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NRL Report 7049

Some General Properties of a Class of Semilinear Hyperbolic Systems Analogous to the Differential-Integral Equations of Gas Dynamics

HOWARD E. CONNER

Mathematics Research Center
Report 70-4
Mathematics and Information Sciences Division

February 25, 1970

NAVAL RESEARCH LABORATORY
Washington, D.C.

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Abstract: Using a background of nonequilibrium theory for gas dynamics, we are investigating the general structure of a class of semilinear hyperbolic systems analogous to the Boltzmann equations for a gas model with discretized velocity states. In this report we develop the global existence of (nonnegative) solutions associated with nonnegative initial data. We also present some auxiliary results on a linear conjugacy of systems and on monotonicity and smoothness properties of general (local) solutions.

1. INTRODUCTION

Using a background of nonequilibrium theory for gas dynamics, we are investigating the general structure of a class of semilinear hyperbolic systems of equations analogous to the Boltzmann equations for a gas model with discrete velocity states.

We begin by first deriving a class of systems, called Boltzmann systems, and then enlarging this class to all systems equivalent to some Boltzmann system. The equivalence between systems is based on the notion of linear conjugacy over the velocity states.

Within this general class we want to locate the systems having structural properties sufficient for a development patterned after a kinetic theory for gas dynamics. The development of kinetic and fluid descriptions for gas dynamics and the problem of finding an interpolation theory connecting them is discussed in a companion report [1].

In particular, we want to locate those systems for which (a) the global existence of nonnegative solutions is guaranteed, (b) an associated H-functional decreases (monotonically) with increasing time to a constant value, (c) a global solution converges with increasing time to an appropriate steady-state solution, and (d) there are (easily) associated conservation equations analogous to those for continuity, fluid motion, and heat flow in gas dynamics.

2. SUMMARY

Section 3 contains an idealized derivation of Boltzmann systems. It is used as a guide for understanding the restrictions imposed on the general systems. Section 5 contains the auxiliary work on linear conjugacy and equivalence. Linear conjugacy is also used to derive the related conservation equations. Section 6 contains some auxiliary work on monotonicity and smoothness properties of local solutions. It also contains the principal result on the global existence of (nonnegative) solutions associated with nonnegative data.

Note: Dr. Conner holds joint appointments at the NRL Mathematics Research Center and the University of Wisconsin.

NRL Problem B01-11; Project RR 003-02-41-6153. This is a final report on one phase of the problem; work is continuing on other phases. Manuscript submitted January 7, 1970.

3. BOLTZMANN SYSTEMS

Background

Let us partition the velocity space of a dilute gas system into many distinct and nonoverlapping cells: $I_1, \ldots, I_n$. At each instant of time t we label each molecule in the system with the index of the cell containing its velocity point at time t. We then let $n_i(t,x)$ be the number of molecules which at time t are in the spatial cell Δ containing x and have velocity points in the cell I_i. To continue, we use a technique very common in the construction of continuous-parameter Markov chains.

Granted that we can obtain constants $\nu_{jk} > 0$, $1 \leqslant j,k \leqslant n$, giving the collision rate between molecules of type j and k over the epoch dt and granted that we can find constants $0 \leqslant \Gamma_{jk}^i \leqslant 1$, $1 \leqslant i,j,k \leqslant n$, giving the probabilities for a j-molecule to have its velocity point scattered from I_j to I_i through collision with a k-molecule, we declare $\Gamma_{jk}^i \nu_{jk} dt$ to be the probability for a (j,k)-collision to generate a (j,i)-scattering in unit time dt.

Setting the time rate of change in n_i equal to the sum of the changes in n_i through streaming and collisions, we have

$$\frac{\partial n_i}{\partial t} = -v_i \cdot \operatorname{grad}_x n_i + \sum_{j,k=1}^{n} (\Gamma_{jk}^i \nu_{jk} n_j n_k - \Gamma_{ik}^j \nu_{ik} n_i n_k), \quad 1 \leqslant i \leqslant n,$$

where v_i is some average velocity point in I_i. This is written in the more general form

$$\frac{\partial n_i}{\partial t} = -v_i \cdot \operatorname{grad}_x n_i + \sum_{j,k=1}^{n} B_{jk}^i \, n_j n_k, \quad 1 \leqslant i \leqslant n,$$

with

$$B_{jk}^i = \frac{1}{2} \nu_{jk} (\Gamma_{jk}^i + \Gamma_{kj}^i - \delta_{ij} - \delta_{ik}).$$

Examples

By specifying the admissible velocity states, a choice of collision transformations, and the associated probabilities, we constiuct several two-state models.

With v and $v' = 1$ or -1, we define the following scattering laws:

elastic scattering: $(v,v') \to (v',v)$ with probability 1.

uniform scattering: $(v,v') \to (vv',v)$ or (v,vv') with equal probabilities.

collinear scattering: $(v,v) = (-v,-v)$ with probability 1.

Assuming the particles are streaming on R and choosing the collision rates $v_{jk} = 1$, we obtain the following examples:

	elastic scattering	uniform scattering	collinear scattering
$\dfrac{\partial u_1}{\partial t} + \dfrac{\partial u_1}{\partial x} =$	0	$u_2^2 - u_1 u_2$	$u_2^2 - u_1^2$
$\dfrac{\partial u_2}{\partial t} - \dfrac{\partial u_2}{\partial x} =$	0	$u_1 u_2 - u_2^2$	$u_1^2 - u_2^2$

Definition of an nth-Order Boltzmann System

Using these models as a pattern, we make the following definition of an nth-order Boltzmann system.

Definition 1. The system of equations

$$\frac{\partial u_i}{\partial t} + v_i \frac{\partial u_i}{\partial x} = \sum_{j,k} B^i_{jk} u_j u_k, \quad 1 \leqslant i \leqslant n, \tag{1}$$

is called an nth-order Boltzmann system with velocity states $v = (v_1,\ldots,v_n)$ and collision form $B = (B^1,\ldots,B^n)$, provided the matrices B^i satisfy the following:

(a) B^i is symmetric

(b) $B^i_{jk} \geqslant 0$, all $j,k \neq i$
$\leqslant 0, j$ or $k = 1$,

(c) $\sum_i B^i_{jk} = 0$, all j,k

(d) $\sum_{j,k} B^i_{jk} = 0$, all i.

The significance of properties c and d should be explained. For example, d implies the positive ray through the vector 1 (all components equal to 1) is a ray of critical points for B; *i.e.*, $B(c1) = 0$, for all $c \geqslant 0$. Similarily, c implies the covector 1 annihilates the range of B: *i.e.*, $\sum_i B^i(\xi) = 0$, for all ξ in R^n. We can rephrase c and d in this more general form:

(c) existance of a positive critical point in R^n for B.

(d) existance of a positive covector annihilating the range of B.

Conditions c and d are redundant. In a recent paper on ordinary quadratic systems, Jenks [2] has announced these and related results.

4. NOTATION, TERMINOLOGY, AND CONVENTION

We use the notation $C(X,Y)$ to denote the class of continuous mappings from a Banach space X into another, Y. The Banach space associated with $C(X,Y)$ by imposing the classical norm is also denoted by $C(X,Y)$ or just C. For example, $C(\overline{R},R^n)$ is the space of bounded uniformly continuous R^n-valued functions f on R with or without $\|f\|^2 = \max_i |f_i|$ being imposed, depending on the context. The classical norm of a scalar f (*i.e.*, f in $C(\overline{R},R)$) is denoted $|f|$. The class of linear operators from X into Y is denoted by $L(X,Y)$. The Banach space associated with $L(X,Y)$ by imposing the operator norm is also denoted by $L(X,Y)$. The open ball in $C = C(\overline{R},R^n)$ with radius r, centered at f, is denoted by $B_r(f)$; its closure by $\overline{B}_r(f)$.

The cone of positive elements in R^n associated with the partial ordering $\xi \leqslant \eta \equiv \xi_i \leqslant \eta_i$, $1 \leqslant i \leqslant n$, is denoted by R^n_+. The cone of positive elements in $C(\overline{R},R^n)$ associated with the ordering $f \leqslant g \equiv f_i(x) \leqslant g_i(x)$, $1 \leqslant i \leqslant n$, x in R, is denoted by Λ; its interior, by Λ^0. A map F of $C(\overline{R},R^n)$ into itself is called positive (strictly positive) if $F(\Lambda) \subset \Lambda$ (Λ^0) and, similarily for F of R^n into itself. Additional information on partial orderings and positive cones, positive and monotone maps, *etc.*, can be found in Ref. 3.

We shall call an equicontinuous semigroup of class C_0 a semigroup.

The canonical imbedding j of R^n into $C(\overline{R},R^n)$ is defined by $\alpha \to j_\alpha$, $j_\alpha(x) = \alpha$, all $x \in R$.

The remainder of our notation is chosen to conform to ordinary usage as evidenced in standard texts such as Refs. 4 and 5.

5. SOME PROPERTIES OF GENERAL QUADRATIC SYSTEMS

In this section some qualitative characteristics of the Boltzmann systems of Definition 1 are examined within the larger class of general systems

$$\frac{\partial u_i}{\partial t} + \sum_{j=1}^{n} a_{ij} \frac{\partial u_j}{\partial x} = \sum_{j,k=1}^{n} Q^i_{jk}\, u_j\, u_k$$

or, in the equivalent vector form,

$$\dot{u} + A\partial u = Q(u), \qquad (2)$$

$$\dot{u}\colon R_+ \times R \to R^n,$$

where

$$Q = (q_1,..,q_n), \quad q_i(u) = \sum_{j,k} Q^i_{jk}\, u_j\, u_k, \quad 1 \leqslant i,j,k \leqslant n, \quad u \text{ in } R^n.$$

(Also, $\dot{u}$ is the vector whose ith component is $\partial u_i/\partial t$; and ∂u is the vector whose ith component is $\partial u_i/\partial x$.)

In the special case of spatial homogeneity ($\partial u \equiv 0$), Eq. (2) reduces to a system of ordinary differential quadratic equations. In addition to the cited work of Jenks the qualitative work of Markus [6] is particularly relevant.

Markus developed a necessary and sufficient condition for the linear conjugacy of two ordinary quadratic systems. Using the forms Q^i to construct a (nonassociative) algebra $A(Q)$, Markus transferred the question of linear conjugacy between two systems to the question of isomorphism between two algebras.

As an immediate application of this work we can obtain a complete set of (abstract) conjugacy invariants over the full linear group $L(n)$ for a general quadratic system, namely, the Jordan form for the streaming matrix A and the Markus algebra $A(Q)$ for Q. The construction of a Markus algebra associated with the quadratic forms (construction of a Markus algebra $A(Q)$ for Q) is given in the following:

Let $e_1, \ldots, e_n$ be a basis for V^n (linear n-space). A multiplication $\circ$ is defined over V^n using the rules

$$e_j \circ e_k = \sum_i Q^i_{jk}\, e_i.$$

Then, V^n equipped with the multiplication $\circ$ is a Markus algebra $A(Q)$ for Q.

We will refer to as (M) the conjugacy invariants mod $L(n)$ for general quadratic systems: *Two systems* (A,Q) *and* $(\widetilde{A},\widetilde{Q})$ *are conjugate with respect to* T *in* $L(n)$ *through the mapping* $u = T\widetilde{u}$ *if and only if (a)* A *and* B *are conjugate using* $T\colon A = T^{-1}BT$ *and (b) the algebras* $A(Q)$ *and* $A(\widetilde{Q})$ *are isomorphic using* T.

Since a proof of (M) is easily made by performing the calculations, we defer from presenting one and refer the interested reader to the work of Markus [6, p. 186].

The result stated in (M) can be used to transform a system into an equivalent system (over $L(n)$) with standard form. For example, using an anistropic scaling followed by a diagonal rotation (a diagonal matrix with $+1$ or -1 entries), a system with a critical point ξ in R^n_+ can be transformed into an equivalent system with critical point 1. More generally, we have the following

COROLLARY: *Suppose two quadratic systems (A,Q) and $(\tilde{A},\tilde{Q})$ are conjugately related through the mapping $u = T\tilde{u}$, T in $L(n)$. Then $\tilde{\xi} = T^{-1}\xi$ and $\tilde{\eta} = T'\eta$ are a critical vector and annihilating covector for $(\tilde{A},\tilde{Q})$ if and only if ξ and η are for (A,Q) (re:the discussion after Eq. (1)).*

This corollary shows we must modify our definition of Boltzmann systems if we want to include the entire class of general systems conjugately related to Boltzmann systems. Therefore we make the following amended definition.

Definition 2. A system given by Eq. (2) is called a kinetic system if it is equivalent over $L(n)$ to a system of the form

$$\dot{u} + v\partial u = K(u),$$

where the K^i_{jk} satisfy properties a through c of Definition 1. We obtain an equivalent definition replacing c with d.

Jenks [2;p. 503] has shown that any system $K = \{K^i_{jk}\}$ satisfying a through c of Definition 1 always has a critical ray in R^n_+. He has also shown that such a system has no critical ray contained in the faces (boundary) of R^n_+ if and only if K is irreducible. K is called irreducible if for any partition of 1, ..., n into I and J there always exists $K^i_{jk} \neq 0$ with i in I and j,k in J.

Even though the concept of conjugacy mod $L(n)$ is useful, it is too sensitive for a generic study of kinetic systems. An examination of the two-state models tabulated at the end of the Examples subsection of Section 2 illustrates this sensitivity: the algebra for an elastic scatterer has a trivial multiplication law, the algebra for a uniform scatterer has exactly one linearily independent nilpotent of order 2, and the algebra for a collinear scatterer has a basis of nilpotents of order 2. All three systems fall into distinct conjugacy classes mod $L(n)$. Furthermore, only the collinear model is irreducible.

Using the previous work on conjugacy mod $L(n)$, we can transform a kinetic system (Eq. (2)) into an equivalent system which displays the associated "conservation equations." The basic idea is that each covector β in R^n satisfying

$$\sum_{i=1}^{n} \beta_i K^i_{jk} = 0. \qquad \text{all } j,k,$$

annihilates the range of

$$K\colon \xi \to K(\xi), \quad K^i(\xi) = \sum_{j,k} K^i_{jk}\xi_j\xi_k$$

and generates a natural projection

$$\pi_\beta\colon C(\overline{R},R^n) \to C(\overline{R},R), \quad (\pi_\beta f)(x) = \sum_{i=1}^{n} \beta_i f_i(x).$$

These projections are analogous to the velocity moment projections defining the mass density, average thermal mass velocity, and internal mass energy for the classical Boltzmann equation.

One of our principle assumptions (Definition 1, property c) implies that the covector 1 is always an annihilating covector for the range of K. There is nothing in these assumptions which either prevents or implies the existance of additional annihilating covectors for the range of K. Without any great loss in generality, we shall assume: (a) 1 is the only (up to a constant multiple) covector annihilating the range of K, and (b) the covectors $V^0 = 1, V^1, V^2, \ldots, V^{n-1}$, $V^k = (v_1^k,\ldots,v_n^k)$, are linearly independent.

Suppose we let T: $R^n \to R^n$ be the (nonsingular) linear transformation whose canonical matrix has the covector V^k in the kth row, $1 \leq k \leq n$; *i.e.*, $t_{kj} = v_j^{k-1}$. Then, using the previous work, we see that the original system is conjugate over T to the system

$$\dot{w}_1 + \frac{\partial}{\partial x} w_2 = 0,$$

$$\dot{w}_i + \frac{\partial}{\partial x} w_{i+1} = \sum_{j,k} \widetilde{K}^i_{jk} w_j w_k, \quad 2 \leq i \leq n-1,$$

$$\dot{w}_n + \sum_{l} v_j^n \, t_{jl}^{-1} \frac{\partial}{\partial x} w_l = \sum_{j,k} \widetilde{K}^n_{jk} w_j w_k$$

$$\left(\widetilde{K}^i_{jk} = \sum_{l,\alpha,\beta} t_{i,l} K^l_{\alpha\beta} t^{-1}_{\alpha j} t^{-1}_{\beta k}\right).$$

The first equation is the conservation equation for the system. Since

$$w_j = \sum_i v_i^j u_i,$$

the above system expresses the time derivative of the jth velocity moment of u in terms of the spatial derivative of its $(j+1)$th velocity moment and an "average" over the collision term,

$$\sum_{i,l,m} v_i^j K^i_{l\,m} u_l u_m = \widetilde{K}^j(w).$$

6. GENERAL PROPERTIES OF SOLUTIONS

A kinetic (Boltzmann) map K is defined to be a map of $C(\overline{R},R^n)$ into itself for which

$$K(f) = (K^1(f),K^2(f),\ldots,K^n(f)),\; K^i(f) = \sum_{j,k} K^i_{jk} f_j f_k,$$

and the K^i_{jk} satisfy conditions a-c (a-d) of Definition 1. A quadratic map Q is defined in the same way, replacing K^i_{jk} with Q^i_{jk} and not imposing restrictions b and c.

Using the Jacobian matrix of Q at f $(\partial q_j(f)/\partial x_i)$ and the mean value theorem for Banach spaces, we can develop the following properties for Q: A quadratic map Q is of class C^∞ on $C(\overline{R},R^n)$, with locally bounded derivatives d^kQ, uniformly in k, and is a local Lipshitz map.

A velocity operator V is defined to be a (linear) map of $C(\overline{R},R^n)$ into itself with $Vf = (v_1 f_1, \ldots, v_n f_n)$, for a given set $v_1, \ldots, v_n$ of velocity states.

The isotropic translation semigroup $e^{t\partial}$ on $C(\overline{R},R^n)$ is defined by $(e^{t\partial})f = (f_1(x+t), \ldots, f_n(x+t))$; the differentiation operator $\partial f = (\partial f_1/\partial x, \ldots, \partial f_n/\partial x)$, with domain $D(\partial)$ consisting of all f in $C(\overline{R},R^n)$ for which ∂f is in $C(\overline{R},R^n)$, is the infinitesimal generator of $e^{t\partial}$.

A velocity operator V generates an anistropic translation semigroup $e^{-tV\partial}$, $(e^{-tV\partial})f = (f_1(x+v_1t),\ldots,f_n(x+v_nt))$, with $V\partial$ as its infinitesimal generator, $D(V\partial) = D(\partial)$.

Using the semigroup $e^{tV\partial}$ and a variant of Duhamel's principle, we transform the differential equations defining a kinetic system into the weaker (abstract) integrated system: u: $R_+ \to C(\overline{R},R^n)$,

$$u(t) = (e^{-tV\partial})f + \int_0^t (e^{-(t+s)V\partial})\ K(u(s))\ ds,\ R_+ \to C(\overline{R},R^n), \tag{3a}$$

$$u(0+) = f \text{ in } C(\overline{R},R^n). \tag{3b}$$

Replacing the semigroup $e^{-tV\partial}$ with $e^{-tA\partial}$ and the kinetic map K with Q, we have the integrated form of a general quadratic system.

The existence of local solutions and the continuous differentiability of these solutions with respect to the initial data is given by

THEOREM 1. *For each s in R_+ and f in $C(\overline{R},R^n)$ there exists an interval I_s $(=[s,s_1])$, a ball $B_r = B_r(f)$, and constant $0 < \lambda < 1$ such that*

$$u(t) = (e^{-(t-s)A\partial})f + \int_s^t (e^{-(t-r)A\partial})Q(u(r))\ dr \tag{4}$$

has a unique continuous solution u: $I_s \times B_{\lambda r} \to B_r \subset C(\overline{R},R^n)$ satisfying $u(s,s,g) = g$ in B_r. Moreover, u is continuously differentiable in g on $B_{\lambda r}$.

Proof. A proof for all but the last assertion is easily made using a classical iteration scheme, the local Lipshitzian character of Q, and growth estimates obtained from the Banach space variant of Gronwell's inequality.

To validate the last assertion, we verify the existence of a map L: $I_s \times B_{\lambda r} \to L(C,C)$ $(C = C(\overline{R},R^n))$, for which

$$\|u(t;s,g+\epsilon h) - u(t;s,g) - \epsilon\ L_g^t h\| = O(\epsilon\|h\|) \tag{5}$$

uniformly on $I_s \times B_{\lambda r}$.

Since dQ is continuous and $u(t;s,g)$ remains inside B_r for t in I_s, $dQ|_{u(t;s,g)}$ is a uniform Lipshitz map of $I_s \times B_{\lambda r} \to L(C,C)$. Using this property, the methods outlined in the previous paragraph provide a unique solution W: $I_s \times C(\overline{R},R^n) \times B_{\lambda r} \to C(\overline{R},R^n)$ satisfying

$$w(t) = (e^{-(t-s)A\partial})h + \int_s^t (e^{-(t-r)A\partial})\ [dQ|_{u(r;s,g)}w(r)]\ dr,$$

$$w(0+) = h.$$

We define L using $L_g^t h = w(t;h,g)$.

Denoting the vector in the left-hand side of Eq. (5) by $\delta(t;h,g)$, adding Eq. (4) for g and $g+\epsilon h$, and subtracting the corresponding equation for $w(t;\epsilon h,g)$, we have

$$\delta(t;h,g) = \int_s^t (e^{-(t-r)A\partial})\ [Q(u(r;s,g+\epsilon h)) - Q(u(r;s,g)) - \epsilon dQ|_{u(r;s,g)}L_g^r h]\ dr. \tag{6}$$

We denote the bracketed expression by $R(r;g,h)$. Since $u(r;s,g+\epsilon h)$ and $u(r;s,g)$ stay inside B_r on I_s for sufficiently small ϵ, we can use the mean value theorem for Banach spaces, the local Lipshitzian character of Q, and some manipulation of entries to show

$$\|R(r;h,g)\| \leqslant a\|\delta(r;h,g)\| + \epsilon b\|h\|$$

on I_s for some $a,b > 0$.

Taking norms in Eq. (6) and using the above estimate, we have

$$\|\delta(t;g,h)\| \leqslant \int_s^t (a\|\delta(r;g,h)\| + \epsilon b\|h\|)\, dr$$

for sufficiently small ϵ. This implies

$$\|\delta(t;g,h)\| \leqslant \frac{\epsilon b\|h\|}{a}\,(e^{(t-s)a}-1)$$

on $I_s \times B_r \times C(\overline{R},R^n)$, completing the proof.

We shall need information about some regularity restrictions on the initial data which are propagated by solutions.

THEOREM 2. *Suppose $u(\ldots;s,f)$ is the solution of Eq. (6). Then $u(t;s,f)$ is in $C^1(\overline{R},R^n)$, on I_s, if f is.*

Proof. Using again an iteration scheme, we can construct a sequence of continuous maps $u^{(n)}\colon I_s \times B_{\lambda r} \to B_r \subset C(\overline{R},R^n)$, $u^{(n)}(0+) = f$, converging to u uniformly on I_s. An induction argument shows $u^{(n)}$ is in $D(\partial)$ ($(A\partial)u^{(n)}$ is defined) and $u^{(n)}$ and $(A\partial)u^{(n)}$ are equibounded uniformly on I_s. So, we can assume s-lim $(A\partial)u^{(n)}$, $n \to \infty$, exists in $C(\overline{R},R^n)$. Since $(A\partial)$ is a closed operator, $u(t) = s$-lim $u^{(n)}(t)$ and the existence of s-lim $(A\partial)u^{(n)}$ forces s-lim $(A\partial)u^{(n)} = (A\partial)u$; *i.e.*, $u(t)$ is in $D(\partial) = C^1(\overline{R},R^n)$ on I_s, completing the proof.

Since a quadratic operator Q is a C^∞ map, we can repeatedly apply Theorem 2 and obtain the following corollary.

COROLLARY: *If f is in $\bigcap_{n\geqslant 0} D(\partial^n)C(\overline{R},R^n)$, then so is $u(t)$, on I_s.*

Nonglobal solutions u are easily constructed for the two-state models with either collinear scattering [7,p. 18] or uniform scattering. This is done by looking for special solutions for which the mass density $u_1(t,x) + u_2(t,x)$ is constant in x for each $t \geqslant 0$. It is significant that the initial data $u(0+)$ of these solutions is not in the positive cone Λ of $C(\overline{R},R^2)$.

Kinetic maps have a natural and useful decomposition into the difference of two positive maps. Writing the ith component $K^i(g)$ of $K(g)$ in the form

$$K^i(g) = \Big(\sum_{jk\neq i} K^i_{jk}g_jg_k - K^i_{ii}g_i^2\Big) - \Big((-2)\sum_{j\neq i} K^i_{ij}g_j\Big)\, g_i \tag{7}$$

and defining the maps P and V of $C(\overline{R},R^n)$ into itself by setting $P_i(g)$ equal to the first quantity in the right-hand side of Eq. (7) and $V_i(g)$ equal to the last, we have $K = P - V$. Since $K^i_{jk} \geqslant 0$ for $j,k \neq i$ and $K^i_{jk} \leqslant 0$ for either j or $k = i$ (Definition 1, condition b), we see that P and V are positive maps over the cone Λ.

LEMMA 1. *Suppose K is a kinetic map on $C(\overline{R},R^n)$. Then, given any bounded set E, there exists a positive (diagonal) operator D on $C(\overline{R},R^n)$ for which $(K + D)$ is monotonic on E. Since $K(0) = 0$, $(K + D)$ is also a positive map on E.*

Proof. It is sufficient to restrict to f and g in some ball B, with $f \leq g$. Using the mean value theorem for Banach spaces and the majorizability of the differentials $dV|_h$ for h in B, we have

$$V(g) - V(f) = L(g-f) \leq M(g-f),$$

where L is in the convex closure of $\{dV|_h : f < h < g\}$, and where M majorizes $\{dV|_h : h \text{ in } B\}$ and is diagonal.

If D is a positive diagonal operator, we have

$$(K+D)(g) - (K+D)(f) \geq (P(g) - P(f)) + (D-M)(g-f).$$

The first term is in Λ. Choosing each d_{ii} to majorize $\max_{ij} m_{ij}$, the second term is also in Λ, completing the proof for the case when E is a ball. The extension to general bounded E presents no difficulty.

Before showing that all solutions $u(t;f)$, with positive initial data f, are positive and exist for all t, we present a fundamental integral identity for the solutions of the kinetic systems of Eqs. (3). Povzner developed this identity for use in his work on the existence and uniqueness of solutions of a modified classical Boltzmann equation, [8,Lemma 3, p. 209].

The notation

$$\langle \alpha,\beta \rangle = \sum_{i=1}^{n} \alpha_i \beta_i, \qquad \alpha,\beta \text{ on } R^n,$$

is used.

LEMMA 2. *Suppose* u: $I \times \{f\} \to C(\overline{R},R^n)$ *is a continuous solution of Eqs. (3) with existence domain* $I = [0,t(f)]$. *Let D be a positive scaling operator, as in Lemma 1. Then the validity of Eqs. (3) over I is equivalent to the validity of*

$$\int_R \langle h(x), u(t,x) \rangle \, dx = \int_R \langle (e^{-tD} e^{tV\partial} h)(x), f(x) \rangle \, dx$$

$$+ \int_R \int_0^t \langle (e^{-(t-s)D} e^{(t-s)V\partial} h)(x), (K+D)(u(s,x)) \rangle \, ds dx \tag{8}$$

over I for all h in $C_0(R,R^n_+)$. *(For any class C of functions,* C_0 *denotes those with compact support.)*

Since Lemma 2 is the keystone in our development of the monotonicity and positivity properties, we present an adaptation of Povzner's proof, applicable to the system of Eqs. (3).

Proof. We present only the essential ideas, letting the reader supply the appropriate integration theorems for connecting the chain of identities.

Multiplying Eq. (3a) by h in $C_0(R,R^n_+)$ and integrating over R, we see that the validity of Eqs. (3) over I is equivalent to the validity of

$$\int_R \langle h(x), u(t,x) \rangle \, dx = \int_R \langle (e^{tV\partial} h)(x), f(x) \rangle \, dx$$

$$+ \int_R \int_0^t \langle (e^{(t-s)V\partial} h)(x), K(u(s,x)) \rangle \, ds dx, \tag{9}$$

over I for all h in $C_0(R,R^n_+)$.

In the same way, multiplying Eq. (9) by φ in $C_0(I,R_+)$, integrating over R_+, and remembering that $C_0(I,R_+)\cdot C_0(\overline{R},R_+^n)$ is dense in $C_0(I\times R,R_+^n)$, we see that the validity of Eqs. (3) over I is equivalent to the validity of

$$\int_{R_+\times R}\langle\widetilde{H}(s,x),u(s,x)\rangle\,dxds=\int_{R_+\times R}\langle(e^{sV\partial}\widetilde{H})(s,x),f(x)\rangle\,dxds$$

$$+\int_{R_+\times R}\left\langle\int_s^\infty(e^{(t-s)V\partial}\widetilde{H})(t,x)\,dt,\,K(u(s,x))\right\rangle dxds$$

over I for all $\widetilde{H}$ in $C_0(I\times R,R_+^n)$. The integration order $dsdxdt$ is changed in the last term to $dtdxds$.

Adding to the above the identity

$$\int_{R_+\times R}\left\langle\int_s^\infty D(e^{(t-s)V\partial}\widetilde{H})(t,x)\,dt,\,u(s,x)\right\rangle dxds$$

$$=\int_{R_+\times R}\left\langle\int_s^\infty(e^{(t-s)V\partial}\widetilde{H})(t,x)\,dt,\,Du(s,x)\right\rangle dxds,$$

we see that the validity of Eqs. (3) over I is equivalent to the validity of

$$\int_{R_+\times R}\left\langle\widetilde{H}(s,x)+\int_s^\infty D(e^{(t-s)V\partial}\widetilde{H})(t,x)\,dt,\,u(s,x)\right\rangle dxds$$

$$=\int_{R_+\times R}\langle(e^{tV\partial}\widetilde{H})(t,x),f(x)\rangle\,dxds$$

$$+\int_{R_+\times R}\left\langle\int_s^\infty(e^{(t-s)V\partial}\widetilde{H})(t,x)\,dt,\,(K+D)(u(s,x))\right\rangle dxds\tag{10}$$

over I for all $\widetilde{H}$ in $C_0(I\times R,R_+^n)$.

If H is related to $\widetilde{H}$ through

$$-\widetilde{H}(s,x)=-H(s,x)+\int_s^\infty De^{-(t-s)D}(e^{(t-s)V\partial}H)(t,x)\,dt,\tag{11}$$

then the reader can easily verify that

$$\int_s^\infty(e^{(t-s)V\partial}\widetilde{H})(t,x)\,dt=\int_s^\infty e^{-(t-s)D}(e^{(t-s)V\partial}H)(t,x)\,dt\tag{12}$$

is true. Applying the operator $\partial/\partial s+V\partial$ to Eq. (12), we obtain Eq. (11); so that Eqs. (11) and (12) are equivalent forms for relating $\widetilde{H}$ and H. Using Eq. (10), we see that $\widetilde{H}$ is in $C_0(I\times R,R_+^n)$ if H is.

Substituting Eq. (11) into the left-hand side of Eq. (10), substituting Eq. (12) evaluated at $s=0$ into the first term of the right-hand side, substituting Eq. (12) into the second term of the right-hand side, and returning to the original integration order $dsdxdt$, we see that the validity of Eqs. (3) over I is equivalent to the validity of

$$\int_{R_+\times R} \langle H(t,x), u(t,x)\rangle\, dxdt = \int_{R_+\times R} \langle e^{-tD}(e^{tV\partial}H)(t,x), f(x)\rangle\, dxdt$$

$$+ \int_{R_+\times R} \int_0^t \langle e^{(t-s)D}(e^{(t-s)V\partial}H)(t,x), (K+D)(u(s,x))\rangle\, dsdxdt$$

over I for all H in $C_0(I \times R, R_+^n)$.

Since $C_0(I,R_+)\cdot C_0(\overline{R},R_+^n)$ is dense in $C_0(I \times R, R_+^n)$, We see that the validity of Eqs. (3) over I implies the validity of Eq. (8) over I for all h in $C_0(R,R_+^n)$, completing the proof.

Using Lemma 2, we prove

THEOREM 3. *Suppose u: $I \times B_{\lambda r}(f) \to B_r(f) \subset C(\overline{R},R^n)$ is a family of local solutions of Eqs. (3), corresponding to initial datum in $B_{\lambda r}(f)$ and existing over $I = [0,t(f)]$. Then $u(t;\ldots)$ is monotonic on $B_{\lambda r}(f)$ over I; i.e., $u(t;h) \geqslant u(t;g)$ on I if $h \geqslant g$, for g,h in $B_{\lambda r}(f)$.*

Proof. Since $K + D$ is monotonic on $B_{\lambda r}(f)$ for a suitable positive scaling operator D (Lemma 1), and since Λ (Λ^0) is invariant under $e^{tV\partial}$ and e^{tD} for all t in R, we see from Eq. (8) that $u(t;h) - u(t;g)$ is weakly nonnegative over I. Therefore using the continuity properties of u we have $u(t;h) \geqslant u(t;g)$, completing the proof.

Since $K(0) = 0$, we have the immediate

COROLLARY: *$u(t)$ is in Λ over I if $u(0)$ is.*

Since $e^{tV\partial}$ cannot map the boundary of Λ into the interior Λ^0, it is not possible to show a strict positivity property for $u(t)$ without placing some additional restriction on the kinetic map K.

We now have sufficient information to show that local positive solutions, associated with positive initial data, exist globally, Without attempting to give the most general result, we consider only kinetic systems with irreducible kinetic maps.

We recall that a kinetic map K is called irreducible when the associated $\{K_{jk}^i\}$ satisfy the following: for any partition $I \cup J$ of the integers 1, ..., n, there exists $K_{jk}^i \neq 0$ with i in I and j,k in J.

COROLLARY: *If K is irreducible and g is in Λ, then $u(t;g)$, $u(0;g) = g$, exists for all t in R_+.*

Proof: Let ξ in R_+^n be a solution of $K(\xi) = 0$, normalized by

$$\sum_{i=1}^{n} \xi_i = 1.$$

Since K is irreducible, we can assume ξ is strictly positive [2,Theorem 4]. For convenience, we let j be the canonical imbedding of R^n into $C(\overline{R},R^n)$ given by $\eta \to j_\eta(x) \equiv \eta$, x in R. Then $\{\lambda j_\xi\colon \lambda > 0\} \subset \Lambda^0$ is a positive critical ray of K.

The statement of Theorem 3 can be easily extended to show that $u(t;g) \leqslant u(t;\lambda j_\xi) \equiv \lambda j_\xi$ over I, where λ is so chosen that $g \leqslant \lambda j_\xi$. Since ξ is strictly positive, any g in Λ can be so bounded above by some λj_ξ.

Since the inequality $u(t;g) \leqslant \lambda j_\xi$, for suitable choice of λ, holds over any existence domain for $u(\ldots;g)$, a standard argument for continuation shows that $u(t;g)$ exists for all t in R_+, completing the proof.

As evidenced from the reasoning employed in the last argument we can also make an assertion about the behavior, $t \to \infty$, of the order-projection π of $u(t)$ onto a unique (positive) critical ray

$$\overline{Oj_\xi} = \left\{\lambda j_\xi \colon \lambda > 0,\ \xi \text{ in } R^n_+,\ K(\xi) = 0,\ \sum_1^n \xi_i = 1\right\}.$$

When K is irreducible (so that it has a unique positive critical ray), the order-projection $\pi \colon C(\overline{R},R^n) \to \overline{Oj_\xi} \subset C(R,R^n)$ is defined as follows: $f \to \pi(f) = \lambda(f)j_\xi$, $\lambda(f)$ is in $C(\overline{R},R_+)$, when $\lambda(f)$ is uniquely determined by $(\lambda(f) - \epsilon)j_\xi < f \leqslant \lambda(f)j_\xi$, $\epsilon > 0$.

Since $|\lambda(g)|j_\xi$, g in Λ, is a (positive) steady-state solution, we have the following

COROLLARY: *Suppose K is irreducible and f is in Λ. Then*

$$u(t;f) \leqslant \pi(u(t;f))j_\xi,\quad t > 0,$$

and

$$|\lambda(u(t;f)| \leqslant |\lambda(u(s;f))|,\quad t > s.$$

7. COMMENTS

The problem of global existence of solutions was previously worked out by Kolodner [7] for the two-state collinear scattering model. Using special properties of a related Ricatti system, he developed the global existence and showed the applicability of his methods to the n-state generalization of collinear scattering; *i.e.*, special K of the form

$$K^i(u) = \sum_{j=1}^n A_{ij}u_j^2,\ 1 \leqslant i \leqslant n.$$

Kolodner also showed that

$$H(t,u) = \int_R \sum_1^2 u_i(t,x) \log u_i(t,x)\ dx$$

is an H-functional for collinear models with doubly stochastic generators $\{A_{ij}\}$.

The problem of determining the appropriate H-functional for the generalized systems of Eq. (2) is still completely unsolved. Some work has been done. Jenks [2] has found suitable Liapunov functions for the quadratic differential systems considered in his work. Using these, he established the convergence of (positive) solutions to steady-state values [2, Theorem 13].

8. REFERENCES

1. Conner, H.E., "The Piecewise Ergodic Problem for Gas Systems," MRC Report 5, NRL Report 7043, March 10, 1970
2. Jenks, R.D., "Quadratic Differential Systems for Interactive Population Models," J. Diff. Eq. **5**, 497 (1969)
3. Krasnoselskii, M.A., "Positive Solutions of Operator Equations," P. Noordhoff Ltd., Groningen, The Netherlands, 1964
4. Loomis and Sternberg, "Advanced Calculus," Addison-Wesley, 1968
5. Yosida, "Functional Analysis," New York, Springer-Verlag, 1968
6. Markus, L., "Quadratic Differential Equations and Nonassociative Algebras," Contributions to the Theory of Nonlinear Oscillations, V, edited by L. Casari, J. LaSalle, and S. Lefshets, Ann. of Math. Studies **45**, 185 (1960), Princeton Univ. Press, Princeton, N.J.
7. Kolodner, I.I., "On the Carleman's Model for the Boltzmann Equation and It's Generalizations," Annali d. Mate., p. 11
8. Povzner, A. Ja., "The Boltzmann Equation in the Theory of Gases," Mat. Sb. (N.S.) **58** (100), 65 (1962) AMS Transl., Ser. 2, **47**, 193 (1965)

☆ U.S. GOVERNMENT PRINTING OFFICE: 1970-397-175/77

Security Classification

DOCUMENT CONTROL DATA - R & D

(Security classification of title, body of abstract and indexing annotation must be entered when the overall report is classified)

1. ORIGINATING ACTIVITY (Corporate author) Naval Research Laboratory Washington, D.C. 20390	2a. REPORT SECURITY CLASSIFICATION Unclassified 2b. GROUP

3. REPORT TITLE

SOME GENERAL PROPERTIES OF A CLASS OF SEMILINEAR HYPERBOLIC SYSTEMS ANALOGOUS TO THE DIFFERENTIAL-INTEGRAL EQUATIONS OF GAS DYNAMICS

4. DESCRIPTIVE NOTES (Type of report and inclusive dates)

Final report on one phase of a continuing NRL problem.

5. AUTHOR(S) (First name, middle initial, last name)

Howard E. Conner

6. REPORT DATE	7a. TOTAL NO. OF PAGES	7b. NO. OF REFS
February 25, 1970	15	8

8a. CONTRACT OR GRANT NO. NRL Problem B01-11 b. PROJECT NO. Project RR 003-02-41-6153 c. d.	9a. ORIGINATOR'S REPORT NUMBER(S) NRL Report 7049 9b. OTHER REPORT NO(S) (Any other numbers that may be assigned this report) Mathematics Research Center Report 70-4

10. DISTRIBUTION STATEMENT

This document has been approved for public release and sale; its distribution is unlimited.

11. SUPPLEMENTARY NOTES	12. SPONSORING MILITARY ACTIVITY Office of Naval Research Department of the Navy Washington, D.C. 20360

13. ABSTRACT

Using a background of nonequilibrium theory for gas dynamics, we are investigating the general structure of a class of semilinear hyperbolic systems analogous to the Boltzmann equations for a gas model with discretized velocity states. In this report we develop the global existence of (nonnegative) solutions associated with nonnegative initial data. We also present some auxiliary results on a linear conjugacy of systems and on monotonicity and smoothness properties of general (local) solutions.

DD FORM 1 NOV 65 1473 (PAGE 1)

S/N 0101-807-6801

Security Classification

UNCLASSIFIED

14. KEY WORDS	LINK A		LINK B		LINK C	
	ROLE	WT	ROLE	WT	ROLE	WT
Gas dynamics						
Boltzmann equations						
Semilinear hyperbolic differential equations						