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On the Accuracy of a Numerical Integration Procedure for Computing Diffraction Fields using Table-Look-Up

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In this report we have analyzed the errors introduced by table-look-up when an electromagnetic diffraction integral is performed numerically. We have shown that, in order to keep the error of both the real and imaginary components to be less than ϵ, the required table should have at least K entries uniformily distributed within a 2π range $K \geqslant 3.85/\epsilon\sqrt{N}$ and N is the number of terms used for numerical integration. A numerical example is presented that confirms this prediction.

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CONTENTS

ON THE ACCURACY OF A NUMERICAL INTEGRATION PROCEDURE FOR COMPUTING DIFFRACTION FIELDS USING TABLE-LOOK-UP

INTRODUCTION

In a classical electromagnetic problem, it is often required to compute the radiation field either radiated directly from a known current source or scattered from a secondary source. In the latter case, the incident field on the scatterer is known or can be determined with the known boundary conditions. The vector potential function of such an electromagnetic field is evaluated by integrating the current source or current distribution on the surface of the scatterer. Then the radiation field in space can be determined from this vector potential function. In general this leads to the evaluation of the integral [1, 2]

$$\mathbf{E}(\theta, \Phi) = \int_s \mathbf{J}(\xi, \eta)\exp[jG(\xi, \eta, \Phi, \theta)]\,d\xi\,d\eta, \tag{1}$$

where $\mathbf{J}$ is the current density excited by the source of the scatterer surface, and ξ, η are the coordinates of the current source; Φ and θ are the coordinates of a field point. It can be shown that the above vector integration can be converted into a scalar integration function such that

$$F(\theta, \Phi) = k\int f(\xi, \eta)\exp[jG(\xi, \eta, \Phi, \theta)]\,d\xi\,d\eta. \tag{2}$$

In general, the current distribution and the geometry of the source or scatterers are very complicated. Except for a very few simple caes, it is impossible to integrate the above equation into a closed form. This integration is usually performed numerically on a digital computer. Although it is straightforward to program such an integration, the required computation time may sometimes be extremely lengthy. As an example, if we assume a geometry for which the current source is not too complicated, a modest 100 mesh points for each of the ξ and η dimensions may be required. Then it requires 10^4 points of summation for this integration for each field point at a given θ and Φ. Now suppose that a 100 by 100 mesh points are needed to map the entire radiation space. Although the $F(\xi, \eta)$ current function does not need to be repeated, the complex phase function (the Green's function) is nevertheless a function of both source coordinate and field coordinate. Thus, computation of 10^8 points is required in this example. At each point a computation of sine and cosine functions is required. In general, computation of this sine and cosine is a time-consuming process for a digital computer. Most fast machines can probably compute these sine and cosine functions and its argument within perhaps 100 μs. To compute one set of output data for this example, a central processor time of 2.8 h is required just for the sines and cosines. Such an exercise is certainly not cheap.

There are many ways to combat this problem. For example, Eq. (2) is a generalized Fourier transformation. Therefore, the fast Fourier transformation (FFT) may be applied. Unfortunately, the exponential function $G(\theta, \Phi, \eta, \xi)$ in general has no linear relation between source coordinates and field point coordinates. In general, a prerequisite of the FFT is that the phase function G computed for any field point (θ, ϕ) must be contained in a set that is finite and the whole set is generated at a certain θ, Φ point. This requirement sometimes puts a contraint on the choice of the integration mesh points that cannot be achieved.

There are other ways. One example is found in the computation of reflector antenna patterns. In this case Galindo-Israel and Mittra [3] proposed to expand the integral with an infinite series as a function of the field point coordinates. The coefficients of such a series are the Fourier transformation of

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the η, ξ coordinates. To compute such coefficients, similar integration of the phase function is required. Furthermore, this series expansion method applies only to this special case.

The third way, perhaps the simplest approach, is table-look-up. In this approach, at the beginning of the computer program, sine and cosine functions with a 2π period are computed and stored, using an adequate number of sampling points. In later computations, the required sine and cosine functions are found from the table, thus avoiding the repeated computation of such functions. However, unless the arguments of these sine and cosine functions coincide with the sampling points in the table, errors will be introduced. The amount of error is a function of the number of sampling points in this sine and cosine table. Therefore, one must decide how much error one can tolerate and then one can determine how big a table one must have. In this report we analyze this problem and present the results.

TABLE-LOOK-UP ERROR

For a numerical integration, Eq. (2) can be transformed into the form

$$F(\theta, \Phi) = \sum_n A_n \exp(jG_n), \tag{3}$$

where $A_n = f(\eta_n, \xi_n) \geqslant 0$

$G_n = G(\eta_n, \xi_n, \theta, \Phi)$.

For simplification, the double summation of η and ξ in Eq. (2) is converted into a single summation index. The function $F(\theta, \Phi)$ is a complex function having both real and imaginary components, thus

$$F(\theta, \Phi) = x + jy.$$

When table-look-up is used to determine the phase of the exponential term, errors are introduced, and Eq. (2) becomes

$$F(\theta, \Phi) = \sum_n A_n \exp(jG_n)\exp(j\delta_n), \tag{4}$$

where the δ_n are independent random variables. Therefore, $A_n \exp(jG_n)\exp(j\delta_n)$ are also independent random variables, and the function $F(\theta, \Phi)$ is the sum of many independent random variables. According to the central limit theorem [4], the probability density function of $F(\theta, \Phi)$ is asymptotically normal. Since a normal distribution is completely determined by the first and second moments, our next task is to find the mean and the variance of the random complex function $F(\theta, \phi)$.

It can be shown that the mean of $F(\theta, \phi)$ is

$$\bar{F}(\theta, \phi) = \phi(1) \sum_n A_n \exp(jG_n), \tag{5}$$

where $\phi(1)$ is the characteristic function of the random variable δ_n, which is defined:

$$\phi(k) = \int p(\delta)\exp(jk\delta)\, d\delta, \tag{6}$$

and $p(\delta)$ is the probability density function of the random variable δ. If the entries of the sine and cosine table are sampled uniformly within a 2π range having a total number of sample points K, then

$$\begin{aligned} p(\delta) &= 1/\alpha \qquad n\alpha < \delta \leqslant (n+1)\alpha \\ &= 0 \qquad \text{otherwise,} \end{aligned} \tag{7}$$

and $\alpha = 2\pi/K$.

There are two ways to use this table. The first is to choose the sine and cosine function of the next index if the argument of the phase angle is larger than the angle $n\alpha$. The second method is to choose the function of the angle $n\alpha$ if the argument is less than $(n + \frac{1}{2})\alpha$. For the former case the characteristic function is

$$\phi(k) = \exp\left(-j\frac{1}{2}k\alpha\right)\frac{\sin\left(\frac{1}{2}k\alpha\right)}{\frac{1}{2}k\alpha}, \tag{8}$$

while for the latter case it is

$$\phi(k) = \frac{\sin\frac{1}{2}k\alpha}{\frac{1}{2}k\alpha}. \tag{9}$$

One may see that the difference between Eqs. (8) and (9) involves a constant phase $(1/2)k\alpha$ which in general is not important because the reference phase of the function $F(\theta,\Phi)$ is arbitrary. However, in the actual programming, the first method is somewhat easier. One has

$$\frac{\sin\frac{1}{2}k\alpha}{\frac{1}{2}k\alpha} \approx 1 - \frac{1}{4}\frac{(k\alpha)^2}{6}. \tag{10}$$

Since α is usually very small, the high-order terms are ignored. By use of this result, Eq. (5) becomes

$$\bar{F}(\theta, \Phi) = (1 - 0.042\alpha^2) \sum_n A_n \exp (G_n). \tag{11}$$

In the appendix it is shown that the variances of the real and imaginary components of $F(\theta, \Phi)$ are respectively:

$$\sigma_x^2 = \frac{1}{2}(1 - \phi^2(1))\sum_n A_n^2 + \frac{1}{2}[\phi(2) - \phi^2(1)]\sum_n A_n^2 \cos(2G_n),$$

and

$$\sigma_y^2 = \frac{1}{2}(1 - \phi^2(1))\sum_n A_n^2 - \frac{1}{2}[\phi(2) - \phi^2(1)]\sum_n A_n^2 \cos(2G_n),$$

while the covariance of x and y is

$$\sigma_{xy} = \frac{1}{2}[\phi(2) - \phi^2(1)]\sum_n A_n^2 \sin(2G_n).$$

The joint probability density function of the real and imaginary components x and y is then:

$$p(xy) = \frac{1}{2\pi\sigma_x\sigma_y\sqrt{1-r^2}} \exp -\left\{\frac{1}{2(1-r^2)}\left[\frac{(x-\bar{x})^2}{\sigma_x^2} - \frac{2r(x-\bar{x})(y-\bar{y})}{\sigma_x\sigma_y} + \frac{(y-\bar{y})^2}{\sigma_y^2}\right]\right\}, \tag{12}$$

where $\bar{x}$ and $\bar{y}$ are respectively the real and imaginary components of $\bar{F}(\theta, \Phi)$, and

$$\sigma_{xy} = r\sigma_x\sigma_y . \tag{13}$$

Our goal is to find the required number of samples, or how big a sine and cosine table is required to achieve a certain prescribed accuracy for both real and imaginary components, x and y. The probability density function of Eq. (12) is too complicated to estimate this relation. However, certain approximations can be made.

Since $\sum_n A_n^2 \sin 2G_n$ and $\sum_n A_n^2 \cos 2G_n$ are in general much smaller than $\sum_n A_n^2$ for a first order estimation, one may assume that

$$\sigma^2 = \sigma_x^2 = \sigma_y^2 = \frac{1}{2}(1 - \phi^2(1)) \sum_n A_n^2, \tag{14}$$

and

$$\sigma_{xy} \approx 0.$$

The probability density then becomes

$$P(x, y) = \frac{1}{2\pi\sigma^2} \exp\left\{-\frac{1}{2\sigma^2}\left[(x - \bar{x})^2 + (y - \bar{y})^2\right]\right\}. \tag{15}$$

This equation implies that the real and imaginary components of $F(\theta, \Phi)$ are two independent variables and have the same variance. Hence, from Eq. (11)

$$\bar{x} = (1 - 0.042\alpha^2)\mathrm{Re} \sum_n A_n \exp(G_n), \tag{16}$$

and

$$\bar{y} = (1 - 0.042\alpha^2)\mathrm{Im} \sum_n A_n \exp(G_n). \tag{17}$$

If the table has enough sampling points and α is small, then the $0.042\alpha^2$ term can be ignored. Both $\bar{x}$ and $\bar{y}$ are then equal to the actual value. The probability density function for either x or y can be written separately as

$$P(x) = \frac{1}{\sqrt{2\pi}\sigma} \exp\left[-\frac{(x - \bar{x})^2}{2\sigma^2}\right]; \tag{18}$$

when α is small, the variance can be approximated by

$$\sigma^2 = \frac{\alpha^2}{24} \sum_n A_n^2.$$

Let us assume that A_n is normalized such that

$$\sum_n A_n = 1. \tag{19}$$

This normalization implies that

$$|F(\theta, \phi)| \leqslant 1.$$

According to Cauchy's inequality,

$$\left[\sum_{n=1}^{N} x_n y_n\right]^2 \leqslant \left[\sum_{n=1}^{N} x_n^2\right]\left[\sum_{n=1}^{N} y_n^2\right].$$

Let $y_n = 1$, then

$$\left(\sum_{n=1}^{N} x_n\right)^2 \leqslant N \sum_{n=1}^{N} x_n^2. \tag{21}$$

Therefore,

$$\sigma \geqslant \sqrt{\frac{1}{24N}} \left(\frac{2\pi}{K}\right). \tag{22}$$

Since $|x - \bar{x}| \leqslant 3\sigma$ with a probability higher than 99%, if one desires that x or y values should not deviate from the actual value by an amount ϵ with a probability of at least 99%, then the number of sampling points of the sine and cosine table should be

$$K \geqslant \frac{3.85}{\epsilon\sqrt{N}}, \tag{23}$$

where N is the number of terms summed in Eq. (3). For the example given in the introduction, where $N = 100 \times 100$, if one desires that the error $\epsilon \leqslant 10^{-6}$, then we find $K \geq 38\ 500$. Rather than compute the sine and cosine terms 10^8 times, a table of the size 40 000 will be adequate to yield results accurate to -120 dB relative to the absolute sum of the integrated terms.

NUMERICAL EXAMPLE

A numerical example is presented here. In this example, the antenna pattern of a uniformly excited line source is plotted. Strictly speaking, this is not quite the same as that predicted by Eq. (23) because in the previous analysis, in the interest of general scattering problems, the scattered field is presented as a complex number with real and imaginary components separated. In an antenna pattern, the amplitude plotted is the square root of the sum of the squares of the real and imaginary components. Probability density of such amplitude is known as Rician distribution. However, the illumination function of a line source is symmetrical and the pattern function contains only the real component. Therefore, the results of Eq. (23) can apply to this case. A 30-wavelength line source pattern is plotted. Since this pattern function is a $\sin x/x$ function, this exact pattern function is used in both Fig. 1 and Fig. 2 as reference. Next, this same pattern is computed by numerical integration. A total of 100 points is used in the integration. Figure 1 shows the pattern plotted with a table of 32 entries. According to Eq. (23), the error should be in the order of 0.012. In Fig. 1 the maximum error occurred at an angle of about 80°. The correct pattern has a -36-dB sidelobe level while the approximate pattern has a peak of -31 dB at the same angle point. At -36 dB the radiation amplitude is 0.0158 while at -31 dB it is 0.0282. The deviation of the amplitude is about 0.0124 which is very close to what Eq. (23) predicts. Figure 2 shows the same antenna pattern, except that the number of table entries increases to 128. The maximum error in this figure is about 0.0038 while Eq. (23) predicts 0.003.

CONCLUSION

In this report we have analyzed the error introduced by table-look-up for the numerical integration of an electromagnetic diffraction integral. We have shown that in order to keep the error of both real and imaginary components to be less than ϵ, the required table should have at least K entries uniformly distributed within a 2π range, and $K \geqslant 3.85/\epsilon\sqrt{N}$ where N is the number of terms used for numerical integration.

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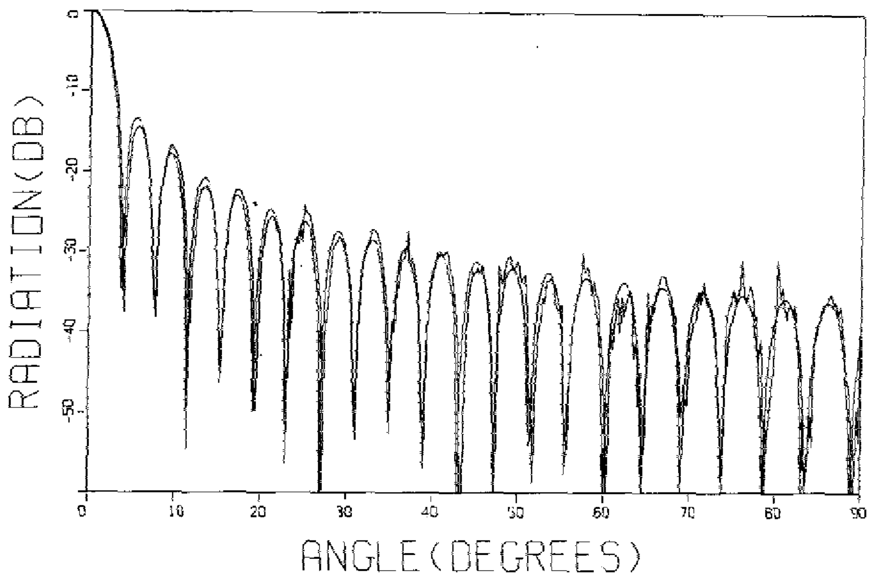

Fig. 1 — Comparison of antenna patterns with an exact solution and the one that uses a table of 32 entries

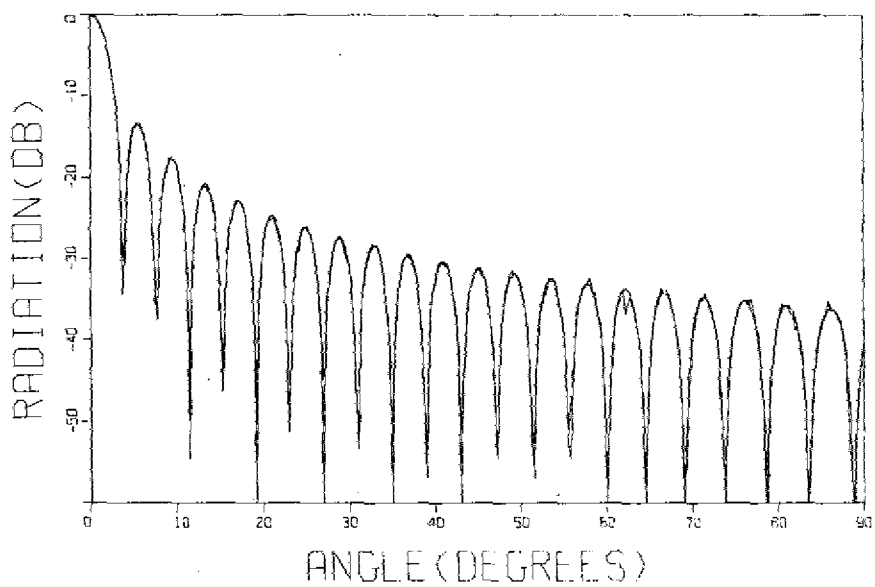

Fig. 2 — Comparison of antenna patterns with an exact solution and the one that uses a table of 128 entries

Appendix
COMPUTATION OF VARIANCE

Let $F(\theta, \Phi) = x + jy$ (A1)

$$E\{|F - \bar{F}|^2\} = \sigma_x^2 + \sigma_y^2 \tag{A2}$$

$$E\{(F - \bar{F})^2\} = \sigma_x^2 - \sigma_y^2 + 2j\sigma_{xy} \tag{A3}$$

$$E\{|F - \bar{F}|^2\} = E\{|F|^2\} - |E(F)|^2 \tag{A4}$$

$$E\{|F|^2\} = \sum_n A_n^2 + \sum_{\substack{n\\n\neq m}}\sum_m A_n A_m \exp[j(G_n - G_m)] \cdot |\phi(1)|^2 \tag{A5}$$

$$|E(F)|^2 = |\phi(1)|^2 \sum_n \sum_m A_n A_m \exp[j(G_n - G_m)] \tag{A6}$$

$$\sigma_x^2 + \sigma_y^2 = (1 - |\phi(1)|^2) \sum_n A_n^2 \tag{A7}$$

$$E\{(F - \bar{F})^2\} = E\{(F)^2\} - \{E(F)\}^2 \tag{A8}$$

$$E\{(F)\}^2 = \sum_n A_n^2 \phi(2) \exp[j2G_n] \tag{A9}$$

$$+ \sum_{\substack{n\\n\neq m}}\sum_m A_n A_m \exp[j(G_n + G_m)]\phi^2(1) \tag{A10}$$

$$\{E(F)\}^2 = \phi^2(1) \sum_n \sum_m A_n A_m \exp[j(G_n + G_m)] \tag{A11}$$

$$\sigma_x^2 - \sigma_y^2 = (\phi(2) - \phi^2(1)) \sum_n A_n^2 \cos(2G_n) \tag{A12}$$

$$2\sigma_{xy} = (\phi(2) - \phi^2(1)) \sum_n A_n \sin(2G_n) \tag{A13}$$