

Cylindrical Bessel Functions for a Large Range of Complex Arguments

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April 29, 1983



NAVAL RESEARCH LABORATORY
Washington, D.C.

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REPORT DOCUMENTATION PAGE		READ INSTRUCTIONS BEFORE COMPLETING FORM
1. REPORT NUMBER NRL Report 8687	2. GOVT ACCESSION NO.	3. RECIPIENT'S CATALOG NUMBER
4. TITLE (and Subtitle) CYLINDRICAL BESSEL FUNCTIONS FOR A LARGE RANGE OF COMPLEX ARGUMENTS		5. TYPE OF REPORT & PERIOD COVERED Interim report on one phase of a continuing NRL problem.
		6. PERFORMING ORG. REPORT NUMBER
7. AUTHOR(s) J. P. Mason		8. CONTRACT OR GRANT NUMBER(s)
9. PERFORMING ORGANIZATION NAME AND ADDRESS Naval Research Laboratory Washington, DC 20375		10. PROGRAM ELEMENT, PROJECT, TASK AREA & WORK UNIT NUMBERS 61153N; RR001-08-41 51-1873-0-3
11. CONTROLLING OFFICE NAME AND ADDRESS Naval Research Laboratory Washington, DC 20375		12. REPORT DATE April 29, 1983
14. MONITORING AGENCY NAME & ADDRESS (if different from Controlling Office)		13. NUMBER OF PAGES 13
		15. SECURITY CLASS. (of this report) UNCLASSIFIED
16. DISTRIBUTION STATEMENT (of this Report) Approved for public release; distribution unlimited.		15a. DECLASSIFICATION/DOWNGRADING SCHEDULE
		17. DISTRIBUTION STATEMENT (of the abstract entered in Block 20, if different from Report)
18. SUPPLEMENTARY NOTES		
19. KEY WORDS (Continue on reverse side if necessary and identify by block number) Bessell functions Cylindrical Bessel function		
20. ABSTRACT (Continue on reverse side if necessary and identify by block number) The evaluation of Bessel functions of the first and second kinds, covering a wide range of complex arguments and integer orders, is required in the determination of the intensity of acoustic reflection from absorbing bodies. Numerical problems associated with the calculations are discussed, and various means by which these problems have been overcome are explained. The numerical methods used in calculating the Bessel functions of the first, second, and third kinds are given, as well as sample results and numerical checks in the form of computer plots and printouts.		

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CYLINDRICAL BESSEL FUNCTIONS FOR A LARGE RANGE OF COMPLEX ARGUMENTS

1. INTRODUCTION

Many problems in physics and engineering require the use of so-called special functions; an important subset known as Bessel functions is essential to the study of scattering of waves by an insonified object. This area of research is of particular interest to those groups concerned with the non-destructive evaluation of materials, underwater acoustics, and elastic wave propagation.

A major difficulty arises when these insonified bodies possess intrinsic absorption properties. Determining the scattering of acoustic waves from submerged absorbing bodies requires the calculation of complex Bessel functions of the first and second kinds, for orders ranging from zero to a value approximately equal to the magnitude or "absolute value" of the argument [1,2]. Due to the longitudinal and shear properties of the viscoelastic materials used, the magnitude of the arguments for which the Bessel functions are to be calculated can be extremely large [3]. To date, no effective way has been available to treat Bessel functions of such large complex arguments for such a wide range of integer orders.

For arguments with large positive or negative imaginary parts, the values of the Bessel functions of the first kind, $J_n(z)$ ("common Bessel functions"), and second kind, $Y_n(z)$ ("Neumann functions"), are very large at the low orders, and for sufficiently big positive or negative imaginary parts $iJ_n(z)$ approaches $Y_n(z)$. These factors create several problems:

- a. when the values of the common Bessel and Neumann functions at low orders become larger and closer to one another, matrix singularities arise in the calculation of the absorption properties, which lead to the introduction of quantities too small to express in fixed length computer words;
- b. to generate an accurate table of Neumann functions by a recurrence relationship, for successive values of the order, a combination of backward and forward recursion would have to be used; and
- c. the function values can become so large as to exceed the permissible floating-point exponent range of the computer.

The next section discusses the steps the author has taken to overcome these problems.

2. APPROACH

To circumvent the first two problems, Bessel functions of the third kind, $H_n^{(i)}(z)$, $i = 1, 2$ ("Hankel functions"), rather than Neumann functions, are calculated when z is large. The values of the Hankel functions are small in magnitude at low orders, thus reducing the chance of a matrix singularity; moreover, for orders of value greater than 1 they can be generated by the forward relationship alone.

As implemented, the quadrature method (Simpson's Rule) used to calculate the Hankel functions does not yield accurate results for small arguments; therefore the absorption program is written to use

Neumann functions for small arguments and Hankel functions for large arguments. A general-purpose subroutine was written, however, that can generate all three functions for any complex argument and any positive integer order. If $Y_n(z)$ for $0 \leq n \leq N$ and large $|z|$ is to be generated, the subroutine first calculates $J_n(z)$ and either $H_n^{(1)}(z)$ or $H_n^{(2)}(z)$. Then it uses the relationship:

$$H_n^{(1)}(z) = J_n(z) + iY_n(z)$$

or

$$H_n^{(2)}(z) = J_n(z) - iY_n(z)$$

to generate $Y_n(z)$. (In the program $H_n^{(1)}(z)$ is calculated when z is in the first or second quadrant; $H_n^{(2)}(z)$ is calculated when z is in the third or fourth quadrant.) Alternately, if $|z|$ is small and $H_n^{(i)}(z)$ is to be generated, $J_n(z)$ and $Y_n(z)$ are calculated first and then one of these two relationships is used to solve for $H_n^{(i)}(z)$.

The subroutine is designed so that a table of values for each of the functions is generated, for successive integer values of n from zero to the order specified. Generally, for functions of order $n > 1$, answers are generated by using the recursion relationship [4,5]:

$$C_{\nu-1}(z) + C_{\nu+1}(z) = (2\nu/z) C_\nu(z).$$

The one exception occurs for $J_2(z)$ to $J_n(z)$ when $|z| < 0.5$ and the maximum order ≤ 5 ; see Part 3b(1), Methods of Computation. For small arguments, entire tables of both $J_n(z)$ and $Y_n(z)$ are calculated; then values of $H_n^{(i)}(z)$ are generated from these. For large arguments, tables of $J_n(z)$ and $H_n^{(i)}(z)$ are calculated; and then the $Y_n(z)$ table is generated by using them. In this way, one avoids the not so simple problem of generating an accurate table of $Y_n(z)$ for extremely large arguments by the recursion operation. The difficulty encountered if one were to attempt to use the recursion relationship to compute a $Y_n(z)$ table arises because the recursion operation works best if each successive function value is greater in magnitude than the one preceding. And $|Y_n(z_{\text{large}})|$ is initially a decreasing function, but at a value of n roughly equal to $|z|$, it begins to increase. (A similar difficulty arises in the recurrence calculation for the spherical Bessel function, $y_n(z)$ [6].) This means that the $Y_n(z)$ table would have to be calculated backward, roughly, from $n \sim |z|$ to $n = 0$, and forward from $n \sim |z|$ to maximum n , and therefore one would have to be able to predict, fairly accurately, where the minimum absolute value of the function occurs. On the other hand, $|J_n(z)|$, $|Y_n(z_{\text{small}})|$, and $|H_n^{(i)}(z)|$ are either monotonically increasing or monotonically decreasing functions, so the recursion operation can begin at $n = 1$ (or $n = n_{\text{max}}$). See Figs. 1 and 2.

Figures 1(a), (b), and (c) show the normalized curves of the real part for each of the three kinds of Bessel functions, for an argument equal to $201 + 150i$ and a range of integer orders from zero to 462. Figure 1(d) shows the three curves superimposed. Unfortunately, if the maximum order, n , were made larger than 462 to show better the oscillations in $Y_n(z)$ and $H_n^{(1)}(z)$ at large orders, then the $Y_n(z)$ values at low orders would appear to vanish because the function values at $n > 462$ are so much larger, relatively. Normalized curves of the imaginary part follow the same general pattern.

Figure 2 gives another illustration of the behavior of these functions. Here $\log |J_n(z)|$, $\log |Y_n(z)|$, and $\log |H_n^{(1)}(z)|$ are plotted for six different values of the argument, z . Note that the turning points for the Neumann functions, in Fig. 2(b), occur at values close to but always smaller than zero, i.e. they have a function value less than one. Also note that when the three sets of plots are superimposed, as in Fig. 2(d), in this scale the Hankel curves appear to be a reflection of the common Bessel curves, and the Neumann curves coincide with the common Bessel curves before the Neumann turning point and with the Hankel curves after the turning point.

The numerical problem of exceeding the exponent range of the computer has been solved by a programmed scaling of each step of the calculation, where necessary, over and above the built-in digital

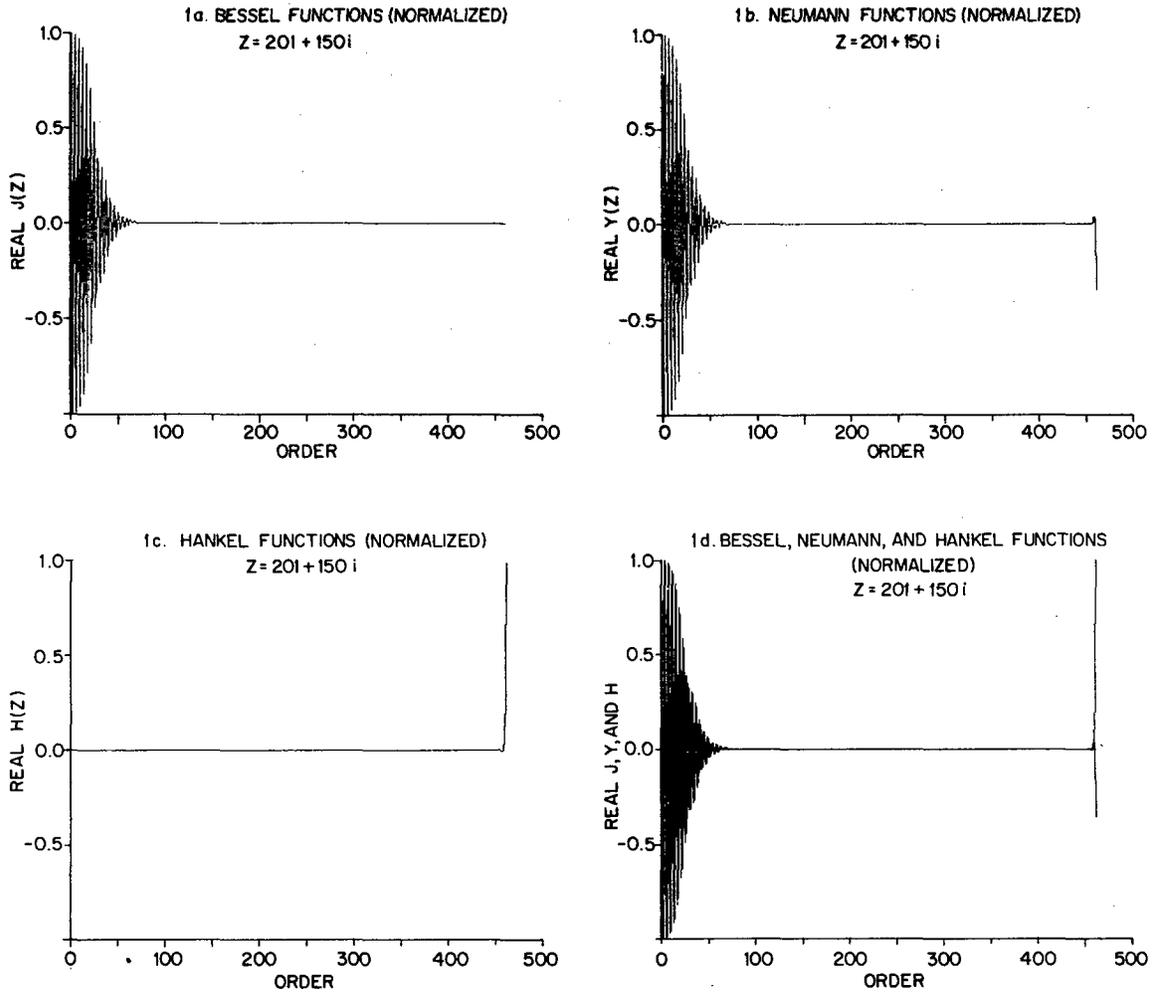


Fig. 1 — Computer-generated plots (normalized), $Z = 201 + 150i$. (a) Bessel functions; (b) Neumann functions; (c) Hankel functions; (d) Bessel, Neumann, and Hankel functions superimposed.

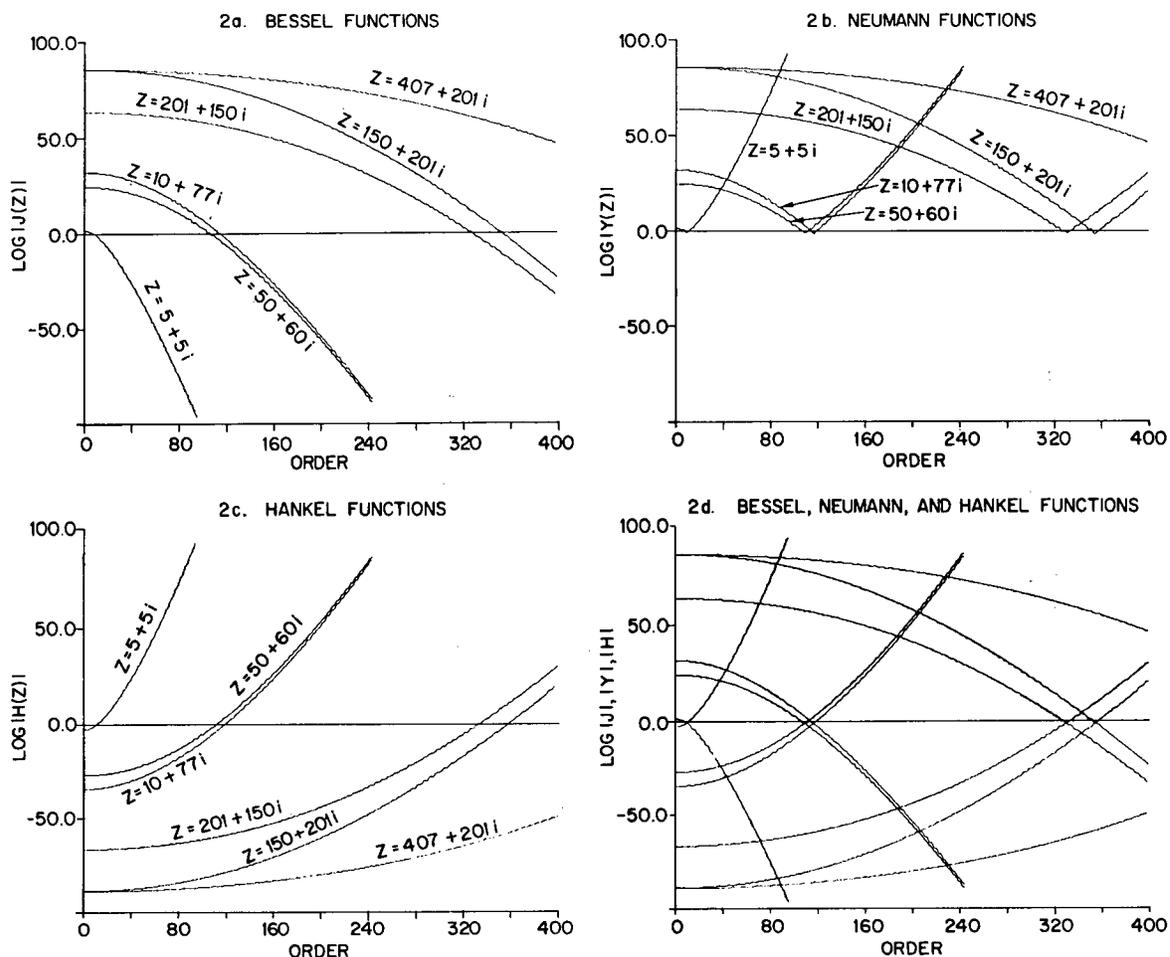


Fig. 2 — Computer-generated plots (logarithmic). (a) Bessel functions; (b) Neumann functions; (c) Hankel functions; (d) Bessel, Neumann, and Hankel functions superimposed.

scaling that is effected automatically by the machine's double-precision floating-point arithmetic. In our case, the values of the functions are scaled by multiples of $10^{\pm 70}$, but the size of this factor depends really upon the exponent range of the particular computer in use.

Theoretically there should be no limits on the values of n , x , or y , but at the present time the subroutines have been tested only for integer n on the interval $0 \leq n \leq 3010$, and on the rectangular region defined by the complex points $\pm 3000 \pm 3000i$.

3. METHODS OF COMPUTATION

a. Procedures

The necessity for using various methods or a combination of methods was indicated, as there appeared to be no single superior one [7,8]. Figure 3 shows the number of different procedures used to generate the three function tables and the portion of the complex plane in which each of these methods is applied. The precise outline of these procedures, including the formulas used, is given in Part 3b. Notice that when $5.0 < |y| \leq 10.0$, there is a choice of one of three possible procedures. The integral method, given in Part 3b(6), is always employed if the Hankel function is to be calculated; otherwise, the power series, Part 3b(2), or the asymptotic expansion, Part 3b(3), is used.

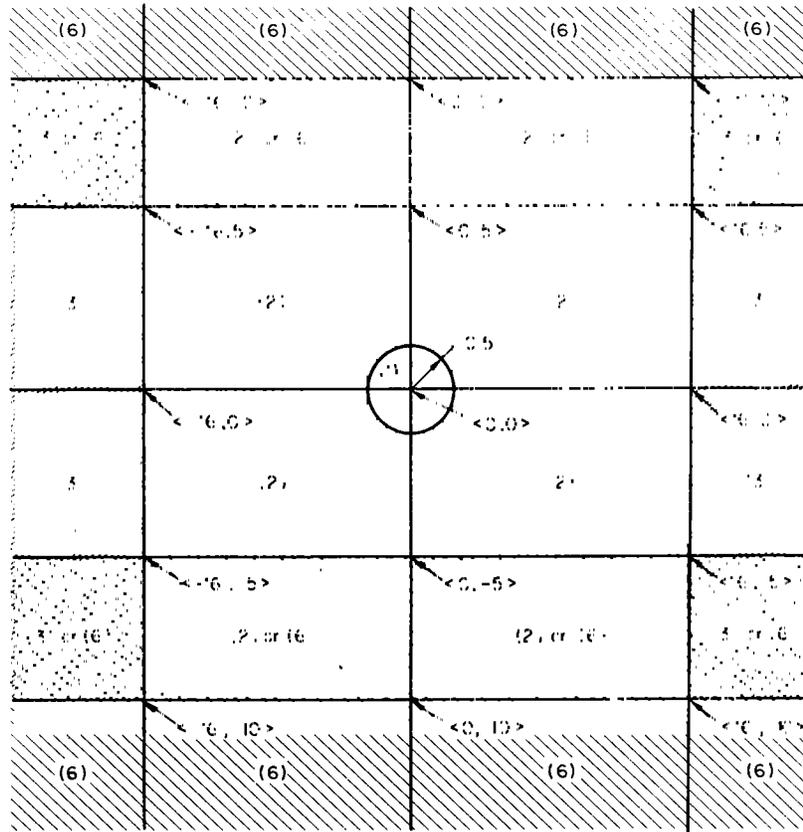


Fig. 3 — Procedural regions in the Z-plane (see Part 3b for definitions of (1), (2), (3), and (6))

b. Definitions

- (1) If $|z| < 0.5$ and maximum order ≤ 5 ; $J_0(z)$, $J_1(z)$, $Y_0(z)$, and $Y_1(z)$ are calculated by means of a power series [9,10]:

$$u_0(\rho, \phi) = \sum_{k=0}^{\infty} (-1)^k \frac{(\rho/2)^{2k}}{(k!)^2} \cos 2k\phi$$

$$v_0(\rho, \phi) = \sum_{k=0}^{\infty} (-1)^k \frac{(\rho/2)^{2k}}{(k!)^2} \sin 2k\phi$$

$$u_1(\rho, \phi) = -\frac{\partial}{\partial \rho} [u_0 \cos \phi + v_0 \sin \phi]$$

$$v_1(\rho, \phi) = \frac{\partial}{\partial \rho} [u_0 \sin \phi - v_0 \cos \phi]$$

$$J_n(\rho e^{i\phi}) = u_n(\rho, \phi) + i v_n(\rho, \phi)$$

$$U_0(\rho, \phi) = \frac{2}{\pi} \left\{ u_0(\rho, \phi) \left[\gamma + \ln \frac{\rho}{2} \right] - \phi v_0(\rho, \phi) \right\} + S_0(\rho, \phi)$$

$$V_0(\rho, \phi) = \frac{2}{\pi} \left\{ v_0(\rho, \phi) \left[\gamma + \ln \frac{\rho}{2} \right] + \phi u_0(\rho, \phi) \right\} + T_0(\rho, \phi)$$

$$\begin{aligned}
 U_1(\rho, \phi) &= \frac{2}{\pi} \left\{ u_1(\rho, \phi) \left[\gamma + \ln \frac{\rho}{2} \right] - \phi \nu_1(\rho, \phi) \right\} - \frac{2}{\pi \rho} \cos \phi - S_1(\rho, \phi) \\
 V_1(\rho, \phi) &= \frac{2}{\pi} \left\{ \nu_1(\rho, \phi) \left[\gamma + \ln \frac{\rho}{2} \right] + \phi u_1(\rho, \phi) \right\} + \frac{2}{\pi \rho} \sin \phi - T_1(\rho, \phi) \\
 S_0(\rho, \phi) &= \frac{2}{\pi} \sum_{k=1}^{\infty} \frac{(-1)^{k+1} \rho^{2k}}{2^{2k} (k!)^2} \left[1 + \frac{1}{2} + \dots + \frac{1}{k} \right] \cos 2k\phi \\
 T_0(\rho, \phi) &= \frac{2}{\pi} \sum_{k=1}^{\infty} \frac{(-1)^{k+1} \rho^{2k}}{2^{2k} (k!)^2} \left[1 + \frac{1}{2} + \dots + \frac{1}{k} \right] \sin 2k\phi \\
 S_1(\rho, \phi) &= \frac{1}{\pi} \sum_{k=0}^{\infty} \frac{(-1)^k \rho^{2k+1}}{2^{2k+1} k!(k+1)!} \left[2 \left[1 + \frac{1}{2} + \dots + \frac{1}{k} \right] + \frac{1}{k+1} \right] \cos [(2k+1)\phi] \\
 T_1(\rho, \phi) &= \frac{1}{\pi} \sum_{k=0}^{\infty} \frac{(-1)^k \rho^{2k+1}}{2^{2k+1} k!(k+1)!} \left[2 \left[1 + \frac{1}{2} + \dots + \frac{1}{k} \right] + \frac{1}{k+1} \right] \sin [(2k+1)\phi] \\
 Y_n(\rho e^{i\phi}) &= U_n(\rho, \phi) + iV_n(\rho, \phi).
 \end{aligned}$$

$J_2(z)$ to $J_n(z)$ are calculated from:

$$J_n(z) = \left(\frac{z}{2} \right)^n \sum_{k=0}^{\infty} \frac{(-z^2/4)^k}{k! \Gamma(n+k+1)}.$$

$Y_2(z)$ to $Y_n(z)$ are calculated by using the forward recursion relationship:

$$Y_{n+1}(z) = (2n/z) Y_n(z) - Y_{n-1}(z).$$

For $H_0^{(i)}(z)$ to $H_n^{(i)}(z)$ one of the following relationships is used:

$$H_n^{(1)}(z) = J_n(z) + iY_n(z)$$

or

$$H_n^{(2)}(z) = J_n(z) - iY_n(z).$$

- (2) If $|z| \geq 0.5$, $|x| < 16.0$, and $|y| \leq 5.0$ or if $|z| < 0.5$ and maximum order > 5 ; $J_0(z)$, $J_1(z)$, $Y_0(z)$, and $Y_1(z)$ are calculated by means of the power series in Part 3b(1).

$J_2(z)$ to $J_n(z)$ are calculated by using the backward recursion relationship [11-13]:

$$J_{n-1}(z) = (2n/z)J_n(z) - J_{n+1}(z).$$

$Y_2(z)$ to $Y_n(z)$ are calculated by forward recursion, as in Part 3b(1). $H_0^{(i)}(z)$ to $H_n^{(i)}(z)$ are calculated by the same method as in Part 3b(1).

- (3) If $|x| \geq 16.0$ and $|y| \leq 5.0$; $J_0(z)$, $J_1(z)$, $Y_0(z)$, and $Y_1(z)$ are calculated by means of an asymptotic expansion [9,10]:

$$\begin{aligned}
 J_\nu(z) &\sim \left(\frac{2}{\pi z} \right)^{1/2} \left\{ P_\nu(z) \cos \left[z - \frac{\nu\pi}{2} - \frac{\pi}{4} \right] - Q_\nu(z) \sin \left[z - \frac{\nu\pi}{2} - \frac{\pi}{4} \right] \right\} \\
 Y_\nu(z) &\sim \left(\frac{2}{\pi z} \right)^{1/2} \left\{ P_\nu(z) \sin \left[z - \frac{\nu\pi}{2} - \frac{\pi}{4} \right] + Q_\nu(z) \cos \left[z - \frac{\nu\pi}{2} - \frac{\pi}{4} \right] \right\}
 \end{aligned}$$

where

$$P_\nu(z) = 1 + \sum_{k=1}^{\infty} \frac{(-1)^k (4\nu^2 - 1^2)(4\nu^2 - 3^2) \dots (4\nu^2 - \{4k - 1\}^2)}{(2k)! 2^{6k} z^{2k}}$$

$$Q_\nu(z) = \sum_{k=1}^{\infty} \frac{(-1)^{k+1} (4\nu^2 - 1^2)(4\nu^2 - 3^2) \dots (4\nu^2 - \{4k - 3\}^2)}{(2k - 1)! 2^{6k-3} z^{2k-1}}$$

$J_2(z)$ to $J_n(z)$ are found by backward recursion; $Y_2(z)$ to $Y_n(z)$ are found by forward recursion. $H_0^{(i)}(z)$ to $H_n^{(i)}(z)$ are calculated by using the same method as in Part 3b(1).

- (4) If $|x| < 16.0$, $5.0 < |y| \leq 10.0$, and the Hankel functions are not needed, the common Bessel and Neumann functions are calculated as in Part 3b(2). If, on the other hand, the Hankel function is to be computed, the common Bessel, Neumann, and Hankel functions are calculated by using the relationships given in Part 3b(6), q.v.
- (5) If $|x| \geq 16.0$, $5.0 < |y| \leq 10.0$, and the Hankel functions are not needed, the common Bessel and Neumann functions are calculated as in Part 3b(3). If, however, the Hankel function is to be computed, the common Bessel, Neumann, and Hankel functions are calculated by using the relationships given in Part 3b(6), q.v.
- (6) If $|y| > 10.0$ (or greater than 5.0, if Hankel functions are needed), $J_0(z)$ and $J_1(z)$ are calculated by:

$$J_n(z) = \frac{1}{\pi} \int_0^\pi \cos(z \sin \theta - n\theta) d\theta.$$

For $H_0^{(i)}(z)$ and $H_1^{(i)}(z)$, where y is positive:

$$H_n^{(1)}(z) = \frac{-2ie^{-in\pi}}{\sqrt{\pi}} \frac{(z/2)^n}{\Gamma(1/2 + n)} \int_0^\infty \frac{t^{2n} e^{iz(t^2+1)^{1/2}}}{(t^2 + 1)^{1/2}} dt,$$

and where y is negative:

$$H_n^{(2)}(z) = \frac{2i(-1)^n}{\sqrt{\pi}} \frac{(z/2)^n}{\Gamma(1/2 + n)} \int_0^\infty \frac{t^{2n} e^{-iz(t^2+1)^{1/2}}}{(t^2 + 1)^{1/2}} dt.$$

For each of the integrals, Simpson's quadrature method is used; the number of subdivisions necessary is determined by convergence criteria.

$J_2(z)$ to $J_n(z)$ are found by backward recursion; $H_2^{(i)}(z)$ to $H_n^{(i)}(z)$ are found by forward recursion. For $Y_0(z)$ to $Y_n(z)$, one of the following relationships is used:

$$Y_n(z) = iJ_n(z) - iH_n^{(1)}(z)$$

or

$$Y_n(z) = -iJ_n(z) + iH_n^{(2)}(z).$$

4. VERIFICATION

Because published tables often are limited in range or lack precision, the accuracy of the derived function values was checked with the Wronskian relationship [4]:

$$J_{n+1}(z) Y_n(z) - J_n(z) Y_{n+1}(z) = 2/(\pi z)$$

$$[J_n(z) H_{n+1}^{(1)}(z) - J_{n+1}(z) H_n^{(1)}(z)]i = 2/(\pi z)$$

or

$$J_{n+1}(z) Y_n(z) - J_n(z) Y_{n+1}(z) = 2/(\pi z)$$

$$-[J_n(z) H_{n+1}^{(2)}(z) - J_{n+1}(z) H_n^{(2)}(z)]i = 2/(\pi z).$$

Table 1 gives an example of the calculated values of $J_n(z)$, $Y_n(z)$, and $H_n^{(2)}(z)$, to 16 digits, and of the corresponding Wronskians for an argument $z = 3000-3000i$ and orders $n = 0, 1, 2, \dots, 9$. The number in the last column is the value of u in the expression 10^{70u} that is used to multiply each value on that line. For example, the fourth line states that

Table 1 — Example of the Calculated Values of $J_n(z)$, $Y_n(z)$, and $H_n^{(2)}(z)$, to 16 Digits, and of the Corresponding Wronskians for an Argument $z = 3000 - 3000i$ and Orders $n = 0, 1, 2, \dots, 9$

	$Z =$	0.3000000000000000D 04	-0.3000000000000000D 04		
	$2/(PI*Z) =$	0.1061032953945969D-03	0	0.1061032953945969D-03 0	
		REAL PART		IMAGINARY PART	
N =	0	J(Z) =	-0.3828646932535671D 41	18	0.2697077698538684D 41 18
		Y(Z) =	0.2697077698538684D 41	18	0.3828646932535671D 41 18
		H(Z) =	-0.2737074579081907D-45	-18	-0.1578462344538983D-44 -18
		WRONSKIAN =	0.1061032953945963D-03	0	0.1061032953947169D-03 0
N =	1	J(Z) =	0.2696533861563628D 41	18	0.3828552653836766D 41 18
		Y(Z) =	0.3828552653836766D 41	18	-0.2696533861563628D 41 18
		H(Z) =	0.1578571076011431D-44	-18	-0.2738617944315793D-45 -18
		WRONSKIAN =	0.1061032953945963D-03	0	0.1061032953947169D-03 0
N =	2	J(Z) =	0.3828269592938247D 41	18	-0.2694902669700219D 41 18
		Y(Z) =	-0.2694902669700219D 41	18	-0.3828269592938247D 41 18
		H(Z) =	0.2743249355316717D-45	-18	0.1578897247632843D-44 -18
		WRONSKIAN =	0.1061032953945963D-03	0	0.1061032953947169D-03 0
N =	3	J(Z) =	-0.2692185080055202D 41	18	-0.3827797075887941D 41 18
		Y(Z) =	-0.3827797075887941D 41	18	0.2692185080055202D 41 18
		H(Z) =	-0.1579440790886165D-44	-18	0.2750972758870223D-45 -18
		WRONSKIAN =	0.1061032953945963D-03	0	0.1061032953947169D-03 0
N =	4	J(Z) =	-0.3827133980942414D 41	18	0.2688382687544276D 41 18
		Y(Z) =	0.2688382687544276D 41	18	0.3827133980942414D 41 18
		H(Z) =	-0.2761794735984449D-45	-18	-0.1580201591147842D-44 -18
		WRONSKIAN =	0.1061032953945963D-03	0	0.1061032953947169D-03 0
N =	5	J(Z) =	0.2683497724497221D 41	18	0.3826278740830077D 41 18
		Y(Z) =	0.3826278740830077D 41	18	-0.2683497724497221D 41 18
		H(Z) =	0.1581179487042897D-44	-18	-0.2775724506400173D-45 -18
		WRONSKIAN =	0.1061032953945963D-03	0	0.1061032953947169D-03 0
N =	6	J(Z) =	0.3825229345915194D 41	18	-0.2677533060102064D 41 18
		Y(Z) =	-0.2677533060102064D 41	18	-0.3825229345915194D 41 18
		H(Z) =	0.2792773934945831D-45	-18	0.1582374269541847D-44 -18
		WRONSKIAN =	0.1061032953945963D-03	0	0.1061032953947169D-03 0
N =	7	J(Z) =	-0.2670492199685187D 41	18	-0.3823983348258451D 41 18
		Y(Z) =	-0.3823983348258451D 41	18	0.2670492199685187D 41 18
		H(Z) =	-0.1583785680794992D-44	-18	0.2812957539660902D-45 -18
		WRONSKIAN =	0.1061032953945963D-03	0	0.1061032953947169D-03 0
N =	8	J(Z) =	-0.3822537866568523D 41	18	0.2662379283823529D 41 18
		Y(Z) =	0.2662379283823529D 41	18	0.3822537866568523D 41 18
		H(Z) =	-0.2836292501756923D-45	-18	-0.1585413412704448D-44 -18
		WRONSKIAN =	0.1061032953945963D-03	0	0.1061032953947169D-03 0
N =	9	J(Z) =	0.2653199087284141D 41	18	0.3820889592037798D 41 18
		Y(Z) =	0.3820889592037798D 41	18	-0.2653199087284141D 41 18
		H(Z) =	0.1587257105228402D-44	-18	-0.2862798677337705D-45 -18
		WRONSKIAN =	0.1061032953945963D-03	0	0.1061032953947169D-03 0

$$J_0(3000-3000i) = (-0.382864 \dots \times 10^{41} + 0.269707 \dots \times 10^{41}i) \times 10^{(70 \times 18)},$$

in other words,

$$J_0(3000-3000i) = -0.382864 \dots \times 10^{1301} + 0.269707 \dots \times 10^{1301}i.$$

The seventh line gives the Wronskian for the calculated function values. This should be compared with the value of $2/(\pi z)$ given in the second line. Table 2 lists the function values and Wronskians for the same argument, but for orders $n = 3000, 3001, 3002, \dots, 3009$.

Although it would be impractical to describe here all the tests that were performed to determine the best range for each of the procedures covered in Part 3, some discussion might be of interest. One set of tests consisted of generating $J_n(z)$ and $Y_n(z)$ (or $J_n(z)$, $Y_n(z)$, and $H_n(z)$) for $0 \leq n \leq 24$ near the points of procedure change, e.g. $z = 0 \pm 5i$, $|z| = 0.5$, $z = \pm 16 \pm 5i$, etc. (see Fig. 3). The computed function values and their corresponding Wronskians were examined at $n = 24$. Whether comparing the function values generated by the integral method to those generated by the series method or to those generated by the asymptotic method, the $J_{24}(z)$'s always agreed to at least 12 figures and frequently to 16 figures. And for $Y_{24}(z)$ agreement ranged from 4 figures at $z = 15.95 + 10.0i$ to 12 figures at $z = 0.01 + 5.01i$. The accuracy of the Wronskians varied from a low of 4 figures for $z = 15.95 + 10.0i$, if the Bessel and Neumann functions were used, to a high of 15 figures for several values of z . Interestingly, when the routine was temporarily modified in such a way that $Y_0(15.95 + 10.0i)$ was calculated by the series, asymptotic, and integral methods, all three calculations agreed to 14 figures for $n = 0$, and $P_0(z)P_1(z) + Q_0(z)Q_1(z)$ equaled 1.0 to 14 figures. The three methods gave values that agreed to 7 figures for $n = 20$. This indicates that the calculation of $Y_0(15.95 + 10.0i)$ by the series method is sufficiently accurate but that as n increases the forward recursion loses accuracy.

The $P_n(z)P_{n+1}(z) + Q_n(z)Q_{n+1}(z) = 1.0$ test was also made using the P and Q results for $z = 150.0 + 4.95i$ at $n = 0$.

The test answer equals $0.1000000000000000 \times 10^1 - 0.6286572655403010 \times 10^{-22}i$.

5. PORTABILITY

The results shown in Tables 1 and 2 were generated by a subroutine written in double-precision FORTRAN and run on the Texas Instruments, Inc. computer, ASC. This computer has a word length of 32 bits, with an exponent range of approximately ± 75 . The subroutine was easily modified for running on a Digital Equipment Corp. PDP-11. This computer also has a word length of 32 bits; however, because the exponent range is approximately ± 36 , the scale factor had to be reduced. Even though the PDP-11 does not provide double-precision complex arithmetic, no accuracy was lost since only real arithmetic is employed in the subroutine. There is less internal storage space in the PDP-11, so the maximum allowable order and argument sizes are approximately $n \leq 230$ and $|z| \leq 230$.

The integral method is always employed if the Hankel function is to be calculated prior to the Neumann function. This approach was taken because, although the asymptotic expansion portion can be modified rather easily to give $H_0(z)$ and $H_1(z)$ (in fact, instructions for doing so are incorporated in the subroutine), scaling here would have presented many more problems than scaling the integral method. If one has access to a computer with a very large exponent range, e.g. the DEC VAX-750, the asymptotic expansion could be used for a much greater range of z -values than shown in Fig. 3.

6. FUTURE USE

Up to the present time, these Bessel function subroutines have been used in the determination of the acoustic reflection from absorbing infinite cylinders. These subroutines will become even more valuable as the work being done on finite cylinders is expanded to include the absorption properties.

Table 2 — Example of the Calculated Values of $J_n(z)$, $Y_n(z)$, and $H_n^{(2)}(z)$, to 16 Digits, and of the Corresponding Wronskians for an Argument $z = 3000 - 3000i$ and Orders $n = 3000, 3001, 3002, \dots, 3009$

Z =		0.3000000000000000D 04		-0.3000000000000000D 04	
2/(PI*Z) =		0.1061032953945969D-03		0 0.1061032953945969D-03	
		REAL PART		IMAGINARY PART	
N =	3000	J(Z) =	-0.4061975773403832D 53	13	0.5106328653828673D 54
		Y(Z) =	0.5106328653828673D 54	13	0.4061975773403832D 53
		H(Z) =	0.1116851357057360D-57	-13	-0.8193785786170054D-58
		WRONSKIAN =	0.1061032953945970D-03	0	0.1061032953947177D-03
N =	3001	J(Z) =	0.2596740771583906D 54	13	0.1527705590799730D 54
		Y(Z) =	0.1527705590799730D 54	13	-0.2596740771583906D 54
		H(Z) =	0.2082766079856745D-57	-13	0.1099101270979637D-57
		WRONSKIAN =	0.1061032953945971D-03	0	0.1061032953947177D-03
N =	3002	J(Z) =	0.1475589103184821D 54	13	-0.9805074759909088D 53
		Y(Z) =	-0.9805074759909088D 53	13	-0.1475589103184821D 54
		H(Z) =	-0.1328586599106263D-58	-13	0.4002306551903667D-57
		WRONSKIAN =	0.1061032953945970D-03	0	0.1061032953947177D-03
N =	3003	J(Z) =	-0.1390067946887262D 53	13	-0.1032293909187689D 54
		Y(Z) =	-0.1032293909187689D 54	13	0.1390067946887262D 53
		H(Z) =	-0.6220688068478915D-57	-13	0.2772926252941398D-57
		WRONSKIAN =	0.1061032953945971D-03	0	0.1061032953947177D-03
N =	3004	J(Z) =	-0.5814087015713586D 53	13	-0.1919645285893830D 53
		Y(Z) =	-0.1919645285893830D 53	13	0.5814087015713586D 53
		H(Z) =	-0.8869749275831106D-57	-13	-0.7453516129256721D-57
		WRONSKIAN =	0.1061032953945971D-03	0	0.1061032953947177D-03
N =	3005	J(Z) =	-0.2509566371905585D 53	13	0.2578895147200668D 53
		Y(Z) =	0.2578895147200668D 53	13	0.2509566371905585D 53
		H(Z) =	0.4802566611042430D-57	-13	-0.1911795601190267D-56
		WRONSKIAN =	0.1061032953945971D-03	0	0.1061032953947177D-03
N =	3006	J(Z) =	0.7171447274088243D 52	13	0.1989089609147737D 53
		Y(Z) =	0.1989089609147737D 53	13	-0.7171447274088243D 52
		H(Z) =	0.3283013943648111D-56	-13	-0.6885732253938286D-57
		WRONSKIAN =	0.1061032953945970D-03	0	0.1061032953947177D-03
N =	3007	J(Z) =	0.1235077600403195D 53	13	0.1327516580290065D 52
		Y(Z) =	0.1327516580290065D 52	13	-0.1235077600403195D 53
		H(Z) =	0.3499273682275780D-56	-13	0.4511425200881057D-56
		WRONSKIAN =	0.1061032953945970D-03	0	0.1061032953947177D-03
N =	3008	J(Z) =	0.3877533088309038D 52	13	-0.6180687491125274D 52
		Y(Z) =	-0.6180687491125274D 52	13	-0.3877533088309038D 52
		H(Z) =	-0.4297527149130133D-56	-13	0.8717963739278029D-56
		WRONSKIAN =	0.1061032953945970D-03	0	0.1061032953947177D-03
N =	3009	J(Z) =	-0.2265733503052480D 52	13	-0.3636812728180477D 52
		Y(Z) =	-0.3636812728180477D 52	13	0.2265733503052480D 52
		H(Z) =	-0.1654947254638636D-55	-13	-0.7920077982609920D-58
		WRONSKIAN =	0.1061032953945971D-03	0	0.1061032953947177D-03

7. ACKNOWLEDGMENTS

The author is grateful to Louis Dragonette of the Naval Research Laboratory and Lawrence Flax of the Naval Coastal Systems Center for encouragement during the preparation of this report.

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