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# Packet-Error Probability Analysis for Unslotted FH-CDMA Systems with Error Control Coding

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In unslotted frequency-hopped (FH) code-division multiple-access (CDMA) systems, a number of users can simultaneously transmit their packets by using quasi-orthogonal FH patterns (codes). The number of interfering users varies throughout the packet duration; thus, the symbol error probability is not constant throughout the transmission, and symbol errors within the packet are not independent. In this report we present an exact analysis of such systems that use Reed-Solomon error-control coding. The computational task for the evaluation of the performance is enormous; thus, bounds and approximations are derived that are easier to compute and that prove to be very close to the exact results for the cases in which the exact results are computable. The results of this study confirm our observation, made also elsewhere, that the widely used threshold-based model of other-user interference is not an accurate one.

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## CONTENTS

# PACKET-ERROR PROBABILITY ANALYSIS FOR UNSLOTTED FH-CDMA SYSTEMS WITH ERROR CONTROL CODING

## INTRODUCTION

In frequency hopping (FH) systems, a code-division multiple-access (CDMA) capability can be achieved where the FH patterns take on the role of codes. Since the codes usually are only quasi-orthogonal rather than truly orthogonal, frequency hits occur and result in loss of data. In the analysis of such systems, the time-slotted case is usually considered in which the packet length is equal to the slot duration. In this case a packet transmission is subjected to a constant number of interfering users throughout its duration. When Reed-Solomon (RS) error control coding is used and all frequency hits are assumed to result in symbol errors, the packet-error probability can be evaluated straightforwardly [1] as

$$Pr\{packet\ error \mid k\ other\ users\} = \sum_{i=\tau+1}^{n} \binom{n}{i} p_k^i \, (1 - p_k)^{n-i}, \tag{1}$$

where $p_k$ is the symbol error probability given $k$ other users are transmitting simultaneously over the same wideband channel, and $\tau = \lfloor (n - v)/2 \rfloor$ is the error correction capability of the RS$(n,v)$ code.

In this report we consider unslotted systems in which the level of interference varies throughout the transmission of the packet because users may begin their transmission at any time. Consequently, the symbol error probability also varies throughout the packet duration, and the analysis becomes much more difficult. The problem is further complicated by the dependence of symbol errors within the packet, since the interference levels experienced by the symbols of a packet are not only time varying, but also they are dependent. In Ref. 2, Pursley bounds the packet error probability of unslotted FH-CDMA systems in which the interference level varies over the packet duration by the packet-error probability of the system with the maximum interference level. Daigle [3] approximates the packet error probability of an unslotted direct sequence CDMA scheme without considering the effect of error control coding.

Often, because of the complexity of the exact analysis, a much simpler other-user interference model is used in the context of CDMA systems. In this model, the probability of packet error is considered to be equal to one if the number of other transmissions throughout the packet duration is greater than $T$, a threshold level, and equal to zero otherwise. This step function channel model has been studied in Ref. 4 for fixed-length messages, and in Ref. 5 for exponentially distributed message lengths.

In this report we present an exact analysis of unslotted FH-CDMA systems that use RS coding. Fixed length packets are assumed. The computational task for this evaluation is enormous; thus it has been possible to evaluate performance only for small codeword sizes. We have also developed upper bounds and close approximations to the packet-error probability that permit the evaluation of more

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practical systems. We have demonstrated, as a result of our analysis, that the bounds that are based on the maximum number of interferers present during the packet duration are rather loose, and that threshold models do not provide a satisfactory characterization of system performance. A preliminary condensed version of this analysis was presented in Ref. 6, and a more detailed development can be found in Ref. 7.

## SYSTEM MODEL

A population of users transmits packets containing a fixed number of symbols on a wideband FH channel consisting of $q$ orthogonal narrowband frequency bins. One $M$-ary symbol, representing $\log_2 M$ bits, is transmitted per hop. The hopping patterns are assumed to be generated by a first-order Markov process, so that the frequency bin for each hop is different from that of the previous hop, but equally likely to be any of the other $q - 1$ frequency bins.

This system is completely asynchronous. It is asynchronous at the packet level in the sense that packet transmission may begin at any time; consequently, packets may overlap for a portion of their duration. It is also asynchronous at the hop level; thus frequency hits may be present for only a portion of the duration of the symbol. However, all frequency hits are assumed to result in symbol errors, even if the interference is present for only a small fraction of the symbol duration. The resulting symbol error probability given that $k$ other users are simultaneously transmitting over the channel is, as discussed in Refs. 1 and 8,

$$p_k = 1 - (1 - 2/q)^k (1 - p_0), \tag{2}$$

where $p_0$ is the probability of symbol error in the absence of other-user interference, i.e., the symbol error probability caused by background noise.

Packet sizes are chosen so that each packet can be encoded as exactly one RS codeword. Each packet consists of $n = M$ $M$-ary symbols. An RS$(n, v)$ code can correct any pattern of no more than $\tau = \lfloor (n - v)/2 \rfloor$ symbol errors in a codeword.* We have considered extended RS codes of rate 1/2, i.e., $(n, n/2)$ codes, which are capable of correcting any pattern of no more than $\tau = \lfloor n/4 \rfloor$ errors.

## PACKET-ERROR PROBABILITY: EXACT ANALYSIS

To determine the exact packet-error probability of the unslotted FH-CDMA system with RS coding, it is necessary to determine the time-varying level of other-user interference present throughout a packet's transmission. We consider the case when $N$, the total number of other active channel users during the transmission of a particular packet, is given. To describe this time-varying interference process, we partition the interferers into two groups: "initial interferers" and "final interferers." Consider a given user, user 1, whose packet transmission begins at time $t_0$ and ends at $t_0 + T$, where $T$ is the fixed packet duration. When this transmission started, $I_i$ packet transmissions were already in progress. These $I_i$ channel users are the initial interferers. Each of these will terminate transmission during the interval $(t_0, t_0 + T)$. Now, during user 1's packet transmission $I_f$ users begin transmission of their packets. These are the final interferers, who will still be transmitting when user 1 ends its transmission (see Fig. 1). Clearly, $I_i + I_f = N$.†

*Undetected codeword error probability is less than $1/\tau!$, which is negligible in many applications.
†It is assumed that the interfering packets cannot arrive simultaneously with the tagged packet. Thus an interfering packet falls unambiguously into either the class of initial or final interferers.

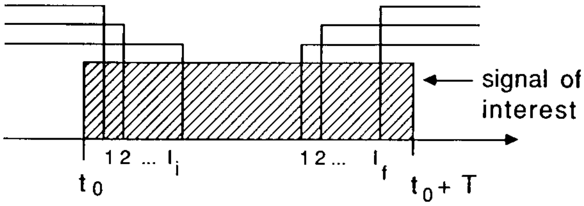


Fig. 1 — $I_i$ initial and $I_f$ final interferers

Now define the departure state $\bar{j} = (j_1, j_2, \ldots, j_n)$, where $j_l$ denotes the number of initial interferers that terminate transmission during the $l$th symbol duration. Note that $\sum_{l=1}^{n} j_l = I_i$. Likewise define the arrival state $\bar{k} = (k_1, k_2, \ldots, k_n)$, where $k_l$ denotes the number of final interferers that begin transmission during the $l$th symbol duration. Again, we have $\sum_{l=1}^{n} k_l = I_f$. For example, for the case of $n = 4$ symbols per packet, $N = 10$ other channel users, and $I_i = 3$ interferers already transmitting when user 1's transmission is started, one possible departure state is (0,1,0,2). That is, one of the $I_i$ interferers finishes transmitting during the second symbol of user 1's packet transmission, and the remaining two interferers finish during the last symbol. In this example, $I_f$ is equal to 7, and one possible arrival state is (3,2,1,1). That is $k_1 = 3$ interferers start transmitting during the first symbol of user 1's transmission, $k_2 = 2$ start during the second symbol, $k_3 = 1$ starts during the third symbol, and $k_4 = 1$ interferer starts during the last symbol. Note that numerous possible departure and arrival state descriptions represent the departure of the the $I_i$ initial interferers and the arrival of the $I_f$ final interferers. Given $I_i$ and $N$, these states are determined by an exhaustive search.

We assume that each initial interferer is equally likely to terminate transmission during any symbol; and similarly, each final interferer is equally likely to start transmission during any symbol. The probability of the departure state $\bar{j}$ and the probability of the arrival state $\bar{k}$, given $I_i$ and $N$, are then given by the multinomial distributions,

$$Pr\{\bar{j} \mid I_i, N\} = \frac{I_i!(1/n)^{I_i}}{j_1!j_2!\ldots j_n!}$$

$$Pr\{\bar{k} \mid I_i, N\} = \frac{I_f!(1/n)^{I_f}}{k_1!k_2!\ldots k_n!}. \tag{3}$$

Let us define the interference state $\bar{x} = (x_1, x_2, \ldots, x_n)$ where $x_l$ denotes the total number of interferers present during the $l$th symbol transmission. The value of $x_l$ is determined by

$$x_l = I_i - \sum_{v=1}^{l-1} j_v + \sum_{v=1}^{l} k_v \qquad l = 1, 2, \ldots, n \tag{4}$$

since the total number of interferers present during the $l$th symbol is equal to the total number of initial interferers $I_i$ less the number of initial interferers that terminated transmission during the first $(l - 1)$ symbols, plus the number of final interferers that arrived during the first $l$ symbols. For simplicity we consider that an interferer is present during the entire symbol in which it starts or terminates transmission, even though we do not require symbol synchronization. This is consistent with our earlier assumption that all hits result in symbol error even if the overlap is a small fraction of the symbol duration. Thus, $x_l$ is actually the maximum number of interferers present during the $l$th symbol. In the above example the interference state $\bar{x}$ that corresponds to the departure and arrival states, $\bar{j} = (0,1,0,2)$ and $\bar{k} = (3,2,1,1)$, is $\bar{x} = (6,8,8,9)$. Note that many different $\bar{j}$ and $\bar{k}$ pairs may combine to produce the same interference state $\bar{x}$.

Now the probability of the interference state $\bar{x}$ is determined from the probabilities of all possible departure and arrival states $\bar{j}$ and $\bar{k}$ that combine to produce $\bar{x}$. Given the state $\bar{x}$, $I_i$, and $N$, we have

$$Pr\{\bar{x} \mid I_i, N\} = \sum_{\bar{j},\bar{k} \in E_x} Pr\{\bar{j} \mid I_i, N\}\, Pr\{\bar{k} \mid I_i, N\}, \tag{5}$$

where $E_x$ is the event that $\bar{j}$ and $\bar{k}$ satisfy, Eq. (4). The probabilities of the departure and arrival states $\bar{j}$ and $\bar{k}$, given $I_i$ and $N$, are determined from Eq. (3).

**Markov Analysis**

A Reed-Solomon $(n,v)$ code can correct any pattern of no more than $\tau = \lfloor (n - v)/2 \rfloor$ $n$-ary symbol errors in a codeword. Thus the probability of packet error given the interference state $\bar{x}$ is equal to the probability that there are more than $\tau$ symbol errors:

$$Pr\{pkt\ error \mid \bar{x}\}^* = \sum_{k=\tau+1}^{n} Pr\{k\ symbol\ errors \mid \bar{x}\}. \tag{6}$$

A Markov analysis approach is used to determine the probability of $k$ symbol errors given $\bar{x}$, $N$, and $I_i$. We consider each symbol in the packet, one by one, starting with the first symbol. As each symbol is considered, we determine the probability that the symbol is in error. We consider the probability of the number of symbol errors in the packet thus far as the number of symbols considered is increased from $w$ to $w + 1$ for $1 < w < n - 1$. Thus we define

$$P_{w+1}(h \mid l) = \tag{7}$$

$$Pr\{h\ symbol\ errors\ in\ the\ first\ w + 1\ symbols \mid l\ symbol\ errors\ in\ the\ first\ w\ symbols\}.$$

Obviously, the only possible transitions from $l$ symbol errors, when one more symbol is considered, are to $h = l$ and $h = l + 1$. Now the transition from $l$ to $l + 1$ symbol errors occurs if the $(w + 1)$th symbol is in error. Given the interference state $\bar{x}$, the probability of the $(w + 1)$th symbol being in error is given as in Eq. (2),

$$P_{w+1}(l+1 \mid l) = Pr\{(w+1)\text{th}\ symbol\ in\ error \mid \bar{x}\}$$

$$= 1 - (1 - 2/q)^{x_{w+1}}(1 - p_0) \tag{8}$$

*Note that "pkt" is used as an abbreviation for "packet" throughout this report.

and thus,

$$P_{w+1}(l \mid l) = Pr\{(w+1)\text{th } symbol\ correct \mid \bar{x}\}$$

$$= (1 - 2/q)^{x_{w+1}}(1 - p_0). \tag{9}$$

Now define

$$p_w(h) = Pr\{h\ symbol\ errors\ in\ the\ first\ w\ symbols \mid \bar{x}\}, \tag{10}$$

where $w = 1, 2, \ldots, n$ and $h = 1, 2, \ldots, n$. Thus,

$$p_w(h) = P_w(h \mid h)\, p_{w-1}(h) + P_w(h \mid h-1)\, p_{w-1}(h-1) \tag{11}$$

with the initial condition $p_0(0) = 1$. The distribution for $p_w(h)$ is then evaluated recursively until we obtain

$$p_n(k) = Pr\{k\ symbol\ errors \mid \bar{x}\}, \tag{12}$$

from which it follows that

$$Pr\{pkt\ error \mid \bar{x}\} = \sum_{k=\tau+1}^{n} p_n(k). \tag{13}$$

The probability of packet error given $N$ and $I_i$, is given by

$$Pr\{pkt\ error \mid I_i, N\} = \sum_{\bar{x}} Pr\{pkt\ error \mid \bar{x}\} \cdot Pr\{\bar{x} \mid I_i, N\}$$

$$= \sum_{\bar{x}} \left[ \sum_{k=\tau+1}^{n} p_n(k) \right] Pr\{\bar{x} \mid I_i, N\}. \tag{14}$$

We assume that $I_i$ is uniformly distributed between 0 and $N$*. Therefore,

$$Pr\{pkt\ error \mid N\} = \frac{1}{N+1} \sum_{I_i=0}^{N} \sum_{\bar{x}} \left[ \sum_{k=\tau+1}^{n} p_n(k) \right] Pr\{\bar{x} \mid I_i, N\}. \tag{15}$$

The computational task for this performance evaluation is enormous. Two methods were used for this performance evaluation. The first method, which did not use memory, required more CPU time than the second method that used memory. However, the amount of memory required by the second method for packet lengths greater than five symbols per packet was prohibitive. Calculations were made, using memory, for packet lengths of $n = 4$ and 5, and without memory for $n = 4, 5, 6, 7, 8$. Table 1 summarizes the amount of CPU time required for these computations. Note that the packet length of $n = 6$ does not correspond to a Reed-Solomon code. Calculation of the performance for this packet length was performed with an arbitrary value of $\tau$ to establish the additional amount of

*This assumption is consistent with Poisson arrival statistics.

CPU time required when the packet length is increased by one symbol. When the packet length is increased by one symbol, a conservative time factor was determined from the computations with and without memory. The CPU time required if infinite memory were available was projected based on the amount of time required for $n = 4$ and 5, and this conservative time factor. Note that even if infinite memory is assumed, the exact performance evaluation of the common packet length of $n = 16$ symbols per packet is not feasible. Thus efforts were made to develop an upper bound and approximation to the packet-error probability.

Table 1 — CPU Time Required for Packet Lengths of $n = 4, 5, 6, 7, 8,$ and 16 Symbols with or Without Memory. Projected CPU Time Required Is Given for Computations with Memory.

| CPU Time Required (yr:h:min:s) | | |
|---|---|---|
| $n$ | Memoryless | With Memory |
| 4 | 00:00:00:59 | 00:00:00:13 |
| 5 | 00:00:05:60 | 00:00:00:58 |
| 6 | 00:00:24:40 | 00:00:04:24* |
| 7 | 00:01:52:24 | 00:00:19:45* |
| 8 | 00:05:46:35 | 00:01:28:51* |
| 16 | | 25:00:00:00* |

*Projected CPU time required

## UPPER BOUND ON THE PACKET-ERROR PROBABILITY

An upper bound on the packet-error probability was developed to permit the evaluation of system performance when practical packet sizes are used. The approach of the upper bound is described in this report. The transmission interval was partitioned into halves, quarters, or eighths, each containing clusters of $n/2$, $n/4$, or $n/8$ symbols respectively. It is assumed that the interference level is constant over the duration of the cluster and equal to the maximum level of interference actually experienced by any symbol in the cluster. That is, this approximation mixes our exact analysis approach with the approximation used in Ref. 4. Note that the smallest partition interval used was eighths, since the computation of the exact analysis indicated that the computational limit was eight symbols per packet. The number of clusters in the packet for a given partition is defined as $b$, where $b = 2$, 4, or 8. Each cluster then consists of $n/b$ symbols. The new interference state $\hat{x} = (\hat{x}_1, \hat{x}_2, \ldots, \hat{x}_b)$ describes the maximum number of possible interferers in the new clusters of the packet, where $\hat{x}_i$ denotes the maximum number of interferers in the $i$th cluster of the packet.

We must recognize that this new interference state $\hat{x}$ is simply the original interference state $\bar{x}$ of the packet with $b$ longer symbols. The length of the new symbols is equal to the length of $n/b$ old symbols. With this knowledge, we realize that the probability of $\hat{x}$ with $n$ symbols per packet, and $n/b$ symbols per cluster, is equal to the probability of $\bar{x}$ with $b$ symbols per packet, which is given in Eq. (5).

A Markov analysis, similar to that used for the exact analysis, has been developed. Here, for a given partition of the packet, $b$ clusters ($b = 2$, 4, or 8), containing $n/b$ symbols each, we consider each cluster of the packet, one by one, starting with the first. As each cluster is considered, we determine the probability that $l$ symbol errors are in the cluster. We consider the probability of the number of symbol errors in the packet as the number of clusters considered is increased from $v$ to $v+1$ for $1 < v < n-1$. Thus we define

$$P_{v+1}(h \mid i) = \tag{16}$$

$$Pr\{h \textit{ symbol errors in the first } v+1 \textit{ clusters} \mid i \textit{ symbol errors in the first } v \textit{ clusters}\}.$$

The only possible transitions from $i$ symbol errors, when one more cluster containing $n/b$ symbols is considered, are to $h = i + l$ symbol errors, where $l = 0, 1, \ldots, n/b$. Now the transition from $i$ to $i + l$ symbol errors occurs if there are $l$ symbol errors in the $v + 1$th cluster. Given the new interference state $\hat{x}$, this is given by

$$P_{v+1}(i+1 \mid i) = Pr\{l \textit{ symbol errors in the } (v+1)\text{th } \textit{cluster} \mid \hat{x}\}$$

$$= \binom{n/b}{l} p_{x_i}^{l} (1 - p_{x_i})^{n/b-l}, \tag{17}$$

where $p_{x_i}$ is the probability of symbol error given $x_i$ interferers given in Eq. (2). As before, let us define

$$p_v(h) = Pr\{h \textit{ symbol errors in the first } v \textit{ clusters} \mid \hat{x}\}, \tag{18}$$

where $v = 1, 2, \ldots, b$ and $h = 1, 2, \ldots, n$. Now,

$$p_v(h) = \sum_{k=0}^{h} P_v(h \mid h-k) p_{v-1}(h-k) \tag{19}$$

with the initial condition $p_0(0) = 1$. The distribution for $p_v(h)$ is then evaluated recursively until we obtain

$$p_b(k) = Pr\{k \textit{ symbol errors in } b \textit{ clusters} \mid \hat{x}\}$$

$$= Pr\{k \textit{ symbol errors in packet} \mid \hat{x}\}, \tag{20}$$

from which we can form

$$Pr\{pkt\ error \mid \hat{x}\} = \sum_{k=\tau+1}^{n} p_b(k). \tag{21}$$

The probability of packet error given $I_i$ and $N$, is thus bounded by

$$Pr\{pkt\ error \mid I_i, N\} \le \sum_{\hat{x}} Pr\{pkt\ error \mid \hat{x}\} Pr\{\hat{x} \mid I_i, N\}$$

$$\le \sum_{\hat{x}} \left[ \sum_{k=\tau+1}^{n} p_b(k) \right] Pr\{\hat{x} \mid I_i, N\}, \tag{22}$$

and thus

$$Pr\{pkt\ error \mid N\} \le \frac{1}{N+1} \sum_{I_i=0}^{N} \sum_{\hat{x}} \left[ \sum_{k=\tau+1}^{n} p_b(k) \right] Pr\{\hat{x} \mid I_i, N\}, \tag{23}$$

since $I_i$ is assumed to be uniformly distributed between 0 and $N$.

## APPROXIMATION OF PACKET-ERROR PROBABILITY

An approximation to the packet-error probability can also be derived to permit the evaluation of the packet error probability for longer packet lengths. In the exact analysis, for a given $I_i$ and $N$, we evaluated the probability of packet error for each possible interference state $\bar{x}$ for the transmitted packet. Averaging over all possible $\bar{x}$ produced the packet-error probability given $I_i$ and $N$. For the approximation, given $I_i$ and $N$, we determine the expected value of symbol-error probability for each symbol of the transmitted packet. Then, incorrectly asssuming that the symbol errors are independent, we evaluate the packet-error probability when the symbol-error probability for each symbol is given by its expected value.

As before, we partition the $N$ other channel users into $I_i$ initial interferers and $I_f$ final interferers. Starting with the initial interferers, we consider each symbol in the packet starting with the first symbol. Let us define

$$i_p = \{number\ of\ initial\ interferers\ in\ the\ p\text{th}\ symbol\}. \tag{24}$$

The probability that $I_i$ initial interferers are in the first symbol is equal to 1, i.e., $Pr\{i_1 = I_i\} = 1.0$. As in the exact analysis, we consider an interferer to be present during the entire symbol in which it starts or terminates transmission; thus there must be $I_i$ initial interferers in the first symbol. Now, the probability that $I_i - k$ initial interferers are in the second symbol is the probability that $k$ of the $I_i$ initial interferers ended their transmission during the first symbol. That probability can be expressed as

$$Pr\{i_2 = I_i - k \mid i_1 = I_i\} = \binom{I_i}{k} \left(\frac{1}{n}\right)^{k} \left(\frac{n-1}{n}\right)^{I_i - k}. \tag{25}$$

Now, in general, the probability that $l-k$ initial interferers are in the $(j+1)$st symbol, given that $l$ initial interferers are in the $j$th symbol, is the probability that $k$ of $l$ initial interferers ended their transmission during the $j$th symbol, that is,

$$Pr\{i_{j+1} = l - k \mid i_j = l\} = \binom{l}{k} \left(\frac{1}{n-j+1}\right)^{k} \left(\frac{n-j}{n-j+1}\right)^{l-k}. \tag{26}$$

Now the probability that $k$ initial interferers are in the $(j+1)$th symbol given $I_i$, and $N$ can be determined from

$$Pr\{i_{j+1} = k \mid I_i, N\} = \sum_{m=0}^{I_i - k} Pr\{i_{j+1} = k \mid i_j = k + m\} \, Pr\{i_j = k + m \mid I_i, N\} \tag{27}$$

and the initial condition $Pr\{i_1 = I_i\} = 1.0$. This yields the distribution of initial interferers in each symbol given $I_i$, and $N$.

To obtain the distribution of final interferers for each symbol, we again consider each symbol in the packet starting with the first symbol. Let us define

$$f_p = \{\textit{number of final interferers in the } p\text{th } \textit{symbol}\}. \tag{28}$$

Now, the probability that no final interferers are transmitting prior to the first symbol is equal to 1.0. We call this the probability that no final interferers are transmitting in the 0th symbol, i.e., $Pr\{f_0 = 0\} = 1.0$. Now, the probability that $k$ final interferers are in the first symbol is the probability that $k$ of the $I_f$ final interferers started transmission during the first symbol. This can be expressed as

$$Pr\{f_1 = k \mid f_0 = 0, N, I_i\} = \binom{I_f}{k} \left(\frac{1}{n}\right)^k \left(\frac{n-1}{n}\right)^{I_f - k}. \tag{29}$$

In general, the probability that $l + k$ final interferers are in the $(j + 1)$th symbol, given that $l$ final interferers are in the $j$th symbol, is the probability that $k$ of the $I_f - k$ final interferers started transmission during the $(j + 1)$th symbol, that is,

$$Pr\{f_{j+1} = l + k \mid f_j = l, N, I_i\} = \binom{I_f - l}{k} \left(\frac{1}{n - j}\right)^k \left(\frac{n - j - 1}{n - j}\right)^{I_f - l - k}. \tag{30}$$

The probability that $k$ final interferers are in the $(j+1)$th symbol, given $I_i$ and $N$, can be determined by

$$Pr\{f_{j+1} = k \mid I_i, N\} =$$
$$\sum_{m=0}^{k} Pr\{f_{j+1} = k \mid f_j = k - m, N, I_i\} \, Pr\{f_j = k - m \mid I_i, N\} \tag{31}$$

and the initial condition $Pr\{f_0 = 0\} = 1.0$. This yields the distribution of final interferers in each symbol given $I_i$ and $N$.

The distributions of initial interferers and final interferers in each symbol are independent (given $I_i$ and $N$); thus the distribution of the total number of interferers in each symbol can be obtained by

$$Pr\{t_j = k \mid I_i, N\} = \sum_{m=0}^{k} Pr\{i_j = m \mid I_i, N\} \, Pr\{f_j = k - m \mid I_i, N\} \tag{32}$$

where $t_j$ is defined as the total number of interferers in the $j$th symbol.

We can now determine the expected value of the probability of symbol error of the $j$th symbol,

$$\hat{p}_j = \sum_{m=0}^{N} Pr\{symbol\ error \mid m\ interferers\}\ Pr\{t_j = m\}. \tag{33}$$

Let us define an average symbol error probability state

$$\hat{p} = (\hat{p}_1, \hat{p}_2, \ldots, \hat{p}_n), \tag{34}$$

where $\hat{p}_i$ is the expected value of the probability of symbol error of the $i$th symbol. Assuming independence of symbol errors, we can determine the probability of packet error given the average symbol error probability state $\hat{p}$ as in the exact analysis. As mentioned above, this assumption is incorrect; since the symbol errors are not independent, this results in only an approximation to the packet-error probability. Now, we need to consider all possible occurrences of $k$ errors in the codeword and form the error state $\overline{e} = (e_1, e_2, \ldots, e_k)$, and the correct symbol state $\overline{d} = (d_1, d_2, \ldots, d_{n-k})$ where $e_i$ denotes the symbol number of the $i$th incorrect symbol, and $d_i$ denotes the symbol number of the $i$th correct symbol.

Now, the probability of $\overline{e}$, given $N$, $I_i$, and $\hat{p}$, is given by

$$Pr\{\overline{e} \mid N, I_i, \hat{p}\} = \prod_{u=1}^{k} \hat{p}_{e_u} \prod_{v=1}^{n-k} (1 - \hat{p}_{d_v}), \tag{35}$$

and thus the probability of packet error given $N$ and $I_i$ can be approximated by

$$\begin{aligned} Pr\{pkt\ error \mid N, I_i\} &= \sum_{k=\tau+1}^{n} \sum_{\overline{e}\,\epsilon\, E_k} Pr\{\overline{e} \mid N, I_i\} \\ &\approx \sum_{k=\tau+1}^{n} \sum_{\overline{e}\,\epsilon\, E_k} \left[ \prod_{u=1}^{k} \hat{p}_{e_u} \prod_{v=1}^{n-k} (1 - \hat{p}_{d_v}) \right], \end{aligned} \tag{36}$$

where $E_k$ is the set of all $\overline{e}$ states corresponding to $k$ symbol errors, and the RS code used has an error-correction capability of $\tau$ symbol errors. Now,

$$Pr\{pkt\ error \mid N\} \approx \frac{1}{N+1} \sum_{I_i=0}^{N} \sum_{k=\tau+1}^{n} \sum_{\overline{e}\,\epsilon\, E_k} \left[ \prod_{u=1}^{k} \hat{p}_{e_u} \prod_{v=1}^{n-k} (1 - \hat{p}_{d_v}) \right], \tag{37}$$

since all partitions of $N$ are equally likely; that is, $I_i$ is assumed to be uniformly distributed between 0 and $N$ (again consistently with the assumption of Poisson arrivals).

Another approximation to the packet-error probability can be developed by calculating the expected number of interferers transmitting during each symbol from the probability distribution of the total number of interferers in each symbol. That is, given $Pr\{t_j = k \mid I_i, N\}$, calculate,

$$\hat{x}_j = \sum_{m=0}^{N} m\ Pr\{t_j = m \mid I_i, N\}. \tag{38}$$

Then the probability of symbol error of the $j$th symbol, given the expected number of interferers in the symbol, could be determined as,

$$\tilde{p}_j = Pr\{symbol\ error \mid \hat{x}_j interferers\} \tag{39}$$

$$= 1 - (1 - 2/q)^{\hat{x}_j}(1 - p_0).$$

Substituting $\tilde{p}_j$ for $\hat{p}_j$ in Eq. (37) yields the second approximation.

## SYSTEM PERFORMANCE

Packet-error probability, given $N$, was evaluated for several rate 1/2 RS codes as $N$ was varied between 1 and 10. In all cases the number of frequency bins was $q = 50$, and a noiseless channel (i.e., $p_0 = 0$) was assumed. Exact performance results were obtained only for the RS(4,2) and RS(8,4) codes. Upper bounds and approximations were obtained for these codes as well as for RS(16,8) and RS(32,16) codes.

The RS(4,2) and RS(8,4) codes have error-correction capabilities of $\tau = 1$ and 2 symbol errors respectively. These are not realistic packet lengths since they permit the transmission of only very little information and since the upper bound on undetected codeword error probabilities for the RS(4,2) and RS(8,4) codes are equal to 1.0 and 0.5 respectively. However, the computation of the exact packet-error probability of the system with the short packet lengths is very useful; it allows comparison with the upper bound and approximation to the packet-error probability. Computational limits prevent the exact evaluation of the packet-error probability for longer packet lengths as discussed earlier. The upper bound, with $n/b$ symbols per cluster, $b = 2$, 4, and 8, was computed for the RS codes mentioned above, as well as the RS(16,8) and RS(32,16) codes having error-correction capabilities of $\tau = 4$ and 8 symbol errors respectively.

Figure 2 shows the results for the unslotted FH-CDMA system with RS(4,2) code. Note that the approximation is extremely close to the exact packet-error probability. When the number of interferers transmitting during the packet $N$ is greater than or equal to 3, the approximation is within 5% of the exact packet-error probability, while it is within 1% for $N \geq 7$.

The upper curve in each of the figures represents a slotted system in which all $N$ interferers are present during the entire packet transmission.

Figure 3 illustrates the performance of the unslotted FH-CDMA system with the RS(8,4) code. As before, the approximation closely resembles the exact packet-error probability. When $N \geq 4$ the approximation is within 10% of the exact packet-error probability, while it is within 3% when $N \geq 7$.

For the system parameters considered, the approximation provides excellent agreement with the exact results. It appears that the proportional difference between the exact packet-error probability and the approximation decreases as $N$ increases. Also note that the packet-error probability as a function of $N$ does not resemble a step-function, thus indicating that a threshold model does not provide a good indication of system performance, as demonstrated in Ref. 9.

Figures 4 and 5 illustrate the upper bound and approximation of the performance for the cases in which the exact performance is not computable. Figure 4 shows the performance of the unslotted FH-CDMA system with the RS(16,8) code. Figure 5 illustrates the approximation and upper bound of the packet-error probability of the system with the RS(32,16) code.

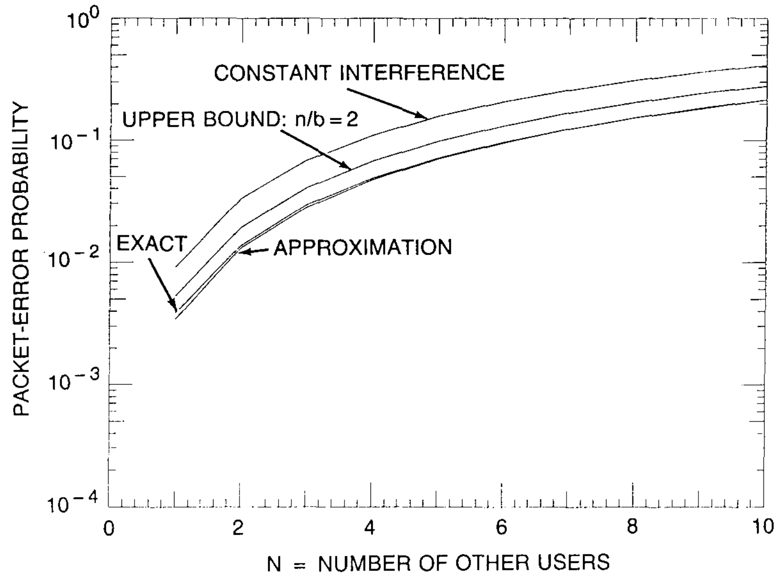


Fig. 2 — Packet-error probability with RS(4,2) coding

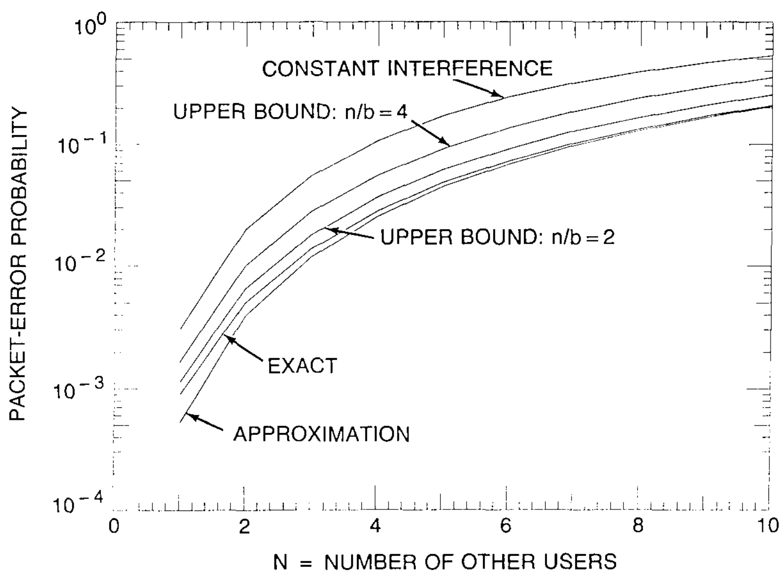


Fig. 3 — Packet-error probability with RS(8,4) coding

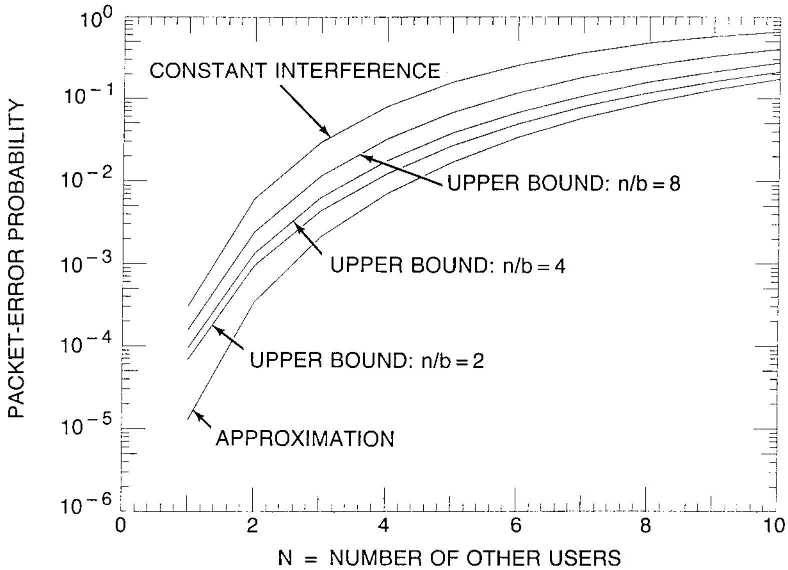


Fig. 4 — Packet-error probability with RS(16,8) coding

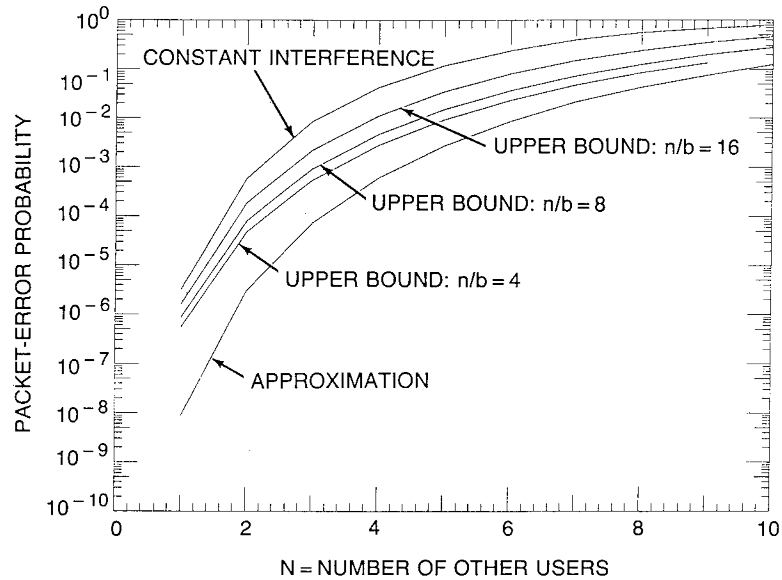


Fig. 5 — Packet-error probability with RS(32,16) coding

It is of interest to compare the packet-error probability of the system with time-varying interference to that of the system with constant interference. Table 2 contains the number of constant interferers that produce the same probability of packet error as 10 interferers whose transmissions start or stop within the packet duration, where we ignore the fact that this number must be an integer. The comparison is given for RS(4,2), RS(8,4), RS(16,8), and RS(32,16) codes and $q = 50$ and 100 frequency bins. For packet lengths of $n = 16$ and 32 symbols, the approximate packet-error probability for $N = 10$ partial interferers was used for the comparison. Note that this number appears to approach 5 as the code length increases.

Table 2 — Comparison of Time-Varying Interference to Constant Interference

| Constant Interference Level Corresponding to 10 Partial Interferers | | |
|---|---|---|
| Code | $q = 50$ | $q = 100$ |
| RS(4,2) | 6.1 | 6.3 |
| RS(8,4) | 5.5 | 5.6 |
| RS(16,8)* | 5.15 | 5.15 |
| RS(32,16)* | 5.1 | 5.1 |

*Approximate packet-error probability for $N = 10$ partial interferers used for comparison.

## CONCLUDING REMARKS

The packet-error probability performance of unslotted FH-CDMA with RS error control coding is evaluated exactly for small packet lengths. Computational limits prevent the exact evaluation of packet-error probability for longer packet lengths, and thus an upper bound and approximation to the packet-error probability are derived and computed. The upper bound calculations are also limited (though not quite as severely) by computational considerations, but the approximation is easily evaluated for large codeword sizes.

In all cases considered, the packet-error probability as a function of the number of other transmitting users *does not* resemble a step function. Thus detailed models that reflect channel characteristics and coding properties are needed to provide an accurate evaluation of system performance. Our previous observation [9] that the step-function channel model does not reliably predict performance of spread spectrum multiple access systems is further confirmed by this study.

## REFERENCES

1. E. A. Geraniotis and M. B. Pursley, "Error Probabilities for Slow-Frequency-Hopped Spread-Spectrum Multiple Access Communications over Fading Channels," *IEEE Trans. Commun.* **COM-30**, 996-1009 (1982).

2. M. B. Pursley, "Frequency-Hop Transmission for Satellite Packet Switching and Terrestrial Packet Radio Networks," *IEEE Trans. Inf. Theory* **IT-32**, 652-667 (1986).

3. J. N. Daigle, "Throughput in Asynchronous CDMA Systems," Conf. Rec. INFOCOM'87, 1079-1089 (Mar. 1987).

4. J. M. Musser and J. N. Daigle, "Throughput Analysis of an Asynchronous Code Division Multiple Access System," *Proc. IEEE Int. Conf. Commun.* **3**, 2F.2.1-2F.2.7 (1982).

5. F. A. Tobagi and J. S. Storey, "Throughput Performance of a Direct Sequence CDMA Packet Radio Network," SEL Tech. Report 85-277, Stanford University, Stanford, CA (1985).

6. J. A. B. Tarr, J. E. Wieselthier, and A. Ephremides, "Packet-Error Probability in Unslotted FH-CDMA Systems: Exact Ananlysis, Bounds, and Approximations," Proc. 22nd Ann. Conf. Inf. Sci. Syst., Princeton University, (Mar. 1988).

7. J. A. B. Tarr, "Packet Error Performance of an Unslotted FH-CDMA Scheme and Performance Evaluation of a Distributed Reservation Scheme," Master's Thesis; Univ. MD, Dec. 1987.

8. B. Hajek, "Recursive Retransmission Control—Application to a Frequency-Hopped Spread-Spectrum System," Proc. 16th Ann. Conf. Inf. Sci. Syst., Princeton University, 116-120, Mar. 1982.

9. J. E. Wieselthier, A. Ephremides, and J. A. B. Tarr, "A Distributed Reservation-Based CDMA Protocol that Does Not Require Feedback Information," *IEEE Trans. Commun.* (1988). Also, NRL Memorandum Report 5976, June 1987.