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A Method for Validating Multidimensional Fast Fourier Transform (FFT) Algorithms

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| <p>A method is described for validating fast Fourier transforms (FFTs) based on the use of simple input functions whose discrete Fourier transforms can be evaluated in closed form. Explicit analytical results are developed for one-dimensional and two-dimensional discrete Fourier transforms. The analytical results are easily generalized to higher dimensions. The results offer a means for validating the FFT algorithm in one, two, or higher dimensional settings. The general motivation for the work comes from the need to validate the FFT algorithm when it is newly implemented on a computer or when new techniques or devices are added to a computer facility to evaluate discrete Fourier transforms.</p> | | | | |
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PREFACE

This work is related to a research program on ocean bottom characterization being conducted by the Acoustics Media Characterization Branch at NRL. The program is motivated by the recent development of a variety of bathymetric measurement systems that resolve aspects of bottom variability heretofore beyond the realm of quantitative study. The general aim of the research is to develop statistical/stochastic representations of ocean bottom variability at scales that are relevant to underwater acoustic systems. Such representations will allow the prediction of acoustic system performance and lead to the better design and use of underwater acoustic systems that either utilize or are sensitive to acoustic interaction with the ocean bottom.

A METHOD FOR VALIDATING MULTIDIMENSIONAL FAST FOURIER TRANSFORM (FFT) ALGORITHMS

INTRODUCTION

In the physical sciences, many problems exist that lead to the analysis of a physical process in terms of the concept of a random process (Yaglom 1962). A key aspect of the theory of random processes is the spectrum of the process, which describes the distribution of energy associated with the process over wave number or frequency. The computation of the spectrum is often carried out by evaluating a discrete Fourier transform using a fast Fourier transform (FFT) algorithm. This report deals with some problems that arise in connection with FFT algorithms.

Many potential difficulties arise in connection with the use of an FFT. 1. The usual situation is that the algorithm already exists as an available subroutine on the computer facility; however, the documentation of the algorithm may be incomplete and there may be questions as to which of several possible definitions of the transform is being utilized. 2. Another situation that arises is the writing of an FFT algorithm for a computer facility, possibly to take advantage of a new technique. In this case, the specific algorithm for the transform is presumably known, but it is desirable to have a means to check the validity of the calculation once the code has been implemented. 3. A further situation that may arise is that an FFT may be added to a computer facility as a built-in feature of an array processor. Again the question arises as to whether the array processor is doing the calculation correctly.

We have mentioned three situations in which it is desirable to have a means to check the validity of the calculation of the FFT. This is especially important because such an algorithm will receive widespread application. This report provides a method for validating FFT algorithms. The method is specifically dealt with for one-dimensional and two-dimensional discrete Fourier transforms. The two-dimensional case demonstrates how the method can be easily generalized to multidimensional transforms of any dimension.

TERMINOLOGY AND BASIC RELATIONS

The discrete Fourier transform is at the core of any procedure that evaluates a spectrum from digital data. In this section we summarize those relationships that will be needed in the following analysis. A more extensive derivation and discussion of discrete Fourier transform properties can be found in several texts (Brigham 1974, Oppenheim and Schaffer 1975).

For one-dimensional data $X(k)$, the discrete Fourier transform is defined as

$$A(m) = \sum_{k=0}^{M-1} X(k) e^{-2\pi i m k / M} \quad (m = 0, 1, \dots, M-1), \quad (1)$$

where it is assumed that the original data consists of M discrete values. It is useful to extend the definition of $A(m)$ and $X(k)$ to all discrete values of m and k by a periodic extension of the previous set of values. The discrete transform $A(m)$ is in general complex. For real data, however, we have

$$X^*(k) = X(k), \quad (2)$$

and this leads to

$$A(m) = A^*(-m) = A^*(M-m), \quad (3)$$

for $m = 0, 1, \dots, M-1$.

A basic relation that exists between the transform and the data is given by Parseval's relation, which states that

$$\sum_{k=0}^{M-1} |X(k)|^2 = \frac{1}{M} \sum_{m=0}^{M-1} |A(m)|^2. \quad (4)$$

This relationship can be thought of as an energy relation and is useful for establishing correspondences between the spectrum of the continuous process and the discrete Fourier transform, which is calculated from equally spaced measurements of the process.

We now consider the two-dimensional case where we have data $X(k, l)$ given over a two-dimensional grid. The discrete Fourier transform is now given by

$$A(m, n) = \sum_{k=0}^{M-1} \sum_{l=0}^{N-1} X(k, l) e^{-2\pi i m k / M} e^{-2\pi i n l / N} \quad (5)$$

where the data $X(k, l)$ is assumed given over a set of data points identified by $k = 0, 1, \dots, M-1$ and $l = 0, 1, \dots, N-1$. It is again useful to regard the transform as defined for all integer values of m, n . When the data $X(k, l)$ are real, we obtain the symmetry properties

$$A^*(-m, -n) = A(m, n) = A^*(M-m, N-n). \quad (6)$$

Parseval's relation for discrete data over a two-dimensional region takes the form

$$\sum_{m=0}^{M-1} \sum_{n=0}^{N-1} |X(m, n)|^2 = \frac{1}{MN} \sum_{k=0}^{M-1} \sum_{l=0}^{N-1} |A(k, l)|^2. \quad (7)$$

An important view to take toward discrete Fourier transforms of data that are defined over a multidimensional array of points is to regard the transform as a series of one-dimensional transforms. For example, the two-dimensional transform defined in Eq. (5) can be written as

$$A(m, n) = \sum_{l=0}^{N-1} \left\{ \sum_{k=0}^{M-1} X(k, l) e^{-2\pi i m k / M} \right\} e^{-2\pi i n l / N}. \quad (8)$$

In this form, it can be regarded as the result of several one-dimensional transforms carried out first on the rows of the data array $X(k, l)$. A series of one-dimensional transforms is then performed on each of the columns of the intermediate array to yield the two-dimensional transform of the original array of data. One could just as well begin with column transforms rather than row transforms in Eq. (8). This point of view is obviously capable of generalization to higher dimensions and is also the basis for computational algorithms (Robinson and Silvia, 1979). Thus one can regard the one-dimensional discrete Fourier transform as the fundamental ingredient for the higher dimensional transforms.

We consider one-dimensional and two-dimensional transforms in this report because there are some aspects of higher dimensional transforms that become clear only when one deals with a transform of data defined over more than one dimension.

DESCRIPTION OF THE METHOD

The essential aspect of the method for validating discrete Fourier transforms is to use simple discrete functions for which the transform can be worked out analytically—by this we mean the summation appearing in the transform relation can be carried out to obtain an expression involving elementary functions whose number does not vary with the number of input values. One then has a

means for checking the output of the transform algorithm. This procedure allows the transform algorithm to be viewed as a *black box*, which may be advantageous in some situations. We shall consider several simple functions that can be used as input and discuss their advantages and disadvantages as a test of the transform algorithm. The one-dimensional and two-dimensional transforms are considered because there are some potential difficulties with the multidimensional transform that do not arise with the one-dimensional transform. Further, the consideration of the two-dimensional transform will demonstrate how the method can be generalized to multidimensional transforms of any dimension.

A simple test is provided by Parseval's relation, which relates a property of the original function to a property of the transform. This relation can be used to check the correctness of the transform algorithm by using a simple function as input. Perhaps the simplest input function is

$$X(k) = \delta(k) \quad (k = 0, 1, \dots, M-1), \quad (9)$$

where

$$\begin{aligned} \delta(k) &= 1 \text{ if } k = 0 \pmod{M} \\ &= 0 \text{ otherwise.} \end{aligned} \quad (10)$$

The discrete Fourier transform is

$$A(m) = \sum_{k=0}^{M-1} \delta(k) e^{-2\pi i m k / M},$$

which becomes

$$A(m) = 1 \quad \text{for } m = 0, 1, \dots, M-1.$$

Parseval's relation states that we must have

$$\sum_{k=0}^{M-1} |X(k)|^2 = \frac{1}{M} \sum_{m=0}^{M-1} |A(m)|^2.$$

The summation on the right is

$$\frac{1}{M} \sum_{m=0}^{M-1} 1 = 1,$$

which is also what the summation of the square of the input data values equals.

This is easily generalized to higher dimensions as is seen for the two-dimensional case where one would use $X(k, l) = \delta(k)\delta(l)$ as the input. The discrete Fourier transform is

$$A(m, n) = \sum_{k=0}^{M-1} \sum_{l=0}^{N-1} \delta(k)\delta(l) e^{-2\pi i m k / M} e^{-2\pi i n l / N},$$

which becomes

$$\begin{aligned} A(m, n) &= 1 \quad \text{for } m = 0, 1, \dots, M-1 \\ &\quad n = 0, 1, \dots, N-1. \end{aligned}$$

Again Parseval's relation requires that

$$\sum_{k=0}^{M-1} \sum_{l=0}^{N-1} |X(k, l)|^2 = \frac{1}{MN} \sum_{m=0}^{M-1} \sum_{n=0}^{N-1} |A(m, n)|^2,$$

where it is easily verified that each side of the equality has the value 1.

These results provide one means of testing the validity of the discrete Fourier transform algorithm by indicating what results should be obtained from an input of a simple form. The disadvantage of this test is that the resultant transforms have all elements equal to one another. It is conceivable that a coding error could produce a constant entry in all array locations that would then yield a

false indication of a correct result for the simple preceding input data. It therefore seems desirable to use a more complicated input function to obtain a transform with sufficient variability that it would be unlikely to provide a false indicator.

It is important to realize that Parseval's relation is a limited test based on a transform property that is not restricted to Fourier transforms. Some further aspects of Parseval's relation that prevent it from serving as a detailed test are that the relation involves magnitudes, which do not preserve phase relations between various coefficients, and the summation is insensitive to a permutation of the order of the coefficients. These features of Parseval's relation imply that the relation is only useful as a global test. The most stringent test of an FFT algorithm will require examination of the specific values of transform coefficients and should involve the use of a fairly general input function.

We now consider the advantages of the linear function

$$X(k) = k \quad (k = 0, 1, \dots, M - 1). \quad (11)$$

The transform is

$$A(m) = \sum_{k=0}^{M-1} k e^{-2imk/M} \quad (m = 0, 1, \dots, M - 1) \quad (12)$$

and can be evaluated explicitly in terms of simple functions as shown in the appendix of this report. The result is that

$$A(0) = \frac{M(M-1)}{2}$$

$$A(m) = -\frac{M}{2} + i \frac{M}{2} \cot \left[\frac{\pi m}{M} \right] \quad (m = 1, 2, \dots, M - 1). \quad (13)$$

It is seen from the expressions in Eq. (13) that the various elements of the transform are different, thus the simple test function provides a useful means for validating one-dimensional discrete Fourier transforms.

The test function appearing in Eq. (11) can easily be generalised to higher dimensions. We explicitly consider the two-dimensional case in which

$$X(k, l) = kl \begin{pmatrix} k = 0, \dots, M - 1 \\ l = 0, \dots, N - 1 \end{pmatrix}. \quad (14)$$

The two-dimensional discrete Fourier transform is given by

$$A(m, n) = \sum_{k=0}^{M-1} \sum_{l=0}^{N-1} k l e^{-2\pi i m k / M} e^{-2\pi i n l / N},$$

$$\begin{pmatrix} m = 0, \dots, M - 1 \\ n = 0, \dots, N - 1 \end{pmatrix},$$

which can be written as

$$A(m, n) = \left[\sum_{k=0}^{M-1} k e^{-2\pi i m k / M} \right] \left[\sum_{l=0}^{N-1} l e^{-2\pi i n l / N} \right], \quad (15)$$

where

$$\begin{pmatrix} m = 0, \dots, M - 1 \\ n = 0, \dots, N - 1 \end{pmatrix}.$$

The two-dimensional transform in Eq. (15) is obviously the product of one-dimensional transforms, which can be written out explicitly using the results obtained in the appendix of this report. The result is

$$A(0,0) = \frac{1}{4} M(M-1)N(N-1), \tag{16}$$

$$A(0,n) = \frac{1}{2} M(M-1) \left[-\frac{N}{2} + i\frac{N}{2} \cot \left(\frac{\pi n}{N} \right) \right] \quad (n = 1,2,\dots,N-1), \tag{17}$$

$$A(m,0) = \frac{1}{2} N(N-1) \left[-\frac{M}{2} + i\frac{M}{2} \cot \left(\frac{\pi m}{M} \right) \right] \quad (m = 1,2,\dots,M-1), \tag{18}$$

and

$$A(m,n) = \left[-\frac{M}{2} + i\frac{M}{2} \cot \left(\frac{\pi m}{M} \right) \right] \left[-\frac{N}{2} + i\frac{N}{2} \cot \left(\frac{\pi n}{N} \right) \right], \tag{19}$$

where

$$\begin{cases} m = 1,2,\dots,M-1 \\ n = 1,2,\dots,N-1 \end{cases}.$$

It is useful to think of the results in Eqs. (16) to (19) as providing various entries in a matrix as shown in Fig. 1.

The two-dimensional test function, just as in the one-dimensional case, leads to a transform with variable entries and provides a useful means for validating two-dimensional FFT algorithms. The test function can easily be generalized to higher dimensions in which case the multidimensional discrete Fourier transform of the test function is given by the product of a number of one-dimensional transforms, entirely analogous to the two-dimensional case we have just considered.

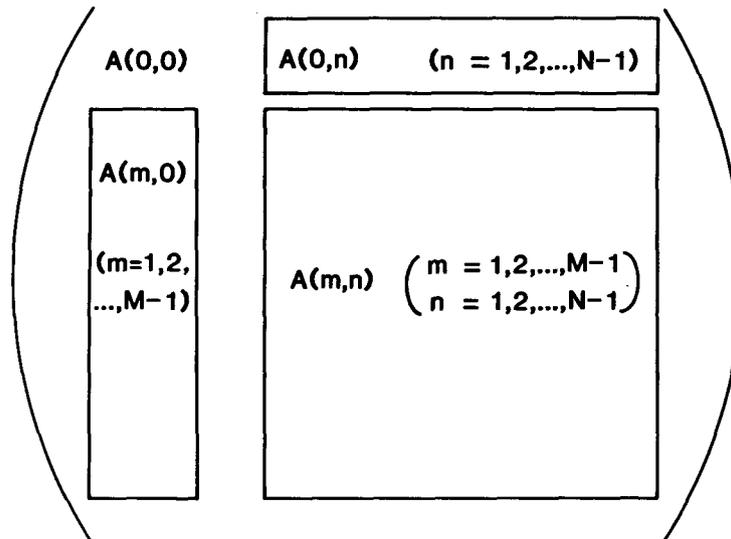


Fig. 1 — Matrix form of the two-dimensional transform $A(m,n)$ that is explicitly written out in Eqs. (16) to (19). The indexes m and n correspond to row and column indicators, respectively. The reason for exhibiting the transform in matrix form is to emphasize the structural aspects of the transform.

The function $X(k, l) = kl$ we have been considering seems adequate for testing the validity of the general case. However, it has some disadvantages when $M = N$ because in that case the transform $A(m, n)$ becomes a symmetric matrix. Thus for the case $M = N$, the test function $X(k, l) = kl$ has properties that are not shared in general by two-dimensional input functions. The symmetry of the transform makes it impossible to test whether or not rows and columns have been incorrectly interchanged in the program. This becomes an important consideration in testing the validity of algorithms that are specifically written for multidimensional input with $M = N$. To have a means for testing this, we need to consider some other input functions that do not lead to symmetric transforms.

Consider the following functions:

$$X(k, l) = ak + bl, \quad (20)$$

$$X(k, l) = ak + b, \text{ and} \quad (21)$$

$$X(k, l) = a + bl. \quad (22)$$

In each of these, a and b are constants. We shall deal with the general case with $M \neq N$ although our main interest is in the case when $M = N$. Again, the functions selected are of a sufficiently simple form that their transforms may be worked out analytically.

The transform of the discrete function given in Eq. (20) is

$$A(m, n) = \sum_{k=0}^{M-1} \sum_{l=0}^{N-1} (ak + bl)e^{-2\pi imk/M} e^{-2\pi inl/N},$$

which can be written as

$$\begin{aligned} A(m, n) = a & \left[\sum_{k=0}^{M-1} ke^{-2\pi imk/M} \right] \left[\sum_{l=0}^{N-1} e^{-2\pi inl/N} \right] \\ & + b \left[\sum_{k=0}^{M-1} e^{-2\pi imk/M} \right] \left[\sum_{l=0}^{N-1} le^{-2\pi inl/N} \right]. \end{aligned} \quad (23)$$

The two-dimensional transform is thus given by evaluating a series of one-dimensional transforms. Note that we have the simple result that

$$\sum_{l=0}^{N-1} e^{-2\pi inl/N} = N\delta(n), \quad (24)$$

where

$$\begin{aligned} \delta(n) &= 1 \text{ if } n = 0 \pmod{N} \\ &= 0 \text{ otherwise.} \end{aligned}$$

The expression in Eq. (24) is the one-dimensional transform of the function that is equal to unity. The result in Eq. (24) allows Eq. (23) to be written as

$$A(m, n) = aN \left[\sum_{k=0}^{M-1} ke^{-2\pi imk/M} \right] \delta(n) + bM \left[\sum_{l=0}^{N-1} le^{-2\pi inl/N} \right] \delta(m). \quad (25)$$

The general structure of this transform is best thought of in terms of a matrix where m designates row location and n designates column location. Figure 2 shows the general form of the result where nonzero entries only occur in the first row or first column. These entries are given by

$$A(0, 0) = aN \frac{M(m-1)}{2} + bM \frac{N(N-1)}{2}, \quad (26)$$

$$A(m, 0) = aN \left[-\frac{M}{2} + i\frac{M}{2} \cot \left[\frac{\pi m}{M} \right] \right] \quad (m = 1, 2, \dots, M-1) \quad (27)$$

and

$$A(0, n) = bM \left[-\frac{N}{2} + i\frac{N}{2} \cot \left[p\frac{\pi n}{N} \right] \right] \quad (n = 1, 2, \dots, N-1), \quad (28)$$

where use has been made of the results obtained in the appendix of this report.

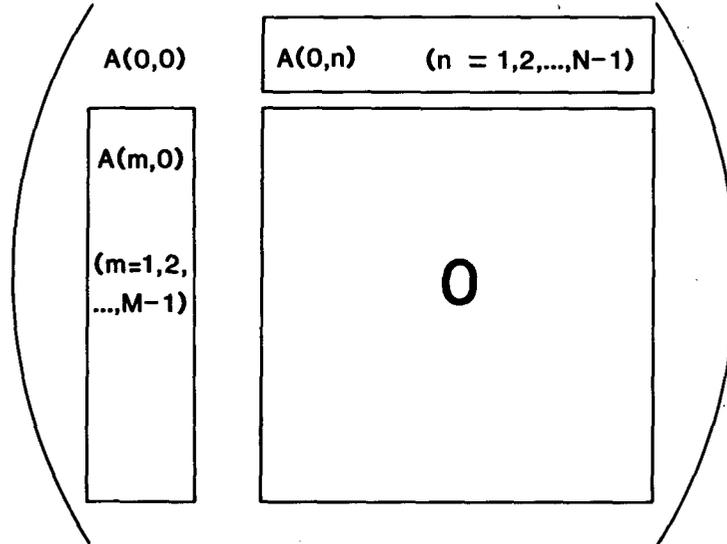


Fig. 2 — Matrix form of the transform $A(m, n)$ of the function $X(k, l) = ak + bl$ showing its structural form. Explicit row and column entries are given by Eqs. (26) to (28).

When $M = N$, the only difference between the first row and first column are in the constants a and b . Thus, while the given test function allows a test of row and column placement, it does not involve much variability in the entries.

The two-dimensional discrete Fourier transforms of the functions in Eqs. (21) and (22) can be evaluated by using the results that were used in the evaluation of the transform of Eq. (20). The transform of Eq. (21) leads to

$$A(m, n) = aN \left[\sum_{k=0}^{M-1} ke^{-2\pi imk/M} \right] \delta(n) + bMN\delta(m)\delta(n), \quad (29)$$

while the transform of Eq. (22) leads to

$$A(m, n) = aMN\delta(m)\delta(n) + bM \left[\sum_{l=0}^{N-1} le^{-2\pi nl/N} \right] \delta(m). \quad (30)$$

The ranges of indexes in Eqs. (29) and (30) are $m = 0, 1, \dots, M-1$ and $n = 0, 1, \dots, N-1$.

Figure 3 is the matrix structure of the transform in Eq. (29). The transform is of a very special form where nonzero entries appear only in the first column. These are given by

$$A(0, 0) = aN\frac{M(M-1)}{2} + bMN \quad (31)$$

$$A(m, 0) = aN \left[-\frac{M}{2} + i\frac{M}{2} \cot \left[\frac{\pi m}{M} \right] \right] \quad (m = 1, 2, \dots, M-1). \quad (32)$$

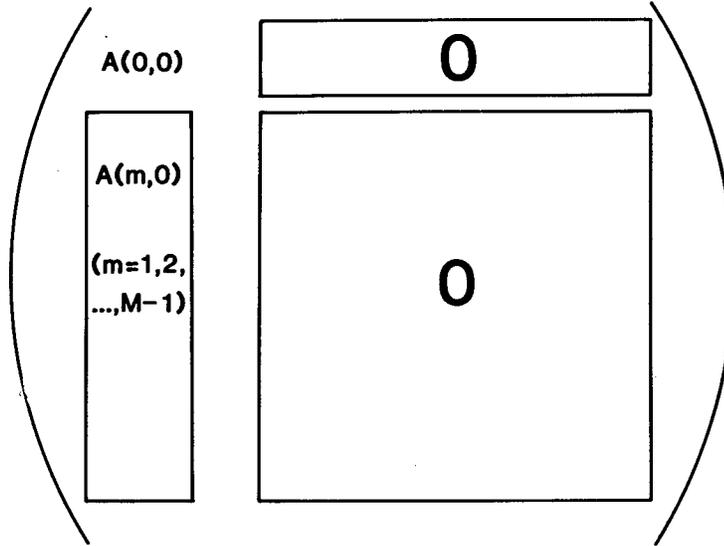


Fig. 3 — Matrix form of the two-dimensional transform $A(m, n)$ of the function $X(k, l) = ak + b$ showing its structural aspects. The elements of the first column are given in Eqs. (31) and (32).

Consequently, even when $M = N$ we obtain a very asymmetric transform with variable entries in the first column.

The matrix structure of the transform in Eq. (30) is similar to that of Eq. (29), except that now we have the nonzero entries appearing in the first row. Figure 4 shows the structure. The elements appearing in the first row are given by

$$A(0, 0) = aMN + bM \frac{N(N-1)}{2}, \text{ and} \quad (33)$$

$$A(0, n) = bM \left[-\frac{N}{2} + i\frac{N}{2} \cot \left(\frac{\pi n}{N} \right) \right] \quad (n = 1, 2, \dots, N-1). \quad (34)$$

It is clear from the results in Eqs. (31) to (34) that the constant terms appearing in the input functions of Eqs. (21) and (22) only contribute to $A(0, 0)$ in each case. They only offer a limited degree of freedom so far as changing the transform and need not be included. The main reason for their inclusion here is to demonstrate what the general linear function leads to under the discrete Fourier transform relation.

CONCLUSIONS

The development of a validation procedure for the FFT algorithm expresses a general attitude of the author that algorithms should be checked by application to a case for which the results are known by other means. This is not always done. In many cases the reason for inadequate tests of an algorithm may be traced to a lack of known analytical results.

We have presented some explicit analytical results for discrete Fourier transforms in the one-dimensional and two-dimensional situations. The results are easily generalized to any multidimensional situation. The primary motivation for the work was to establish a means for validating multidimensional FFT algorithms by establishing analytical results that can be compared to the output of FFT algorithms. The need for validating FFT algorithms arises in many contexts, such as when such an algorithm is first implemented on a computer facility or when modifications are made to take advantage of new techniques or new devices for the calculation of the discrete Fourier transform.

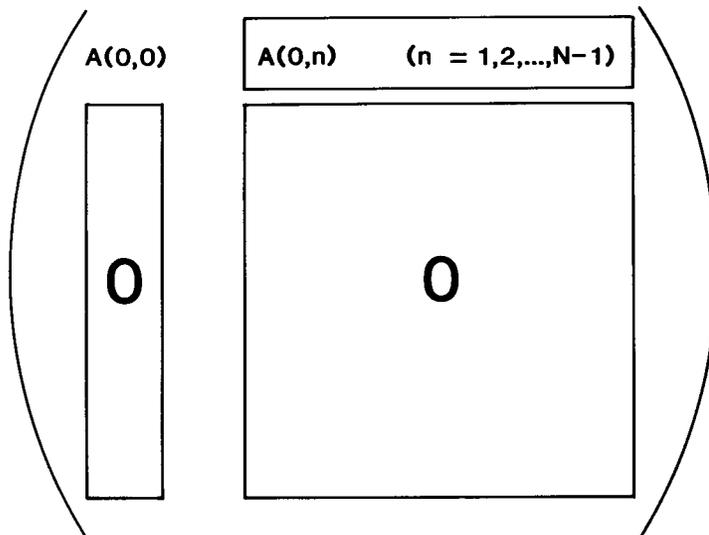


Fig. 4 — Matrix form of the two-dimensional transform $A(m, n)$ of the function $X(k, l) = a + bl$ showing the structural aspects of the transform. Elements appearing in the first row are given by Eqs. (33) and (34).

The general idea discussed in this report is that of using simple discrete functions whose discrete Fourier transforms can be evaluated by direct analytical means. Several functions were considered explicitly for the one-dimensional and two-dimensional cases, and their advantages and disadvantages were examined in terms of a validation procedure. We note that such a procedure allows the algorithm to be viewed as a *black box* with only the input and output being of primary interest.

The types of functions we have dealt with are of the form

$$X(k) = ak + b \quad (k = 0, 1, \dots, M - 1) \text{ and}$$

$$X(k, l) = ak l + bk + cl + d \quad \left[\begin{array}{l} k = 0, 1, \dots, M - 1 \\ l = 0, 1, \dots, N - 1 \end{array} \right],$$

where one or another of the constants may be zero. These functional forms appear to offer sufficiently general tests of FFT algorithms in one and two dimensions.

One could generalize the procedure by dealing with functions of the form

$$X(k) = f(k) \quad (k = 0, 1, \dots, M - 1) \text{ and}$$

$$X(k, l) = f(k)g(l) \quad \left[\begin{array}{l} k = 0, 1, \dots, M - 1 \\ l = 0, 1, \dots, N - 1 \end{array} \right],$$

and so on for higher dimensions. A general property of such functions is that the multidimensional transform of such separable functions will be found to reduce to a product of one-dimensional transforms. One would then select $f(k), g(l)$ so that their one-dimensional transforms can be explicitly evaluated in analytical terms. The only difficulty in carrying out such a procedure is the analytical evaluation of the one-dimensional transforms.

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Appendix

ANALYTICAL EVALUATION OF A DISCRETE FOURIER TRANSFORM

We here consider the analytical evaluation of a discrete Fourier transform of a specified function $X(k)$ for $k = 0, 1, \dots, M-1$. Despite the treatment of the general theoretical aspects of such transforms, the evaluation of specific results is rarely treated and there appears to be no extensive tabulation of transforms of simple functions that can be worked out analytically. The particular function we are interested in is given by

$$X(k) = k \quad (k = 0, 1, \dots, M-1).$$

It is desired to evaluate the transform that can be written as

$$A(m) = \sum_{k=0}^{M-1} k e^{-2\pi i k m / M} \quad (m = 0, 1, \dots, M-1). \quad (\text{A1})$$

For the sake of later convenience, we note the summation in Eq. (A1) can be easily evaluated for the special case $m = 0$ since

$$A(0) = \sum_{k=0}^{M-1} k = \frac{M(M-1)}{2}. \quad (\text{A2})$$

Thus we now have to evaluate the expression in Eq. (A1) for $m = 1, 2, \dots, M-1$.

To carry out the summation in Eq. (A1), it is useful to regard the discrete functions in the summand as defined for all values of k . Thus for each value of m we have a summation problem where it is desired to obtain a simple, analytical expression for the sum. Define the finite difference operator Δ by

$$\Delta\mu(k) = \mu(k+1) - \mu(k),$$

and note that it has the property

$$\Delta(\mu\nu) = \nu\Delta\nu + \nu(k+1)\Delta\mu, \quad (\text{A3})$$

where it is understood the independent variable is k in those terms where it is not indicated explicitly. We specify that

$$\mu(k) = k, \quad (\text{A4})$$

and

$$\Delta\nu(k) = e^{-2\pi i k m / M} \quad (k = 0, 1, \dots, M). \quad (\text{A5})$$

This allows the summation problem in Eq. (A1) to be expressed as

$$A(m) = \sum_{k=0}^{M-1} \mu(k)\Delta\nu,$$

which, by use of Eq. (A3), can be written in the form

$$A(m) = \sum_{k=0}^{M-1} \Delta(\mu\nu) - \sum_{k=0}^{M-1} \nu(k+1),$$

or, since the first summation can be summed explicitly, we have

$$A(m) = \mu(M)\nu(M) - \mu(0)\nu(0) - \sum_{k=0}^{M-1} \nu(k+1). \quad (\text{A6})$$

The reason for the change in summation is that the summation problem in Eq. (A6) will be able to be done in explicit terms. The method being used is analogous to integration by parts for continuous functions.

We now determine $\nu(k)$ which satisfies Eq. (A5). We can define

$$\nu(0) = 0 \quad (\text{A7})$$

since Eq. (A5) is only a requirement on differences of $\nu(k)$.

Then, we find by evaluating the first few terms of the expression in Eq. (A5) that

$$\nu(1) = e^{-2\pi i k m / M} \Big|_{k=0},$$

and

$$\nu(2) = e^{-2\pi i k m / M} \Big|_{k=0} + e^{-2\pi i k m / M} \Big|_{k=1}.$$

On the basis of these results we can guess that

$$\nu(k) = \sum_{l=0}^{k-1} e^{-2\pi i m l / M} \quad (k = 1, 2, \dots, M). \quad (\text{A8})$$

This can be verified by forming

$$\Delta \nu(k) = \nu(k+1) - \nu(k),$$

which gives

$$\Delta \nu(0) = \nu(1) = e^{-2\pi i m l / M} \Big|_{l=0}, \quad (\text{A9})$$

and

$$\nu(k+1) - \nu(k) = \sum_{l=0}^k e^{-2\pi i m l / M} - \sum_{l=0}^{k-1} e^{-2\pi i m l / M},$$

or

$$\Delta \nu(k) = \nu(k+1) - \nu(k) = e^{-2\pi i k m / M} \quad (k = 1, 2, \dots, M). \quad (\text{A10})$$

It is clear that the functions defined by Eqs. (A7) and (A8) solve the difference equation exhibited in Eq. (A5).

We now find an explicit expression for the summation in Eq. (A8). Write Eq. (A8) as

$$\nu(k) = \sum_{l=0}^{k-1} \{ \cos(2\pi m l / M) - i \sin(2\pi m l / M) \} \quad (k = 1, 2, \dots, M). \quad (\text{A11})$$

The summations appearing in Eq. (A11) are known (Hamming 1962, p. 44) and $m \neq 0$, can be written as

$$\sum_{l=0}^{k-1} \cos(2\pi m l / M) = \frac{\sin \frac{2\pi m}{M} \left[k - 1 + \frac{1}{2} \right] + \sin \left[\frac{\pi m}{M} \right]}{2 \sin \left[\frac{\pi m}{M} \right]}, \quad (\text{A12})$$

and

$$\sum_{l=0}^{k-1} \sin(2\pi m l / M) = \frac{\cos \left[\frac{2\pi m}{M} \left[k - 1 + \frac{1}{2} \right] \right] + \cos \left[\frac{\pi m}{M} \right]}{2 \sin \left[\frac{\pi m}{M} \right]}, \quad (\text{A13})$$

where $k = 1, 2, \dots, M$. This completes the explicit determination of the function $\nu(k)$.

We now return to Eq. (A6) to find what simplifications can be made. We note that $\mu(0)$ and $\nu(0)$ both vanish. Also, $\nu(M)$ can be evaluated from Eqs. (A11) to (A13). We have

$$\nu(M) = \frac{\sin \frac{2\pi m}{M} \left[M - 1 + \frac{1}{2} \right] + \sin \left[\frac{\pi m}{M} \right] + i \cos \left[\frac{2\pi m}{M} \left[M - 1 + \frac{1}{2} \right] \right] - i \cos \frac{\pi m}{M}}{2 \sin \frac{\pi m}{M}},$$

and on simplification of the trigonometric terms, the result is obtained that

$$\nu(M) = 0.$$

Consequently, Eq. (A6) takes the form

$$A(m) = - \sum_{k=0}^{M-1} \nu(k+1) = - \sum_{k=0}^{M-1} \left[\frac{\sin \frac{2\pi m}{M} \left[k + \frac{1}{2} \right] + \sin \frac{\pi m}{M}}{2 \sin \left[\frac{\pi m}{M} \right]} + i \frac{\cos \frac{2\pi m}{M} \left[k + \frac{1}{2} \right] - \cos \left[\frac{\pi m}{M} \right]}{2 \sin \frac{\pi m}{M}} \right]. \quad (\text{A14})$$

We now consider each term in Eq. (A14) separately. We obtain

$$\sum_{k=0}^{M-1} \sin \frac{2\pi m}{M} \left[k + \frac{1}{2} \right] = - \frac{\cos \left[\frac{2\pi m}{M} \left[M - 1 + \frac{1}{2} \right] + \frac{\pi m}{M} \right] - \cos \left[-\frac{\pi m}{M} + \frac{\pi m}{M} \right]}{2 \sin \frac{\pi m}{M}},$$

which is found to vanish on simplification. Likewise, we get

$$\sum_{k=0}^{M-1} \cos \frac{2\pi m}{M} \left[k + \frac{1}{2} \right] = 0.$$

Eq. (A14) then simplifies to

$$A(m) = - \sum_{k=0}^{M-1} \left[\frac{1}{2} - i \frac{1}{2} \cos \left[\frac{\pi m}{M} \right] \right]$$

or

$$A(m) = - \frac{M}{2} + i \frac{M}{2} \cot \left[\frac{\pi m}{M} \right] \quad (m = 1, 2, \dots, M - 1). \quad (\text{A15})$$

This is to be supplemented by the results indicated in Eq. (A2) that

$$A(0) = \frac{M(M-1)}{2}. \quad (\text{A16})$$

The expressions given in Eqs. (A15) and (A16) provide the explicit, analytical expression for the discrete Fourier transform of the function $X(k) = k, (k = 0, 1, \dots, M - 1)$, which is the result desired. It should be clear that even relatively simple discrete functions lead to rather difficult summation problems when it is desired to obtain an explicit representation.

Figure A1 shows a plot of the imaginary part of the discrete Fourier transform. When normalized by M the transform values fall on a continuous curve that is independent of M . The real part of the transform is of a simple form and is not exhibited in graphical form.

REFERENCE

Hamming, R.W., *Numerical Methods for Scientists and Engineers* (McGraw-Hill Book Co., Inc., New York, 1962).

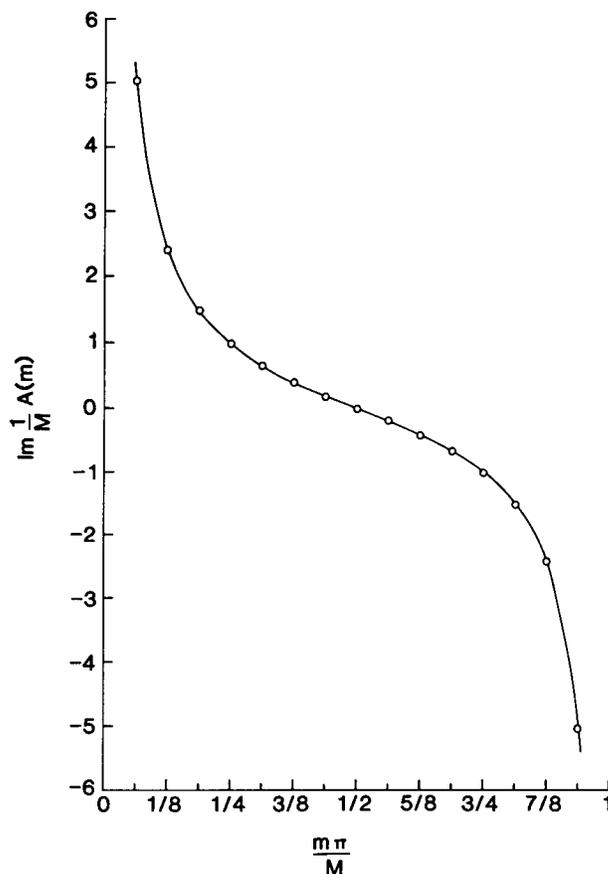


Fig. A1 — The imaginary part of the one-dimensional discrete Fourier transform $A(m)$ of the discrete function $X(k) = k$ ($k = 0, 1, \dots, M - 1$). It is plotted in the form $\text{Im} \left[\frac{1}{M} A(m) \right]$ because when normalized by M the discrete values fall on a single continuous curve that is independent of M . The solid circles are discrete values of the imaginary part of the normalized transform for the case when $M = 2^n$, where $n \geq 4$. For other choices of M , the values would still be along the continuous curve but not always at the circled points. The imaginary part of the transform always vanishes at $m = 0$. The continuous curve is given by $\cot \left[\frac{\pi m}{M} \right]$ with m regarded as continuous. Values of the imaginary part of the normalized transform outside the domain can be obtained by periodic extension of the values shown.