NRL Report 7377

Scattering From a Periodic Corrugated Surface

Part 4
Finite-Depth Alternately Filled Plates with Hard Boundaries

JOHN A. DESANTO

Propagation Branch
Acoustics Division

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NAVAL RESEARCH LABORATORY
Washington, D.C.

PREVIOUS REPORTS IN THIS SERIES

"Part 1 – Semi-Infinite Inhomogeneously Filled Plates with Soft Boundaries," John A. DeSanto, NRL Report 7320, Nov. 11, 1971

"Part 2 – Semi-Infinite Inhomogeneously Filled Plates with Hard Boundaries," John A. DeSanto, NRL Report 7321, Nov. 15, 1971

"Part 3 – Finite-Depth Alternately Filled Plates with Soft Boundaries," John A. DeSanto, NRL Report 7375, May 18, 1972

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ABSTRACT

An incident plane wave is scattered from a periodic corrugated surface consisting of finite-depth parallel plates. Each period is further divided by an additional finite-depth parallel plate into two regions—one with the same density and wavenumber values as the free-space region above the plates, and the second with different (but constant) density and wavenumber values. The plates and bottoms have hard (Neumann) boundaries.

Solutions of the Helmholtz equation, with unknown amplitude coefficients, are assumed in the various geometric regions. By requiring that the pressure and velocity be continuous functions at the boundaries, sets of linear equations are obtained that relate the amplitudes for arbitrary incident angles. Equations for normal incidence are solved using a variation of the modified residue calculus technique involving two zero shifts, and the results yield the amplitudes as values or residues of a meromorphic function. With the exception of the finite depth, this paper is similar to NRL Report 7321.

PROBLEM STATUS

This is an interim report on the problem; work continues.

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SCATTERING FROM A PERIODIC CORRUGATED SURFACE

Part 4—Finite-Depth Alternately Filled Plates with Hard Boundaries

1. INTRODUCTION

The problem considered in this report is the calculation of the scattered and diffracted fields which result when plane waves are incident on a periodic corrugated surface, such as that illustrated in Fig. 1. The surface consists of infinitesimally thin finite-depth parallel plates with lossless bottoms. The periodicity interval is 2ℓ, and the distance between two adjacent plates containing a "homogeneous" or "free-space" region is $2a$. "Free space" means that the region has the same density and wavenumber properties as the region A above the plates (see Fig. 1), which is "free" of plates. For this problem, a convenient thickness parameter is defined by $t = \ell/a$. A second region of "inhomogeneous" density and wavenumber structure (region C), with a width $2(\ell - a)$, consists of density and wavenumber values which differ from those in the free-space region. Both the plates and the bottoms have hard (i.e., Neumann) boundary conditions imposed on the velocity potential ψ. This report is a continuation of a series of papers and reports (1-5) in which additional references to the literature can be found.

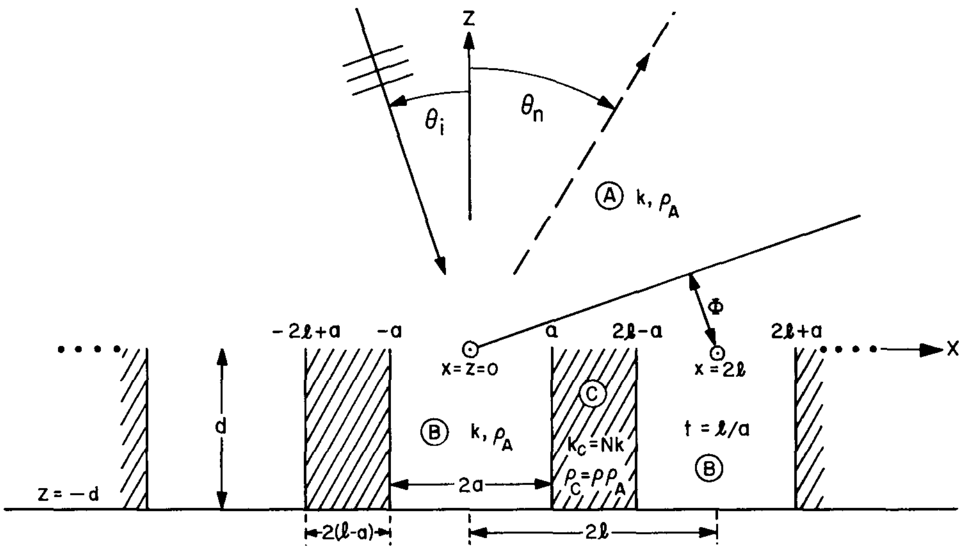

Fig. 1—Plane wave incident at an angle θ_i on a finite-depth (d) corrugated surface which is periodic (period 2ℓ). Shading indicates regions of density and wavenumber inhomogeneity (region C) as distinct from the free-space regions A and B. The discrete scattering angles are indicated by θ_n.

Section 2 presents the basic formalism and the assumed forms of ψ in each geometric region of the problem. Each potential ψ contains unknown amplitude coefficients.

The requirement that the pressure and velocity be continuous across the common interfaces yields, in Sec. 3, linear equations relating the various amplitude coefficients. The equations are general in the sense that the incident angle is arbitrary. The equations for the special case of normal incidence are solved in Sec. 4 using a variation of the modified residue calculus method involving two sets of zero shifts. The amplitudes are shown to be given in terms of values or residues of a meromorphic function.

A summary is presented in Sec. 5. This present report is confined to analytic results only.

2. BASIC FORMALISM

The formalism is similar to that present in previous papers and reports (1-5). Details will often be omitted, and results which can be derived by previous methods will merely be stated. Particular reference is made to Ref. 4 because of the similarity of "thickness" and boundary conditions to this report.

The problem is to calculate the velocity potential ψ_γ satisfying the two-dimensional Helmholtz equation*

$$\left\{\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial z^2} + k_\gamma{}^2\right\} \psi_\gamma(x,z) = 0 \tag{2.1}$$

for plane waves ψ_i incident at angles θ_i on a periodic surface (period 2ℓ) consisting of plates of finite-depth d with lossless bottoms. The periodicity interval is further divided by a parallel plate into two regions B and C which are, respectively, "homogeneous," or "free space," and "inhomogeneous," or filled with density and wavenumber structure differing from the free-space region. The free-space regions A and B are specified by the density ρ_A and the wavenumber k. Two plates without structure are separated by a distance $2a$. The three geometric regions of the problem are indicated by the label $\gamma = A$, B, or C. The wavenumber structure, as in Refs. 3-5, is given by $k_A = k_B = k$ and $k_C = Nk$ where N is the constant wavenumber structure parameter. The density structure, as in Refs. 3-5, is given by $\rho_A = \rho_B$ and $\rho_C = \rho\rho_A$ where ρ is the density structure parameter. Region A is the region for which $z \geqslant 0$. For both the regions B and C, the condition $-d \leqslant z \leqslant 0$ exists. In addition, for B, $-a \leqslant x + 2m\ell \leqslant a$, while for C, $a \leqslant x + 2m\ell \leqslant 2\ell - \alpha$, where $m = 0, \pm 1, \ldots$. The surface $z = S(x)$ is given by

$$S(x) = \begin{cases} 0, & x = a + 2m\ell \\ -d, & x \neq a + 2m\ell \quad (m = 0, \pm 1, \ldots) \end{cases} \tag{2.2}$$

and the hard or Neumann boundary condition is given by

$$\frac{\partial \psi_\gamma}{\partial n}\,[x, S(x)] = 0 \tag{2.3}$$

*The factor $e^{-i\omega t}$ is suppressed throughout this report.

where n is the normal to the surface $S(x)$. In addition, ψ_γ has the following properties:

a. ψ_γ and $|\nabla\psi_\gamma|$ are finite and continuous in each region, except at the plate edge where $|\nabla\psi_\gamma| = 0(r^{-(1/2)+\epsilon})$ and $\epsilon = -\pi^{-1}\sin^{-1}(\sigma/2)$, as in Ref. 4. The parameter σ is defined later.

b. The pressure $p_\gamma = -i\omega\rho_\gamma\psi_\gamma$ and the normal velocity $v_\gamma = -\partial\psi_\gamma/\partial z$ are continuous at $z = 0$.

c. The quantity $\psi_A - \psi_i$ represents upgoing waves as $z \to \infty$.

The field representations are given by

a. $$\psi_A(x,z) = e^{ik(\alpha_0 x - \beta_0 z)} + \sum_{n=-\infty}^{\infty} A_n^h e^{ik(\alpha_n x + \beta_n z)} \tag{2.4}$$

where the first term is the incident plane wave ψ_i, $\alpha_n = \sin\theta_n = \alpha_0 + n\Lambda$ (grating equation), θ_n is the scattering angle, $\Lambda \equiv \lambda/2\ell$, λ is the incident wavelength, and the superscript h stands for hard.

b. For $|x| \leqslant a$,

$$\psi_B(x,z) = \sum_{j=0}^{\infty} B_j^h \cos(j\pi(x+a)/2a)\cos[kq_j(z+d)] \tag{2.5}$$

where $p_j^2 + q_j^2 = 1$ and $p_j \equiv j\pi/2ka \equiv j\Lambda t/2(t = \ell/a)$; and

c. for $a \leqslant x \leqslant 2\ell - a$

$$\psi_0(x,z) = \sum_{j=0}^{\infty} C_j^h \cos[jku(x-a)]\cos[kr_j(z+d)] \tag{2.6}$$

where $r_j^2 + (ju)^2 = N^2$, $u \equiv \Lambda t/2(t-1)$, and $u_j \equiv r_j$ when $N = 1$. The potentials ψ_B and ψ_C satisfy Eq. (2.3). Field representations outside these regions are given by the Floquet conditions in Ref. (4), Eq. (2.10).

3. GENERAL LINEAR EQUATIONS AND FLUX CONSERVATION

To derive the linear equations relating A_n^h and B_n^h, we require the continuity of pressure and velocity across the interface $z = 0$, $|x| \leqslant a$. This yields

$$\psi_A(x,0) = \psi_B(x,0)$$

and

$$\frac{\partial\psi_A}{\partial z}(x,0) = \frac{\partial\psi_B}{\partial z}(x,0).$$

Substituting appropriate field expressions into these relations, solving for the B_j^h amplitudes in terms of the A_n^h, and manipulating the resulting equations yields the set of linear equations

$$\sum_{n=-\infty}^{\infty} \alpha_n A_n^h I_{nm} \left(\frac{e^{-ikdq_m}}{\beta_n - q_m} \pm \frac{e^{ikdq_m}}{\beta_n + q_m} \right)$$

$$-\alpha_0 I_{0m} \left(\frac{e^{-ikdq_m}}{\beta_0 + q_m} \pm \frac{e^{ikdq_m}}{\beta_0 - q_m} \right) = (2\pi i/\Lambda t) q_m \tau_m B_m^h \quad \begin{Bmatrix} 0 \\ 1 \end{Bmatrix} \tag{3.1}$$

where

$$I_{nm} = e^{-\pi i \alpha_n/\Lambda t} - (-)^m e^{\pi i \alpha_n/\Lambda t} \tag{3.2}$$

and

$$\tau_m = \begin{Bmatrix} 2, \, m = 0 \\ 1, \, m > 0 \end{Bmatrix}.$$

Equations relating A_n^h and C_m^h follow from the continuity of pressure and velocity at $z = 0$, $a \leqslant x \leqslant 2\ell - a$, which are

$$\rho_A \psi_A(x,0) = \rho_C \psi_C(x,0)$$

and

$$\frac{\partial \psi_A}{\partial z}(x,0) = \frac{\partial \psi_C}{\partial z}(x,0).$$

The equations are given by

$$\sum_{n=-\infty}^{\infty} \frac{\alpha_n A_n^h J_{nm}}{\beta_n \mp u_m} - \frac{\alpha_0 J_{0m}}{\beta_0 \mp u_m} - \frac{\pi i}{2u} \tau_m \sigma_m^{h(\pm)} C_m^h = 0 \tag{3.3}$$

where

$$J_{nm} = e^{\pi i \alpha_n/\Lambda t} (1 - (-)^m e^{\pi i \alpha_n/u}) \tag{3.4}$$

and

$$\sigma_m^{h(\pm)} = i r_m \sin(kdr_m) \pm \rho u_m \cos(kdr_m). \tag{3.5}$$

The flux conservation relation can be simply written in terms of the reflection coefficient R

$$R \equiv \sum_n R_n \equiv \sum_n |A_n^h|^2 (\beta_n/\beta_0) = 1 \tag{3.6}$$

since there is no energy loss in transmission. The sum is over integers n such that β_n is real. The R_n terms are the individual spectral reflection coefficients.

Equations (3.1) and (3.3) are the equations for the most general case of incident angle (arbitrary $\alpha_0 = \sin\theta_i$). This most general case apparently cannot be solved by the methods presented. Instead, the case of normal incidence ($\alpha_0 = 0$), with t, p, and the wavenumber parameter N arbitrary, is presented in the next section.

4. CASE OF NORMAL INCIDENCE ($\alpha_0 = 0$)

The case of arbitrary incident angle and $t = 1$ was solved in Ref. 2. The case of normal incidence ($\alpha_0 = 0$) and arbitrary values of t is discussed here. The problem is thus a generalization of some problems due to Deryugin (6), which can also be found in a book by Beckmann and Spizzichino (7). The method used is a variation of the modified residue calculus method (8). For $\alpha_0 = 0$, and excluding values of t for which $\sin(\pi n/t) = 0$, Eqs. (3.1) and (3.3) reduce to

$$\begin{aligned}\sum_{n=1}^{\infty} n\, A_n^h \sin(\pi n/t) & \left(\frac{e^{-ikdq_m}}{\beta_n - q_m} \pm \frac{e^{ikdq_m}}{\beta_n + q_m}\right) \\ & - (\pi/\Lambda^2 t)\, \delta_{m0}\, (A_0^h e^{-ikd} \pm e^{ikd}) \\ & + (\pi/2\Lambda^2 t)\, q_m\, \tau_m\, B_m^h \begin{Bmatrix} 0 \\ 1 \end{Bmatrix} = 0 \ (m \text{ even})\end{aligned} \tag{4.1}$$

where $B_m^h = 0$ (m odd), $\beta_n = \beta_{-n}$, $A_n^h = A_{-n}^h$ from the field symmetry, and

$$\sum_{n=1}^{\infty} \frac{n\, A_n^h \sin(\pi n/t)}{\beta_n \mp u_m} + \frac{\pi}{2u\Lambda}\, \delta_{m0} \begin{Bmatrix} A_0^h \\ -1 \end{Bmatrix} - (\pi/8u\Lambda)\tau_m\, \sigma_m^{h(\pm)}\, C_m^h = 0 \ (m \text{ even}) \tag{4.2}$$

where $C_m^h = 0$ (m odd).

Integrals of the form

$$(2\pi i)^{-1} \int_C G(\omega) \left(\frac{e^{-ikdq_m}}{\omega - q_m} \pm \frac{e^{ikdq_m}}{\omega + q_m}\right) d\omega$$

and

$$(2\pi i)^{-1} \int_C \frac{G(\omega)\, d\omega}{\omega \mp u_m}, \ (m = 0, 2, 4, \ldots)$$

where C is a closed contour at infinity, yield residue series which match Eqs. (4.1) and (4.2), respectively, if we choose the meromorphic function $G(\omega)$ to have the following properties:

a. $G(\omega)$ has simple poles at $\omega = \beta_n$ ($n = 1, 2, 3, \ldots$).

b. $G(\omega)$ has simple zeroes at $\omega = q'_m \equiv q_m + \delta_m$ ($m = 2, 4, \ldots$) and at $\omega = u'_m \equiv u_m + v_m$ ($m = 2, 4, \ldots$). These zeroes are shifted from known values at q_m and u_m. The δ_m shifts are found numerically from the symmetry condition

$$G(q_m) = -e^{2ikdq_m} G(-q_m), \tag{4.3}$$

and the v_m shifts are obtained from the symmetry condition

$$G(u_m) = \sigma_m{}^h G(-u_m) \tag{4.4}$$

where $\sigma_m^h \equiv \sigma_m^{h+}/\sigma_m^{h-}$. The asymptotic values

$$\delta = \lim_{m\to\infty} \delta_m = 0 \text{ and } v = \lim_{m\to\infty} v_m = -(2iu/\pi)\sin^{-1}(\sigma/2),$$

where

$$\sigma = \lim_{m\to\infty} \sigma_m^h = \left\{ \begin{matrix} 1, & N = \infty \\ \frac{1-\rho}{1+\rho}, & N = \text{finite} \end{matrix} \right\}, \tag{4.5}$$

can be derived as in Refs. 2 and 4. The $\rho = \infty$ case is that of Deryugin (6,7).

c. $G(\omega) = 0(\omega^{-(1/2)-\epsilon})$ as $|\omega| \to \infty$, where $\epsilon = v/2iu$ and, as an edge is approached, $|\nabla\psi_\gamma| = 0(r^{-(1/2)+\epsilon})$.

$G(\omega)$ is explicitly given by

$$G(\omega) = G(-1)\,\frac{\Pi_e(\omega,q')\,\Pi_e(\omega,u')\,\Pi_1(-1,\beta)}{\Pi_e(-1,q')\,\Pi_e(-1,u')\,\Pi_1(\omega,\beta)}\,e^{-i(1+\omega)H} \tag{4.6}$$

where

$$H = (\Lambda t)^{-1}(t\ln t - (t-1)\ln(t-1)).$$

The constant $G(-)$ is given below, and the infinite products are discussed in Refs. 2 and 4. In order that the residue match Eqs. (4-1) and (4-2), the following identifications (which follow after some algebraic manipulations) must be made:

$$R(\beta_n) = nA_n^h \sin(\pi n/t)\ (n \geqslant 1) \tag{4.7}$$

(where $R(\beta)$ is the residue of $G(\omega)$ at $\omega = \beta$),

$$B_m^h = (4\Lambda^2 t/\pi q_m)e^{-ikdq_m} G(q_m),\ (m = 2, 4, \ldots) \tag{4.8}$$

$$C_m^h = -(8u\Lambda/\pi\sigma_m^{h+})\,G(u_m),\ (m = 2, 4, \ldots) \tag{4.9}$$

$$G(-1) = -(\pi/2u\Lambda) - (\pi/4u\Lambda)\sigma_0^{h-}C_0^h, \tag{4.10}$$

$$C_0^h = 2(1 + GA_0^h)/(G\sigma_0^{h+} - \sigma_0^{h-}), \tag{4.11}$$

$$A_0^h = \frac{\exp(2ikd)\,[G\sigma_0^{h+} - \sigma_0^{h-}] + K_1\sigma_0^{h+}}{G\sigma_0^{h+} - \sigma_0^{h-} - K_1\,\sigma_0^{h-}} \tag{4.12}$$

and

$$B_0^h = e^{ikd}(1+K) + A_0^h(1-K)e^{-ikd}, \tag{4.13}$$

with the definitions

$$G \equiv \frac{G(-1)}{G(+1)} = \frac{\Pi_e(-1,q')\,\Pi_e(-1,u')\,\Pi_1(1,\beta)}{\Pi_e(1,q')\,\Pi_e(1,u')\,\Pi_1(-1,\beta)}\,e^{+2iH} \tag{4.14}$$

$$K_1 \equiv (t-1)(1+Ge^{2ikd}), \tag{4.15}$$

and

$$K \equiv (1-Ge^{2ikd})/(1+Ge^{2ikd}). \tag{4.16}$$

Note that substituting Eq. (4.12) into Eq. (4.11), and the resulting expression for Eq. (4.11) into Eq. (4.10), yields $G(-1)$ in terms of known quantities. Hence $G(\omega)$ and, thus, all the amplitudes are well defined once the zero shifts are known. From Eqs. (4.7)-(4.9), for $n \geqslant 1$, the forms of the amplitudes are

$$A_n^h = \frac{(-)^{n+1}n\Lambda^2 G(-1)\Pi_e(\beta_n,q')\,\Pi_e(\beta_n,u')\,\Pi_1(-1,\beta)}{\beta_n\,\sin(\pi n/t)\,\Pi_e(-1,q')\,\Pi_e(-1,u')\,\Pi_1(1,\beta)}\,e^{-i(1+\beta_n)H}, \tag{4.17}$$

$$B_m^h = \frac{4\Lambda^2 G(-1)\,\Pi_e(q_m,q')\,\Pi_e(q_m,u')\,\Pi_1(-1,\beta)}{\pi q_m\,\Pi_e(-1,q')\,\Pi_e(-1,u')\,\Pi_1(q_m,\beta)}\,e^{-i(1+q_m)H-ikdq_m}, \tag{4.18}$$

and

$$C_m^h = \frac{-8u\Lambda G(-1)\,\Pi_e(u_m,q')\,\Pi_e(u_m,u')\,\Pi_1(-1,\beta)}{\pi\sigma_m^{h+}\,\Pi_e(-1,q')\,\Pi_e(-1,u')\,\Pi_1(u_m,\beta)}\,e^{-i(1+u_m)H}\quad (m = 2, 4, \ldots). \tag{4.19}$$

Values for A_0^h, B_0^h, and C_0^h are given by Eqs. (4.12), (4.13), and (4.11) respectively.

The numerical calculation of the zero shifts is different from the calculations given in Ref. 1-4. Both the calculation and the numerical results for the reflection coefficient R_n will be developed in a future paper.

5. SUMMARY

General linear equations have been derived which relate the amplitude coefficients in the various geometrical regions of a periodic, inhomogeneously loaded, finite-depth parallel plate surface with hard boundaries when a plane wave is reflected from the surface. For the special case of normal incidence ($\alpha_0 = 0$) the equations were solved using a variation of the modified residue calculus method. The edge behavior of the fields was presented and shown to be similar to the results given in Ref. 4. Only the analytic results are presented in this report. The numerical procedure used, and the numerical results, are being worked on and will be published at a later date.

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13. ABSTRACT

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