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Pulse Compression Degradation Due to Open Loop Adaptive Cancellation, Part III

KARL GERLACH

Target Characteristics Branch
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Karl Gerlach

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13. ABSTRACT *(Maximum 200 words)*

An exact expression for the perturbed range sidelobe level of a compressed pulse that has been preprocessed through an adaptive canceller is derived. This result is a generalization of past research (parts I and II, NRL Reports 9107 and 9309) where the signal was assumed to be completely contained within the canceller's processing batch. In this report, we allow the signal to extend over an arbitrary number of canceller processing batches. A good approximate expression is also obtained for evaluating the perturbed range sidelobe level. This report derives the number of independent samples per channel (main and auxiliaries) necessary so that the average adaptive range sidelobe level is within 3 dB of the quiescent range sidelobe level. Furthermore, the same analysis is used to predict the canceller noise power level that is induced by the desired signal's presence in the canceller weight calculation. Placement of the pulse compressor before or after the canceller is also considered. It is shown that if the desired waveform's code length L is less than or equal to the canceller's processing batch width K, it is desirable to place the pulse compression after the adaptive canceller. If $L > K$, then the issue is not so clear-cut, and a trade-off study is necessary.

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PULSE COMPRESSION DEGRADATION DUE TO OPEN LOOP ADAPTIVE CANCELLATION, PART III

1. INTRODUCTION

An exact expression for the perturbed sidelobe level of a compressed pulse that has been preprocessed through an adaptive canceller is derived in Ref. 1. The pertinent assumptions of that analysis are

1. the adaptive canceller is implemented using the Sampled Matrix Inversion (SMI) algorithm [2] or its equivalent, the Gram-Schmidt canceller [3]

2. the input noises are temporally independent and Gaussian

3. the desired signal's input vector (or code) is completely contained within the samples that were used to calculate the adaptive weights and is only present in the main channel, and

4. the adaptive weights are computed from the same data set to which they are applied (concurrent processing).

Earlier research has shown that because of finite sampling, the quiescent compressed pulse sidelobe levels are degraded by preprocessing the main channel input data stream (the uncompressed pulse) through the adaptive canceller. It was also shown that the level of degradation is independent of whether pulse compression occurs before or after the adaptive canceller under assumption 3.

The exact expression [1] for pulse compression degradation requires computer assistance to evaluate this expression. In Ref. 4, we derived a "rule of thumb" expression that is a good approximation of the exact expression.

This report considers the case where the desired signal input waveform (or code) can extend over any number of processing batches of the adaptive canceller. An exact result for the adaptive range sidelobe level is derived and its associated good approximation is given. In addition, it is shown that the same analysis can be used to predict the canceller noise power level that is induced by having the desired signal present in the canceller weight calculation.

2. BACKGROUND

Figure 1 shows a functional block diagram of an adaptive canceller followed by a pulse compressor. The adaptive canceller linearly weights the auxiliary channels with weights that are calculated from a batch of sampled input data. The main channel consists of desired signal plus noise

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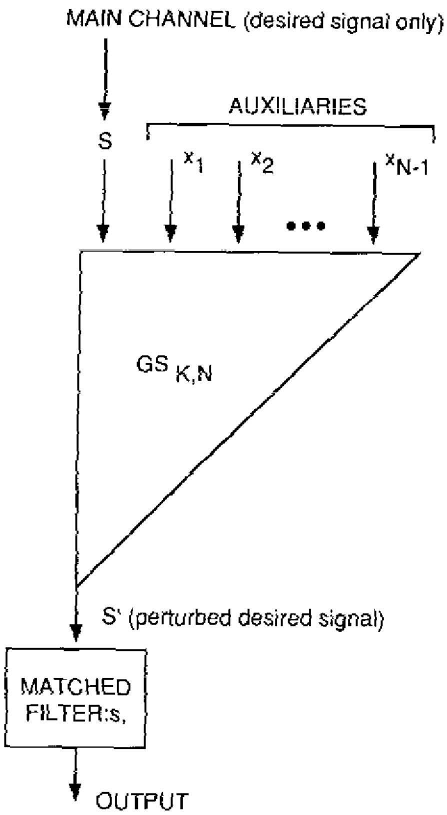

Fig. 1 — GS canceller followed by a matched filter

that may or may not be correlated with the auxiliary channels. It was shown [1] that when analyzing the pulse compression degradation, it is only necessary to consider the interaction of the main channel's desired signal with the random variables in the auxiliary channels (Fig. 1). Thus, for analysis purposes, the adaptive weights of x_n, $n = 1, 2, \ldots, N-1$ are only a function of the desired signal s and the samples of x_n. Furthermore, as the number of independent samples goes to infinity, the auxiliary adaptive weights go to zero [1].

In Fig. 1, $\mathbf{s}$ represents the desired signal vector (or code), and $\mathbf{x}_n$, $n = 1, 2, \ldots, N-1$ represents the nth auxiliary random data vector of length K. The canceller used is the Gram-Schmidt (GS) algorithm [3]. We denote it by $GS_{K,N}$, where K is the number of samples per channel used to calculate the canceller weights and N is the number of input channels (main and auxiliaries).

The pulse compressor is essentially the matched filter for a given radar waveform. Most of the energy in the received radar waveform is compressed into a given single-range cell and, thus, the signal level can be increased significantly for detection purposes. However, some energy does leak into the sidelobes of the compressed pulse response, resulting in low gain in range cells outside of the given range cell. If a target or piece of clutter is large enough, it can break through and be detected in these range sidelobes, falsely indicating a target detection or masking a real target. Thus, it is highly desirable to maintain a low sidelobe response.

Let **r** equal the $2L - 1$ output vector of the pulse compressor. If no adaptive canceller is used then it is straightforward to show that

$$\mathbf{r} = S^t \mathbf{s}, \tag{1}$$

where

$$\mathbf{s} = (s_1, s_2, \ldots, s_L)^T,$$

$$S^T = \begin{bmatrix} s_L & 0 & 0 & \cdots & 0 \\ s_{L-1} & s_L & 0 & \cdots & 0 \\ s_{L-2} & s_{L-1} & s_L & \cdots & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ s_1 & s_2 & s_3 & \cdots & s_L \\ 0 & s_1 & s_2 & \cdots & s_{L-1} \\ 0 & 0 & s_1 & \cdots & s_{L-2} \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & \cdots & s_1 \end{bmatrix} \tag{2}$$

and T, t denotes transpose and complex conjugate transpose, respectively. S is a $L \times (2L - 1)$ matrix called the autocorrelation function (ACF) matrix of **s**.

We assume for this analysis that the GS canceller processes data in blocks of K data samples per channel. Thus, the desired signal vector may be spread across a number of sample blocks. To analyze the resultant GS canceller output for the desired signal, we must subdivide the L length code into M subcodes each of length K where the first and last subcodes may be partially zero-filled. Define an augmented vector $\mathbf{s}_{aug}$ such that

$$\mathbf{s}_{aug} = (\mathbf{s}^{(1)}, \mathbf{s}^{(2)}, \ldots, \mathbf{s}^{(M)})^T, \tag{3}$$

where each subcode vector $\mathbf{s}^{(m)}$, $m = 1, 2, \ldots, M$ is of length K. The leftmost elements of $\mathbf{s}^{(1)}$ and the rightmost elements of $\mathbf{s}^{(M)}$ may be partially zero filled. If K_1 and K_2 are the number of nonzero filled elements of $\mathbf{s}^{(1)}$ and $\mathbf{s}^{(M)}$, respectively, then for $M \geq 2$

$$L = (M - 2)\,K + K_1 + K_2. \tag{4}$$

For $M = 1$,

$$L = K_1. \tag{5}$$

For example, if $\mathbf{s} = (1, -1, -1, 1, 1, -1, -1, 1, 1)$ where $L = 9$, $K = 3$, then the input signal vectors into a $GS_{3,N}$ canceller could be $\mathbf{s}^{(1)} = (0, 0, 1)$, $\mathbf{s}^{(2)} = (-1, -1, 1)$, $\mathbf{s}^{(3)} = (1, -1, -1)$, and $\mathbf{s}^{(4)} = (1, 1, 0)$. Here, $M = 4$ and

$$\mathbf{s}_{aug} = (0, 0, 1, -1, -1, 1, 1, -1, -1, 1, 1, 0)^T .$$

Each subcode vector is input to the GS canceller one at a time. Let $\mathbf{s}^{(m)\prime}$ be the resultant output vector of the canceller for each input $\mathbf{s}^{(m)}$, $m = 1, 2, \ldots, M$, and $\mathbf{s}'_{aug}$ be the resultant augmented output vector. Thus,

$$\mathbf{s}'_{aug} = (\mathbf{s}^{(1)\prime}, \mathbf{s}^{(2)\prime}, \ldots, \mathbf{s}^{(M)\prime})^T \tag{6}$$

is the total result output vector of length KM. This resultant output vector is then inputted to the matched filter of the vector $\mathbf{s}$, or equivalently, $\mathbf{s}_{aug}$. If we set $\mathbf{r}'$ equal to the response of $\mathbf{s}'_{aug}$ match filtered with $\mathbf{s}_{aug}$ then

$$\mathbf{r}' = S^t_{aug}\mathbf{s}'_{aug} , \tag{7}$$

where S_{aug} is defined as the $KM \times (2KM - 1)$ augmented ACF matrix of s_{aug}.

The results and derivations presented are the same whether we use the augmented or non-augmented notation. Hence, we assume that all vectors are augmented and drop the augmented designation.

Vector $\mathbf{s}$ is often chosen so that the matched filter response has low sidelobes (i.e., $r(m) << r(0)$ for $m \neq 0$). However, if the desired signal is passed through a GS canceller structure, the desired signal vector is perturbed and degradations occur in the matched filter response. Examples of codes that have high compression ratios and low sidelobes are the Frank [5], Lewis and Kretschmer's P1-P4 [6], and shift register codes [7]. All of these codes have an ACF with all sidelobes well below the matched response. Figure 2, for example, shows the ACF of the 100-element Frank code.

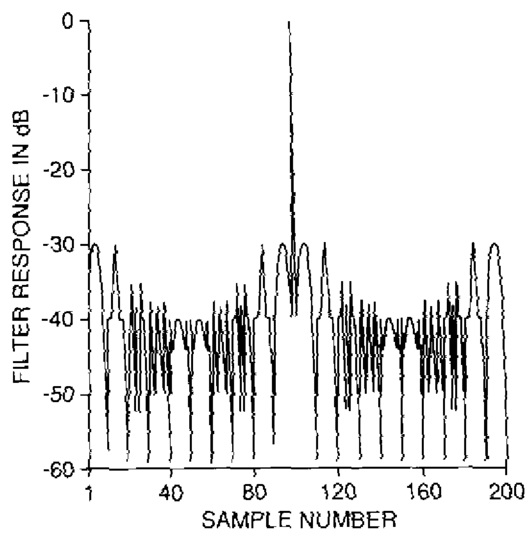

Fig. 2 — Frank code autocorrelation function $L = 100$, zero Doppler shift and no bandwidth limitation

Under the assumption that the signal vector is completely contained within a block of K samples from which the adaptive weights are calculated ($L \leq K$) [1], it was shown that the average pulse-compressed sidelobe level after adaptive cancellation is given by

$$SL_a(l) = \frac{K(K+1)A_{11}(K,N)}{(K-N+1)(K-N+2)} SL_q(l) + \frac{K(K+1)}{(K-N+1)(K-N+2)} \cdot A_{12}(K,N) \|\mathbf{s}_c(l)\|^2, \quad (8)$$

where

$SL_a(l)$ is average pulse-compressed sidelobe level after adaptive cancellation of the lth range sidelobe (sidelobes are numbered $\pm l$, $l = 1, 2, \cdots$; these can be related directly to the elements of $\mathbf{r}'$; for example, $l = \pm 1$ are the sidelobes adjacent the match point)

$SL_q(l)$ is quiescent pulse-compressed sidelobe level of the lth sidelobe ($K = \infty$ or equivalently no adaptive cancellation before pulse compression; these can be related directly to the elements of $\mathbf{r}$)

K is number of independent samples per channel used to calculate the adaptive canceller weights

N is number of channels (main and auxiliaries)

$\mathbf{s}_c(l)$ is $K - l$th column of the augmented ACF matrix, S_{aug}, $l \neq K$, and

$\|\mathbf{s}_c(l)\|^2 = \mathbf{s}_c^t(l)\mathbf{s}_c(l)$.

We note that $SL_a(l)$ and $SL_q(l)$ are normalized to the mainlobe pulse compression gain (adapted or quiescent, respectively) that is set equal to one or 0 dB.

The scalars $A_{11}(K,N)$ and $A_{12}(K,N)$ are computed as follows. Consider the two parallel adaptive cancellers shown in Fig. 3. Define

$\mathbf{u}_0, \mathbf{v}_0$ = arbitrary K-length main channel input vectors,

$\mathbf{u}_N, \mathbf{v}_N$ = K-length main channel output vectors,

$\mathbf{x}_n = (x_n(1), x_n(2), \ldots, x_n(K))^T$, $n = 1, 2, \ldots, N-1$, K-length random data vector of the nth auxiliary channel.

The elements of $\mathbf{x}_n$, $n = 1, 2, \ldots, N-1$ are assumed to have the following characteristics:

1. $x_n(k)$, $n = 1, \ldots, N-1$, $k = 1, \ldots, K$ are identically distributed circular Gaussian complex random variables (r.v.)

2. $E\{x_n(k)\} = 0$, $E\{|x_n(k)|^2\} = 1$, where $E\{\cdot\}$ denotes expectation and $|\cdot|$ denotes magnitude.

3. $E\{x_{n_1}(k_1)x^*_{n_2}(k_2)\} = 0$ unless $n_1 = n_2$ and $k_1 = k_2$.

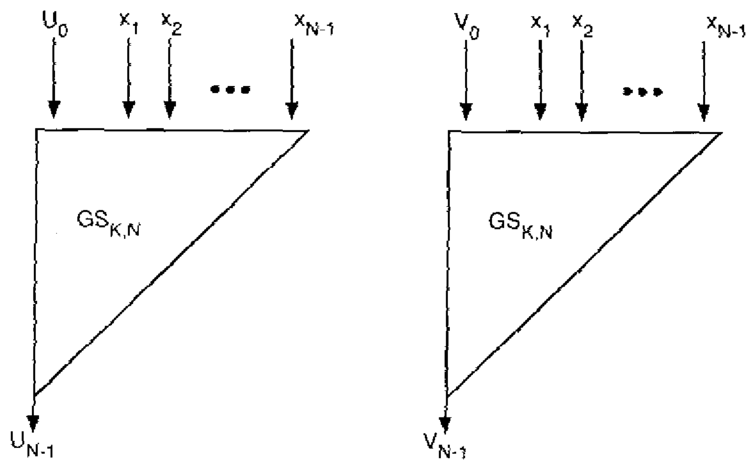

Fig. 3 — Parallel N-input GS cancellers

Define

$$a_n = 1 - \frac{2}{K-n} + \frac{1}{(K-n)(K-n+1)}, \; n = 0, 1, \ldots, N-2, \text{ and} \tag{9}$$

$$b_n = \frac{1}{(K-n)(K-n+1)}. \tag{10}$$

It is shown in Ref. 1 that

$$\begin{bmatrix} E\{|\mathbf{u}_{N-1}^t \mathbf{v}_{N-1}|^2\} \\ E\{\|\mathbf{u}_{N-1}\|^2 \|\mathbf{v}_{N-1}\|^2\} \end{bmatrix} = \begin{bmatrix} A_{11}(K,N) & A_{12}(K,N) \\ A_{21}(K,N) & A_{22}(K,N) \end{bmatrix} \begin{bmatrix} |\mathbf{u}_0^t \mathbf{v}_0|^2 \\ \|\mathbf{u}_0\|^2 \|\mathbf{v}_0\|^2 \end{bmatrix} \tag{11}$$

where

$$\begin{bmatrix} A_{11}(K,N) & A_{12}(K,N) \\ A_{21}(K,N) & A_{22}(K,N) \end{bmatrix} = \prod_{n=0}^{N-2} \begin{bmatrix} a_n & b_n \\ b_n & a_n \end{bmatrix}. \tag{12}$$

Equations (11) and (12) resulted from solving the following coupled recursive relationships that were derived in Ref. 1:

$$E\{|\mathbf{u}_{n+1}^t \mathbf{v}_{n+1}|^2\} = E\{|\mathbf{u}_n^t \mathbf{v}_n|^2\} \left[1 - \frac{2}{K-n} + \frac{1}{(K-n)(K-n+1)}\right]$$

$$+ E\{\|\mathbf{u}_n\|^2 \|\mathbf{v}_n\|^2\} \left[\frac{1}{(K-n)(K-n+1)}\right], \tag{13}$$

$$E\{\|\mathbf{u}_{n+1}\|^2\|\mathbf{v}_{n+1}\|^2\} = E\{|\mathbf{u}_n^t\mathbf{v}_n|^2\}\left[\frac{1}{(K-n)(K-n+1)}\right]$$

$$+ E\{\|\mathbf{u}_n\|^2\|\mathbf{v}_n\|^2\}\left[1 - \frac{2}{K-n} + \frac{1}{(K-n)(K-n+1)}\right], \tag{14}$$

where $n = 0, 1, \ldots, N-1$.

Reference 4 derived a good approximation of $SL_a(l)$. It was shown that good approximations of $A_{11}(K,N)$ and $A_{12}(K,N)$ are given by

$$A_{11}(K,N) \doteq \left[1 - \frac{N-1}{K}\right]^2 \tag{15}$$

and

$$A_{12}(K,N) \doteq \frac{(K-N+2)(N-1)}{K^2(K+1)}. \tag{16}$$

In Ref. 1 it was shown that

$$A_{11}(K,N) + A_{12}(K,N) = 1 - \frac{2(N-1)}{K} + \frac{N(N-1)}{K(K+1)}, \tag{17}$$

$$A_{11}(K,N) = A_{22}(K,N), \tag{18}$$

and

$$A_{12}(K,N) = A_{21}(K,N). \tag{19}$$

3. SIDELOBE DEGRADATION: SIGNAL SEGMENTATION

In this section, we consider desired signals that are segmented and processed through the GS canceller. We assume that the set of GS weights computed for each $K \times N$ data block is statistically independent from block-to-block. Let the desired signal's input and output vectors $\mathbf{s}_0$ and $\mathbf{s}_{N-1}$ of a canceller be segmented into M vectors such that

$$\mathbf{s}_0 = (\mathbf{s}_0^{(1)}, s_0^{(2)}, \ldots, \mathbf{s}_0^{(M)})^T, \tag{20}$$

and

$$\mathbf{s}_{N-1} = (\mathbf{s}_{N-1}^{(1)}, \mathbf{s}_{N-1}^{(2)}, \ldots, \mathbf{s}_{N-1}^{(M)})^T. \tag{21}$$

Note that we have set $s_0 = s$ and $s_{N-1} = s'$. Each $s_0^{(m)}$ and $s_{N-1}^{(m)}$, $m = 1,2, \ldots, M$ is of length K where the end vectors may be augmented by zeros to fill out the K-length vector. Note that s_0 or s_{N-1} can be considered augmented so that their length is KM, and that s_0 is normalized so that $\|s_0\|^2 = 1$. Similarly, let s_c be a column of the augmented ACF matrix defined as

$$s_c = (s_c^{(1)}, s_c^{(2)}, \ldots, s_c^{(M)})^T. \tag{22}$$

Thus, an expression representing an output r of the matched filter can be given by

$$r = s_c^t s_{N-1}, \tag{23}$$

and the average adaptive pulse compression level associated with s_c is given by

$$SL_a = \frac{E\{|s_c^t s_{N-1}|^2\}}{E\{|s_{N-1}^t s_{N-1}|^2\}}. \tag{24}$$

We will derive good approximations of the numerator and denominator of this expression. The above expectations are a function of two kinds of randomness: the first is the auxiliary channel data and the second is where in time the code s begins with respect to the first segment. We evaluate the above expectations first with respect to the auxiliary channel data and denote this expectation by $E_x\{\cdot\}$.

To this end, $E_x\{|s_0^t s_{N-1}|^2\}$ is decomposed into terms dependent on the individual segments as

$$\begin{aligned} E_x\{|s_c^t s_{N-1}|^2\} &= E_x\left\{\left|\sum_{m=1}^{M} s_c^{(m)t} s_{N-1}^{(m)}\right|^2\right\} \\ &= \sum_{m=1}^{M} E_x\{|s_c^{(m)t} s_{N-1}^{(m)}|^2\} - \sum_{n=1}^{M} |E_x\{s_c^{(m)t} s_{N-1}^{(m)}\}|^2 \\ &+ \sum_{m_1=1}^{M}\sum_{m_2=1}^{M} E_x\{s_c^{(m_1)t} s_{N-1}^{(m_1)}\}\, E_x^*\{s_c^{(m_2)t} s_{N-1}^{(m_2)}\}. \end{aligned} \tag{25}$$

We used the fact that the auxiliary random variables are assumed independent from one batch of K, N-length sample vectors to the next to separate the expectations in this double summation given in Eq. (25).

It is shown in Appendix A that

$$E_x\{s_c^{(m)t} s_{N-1}^{(m)}\} = \left(1 - \frac{N-1}{K}\right) s_c^{(m)t} s_0^{(m)}. \tag{26}$$

Thus, Eq. (25) simplifies to

$$E_x\{|\mathbf{s}_c^t\mathbf{s}_{N-1}|^2\} = \sum_{m=1}^{M} E_x\{|\mathbf{s}_c^{(m)t}\mathbf{s}_{N-1}^{(m)}|^2\} - \left(1 - \frac{N-1}{K}\right)^2 \sum_{m=1}^{M} |\mathbf{s}_c^{(m)t}\mathbf{s}_0^{(m)}|^2$$

$$+ \left(1 - \frac{N-1}{K}\right)^2 \sum_{m_1=1}^{M}\sum_{m_2=1}^{M} [\mathbf{s}_c^{(m_1)t}\mathbf{s}_0^{(m_1)}]\,[\mathbf{s}_c^{(m_2)t}\mathbf{s}_0^{(m_2)}]^*. \tag{27}$$

However, it can be shown that

$$\sum_{m_1=1}^{M}\sum_{m_2=1}^{M} [\mathbf{s}_c^{(m_1)t}\mathbf{s}_0^{(m_1)}]\,[\mathbf{s}_c^{(m_2)t}\mathbf{s}_0^{(m_2)}]^* = |\mathbf{s}_c^t\mathbf{s}_0|^2. \tag{28}$$

Using Eq. (11), it is straightforward to show that

$$\sum_{m=1}^{M} E_x\{|\mathbf{s}_c^{(m)t}\mathbf{s}_{N-1}^{(m)}|^2\} = A_{11}(K,N) \sum_{m=1}^{M} |\mathbf{s}_c^{(m)t}\mathbf{s}_0^{(m)}|^2$$

$$+ A_{12}(K,N) \sum_{m=1}^{M} \|\mathbf{s}_c^{(m)}\|^2 \cdot \|\mathbf{s}_0^{(m)}\|^2. \tag{29}$$

Substituting Eqs. (28) and (29) into Eq. (27) and taking the total expectation over all random variables results in

$$E\{|\mathbf{s}_c^t\mathbf{s}_{N-1}|^2\} = \left[A_{11}(K,N) - \left(1 - \frac{N-1}{K}\right)^2\right] E\left\{\sum_{m=1}^{M} |\mathbf{s}_c^{(m)t}\mathbf{s}_0^{(m)}|^2\right\}$$

$$+ \left(1 - \frac{N-1}{K}\right)^2 |\mathbf{s}_c^t\mathbf{s}_0|^2$$

$$+ A_{12}(K,N)\, E\left\{\sum_{m=1}^{M} \|\mathbf{s}_c^{(m)}\|^2 \|\mathbf{s}_0^{(m)}\|^2\right\}. \tag{30}$$

At the match point, $\mathbf{s}_c = \mathbf{s}_0$ and the summations seen in Eq. (30) are equal so that

$$E\{|\mathbf{s}_{N-1}^t\mathbf{s}_{N-1}|^2 = \left(1 - \frac{N-1}{K}\right)^2 |\mathbf{s}_0^t\mathbf{s}_0|^2$$

$$+ \left[A_{11}(K,N) + A_{12}(K,N) - \left(1 - \frac{N-1}{K}\right)^2\right] E\left\{\sum_{m=1}^{M} \|\mathbf{s}_0^{(m)}\|^4\right\}. \tag{31}$$

However, in lieu of Eq. (17) and $\mathbf{s}_0^t \mathbf{s}_0 = 1$,

$$E\{|\mathbf{s}_{N-1}^t \mathbf{s}_{N-1}|^2\} = \left(1 - \frac{N-1}{K}\right)^2 + \frac{(K-N+1)(N-1)}{K^2(K+1)} E\left\{\sum_{m=1}^{M} \|\mathbf{s}_0^{(m)}\|^4\right\}. \tag{32}$$

The expectations seen in Eqs. (30) and (32) are dependent on the signal code. For a signal code that has uniform amplitude elements, it is shown in Appendix B that

$$E\left\{\sum_{m=1}^{M} \|\mathbf{s}_0^{(m)}\|^4\right\} = \begin{cases} \dfrac{K}{L} - \dfrac{1}{3}\left(\dfrac{K}{L}\right)^2 + \dfrac{1}{3L^2}, & L > K \\[2ex] 1 - \dfrac{L}{3K} + \dfrac{1}{3KL}, & L \leq K. \end{cases} \tag{33}$$

Note that if either expression given in Eq. (33) is substituted into Eq. (32), the second term of Eq. (32) is small with respect to the first term. Thus, a good approximation of $E\{|\mathbf{s}_{N-1}^t \mathbf{s}_{N-1}|^2\}$ is given by

$$E\{|\mathbf{s}_{N-1}^t \mathbf{s}_{N-1}|^2\} \doteq \left(1 - \frac{N-1}{K}\right)^2. \tag{34}$$

For uniform amplitude elements, the expectations seen in Eq. (30) may be upper-bounded by $E\left\{\sum_{m=1}^{M} \|\mathbf{s}_0^{(m)}\|^4\right\}$. In fact, this upper bound is a good approximation of the second expectation for the near-in range sidelobes (small l). It may not be a good approximation of the first expectation which is expected to be much smaller than the upper bound. Note also the form of the approximation of $A_{11}(K,N)$ given by Eq. (15). As a result, it can be shown that the first term in Eq. (30) is small with respect to the sum of the second and third terms of the equation for the near-in range sidelobe case. Hence, we delete this term from our approximation.

Close upper bounds to $A_{11}(K, N)$ and $A_{12}(K, N)$ are given by Eqs. (15) and (16), respectively. If these are substituted into Eq. (30), then for the near-in range sidelobes

$$E\{|\mathbf{s}_c^t \mathbf{s}_{N-1}|^2\} \doteq \left(1 - \frac{N-1}{K}\right)^2 |\mathbf{s}_c^t \mathbf{s}_0|^2 + \frac{(K-N+2)(N-1)}{K^2(K+1)} E\left\{\sum_{m=1}^{M} \|\mathbf{s}_0^{(m)}\|^4\right\}. \tag{35}$$

Dividing by $E\{|\mathbf{s}_{N-1}^t \mathbf{s}_{N-1}|^2\}$ as given in Eq. (34) results in

$$SL_a \doteq |\mathbf{s}_c^t \mathbf{s}_0|^2 + \frac{(K-N+2)(N-1)}{(K-N+1)^2(K+1)} E\left\{\sum_{m=1}^{M} \|\mathbf{s}_0^{(m)}\|^4\right\}. \tag{36}$$

We approximate

$$\frac{(K - N + 2)(N - 1)}{(K - N + 1)^2(K + 1)} \doteq \frac{N - 1}{(K - N + 1)K}. \tag{37}$$

Furthermore, $E\left\{\sum_{m=1}^{M} \|s_0^{(m)}\|^4\right\}$ can be approximated by a close upper-bound using Eq. (33). This is

$$E\left\{\sum_{m=1}^{M} \|s_0^{(m)}\|^4\right\} \doteq \begin{cases} \frac{K}{L}, & L > K \\ 1, & L \leq K. \end{cases} \tag{38}$$

Thus, substituting Eqs. (38) and (37) into Eq. (36) results in

$$SL_a(l) \doteq SL_q(l) + \frac{N - 1}{(K - N + 1)L}, \quad L > K, \tag{39}$$

and

$$SL_a(l) \doteq SL_q(l) + \frac{N - 1}{(K - N + 1)K}, \quad L \leq K, \tag{40}$$

where $SL_q(l)$ and $SL_a(l)$ were previously defined and l is small (near-in range sidelobe case).

We note that if the above approximations do not suffice in some cases (for example, $l >> 1$), one can always use the exact formulation of $SL_a(l)$ given by the ratio of the expressions given by Eqs. (30) and (31).

4. RESULTS

In this section, we calculate the number of independent samples per channel $K_{3\text{dB}}$ necessary for the average transient sidelobe level of the maximum quiescent sidelobe level defined by $\overline{SL}_q$ to be within 3 dB of $\overline{SL}_q$. We assume that the maximum quiescent sidelobe level occurs in the near-in range sidelobes (which is normally the case), so that the approximations given in the previous section are valid. We use this as a performance measure of convergence. If the average adaptive sidelobe level $\overline{SL}_a$ were plotted vs K, it would be found that $\overline{SL}_a$ monotonically decreases with K and is asymptotic with $\overline{SL}_q$ as $K \to \infty$. The $K = K_{3\text{dB}}$ point is representative of the "knee" of this curve (where $\overline{SL}_a$ decreases slowly with increases in K).

To find $K_{3\text{dB}}$, the following two equations (which result from Eqs. (39) and (40), respectively) are solved for $K_{3\text{dB}}$:

$$2\,\overline{SL}_q = \overline{SL}_q + \frac{N - 1}{(K_{3\text{dB}} - N + 1)L}, \quad L > K_{3\text{dB}} \tag{41}$$

and

$$2\,\overline{SL}_q = \overline{SL}_q + \frac{N-1}{(K_{3\mathrm{dB}} - N + 1)K}, \quad L \le K_{3\mathrm{dB}}. \tag{42}$$

Solving Eqs. (41) or (42) for $K_{3\mathrm{dB}}$ results in

$$K_{3\mathrm{dB}} = \left[1 + \frac{1}{L \cdot \overline{SL}_q}\right](N-1), \quad L > K_{3\mathrm{dB}} \tag{43}$$

and

$$K_{3\mathrm{dB}} = \left[\frac{1}{2} + \sqrt{\frac{1}{2} + \frac{1}{(N-1)\cdot\overline{SL}_q}}\right](N-1), \quad L \le K_{3\mathrm{dB}}. \tag{44}$$

Note that the solution for $K_{3\mathrm{dB}}$ depends on this solution satisfying the inequalities given with each of the above solutions. If both inequalities are satisfied, then obviously the first solution given by Eq. (43) is chosen because this solution is less than the solution given by Eq. (44). Appendix C shows that at least one of the solutions given above is valid.

It is also shown in Appendix C that Eq. (43) is the solution for $K_{3\mathrm{dB}}$ if

$$\overline{SL}_q > \frac{N_{aux}}{L(L - N_{aux})} \quad \text{and} \quad L > N_{aux} \tag{45}$$

where $N_{aux} = N - 1$. If either condition given by Eq. (45) is not true, then the solution given by Eq. (44) is valid.

We can rewrite Eqs. (43) and (44) as

$$\frac{K_{3\mathrm{dB}}}{N_{aux}} = 1 + \frac{1}{L \cdot \overline{SL}_q}, \quad L > K_{3\mathrm{dB}}, \tag{46}$$

and

$$\frac{K_{3\mathrm{dB}}}{N_{aux}} = \frac{1}{2} + \sqrt{\frac{1}{2} + \frac{1}{N_{aux}\cdot\overline{SL}_q}}, \quad L \le K_{3\mathrm{dB}}. \tag{47}$$

In Fig. 4, $K_{3\mathrm{dB}} / N_{aux}$ is plotted vs $L \cdot \overline{SL}_q$. Again, this solution is valid if Eq. (45) holds. In Fig. 5, $K_{3\mathrm{dB}} / N_{aux}$ is plotted vs $N_{aux} \cdot \overline{SL}_q$. This solution is valid if Eq. (45) does not hold.

For example, let $\overline{SL}_q = 10^{-3}$ (or -30 dB), $N_{aux} = 10$, and $L = 100$. In this case, Eq. (45) does not hold, so we use Fig. 5 to find $K_{3\mathrm{dB}} / N_{aux}$, which approximately equals 11. As another example, let $\overline{SL}_q = 10^{-2}$ (or -20 dB), $N_{aux} = 10$, and $L = 100$. In this case, the conditions given by Eq. (45) hold. We use Fig. 4 to find $K_{3\mathrm{dB}} / N_{aux} = 2$.

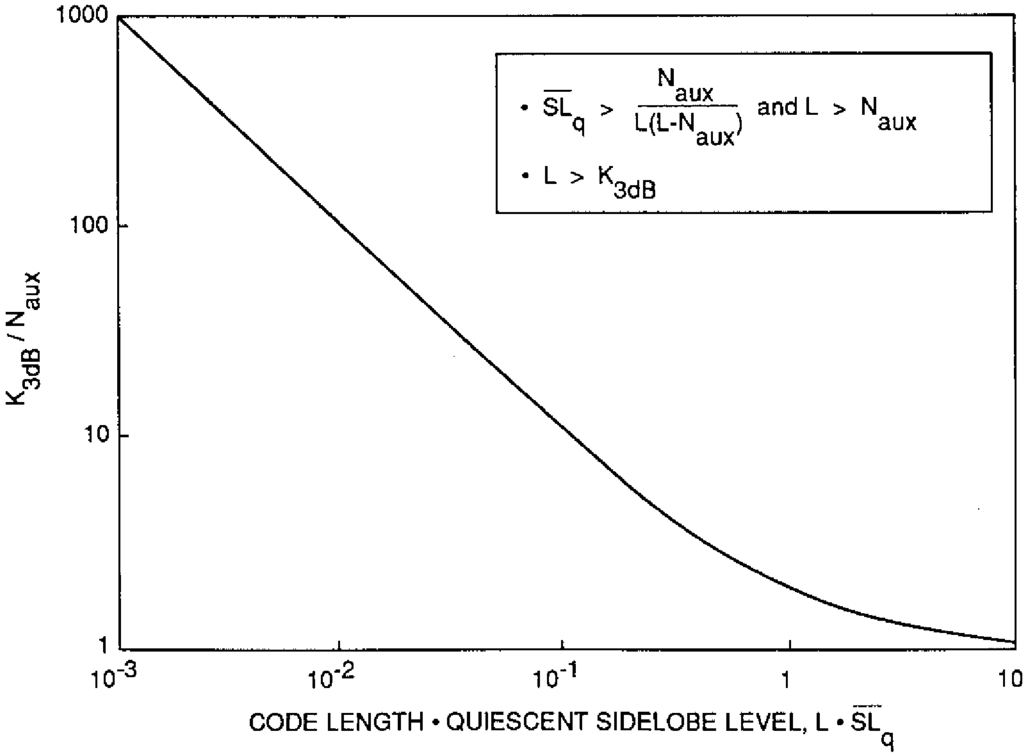

Fig. 4 — $K_{3\,dB}$ vs $L \cdot \overline{SL}_q$

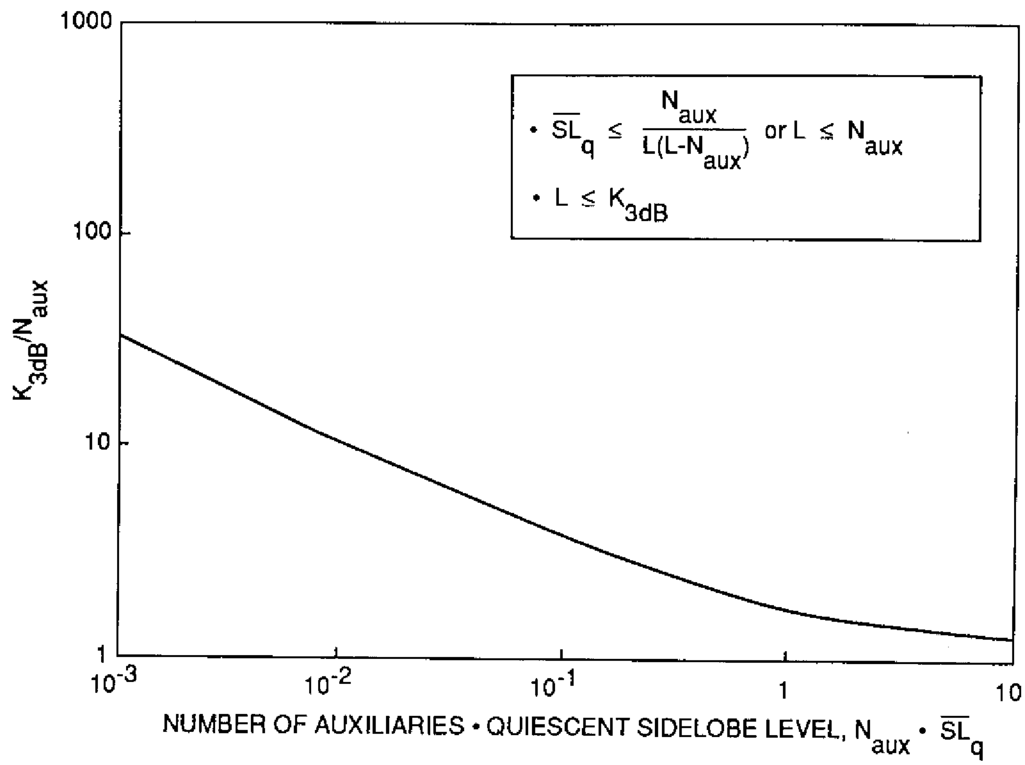

Fig. 5 — $K_{3\,dB}/N_{aux}$ vs $N_{aux} \cdot \overline{SL}_q$

By examining the solutions for K_{3dB} / N_{aux} given in Figs. 4 and 5, we make the observations that for $K_{3dB} / N_{aux} \approx 2$, either $N_{aux} \cdot SL_q$ or $L \cdot SL_q$ must be approximately equal to one.

As noted in Refs. 1 and 4, the preceding analysis of pulse compression and canceller interactions can also be applied to quantify the canceller degradation caused by the presence of a desired signal in the samples used to calculate the adaptive canceller weights. Set

$$\Delta SL_a(K,N) = \begin{cases} \dfrac{N-1}{(K-N+1)L}, \; L > K & (48) \\[2ex] \dfrac{N-1}{(K-N+1)K}, \; L \leq K. & (49) \end{cases}$$

If the desired signal has the power σ_s^2, after pulse compression, then the maximum of the average power residue caused by signal in the $K-1$ range bins not containing the signal can be shown from our analysis to approximately equal $\sigma_s^2 \Delta SL_a(K,N)$ plus possibly the signal power caused by the quiescent compressed sidelobes. Let $\sigma_{\min}^2$ be the quiescent output noise power level of the canceller (no desired signal). Define

$$\delta = \frac{\sigma_s^2}{\sigma_{\min}^2} \Delta SL_a(K,N). \tag{50}$$

If $\delta > 1$, then the signal induced power will be greater than the quiescent output noise power of the canceller. Hence, it is desirable to choose the number of independent input samples K so that $\delta \leq 1$. Set $K = K_0$ for when $\delta = 1$. It is straightforward to show that

$$K_0 = \begin{cases} \dfrac{N-1}{L} \left(\dfrac{\sigma_s^2}{\sigma_{\min}^2} + L \right), \; L > K_0 & (51) \\[2ex] \dfrac{N-1}{2} + \sqrt{\left(\dfrac{N-1}{2} \right)^2 + (N-1) \dfrac{\sigma_s^2}{\sigma_{min}^2}}, \; L \leq K_0. & (52) \end{cases}$$

We note that $\sigma_s^2 / \sigma_{\min}^2$ equals the output signal-to-noise power ratio $(S/N)_{\text{out}}$ of the adaptive canceller. Thus, Eqs. (51) and (52) reduce to

$$K_0 = \begin{cases} \dfrac{N-1}{L} \left(\left(\dfrac{S}{N} \right)_{\text{out}} + L \right), \; L > K_0 & (53) \\[2ex] \dfrac{N-1}{2} + \sqrt{\left(\dfrac{N-1}{2} \right)^2 + (N-1) \left(\dfrac{S}{N} \right)_{\text{out}}}, \; L \leq K_0. & (54) \end{cases}$$

For the radar designer there is the choice of where to put the pulse compressor: before or after the canceller. A disadvantage of placing it before the canceller is that a pulse compressor must be placed in each antenna channel (main and auxiliaries) to maintain channel match (mismatched channels degrade canceller performance). Another disadvantage is that the pulse compressor must have the dynamic range of the interference (possibly clutter and jamming) that has yet to be cancelled. These disadvantages do not occur if the compressor is placed after the canceller. However, as we have seen, a disadvantage of placing the compressor after the canceller is that the range sidelobes of the compressed pulse increase because a finite number of samples are used to compute the canceller weights.

It should be pointed out, however, that this effect also occurs if the desired waveform is compressed before the canceller. In this case, it was shown [4] that the ratio of signal-induced power to the quiescent-noise power level is given by

$$\delta(\text{pc before}) = \frac{\sigma_s^2}{\sigma_{\min}^2} \frac{N-1}{(K-N+1)K}, \quad L \leq K. \tag{55}$$

Note that this is identical for the expression of δ (pc after) if $L \leq K$ (see Eqs. (49) and (50)). Hence, for waveform codes that have length less than the processing batch length ($L \leq K$), it is desirable to pulse-compress after cancellation.

However, for $L > K$, the issue is not so clear-cut. Even though δ (pc after) $<$ δ (pc before) for $L > K$, we must remember that the signal induces noise over $KM \approx L$ samples of output data. Thus, for $M \geq 2$, more samples are affected by degradation caused by performing pulse compression after cancellation. As a result, for $L > K$, a trade-off study is necessary to determine whether one does pulse compression before or after cancellation. The cost function associated with this trade-off study will depend directly on the user's system parameters and needs.

One final note. For some applications, the matched filter is replaced by a filtering scheme whereby the range sidelobes are reduced at the expense of signal gain at the match point. However, the results derived in this report are also valid for the use of any filter other than the matched filter $\mathbf{s}_0$. We could replace the $\mathbf{s}_0$ seen in the "matched filter" block in Fig. 1 with a general weighting function given by the L length vector $\mathbf{a}$ with elements $a_0, a_1, \ldots, a_{L-1}$. In our analysis, we would replace the S matrix defined by Eq. (2) with an A matrix whose elements are given by replacing the s s with a s in Eq. (2). The vector $\mathbf{s}_c$ then would be taken to be any column in A and the analysis follows as given.

5. SUMMARY

This report has presented an exact expression for the perturbed range sidelobe level of a compressed pulse that has been preprocessed through an adaptive canceller. This result is a generalization of Refs. 1 and 4 where the signal was assumed to be completely contained within the canceller's processing batch. In this report, we allow the signal to extend over an arbitrary number of canceller processing batches. A good approximate expression was also obtained for evaluating the perturbed range sidelobe level. The number of independent samples per channel (main and auxiliaries) necessary so that the average adaptive range sidelobe level is within 3 dB of the quiescent range sidelobe level was derived. Furthermore, the same analysis was used to predict the canceller noise power level that is induced by having the desired signal present in the canceller weight calculation. Placement of the pulse compressor before or after the canceller was also considered. It was shown that if the desired waveform's code length L is less than or equal to the canceller's processing batch width K, it is desirable to place the pulse compression after the adaptive canceller. If $L > K$, the issue is not so clear-cut, and a trade-off study is necessary.

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Appendix A
DERIVATION OF EQ. (26)

It is shown in Ref. 8 that if $\mathbf{x}_0$ is the main channel K-length vector, then the resultant output vector $\mathbf{y}_0$ through a $GS_{K,N}$ canceller can be represented as

$$\mathbf{y}_0 = Gx_0, \tag{A1}$$

where G is the GS complementary projection matrix and is given by

$$G = I_K - \frac{\mathbf{z}_1\mathbf{z}_1^t}{\mathbf{z}_1^t\mathbf{z}_1} - \frac{\mathbf{z}_2\mathbf{z}_2^t}{\mathbf{z}_2^t\mathbf{z}_2} - \cdots - \frac{\mathbf{z}_{N-1}\mathbf{z}_{N-1}^t}{\mathbf{z}_{N-1}^t\mathbf{z}_{N-1}}. \tag{A2}$$

In Eq. (A2), I_K is the $K \times K$ identity matrix and $\mathbf{z}_n$, $n = 1, 2, \ldots, N - 1$ is a set of orthogonal vectors that is an orthogonal basis for the original auxiliary K-length input vectors. If we assume that the input samples are zero mean independent, identically distributed r.v.s, it is straightforward to show

$$E\left\{\frac{\mathbf{z}_n\mathbf{z}_n^t}{\mathbf{z}_n^t\mathbf{z}_n}\right\} = \frac{1}{K} I_K \quad \text{for all } n. \tag{A3}$$

Thus

$$E\{G\} = \left(1 - \frac{N-1}{K}\right) I_K. \tag{A4}$$

Thus, for arbitrary K-length vector $\mathbf{u}$

$$\begin{aligned} E\{\mathbf{u}^t\mathbf{y}_0\} &= E\{\mathbf{u}^t G\mathbf{x}_0\} \\ &= \mathbf{u}^t E\{G\}\mathbf{x}_0 \\ &= \left(1 - \frac{N-1}{K}\right) \mathbf{u}^t\mathbf{x}_0. \end{aligned} \tag{A5}$$

Appendix B

DERIVATION OF EQ. (33)

In this appendix we derive the expected value of $\sum_{m=1}^{M} \|s_0\|^4$ assuming the code element amplitudes are uniform. For any M we can write

$$\beta = E\left\{\sum_{m=1}^{M} \|s_0^{(m)}\|^4\right\} = \begin{cases} E\left\{(M-2)\dfrac{K^2}{L^2} + \dfrac{K_1^2}{L^2} + \dfrac{K_2^2}{L^2}\right\}, & M \geq 2 \\ 1 & M = 1 \end{cases} \tag{B1}$$

We distinguish between the two cases, $L > K$ and $L \leq K$.

Case 1: $L > K$

For this case $M \geq 2$. Thus from Eq. (B1), if we find expressions for $E\{M\}$, $E\{K_1^2\}$ and $E\{K_2^2\}$, we can find β. Now

$$L = (M-2)K + K_1 + K_2. \tag{B2}$$

Therefore

$$E\{M-2\} = \frac{1}{K}(L - E\{K_1\} - E\{K_2\}). \tag{B3}$$

Thus, if we can obtain $E\{K_1\}$ and $E\{K_2\}$, then $E\{M-2\}$ can be found by using Eq. (B3). By symmetry

$$E\{K_1\} = E\{K_2\}, \quad E\{K_1^2\} = E\{K_2^2\}. \tag{B4}$$

Let $\mathrm{Prob}_{K_1}\{\nu\}$ be the probability that $K_1 = \nu$, where ν can range from 1, 2, ..., K. The starting position of the code within the first code segment is uniformly distributed, so that

$$\mathrm{Prob}_{K_1}\{\nu\} = \frac{1}{K}. \tag{B5}$$

Thus

$$E\{K_1\} = \sum_{k=1}^{K} k \ \text{Prob}_{K_1}\{k\} = \sum_{k=1}^{K} \frac{k}{K} = \frac{1}{2}(K + 1), \tag{B6}$$

$$E\{K_1^2\} = \sum_{k=1}^{K} k^2 \ \text{Prob}_{K_1}\{k\} = \sum_{k=1}^{K} \frac{k^2}{K} = \frac{1}{6}(K + 1)(2K + 1). \tag{B7}$$

Using Eqs. (B6) and (B7) in Eq. (B3),

$$E\{M - 2\} = \frac{L - 1}{K} - 1. \tag{B8}$$

Using Eqs. (B8), (B7), and Eq. (B4), we see that

$$\beta = \frac{K^2}{L^2} \left(\frac{L - 1}{K} - 1 \right) + \frac{1}{3} \frac{(K + 1)(2K + 1)}{L^2}, \tag{B9}$$

$$= \frac{K}{L} - \frac{1}{3} \left(\frac{K}{L} \right)^2 + \frac{1}{3L^2}.$$

Case 2: $L \leq K$

For this case $M = 1$ or 2. For $L \leq K$, we start by computing two probabilities: the probability that $K_1 = L$ (or equivalently, $M = 1$) denoted by $\text{Prob}\{K_1 = L\}$ and the probability that $K_1 = \nu$ where ν is a positive integer less than L (or equivalently, $M = 2$) denoted by $\text{Prob}\{\nu \text{ and } \nu < L\}$. It is straightforward to show that

$$\text{Prob}\{K_1 = L\} = \frac{K - L + 1}{K}, \tag{B10}$$

and

$$\text{Prob}\{\nu \text{ and } \nu < L\} = \text{Prob}\{\nu \mid \nu < L\} \cdot \text{Prob}\{\nu < L\}$$

$$= \frac{1}{L - 1} \cdot \frac{L - 1}{K}$$

$$= \frac{1}{K}. \tag{B11}$$

Thus

$$E\{K_1^2\} = L^2 \cdot \frac{K - L + 1}{K} + \sum_{k=1}^{L-1} k^2 \frac{1}{K}$$

$$= L^2 \left(\frac{K - L + 1}{K} \right) + \frac{1}{K} \frac{1}{6} (L - 1)L (2L - 1). \tag{B12}$$

Note for $L \leq K$ that Eq. (B4) does not hold. Let $\mathrm{Prob}_{K_2}\{v\}$ be the probability that $K_2 = v$ where $v = 0, 1, \ldots, L - 1$. It is apparent that

$$\mathrm{Prob}_{K_2}\{0\} = \mathrm{Prob}\{K_1 = L\} = \frac{K - L + 1}{K}, \tag{B13}$$

and for $v > 0$,

$$\mathrm{Prob}_{K_2}\{v\} = \mathrm{Prob}\{v \mid v > 0\} \cdot \mathrm{Prob}\{v > 0\},$$

$$= \frac{1}{L - 1} \cdot \frac{L - 1}{K} = \frac{1}{K}. \tag{B14}$$

Thus

$$E\{K_2^2\} = \sum_{k=1}^{L-1} k^2 \frac{1}{K} = \frac{1}{K} \frac{1}{6}(L - 1)L\,(2L - 1). \tag{B15}$$

Using Eqs. (B12) and (B15), it can be shown that

$$\beta = E\left\{\frac{K_1^2 + K_2^2}{L^2}\right\}$$

$$= \frac{K - L + 1}{K} + \frac{1}{3} \frac{(L - 1)L\,(2L - 1)}{KL^2}$$

$$= 1 - \frac{1}{3} \frac{L}{K} + \frac{1}{3KL}. \tag{B16}$$

Appendix C
CONDITIONS FOR CONVERGENCE SOLUTIONS

If the solution given by Eq. (43) is valid, then L must be greater than $K_{3\mathrm{dB}}$. Thus

$$K_{3\mathrm{dB}} = N_{aux} \left(1 + \frac{1}{L \cdot \overline{SL}_q} \right) < L. \tag{C1}$$

Reducing Eq. (C1) further

$$\frac{1}{L \cdot \overline{SL}_q} < \frac{L}{N_{aux}} - 1. \tag{C2}$$

Now if $L \leq N_{aux}$, Eq. (C2) does not hold. Thus for Eq. (43) to be a valid solution, $L > N_{aux}$. Equation (C2) can be further simplified to show that

$$\overline{SL}_q > \frac{N_{aux}}{L(L - N_{\mathrm{aux}})}. \tag{C3}$$

We show that one of the solutions given by Eqs. (43) or (44) is valid. We do this by showing that if no solution exists, a contradiction results. Assume no solution exists. Thus

$$K_{3\mathrm{dB}} = N_{aux} \left(1 + \frac{1}{L \cdot \overline{SL}_q} \right) \geq L, \tag{C4}$$

and by using Eq. (44) with $N_{aux} = N - 1$,

$$K_{3\mathrm{dB}} = N_{aux} \left[\frac{1}{2} + \sqrt{\frac{1}{2} + \frac{1}{N_{aux} \cdot \overline{SL}_q}} \right] < L. \tag{C5}$$

Using Eq. (C5), we can show that $L/N_{aux} > 1$.

Solving for $\overline{SL}_q$ in Eq. (C4) results in

$$\overline{SL}_q \leq \frac{1}{L\left(\dfrac{L}{N_{aux}} - 1 \right)}. \tag{C6}$$

and solving for $\overline{SL}_q$ in Eq. (C5) results in

$$\frac{1}{N_{aux}} \frac{1}{\left(\dfrac{L}{N_{aux}} - \dfrac{1}{2}\right)^2 - \dfrac{1}{2}} < \overline{SL}_q. \tag{C7}$$

Thus, Eqs. (C6) and (C7) imply that

$$\frac{1}{L\left(\dfrac{L}{N_{aux}} - 1\right)} > \frac{1}{N_{aux}} \frac{1}{\left(\dfrac{L}{N_{aux}} - \dfrac{1}{2}\right)^2 - \dfrac{1}{2}}. \tag{C8}$$

Equation (C8) can be simplified to

$$\frac{L}{N_{aux}} \left(\frac{L}{N_{aux}} - 1\right) < \left(\frac{L}{N_{aux}} - \frac{1}{2}\right)^2 - \frac{1}{2}. \tag{C9}$$

Set $\alpha = L / N_{aux}$. Thus

$$\alpha(\alpha - 1) < \left(\alpha - \frac{1}{2}\right)^2 - \frac{1}{2}. \tag{C10}$$

This inequality results in the contradicting inequality, $0 < -\frac{1}{4}$. Hence, the original assumption that no solution exists for $K_{3\text{dB}} / N_{aux}$ must be false.