'_k = \sum_{j=0}^{N-1} Y_j W^{-jk}, \quad k = 0, 1, \ldots, N-1. \tag{13}$$

This is because the FFT algorithm is essentially a reformulation of Eq. 13 which results in a reduced number of arithmetic operations. The DFT of Y then, is given by

$$B_k = \frac{1}{N} B'_k, \quad k = 0, 1, \ldots, N-1. \tag{14}$$

The same computer program can be utilized to calculate the IDFT of Y by first conjugating each Y_k, then using the computer program to calculate

$$\bar{Z}_k = \sum_{j=0}^{N-1} \bar{Y}_j W^{-jk}, \quad k = 0, 1, \ldots, N-1. \tag{15}$$

Next, conjugate the $\bar{Z}_k$ to get

$$\begin{aligned} Z_k &= \overline{\sum_{j=0}^{N-1} \bar{Y}_j W^{-jk}} \\ &= \sum_{j=0}^{N-1} Y_j W^{jk}, \quad k = 0, 1, \ldots, N-1. \end{aligned} \tag{16}$$

Then the sequence Z_k $(k = 0, 1, \ldots, N-1)$ is the IDFT of Y.

Before proceeding, let us develop some notation. For a given sequence $X = \{X_0, X_1, \ldots, X_{N-1}\}$ of complex numbers, let the sequence $A = \{A_0, A_1, \ldots, A_{N-1}\}$ be the DFT of X. That is, let the numbers A_j be given by

$$A_j = \frac{1}{N}\sum_{k=0}^{N-1} X_k W^{-jk} \ , \ j = 0,1, \ldots, N-1 \ . \tag{17}$$

Let F denote the operator defined on finite sequences of complex numbers by the DFT and write

$$A = F(X). \tag{18}$$

Letting F^{-1} denote the IDFT operator,

$$X = F^{-1}(A). \tag{19}$$

It is easy to show that F and F^{-1} are inverse operators, so we may write

$$X = F^{-1}F(X) \ , \ A = FF^{-1}(A). \tag{20}$$

Define kX, where k is an integer, to be the sequence

$$kX = \{kX_0, kX_1, \ldots, kX_{N-1}\}. \tag{21}$$

and $\bar{X}$ by

$$\bar{X} = \{\bar{X}_0, \bar{X}_1, \ldots, \bar{X}_{N-1}\}. \tag{22}$$

Then (15) becomes

$$\bar{Z} = \overline{F^{-1}(Y)} = NF(\bar{Y}). \tag{23}$$

so that

$$F^{-1}(Y) = N\overline{F(\bar{Y})}. \tag{24}$$

From Eq. 17 it follows that

$$\begin{aligned} \bar{A}_j &= \overline{\frac{1}{N}\sum_{k=0}^{N-1} X_k W^{-jk}} \\ &= \frac{1}{N}\sum_{k=0}^{N-1} \bar{X}_k \overline{W^{-jk}} \\ &= \frac{1}{N}\sum_{k=0}^{N-1} \bar{X}_k W^{jk} \ , \ j = 0,1, \ldots, N-1. \end{aligned} \tag{25}$$

so

$$F(X) = \frac{1}{N} \overline{F^{-1}(\overline{X})}. \tag{26}$$

CONVOLUTION

Consider two sequences X_j and Y_j $(j = 0, 1, \ldots, N-1)$ of complex numbers. A cyclic convolution of these two sequences is defined as

$$Z_r = \sum_{j=0}^{N-1} X_j Y_{r-j}, \ r = 0, 1, \ldots, N-1. \tag{27}$$

The term cyclic refers to the assumption that the Y_j is defined for all integral values of j, with the values $Y_0, Y_0, \ldots, Y_{N-1}$ being repeated every N values. Thus,

$$\text{if } m \equiv n \,(\text{mod } N), \text{ then } Y_m = Y_n \tag{28}$$

where $0 \le n \le N-1$. In order to avoid modular arithmetic it is sufficient to note that when the subscript $r-j$ in Eq. 27 is negative, one may use the relationship

$$Y_{r-j} = Y_{r-j+N}. \tag{29}$$

For example,

$$\begin{aligned} Y_{-1} &= Y_{N-1} \\ Y_{-2} &= Y_{N-2} \\ &\vdots \\ Y_{r-N+1} &= Y_{r+1}. \end{aligned} \tag{30}$$

Therefore, in Eq. 27 as j is incremented from 0 to $N-1$, Y_{r-j} will follow the sequence

$$Y_r, Y_{r-1}, \ldots, Y_0, Y_{N-1}, Y_{N-2}, \ldots, Y_{r+1}. \tag{31}$$

It has been shown (5, 6) that

$$Z_r = N \sum_{k=0}^{N-1} A_k B_k W^{rk} \tag{32}$$

where $A = F(X)$ and $B = F(X)$. Now for any integer m, there exist integers p and n, where $0 \le n \le N-1$, such that $m = n + pN$. Thus,

$$m-j = n + pN - j = n - j + pN. \tag{33}$$

Since the values of Y are repeated every N values it follows that $Y_{m-j} = Y_{n-j}$. So

$$Z_m = \sum_{j=0}^{N-1} X_j Y_{m-j} = \sum_{j=0}^{N-1} X_j Y_{n-j} = Z_n, \tag{34}$$

and hence the values

$$Z_r = N \sum_{k=0}^{N-1} A_k B_k W^{rk}, \; r = 0, 1, \ldots, N-1 \tag{35}$$

are repeated every N values of the index r. Letting

$$C_k = A_k B_k \; , k = 0, 1, \ldots, N-1 \tag{36}$$

Eq. 35 becomes

$$Z = NF^{-1}(C). \tag{37}$$

In review, the sequence Z_r ($r = 0, 1, \ldots, N-1$) may be generated by first taking the DFT of both X and Y, then multiplying the results [A and B] together, term by term. Finally, take the IDFT of the result and multiply each term by N to get the desired Z. Another expression for Z may be obtained by noting that from Eq. 26

$$\begin{aligned} F^{-1}(\overline{Y}) &= N\overline{F(Y)} = N\overline{B} \\ F^{-1}(\overline{X}) &= N\overline{F(X)} = N\overline{A}. \end{aligned} \tag{38}$$

Therefore

$$\begin{aligned} \overline{Z}_r &= N \sum_{k=0}^{N-1} \overline{A}_k \overline{B}_k W^{-rk} \\ &= \frac{1}{N} \sum_{k=0}^{N-1} N\overline{A}_k N\overline{B}_k W^{-rk} \\ &= \frac{1}{N} \sum_{k=0}^{N-1} N^2 \overline{C}_k W^{-rk}, \end{aligned} \tag{39}$$

so that

$$\overline{Z} = F(N^2\overline{C})$$

and

$$Z = \overline{F(N^2\overline{C})}. \tag{40}$$

Therefore Z may also be obtained by first taking the IDFT of both $\overline{X}$ and $\overline{Y}$, then multiplying term by term. Next, one may either take the DFT of the resultant products and conjugate to get Z, or first conjugate the resultant products, take the IDFT, and divide by N. If X and Y are real, then Z will be real, in which case conjugation may be ignored. Graphically,

$$
\begin{tikzcd}[column sep=huge, row sep=large]
N\bar{A} & \overline{X} \arrow[l, "\text{IDFT}"'] & X \arrow[l, "\text{Conjugate}"'] \arrow[r, "\text{DFT}"] & A \\
N\bar{B} \arrow[d, Rightarrow, "\substack{\text{Multiply}\\ \text{term by term}}"'] & \overline{Y} \arrow[l, "\text{IDFT}"'] & Y \arrow[l, "\text{Conjugate}"'] \arrow[r, "\text{DFT}"] \arrow[d, Rightarrow, "\text{Convolution}"] & B \arrow[d, Rightarrow, "\substack{\text{Multiply}\\ \text{term by term}}"] \\
N^2\overline{C} \arrow[rr, "\substack{\text{DFT and}\\ \text{conjugate}\\ \text{or}\\ \text{Conjugate and}\\ \text{IDFT and divide by } N}"'] & & Z & C \arrow[l, "\substack{\text{IDFT and}\\ \text{multiply}\\ \text{by } N}"]
\end{tikzcd}
$$

The definitions of the DFT and IDFT vary in the scientific literature. This may lead to confusion when attempting to implement the technique of some article, especially if Fourier transforms are used to perform convolutions. The above graph demonstrates two ways in which the transforms may be used to convolve data.

APPLICATIONS OF THE DFT

Digital Filtering

Digital filtering consists of convolving a predetermined set of coefficients C_k, $k = 0,1,\ldots,L-1$, with a set of samples Y_j, $j = 0,1,\ldots,N-1$, from some time series, with $L \leq N$. The output, or filtered, values are calculated from

$$Z_r = \sum_{k=0}^{L-1} C_k Y_{r-k}. \tag{41}$$

Since the values Y_k are not considered as being cyclic with $Y_0, Y_1, \ldots, Y_{L-1}$ representing one period, Eq. 41 is only valid for $r = L-1, L, \ldots, N-1$. The transform methods of the previous section yield values of Z_r for $r = 0, 1, \ldots, L-1$, so only Z_{L-1} could be calculated in this way. However, consider appending zeros to the sequence of coefficients. That is, if one defines

$$X_k = \begin{cases} C_k, & k = 0,1,\ldots,L-1 \\ 0, & k = L, L+1, \ldots, N-1 \end{cases} \tag{42}$$

Eq. 41 becomes

$$Z_r = \sum_{k=0}^{N-1} X_k Y_{r-k} \tag{43}$$

which will be valid for $r = L-1, L, \ldots, N-1$. Now the transform methods of the previous section may be applied.

Several problems arise when attempting to apply the transform methods of the previous section to digital filtering. First, most FFT computer programs for the rapid calculation of the DFT and IDFT require that N be a power of 2. If N is not a power of 2, the difficulty may be overcome by appending additional zeros to X and Y, or by disregarding some of the last terms of Y. Second, the resulting power of 2 may be so large as to make computation by a digital computer impracticable because of computer storage restrictions. This may be overcome with Helms' "select saving" method (5) in applying the before-mentioned transform methods. Third, the transform method of calculating Eq. 43 forces the shifting of the filter coefficients along Y in increments of one data point. That is, Z_r is calculated for all values of $r=0, 1, \ldots, N-1$. When not every Z_r is desired, it may be more practicable to evaluate Eq. 41 directly. This problem usually arises when only every n^{th} output value Z_r is desired. The critical value of n depends on N and L (5). If X and Y are real, this value appears to be about 10. Direct evaluation of Eq. 41 versus the transform method of digital filtering is investigated in the last section of this report.

Lagged Products

The cyclic lagged product with lag r for two (perhaps identical) sequences X_j and Y_j, $j=0,1,\ldots,N-1$, of complex numbers is defined as

$$U_r = \sum_{j=0}^{N-1} X_j Y_{r+j}, \quad r=0,1,\ldots,N-1. \tag{44}$$

Again, the term cyclic refers to the assumption that the values $Y_0, Y_1, \ldots, Y_{N-1}$ are repeated every N values. Thus as the index j in Eq. 44 is incremented from 0 to $N-1$, Y_{j+1} will follow the sequence

$$Y_r, Y_{r+1}, \ldots, Y_{N-1}, Y_0, \ldots, Y_{r-1}. \tag{45}$$

Defining the sequence X' by

$$X' = \{X_0, X_{N-1}, X_{N-2}, \ldots, X_1\}, \tag{46}$$

Eq. 44 can be written

$$\begin{aligned} U_r &= \sum_{j=0}^{N-1} X_j Y_{r+j} \\ &= X_0 Y_r + X_1 Y_{r+1} + \ldots + X_{N-1} Y_{r-1} \\ &= X_0 Y_r + X_{N-1} Y_{r-1} + X_{N-2} Y_{r-2} + \ldots + X_1 Y_{r+1} \\ &= X'_0 Y_r + X'_1 Y_{r-1} + \ldots + X'_{N-1} Y_{r+1} \\ &= \sum_{j=0}^{N-1} X'_j Y_{r-j}. \end{aligned} \tag{47}$$

Thus, the cyclic lagged product can be considered as a convolution. From Eq. 32,

$$U_r = N \sum_{k=0}^{N-1} A'_k B_k W^{rk}, \tag{48}$$

where $A' = F(X')$ and $B = F(Y)$. Note that

$$\begin{aligned} NA'_k &= \sum_{j=0}^{N-1} X'_j W^{-jk} \\ &= X'_0 W^0 + X'_1 W^{-k} + \ldots + X'_{N-1} W^{-(N-1)k} \\ &= X_0 W^0 + X_{N-1} W^{-k} + \ldots + X_1 W^{-(N-1)k} \\ &= X_0 W^0 + X_1 W^{-(N-1)k} + \ldots + X_{N-1} W^{-k} \\ &= \sum_{j=0}^{N-1} X_j W^{-(N-j)k}, \quad k = 0,1, \ldots, N-1 \end{aligned} \tag{49}$$

since $W^0 = 1 = W^{-Nk}$, by Eq. (3). But also by Eq. 3

$$W^{-(N-j)} = W^{jk} = W^{-j(N-k)}, \tag{50}$$

so Eq. 49 becomes

$$\begin{aligned} NA'_k &= \sum_{j=0}^{N-1} X_j W^{-j(N-k)} \\ &= NA_{N-k}, \quad k = 0,1, \ldots, N-1 \end{aligned} \tag{51}$$

where $A = F(X)$. Therefore,

$$A'_k = A_{N-k}, \quad k = 0,1, \ldots, N-1 \tag{52}$$

and in particular

$$\begin{aligned} A'_0 = A_N &= \frac{1}{N} \sum_{j=0}^{N-1} X_j W^{-jN} = \frac{1}{N} \sum_{j=0}^{N-1} X_j \\ &= \frac{1}{N} \sum_{j=0}^{N-1} X_j W^{0j} \\ &= A_0. \end{aligned} \tag{53}$$

Now Eq. 48 becomes

$$U_r = N \sum_{k=0}^{N-1} A_{N-k} B_k W^{rk}, \quad r = 0, 1, \ldots, N-1 \tag{54}$$

where $B = F(Y)$, $A = F(X)$, and $A_N = A_0$. Thus when computing lagged products by the technique described in the previous section on convolution, the transforms A and B must be multiplied together as in Eq. 54.

As was the case with digital filtering, there are several problems in attempting to apply the Fourier transform to compute lagged products. These problem include requirements of existing FFT computer programs that N be a power of 2, computer storage restrictions on the size of N, and the desirability of computing for only selected indices. Possible solutions to these problems are the same as those in the case of digital filtering.

Also, lagged products are often desired in such areas as correlation and spectrum analysis (7) where X and Y are samples, perhaps identical, from time series. In such cases Y is usually considered as not being cyclic. One possible way to account for Y not being cyclic is to compute

$$U_r = \sum_{j=0}^{N-r-1} X_j Y_{j+r}, \quad r = 0, 1, \ldots, N-1. \tag{55}$$

Unfortunately, it is found in most applications of lagged products that Eq. (55) is only useful for $r = 0, 1, \ldots, L$ where L is approximately 10% of N (7). Also, note that in Eq. 55 the subscript j has a variable upper bound of $N-r-1$. This fact inhibits the application of the Fourier transform methods used for computing a sequence of lagged products as defined by Eq. 44, where the upper bound is constant.

But if L zeros are appended to each of the sequences X and Y, Eq. 55 becomes

$$U_r = \sum_{j=0}^{N+L-1} X_j Y_{j+r}, \quad r = 0, 1, \ldots, N+L-1 \tag{56}$$

which has the form of Eq. 44 so that the Fourier methods previously discussed in this section may now be used. However, since the transform method of evaluating Eq. 44 yields more U_r than the desired 10% of N, direct computation of Eq. 55 may be warranted. This question is discussed more fully in the last section of this report.

Amplitude Spectrum

A real-valued periodic function Y of period T that satisfies certain conditions (The Dirichlet conditions) that are usually met in practice can be expressed by a Fourier series

$$Y(t) = a_o/2 + \sum_{j=1}^{\infty} \left(a_j \cos\left(j\, 2\pi t/T\right) + b_j \sin\left(j\, 2\pi t/T\right)\right) \tag{57}$$

in which the coefficients are given by

$$a_j = \frac{2}{T}\int_0^T Y(t)\ \cos\left(j\,\frac{2\pi t}{T}\right)dt,\ j \geq 0.$$

$$b_j = \frac{2}{T}\int_0^T Y(t)\ \sin\left(j\,\frac{2\pi t}{T}\right)dt,\ j > 0. \tag{58}$$

If Y is not periodic, Eqs. 57 and 58 can still be used to represent Y on an interval of length T. If we define an amplitude c_j and a phase ϕ_j by the relations

$$c_j = \sqrt{a_j^2 + b_j^2}\ ;\ \sin\phi_j = a_j/c_j,\ \text{and}\ \cos\phi_j = b_j/c_j, \tag{59}$$

then Eq. 57 can be rewritten as

$$Y(t) = \frac{1}{2}a_o + \sum_{j=1}^{\infty} c_j\ \sin\left[(j2\pi t/T) + \phi_j)\right]. \tag{60}$$

Thus, Y may be thought of as a sum of sine waves, the j^{th} component having an amplitude of c_j, a (cyclical) frequency of $f_j = j/T$, and a phase of ϕ_j (8).

A more convenient form of Eq. 57 can be obtained by replacing the sine and cosine functions with complex exponentials according to the Euler definitions

$$\cos x = (e^{ix} + e^{-ix})/2$$

and

$$\sin x = (e^{ix} - e^{-ix})/2i. \tag{61}$$

After some rearrangement of terms, Y is found to be represented by the complex Fourier series

$$Y(t) = \sum_{j=-\infty}^{\infty} C_j\ \exp\ (i2\pi j\,t/T), \tag{62}$$

in which the coefficients are given by

$$C_j = \frac{1}{T}\int_0^T Y(t)\ \exp\ (-i2\pi j\,t/T)dt. \tag{63}$$

Comparing Eqs. 59 and 63 we find that

$$\sqrt{C_j C_{-j}} = \begin{cases} |C_j| = \frac{1}{2}\sqrt{a_j^2 + b_j^2} = \frac{1}{2}c_j,\ j > 0 \\ |C_o| = \frac{1}{2}a_o = c_o,\ j = 0. \end{cases} \tag{64}$$

The phases ϕ_j are found to satisfy

$$\sin\phi_j = \frac{Re(C_j)}{|C_j|}\ ,\ \cos\phi_j = -\frac{Im(C_j)}{|C_j|}\ ,\ j > 0. \tag{65}$$

The set of coefficients C_j, when paired off with the corresponding frequencies f_j, form the amplitude spectrum of Y. This spectrum is a "line" spectrum since the frequency f takes on only the discrete values $f_j = j/T$. The phase spectrum is defined in a similar way. In many applications the amplitude and phase spectra of a time series play an important role in understanding and describing the underlying physical process that generates Y in time. We will now show how the amplitude spectrum can be estimated from a sample of Y.

Suppose N samples are taken of Y such that the time between samples is given by

$$\Delta t = T/N. \tag{66}$$

Then the time span over which the N sample values are obtained from Y is $(N-1)T/N$. Without loss of generality it can be assumed that the first sample Y_0 was taken at $t = 0$, so the time t_k at which the $k+1^{\text{th}}$ sample Y_k was obtained is

$$t_k = k\Delta t = kT/N. \tag{67}$$

From Eq. 62, Y_k can then be represented as

$$Y_k = Y(t_k) = \sum_{j=-\infty}^{\infty} C_j \exp(i2\pi j t_k/T)$$

$$= \sum_{j=-\infty}^{\infty} C_j \exp(i2\pi f_j t_k). \tag{68}$$

Although the sample values Y_k are real, they can be considered as complex numbers with zero imaginary parts. Therefore the sample value Y_k can also be represented by

$$Y_k = \sum_{j=0}^{N-1} A_j W^{jk},\ k = 0, 1, \ldots, N-1. \tag{69}$$

where the coefficients A_j are found by taking the DFT of the observations Y_k. In order to bring Eq. 69 into a form more like Eq. 68, we let N' be the largest integer less than or equal to $N/2$ so that Eq. 69 becomes

$$Y_k = \sum_{j=0}^{N'} A_j W^{jk} + \sum_{j=N'+1}^{N-1} A_j W^{jk}. \tag{70}$$

Changing the index of summation in the second sum by letting $j = N-m$, we have

$$Y_k = \sum_{j=0}^{N'} A_j W^{jk} + \sum_{m=1}^{(N-1)-N'} A_{N-m} W^{(N-m)k}$$

$$= \sum_{j=0}^{N'} A_j W^{jk} + \sum_{m=1}^{(N-1)-N'} A_{N-m} W^{-mk}. \tag{71}$$

Now define A_{-m} by

$$A_{-m} = A_{N-m}. \tag{72}$$

Then Eq. 71 becomes

$$Y_k = \sum_{j=0}^{N'} A_j W^{jk} + \sum_{m=1}^{(N-1)-N'} A_{-m} W^{-mk} \tag{73}$$

Finally, let $j = -m$ in the second sum. Then

$$\begin{aligned}
Y_k &= \sum_{j=0}^{N'} A_j W^{jk} + \sum_{j=-[(N-1)-N']}^{-1} A_j W^{jk} \\
&= \sum_{j=-(N-N'-1)}^{N'} A_j W^{jk} \\
&= \sum_{j=-(N-N'-1)}^{N'} A_j \exp(i 2\pi jk/N) \\
&= \sum_{j=-(N-N'-1)}^{N'} A_j \exp[i 2\pi (j/T)(kT/N)] \\
&= \sum_{j=-N'+EN}^{N'} A_j \exp(i 2\pi f_j t_k)
\end{aligned} \tag{74}$$

where $EN = 1$ if N is even and 0 if N is odd.

Now Eq. 74 can be considered as an evaluation of the function

$$Y'(t) = \sum_{j=-N'+EN}^{N'} A_j \exp(i 2\pi f_j t) \tag{75}$$

at the time

$$t_k = k\Delta t = kT/N. \tag{76}$$

Comparing Eqs. 68 and 74, we see that $Y'(t)$ can be interpreted as an approximation to $Y(t)$ i.e.,

$$\sum_{j=-\infty}^{\infty} C_j \exp(i2\pi f_j t) \approx \sum_{j=-N'+EN}^{N'} A_j \exp(i2\pi f_j t) \tag{77}$$

with the coefficients A_j serving as estimates of the complex amplitudes C_j. The amplitudes C_j are obtained from the C_j by means of Eq. 64.

We do not obtain estimates of the amplitude for all frequencies in the spectrum. If we define the sampling frequency fs by

$$fs = N/T, \tag{78}$$

then we see that we obtain estimates only at the frequencies

$$f_j = \frac{j}{T} = \frac{j}{N}\frac{N}{T} = j\frac{fs}{N}, \; j = 0, 1, \ldots, N'. \tag{79}$$

In the special case where N is a power of 2 (and hence even), estimates are obtained for the frequencies

$$f = 0, \; fs/N, \; 2fs/N, \ldots, \; fs/2. \tag{80}$$

The fact that only N' sine wave amplitudes are approximated is related to the Nyquist sampling theorem (9) which says in part that one can not expect to detect frequencies greater than $fs/2$.

It is not unreasonable to question the validity of approximating an infinite sum (Eq. 68) by a finite sum (Eq. 74). However, the physical source or medium of Y may warrant the assumption that there are no contributing since waves having a frequency greater than twice the sampling frequency, or if there are any, they are insignificant. In such a case, an approximation of the time series $Y(t)$ by the finite sum (Eq. 75) may not be so unreasonable. Also, one may only be interested in the amplitudes and phases corresponding to some finite range of frequencies. In this case the contributing sine waves of the undesired frequencies may be filtered out of Y before it is spectrum analyzed.

The estimates for C_j may be further refined by applying "smoothing" coefficients to the sequence of estimates. Hamming and Hanning coefficients (7) are two such examples. Another procedure is to generate more than one set of estimates and average.

ADVANTAGES AND DISADVANTAGES OF FOURIER TRANSFORM METHODS

The FFT algorithm provides such rapid calculation of the DFT and IDFT as to promote the use of Fourier transform methods by computers. However, the use of these transformations is not without its drawbacks. For example, the requirement by most existing FFT programs that the number of values to be transformed by a power of 2 may not be convenient for the user. In addition, the transform methods provide the evaluation of cyclic convolution and lagged product formulae, whereas noncyclic formulae are usually desired. This produces extraneous results in addition to the desired results. Also not to be forgotten are the usual computer restrictions such as amount of available storage and input-output speeds.

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